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Research Article

The Point of Coincidence and Common Fixed Point for Three Mappings in Cone Metric Spaces

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The aim of this paper is to present the point of coincidence and common fixed point for three mappings in cone metric spaces over normal cone which satisfy a different contractive condition. Our result generalizes the recent related results proved by Stojan Radenović (2009) and Rangamma and Prudhvi (2012).

1. Introduction and Preliminaries

It is well known that the classical contraction mapping principle of Banach is a fundamental result in fixed point theory. Several authors have obtained various extensions and generalizations of Banach's theorems by considering contractive mappings on different metric spaces. Huang and Zhang [1] have replaced real numbers by ordering Banach space and have defined a cone metric space. They have proved some fixed point theorems of contractive mappings on cone metric spaces. Further generalizations of Huang and Zhang were obtained by Abbas and Jungck [2]. In 2009 Radenović [3] has obtained coincidence point result for two mappings in cone metric spaces which satisfy new contractive conditions. Recently, in this paper we generalized the coincidence point results of Radenović [3] for three maps with different contractive condition.

We recall some definitions and results that will be needed in what follows.

Definition 1. Let E be a real Banach space and P be a subset of E. Then P is called a cone if

(1) P is closed, nonempty and satisfies $P \neq \{0\}$,

(2) $a, b \in R$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$,

(3) $x \in P$ and $-x \in P$ imply $x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ if and only if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ is the interior of P. A cone P is called normal if there is a number $K > 0$ such that, for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P.

In the following we suppose that E is a real Banach space and P is a cone in E with $\operatorname{int} P \neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 3. Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq R^2$, $X = R^2$, and $d : X \times X \rightarrow E$ be defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = [\max(|x_1 - y_1|, |x_2 - y_2|), \alpha \max(|x_1 - y_1|, |x_2 - y_2|)]$, where $\alpha \geq 0$ is a constant; then (X, d) is a cone metric space.

Definition 4. Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ converges to x if for every c lies in E with $0 \ll c$ there is an N such that for all $n > N$, $d(x_n, x) \ll c$. One denotes this by $x_n \to x$ as $n \to \infty$.

Definition 5. Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If for every c lies in E with $0 \ll c$ there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 6. A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 7. *Let (X, d) be a cone metric space and P be a normal cone. Let $\{x_n\}$ be a sequence in X. One has the following.*

(i) *$\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$.*

(ii) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

(iii) *$\{x_n\}$ converges to $x \in X$ and $\{x_n\}$ converges to $y \in X$. Then $x = y$.*

Definition 8. Let f and g be self-maps on set X. If $fx = gx = w$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 9. Two self-mappings f and g of a cone metric space X are said to be weakly compatible if $fgx = gfx$ whether $fx = gx$.

2. Main Result

In this section, we give fixed point theorems for mappings defined on cone metric space with generalized contractive condition.

Theorem 10. *Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Suppose that the mappings f, g, and $h : X \to X$ satisfy the condition*

$$\begin{aligned}\|d(fx, gy)\| \le{} & a\|d(hx, hy)\| + b\|d(fx, hx)\| \\ & + c\|d(gy, hy)\| + \lambda\{\|d(hx, gy)\| \\ & + \|d(fx, hy)\|\}\end{aligned} \tag{1}$$

for all $x, y \in X$, where a, b, c, and λ are nonnegative real numbers satisfying $a + b + c + 2\lambda < 1$. If the range of h contains range of f and also range of g and $h(X)$ is a complete subspace of X, then f, g, and h have a unique point of coincidence in X. Moreover, if (f, h) and (g, h) are weakly compatible, then f, g, and h have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X)$ and $g(X)$ are contained in $h(X)$, there exists $x_1 \in X$ such that $y_0 = fx_0 = hx_1$, and also there exists $x_2 \in X$ such that $y_1 = gx_1 = hx_2$. Continuing this process, a sequence $\{y_n\}$ can be chosen such that $y_{2n} = fx_{2n} = hx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$, for $n = 0, 1, 2, \ldots$; then

$$\begin{aligned}\|d(y_{2n}, y_{2n+1})\| &= \|d(fx_{2n}, gx_{2n+1})\| \\ &\le a\|d(hx_{2n}, hx_{2n+1})\| \\ &\quad + b\|d(fx_{2n}, hx_{2n})\| \\ &\quad + c\|d(gx_{2n+1}, hx_{2n+1})\| \\ &\quad + \lambda\{\|d(hx_{2n}, gx_{2n+1})\| \\ &\qquad + \|d(fx_{2n}, hx_{2n+1})\|\} \\ &= a\|d(y_{2n-1}, y_{2n})\| \\ &\quad + b\|d(y_{2n}, y_{2n-1})\| + c\|d(y_{2n+1}, y_{2n})\| \\ &\quad + \lambda\{\|d(y_{2n-1}, y_{2n+1})\| + \|d(y_{2n}, y_{2n})\|\} \\ &\le a\|d(y_{2n-1}, y_{2n})\| + b\|d(y_{2n}, y_{2n-1})\| \\ &\quad + c\|d(y_{2n+1}, y_{2n})\| + \lambda\{\|d(y_{2n-1}, y_{2n})\| \\ &\qquad + \|d(y_{2n}, y_{2n+1})\|\} \\ &= (a + b + \lambda)\|d(y_{2n-1}, y_{2n})\| \\ &\quad + (c + \lambda)\|d(y_{2n}, y_{2n+1})\|.\end{aligned} \tag{2}$$

This implies that $\|d(y_{2n}, y_{2n+1})\| \le ((a + b + \lambda)/(1 - (c + \lambda)))\|d(y_{2n-1}, y_{2n})\|$.

Thus

$$\|d(y_{2n}, y_{2n+1})\| \le \eta\|d(y_{2n-1}, y_{2n})\|, \tag{3}$$

where $\eta = (a+b+\lambda)/(1-(c+\lambda)) \in [0, 1)$, as $a+b+c+2\lambda < 1$.

Writing $d_n = \|d(y_n, y_{n+1})\|$, we obtain

$$d_{2n} \le \eta d_{2n-1}. \tag{4}$$

Again

$$\begin{aligned}\|d(y_{2n+2}, y_{2n+1})\| &= \|d(fx_{2n+2}, gx_{2n+1})\| \\ &\le a\|d(hx_{2n+2}, hx_{2n+1})\| \\ &\quad + b\|d(fx_{2n+2}, hx_{2n+2})\| \\ &\quad + c\|d(gx_{2n+1}, hx_{2n+1})\| \\ &\quad + \lambda\{\|d(hx_{2n+2}, gx_{2n+1})\| \\ &\qquad + \|d(fx_{2n+2}, hx_{2n+1})\|\}\end{aligned}$$

$$\begin{aligned}
&= a\left\|d\left(y_{2n+1}, y_{2n}\right)\right\| \\
&\quad + b\left\|d\left(y_{2n+2}, y_{2n+1}\right)\right\| \\
&\quad + c\left\|d\left(y_{2n+1}, y_{2n}\right)\right\| \\
&\quad + \lambda\left\{\left\|d\left(y_{2n+1}, y_{2n+1}\right)\right\|\right. \\
&\qquad \left.+\left\|d\left(y_{2n+2}, y_{2n}\right)\right\|\right\} \\
&\leq a\left\|d\left(y_{2n+1}, y_{2n}\right)\right\| \\
&\quad + b\left\|d\left(y_{2n+2}, y_{2n+1}\right)\right\| \\
&\quad + c\left\|d\left(y_{2n+1}, y_{2n}\right)\right\| \\
&\quad + \lambda\left\{\left\|d\left(y_{2n+2}, y_{2n+1}\right)\right\|\right. \\
&\qquad \left.+\left\|d\left(y_{2n+1}, y_{2n}\right)\right\|\right\} \\
&= (a + c + \lambda)\left\|d\left(y_{2n+1}, y_{2n}\right)\right\| \\
&\quad + (b + \lambda)\left\|d\left(y_{2n+2}, y_{2n+1}\right)\right\|.
\end{aligned} \tag{5}$$

This implies that $\|d(y_{2n+2}, y_{2n+1})\| \leq ((a + c + \lambda)/(1 - (b + \lambda)))\|d(y_{2n+1}, y_{2n})\|$.

Thus

$$\left\|d\left(y_{2n+2}, y_{2n+1}\right)\right\| \leq \mu\left\|d\left(y_{2n+1}, y_{2n}\right)\right\|, \tag{6}$$

where $\mu = (a+c+\lambda)/(1-(b+\lambda)\ ) \in [0, 1)$, as $a+b+c+2\lambda < 1$.

Therefore

$$d_{2n+1} \leq \mu\ d_{2n}. \tag{7}$$

From (4) and (7) we get

$$\begin{aligned}
d_{2n} &\leq \eta d_{2n-1} \leq \eta\mu d_{2n-2} \leq \cdots \leq \eta^{n}\mu^{n} d_{0,} \\
d_{2n+1} &\leq \mu d_{2n} \leq \eta\mu d_{2n-1} \leq \cdots \leq \eta^{n}\mu^{n+1} d_{0}.
\end{aligned} \tag{8}$$

Therefore

$$d_{2n} + d_{2n+1} \leq \eta^{n}\mu^{n}\left(1 + \mu\right) d_{0}, \tag{9}$$

$$d_{2n+1} + d_{2n+2} \leq \eta^{n}\mu^{n+1}\left(1 + \eta\right) d_{0}. \tag{10}$$

Now we will show that $\{y_n\}$ is a Cauchy sequence. By triangle inequality for $m > n$, we have

$$\begin{aligned}
d\left(y_{n}, y_{m}\right) &\leq d\left(y_{n}, y_{n+1}\right) + d\left(y_{n+1}, y_{n+2}\right) \\
&\quad + \cdots + d\left(y_{m-1}, y_{m}\right).
\end{aligned} \tag{11}$$

Hence, as P is normal cone with normal constant K,

$$\begin{aligned}
\left\|d\left(y_{n}, y_{m}\right)\right\| &\leq K\left\|d\left(y_{n}, y_{n+1}\right) + d\left(y_{n+1}, y_{n+2}\right)\right. \\
&\quad \left.+ \cdots + d\left(y_{m-1}, y_{m}\right)\right\| \\
&\leq K\left\{\left\|d\left(y_{n}, y_{n+1}\right)\right\| + \left\|d\left(y_{n+1}, y_{n+2}\right)\right\|\right. \\
&\quad \left.+ \cdots + \left\|d\left(y_{m-1}, y_{m}\right)\right\|\right\}.
\end{aligned} \tag{12}$$

If n is even, then from (9) and (12) we have

$$\begin{aligned}
\left\|d\left(y_{n}, y_{m}\right)\right\| &\leq K\left\{d_{n} + d_{n+1} + d_{n+2} + d_{n+3} + \cdots\right\} \\
&\leq K\left\{\eta^{n/2}\mu^{n/2}(1 + \mu) d_{0}\right. \\
&\qquad \left. + \eta^{(n+2)/2}\mu^{(n+2)/2}(1 + \mu) d_{0} + \cdots\right\} \\
&= K\frac{(\eta\mu)^{n/2}(1 + \mu)}{1 - \eta\mu} d_{0}.
\end{aligned} \tag{13}$$

If n is odd, then from (10) and (12) we have

$$\begin{aligned}
\left\|d\left(y_{n}, y_{m}\right)\right\| &\leq K\left\{d_{n} + d_{n+1} + d_{n+2} + d_{n+3} + \cdots\right\} \\
&\leq K\left\{\eta^{(n-1)/2}\mu^{(n-1)/2+1}(1 + \eta) d_{0} + \eta^{(n+1)/2}\right. \\
&\qquad \left. \times\mu^{(n+1)/2+1}(1 + \eta) d_{0} + \cdots\right\} \\
&= K\frac{(\eta\mu)^{(n-1)/2}(1 + \eta)\mu}{1 - \eta\mu} d_{0}.
\end{aligned} \tag{14}$$

Since $\eta < 1$, $\mu < 1$, therefore $\eta\mu < 1$, so in both cases, $\|d(y_n, y_m)\| \to 0$ as $n \to \infty$.

From Lemma 7, it follows that $\{y_n\} = \{hx_{n+1}\}$ is a Cauchy sequence. Since $h(X)$ is a complete subspace of X, there exists q in $h(X)$ such that $\{hx_{n+1}\} \to q$ as $n \to \infty$; consequently we can find p in X such that $hp = q$. We shall show that $hp = fp = gp$.

Now using contractive condition (1), we can write

$$\begin{aligned}
\left\|d\left(fp, gx_{2n+1}\right)\right\| &\leq a\left\|d\left(hp, hx_{2n+1}\right)\right\| \\
&\quad + b\left\|d\left(fp, hp\right)\right\| \\
&\quad + c\left\|d\left(gx_{2n+1}, hx_{2n+1}\right)\right\| \\
&\quad + \lambda\left\{\left\|d\left(hp, gx_{2n+1}\right)\right\|\right. \\
&\qquad \left.+\left\|d\left(fp, hx_{2n+1}\right)\right\|\right\}.
\end{aligned} \tag{15}$$

Taking $n \to \infty$, we have

$$\begin{aligned}
\left\|d\left(fp, q\right)\right\| &\leq a\left\|d\left(hp, q\right)\right\| + b\left\|d\left(fp, hp\right)\right\| \\
&\quad + c\left\|d\left(q, q\right)\right\| + \lambda\left\{\left\|d\left(hp, q\right)\right\| + \left\|d\left(fp, q\right)\right\|\right\} \\
&= (b + \lambda)\left\|d\left(fp, q\right)\right\|, \quad \text{since } hp = q.
\end{aligned} \tag{16}$$

Hence, $fp = q$, since $a + b + c + 2\lambda < 1$ and $a, b, c, \lambda \geq 0$.

Again from (1), we can write

$$\begin{aligned}
\left\|d\left(fx_{2n}, gp\right)\right\| &\leq a\left\|d\left(hx_{2n}, hp\right)\right\| \\
&\quad + b\left\|d\left(fx_{2n}, hx_{2n}\right)\right\| + c\left\|d\left(gp, hp\right)\right\| \\
&\quad + \lambda\left\{\left\|d\left(hx_{2n}, gp\right)\right\| + \left\|d\left(fx_{2n}, hp\right)\right\|\right\}.
\end{aligned} \tag{17}$$

Taking $n \to \infty$, we have

$$\begin{aligned}\|d(q,gp)\| &\le a\|d(q,hp)\| + b\|d(q,q)\| \\ &\quad + c\|d(gp,hp)\| \\ &\quad + \lambda\{\|d(q,gp)\| + \|d(q,hp)\|\} \\ &= (c+\lambda)\|d(gp,q)\|, \quad \text{since } hp = q.\end{aligned} \tag{18}$$

Hence, $gp = q$, since $a+b+c+2\lambda < 1$ and $a,b,c,\lambda \ge 0$.

So we get

$$hp = gp = fp = q. \tag{19}$$

Therefore p is a coincidence point of f, g, and h.

Now we show that f, g, and h have a unique point of coincidence. For this, assume that there exists another point of coincidence r in X such that $fp_1 = gp_1 = hp_1 = r$.

Consider

$$\begin{aligned}\|d(gp,gp_1)\| = \|d(fp,gp_1)\| &\le a\|d(hp,hp_1)\| \\ &\quad + b\|d(fp,hp)\| + c\|d(gp_1,hp_1)\| \\ &\quad + \lambda\{\|d(hp,gp_1)\| + \|d(fp,hp_1)\|\} \\ &= (a+2\lambda)\|d(gp,gp_1)\|.\end{aligned} \tag{20}$$

Since $a+b+c+2\lambda < 1$ and $a,b,c,\lambda \ge 0$, so from (20), $gp = gp_1$.

Therefore, $q = fp = hp = p = gp_1 = fp_1 = hp_1 = r$, and hence f, g, and h have unique point of coincidence in X.

Now from (1) we have

$$\begin{aligned}\|d(ffp,fp)\| &= \|d(ffp,gp)\| \\ &\le a\|d(hfp,hp)\| + b\|d(ffp,hfp)\| \\ &\quad + c\|d(gp,hp)\| \\ &\quad + \lambda\{\|d(hfp,gp)\| + \|d(ffp,hp)\|\}.\end{aligned} \tag{21}$$

As (f,h) is weakly compatible, therefore from (19) and (21) we can write

$$\|d(ffp,fp)\| \le (a+2\lambda)\|d(ffp,fp)\|. \tag{22}$$

As $a+b+c+2\lambda < 1$ and $a,b,c,\lambda \ge 0$, so from (22), $ffp = fp$.

Therefore,

$$fq = q. \tag{23}$$

Also,

$$q = fp = ffp = fhp = hfp = hq. \tag{24}$$

Again from (1) we have

$$\begin{aligned}\|d(gp,ggp)\| &= \|d(fp,ggp)\| \\ &\le a\|d(hp,hgp)\| + b\|d(fp,hp)\| \\ &\quad + c\|d(ggp,hgp)\| \\ &\quad + \lambda\{\|d(hp,ggp)\| + \|d(fp,hgp)\|\}.\end{aligned} \tag{25}$$

As (g,h) is weakly compatible, therefore from (19) and (25) we can write

$$\|d(gp,ggp)\| \le (a+2\lambda)\|d(gp,ggp)\|. \tag{26}$$

As $a+b+c+2\lambda < 1$ and $a,b,c,\lambda \ge 0$, so from (26), $ggp = gp$.

Hence,

$$gq = q. \tag{27}$$

From (23), (24), and (27), it follows that q is common fixed point for f, g, and h.

Now we shall prove the uniqueness of common fixed point for f, g, and h. Suppose r is another common fixed point for f, g, and h.

Consider

$$\begin{aligned}\|d(q,r)\| &\le a\|d(hq,hr)\| + b\|d(fq,hq)\| \\ &\quad + c\|d(gr,hr)\| \\ &\quad + \lambda\{\|d(hq,gr)\| + \|d(fq,hr)\|\} \\ &= (a+2\lambda)\|d(q,r)\|.\end{aligned} \tag{28}$$

Therefore, $q = r$, since $a+b+c+2\lambda < 1$ and $a,b,c,\lambda \ge 0$. Thus f, g, and h have unique common fixed point in X. □

Remark 11. (i) If we take $b = c = \lambda = 0$, $a = k$ in Theorem 10, then

$$\|d(fx,gy)\| \le k\|d(hx,hy)\|, \quad \text{where } k \in [0,1). \tag{29}$$

(ii) If we take $a = \lambda = 0$, $b = c = k$ in Theorem 10, then

$$\|d(fx,gy)\| \le k\{\|d(fx,hx)\| + \|d(gy,hy)\|\}, \quad \text{where } k \in \left[0,\frac{1}{2}\right). \tag{30}$$

(iii) If we take $a = b = c = 0$, $\lambda = k$ in Theorem 10, then

$$\|d(fx,gy)\| \le k\{\|d(hx,gy)\| + \|d(fx,hy)\|\}, \quad \text{where } k \in \left[0,\frac{1}{2}\right). \tag{31}$$

From Remark 11, it is clear that Theorem 2.1 in [4] is a special case of Theorem 10 with $a = k$ and $b = c = \lambda = 0$, where $k \in [0,1)$, and Theorem 2.3 in [4] is a special case of Theorem 10 with $a = \lambda = 0$ and $b = c = k$, where $k \in [0,1/2)$. Therefore, we can say that Theorem 10 has generalized and unified the main results in [4].

In Theorem 10 if we take $g = f$, then as immediate consequence of Theorem 10 we obtain the following corollary.

Corollary 12. *Let (X,d) be a cone metric space and P be a normal cone with normal constant K. Suppose that the mappings $f,h : X \to X$ satisfy the condition*

$$\begin{aligned}\|d(fx,fy)\| &\le a\|d(hx,hy)\| + b\|d(fx,hx)\| \\ &\quad + c\|d(fy,hy)\| \\ &\quad + \lambda\{\|d(hx,fy)\| + \|d(fx,hy)\|\},\end{aligned} \tag{32}$$

for all $x, y \in X$, where a, b, c, and λ are nonnegative real numbers satisfying $a+b+c+2\lambda < 1$. If the range of h contains the range of f and $h(X)$ is a complete subspace of X, then f and h have a unique point of coincidence in X. Moreover, if (f, h) is weakly compatible, then f and h have a unique common fixed point.

Remark 13. (i) If we take $b = c = \lambda = 0$, $a = k$ in Corollary 12, then

$$\|d(fx, fy)\| \leq k \|d(hx, hy)\|, \quad \text{where } k \in [0, 1). \tag{33}$$

(ii) If we take $a = \lambda = 0$, $b = c = k$ in Corollary 12, then

$$\|d(fx, fy)\| \leq k \{\|d(fx, hx)\| + \|d(fy, hy)\|\}, \quad \text{where } k \in \left[0, \frac{1}{2}\right). \tag{34}$$

(iii) If we take $a = b = c = 0$, $\lambda = k$ in Corollary 12, then

$$\|d(fx, fy)\| \leq k \{\|d(hx, fy)\| + \|d(fx, hy)\|\}, \quad \text{where } k \in \left[0, \frac{1}{2}\right). \tag{35}$$

From Remark 13 it is clear that Theorem 2.3 [3] is a special case of Corollary 12. Therefore we can say that Theorem 10 has generalized and unified the main result of Radenović in [3].

We present now some nontrivial examples that illustrate how general and important is the result given by Theorem 10.

Example 14. Let $E = R^2$, with the norm $\|(x, y)\| = |x| + |y|$, be a real Banach space and let $P = \{(x, y) \in E : x, y \geq 0\}$. If we consider $X = \{\alpha, \beta, \gamma, \delta\}$ and define $d : X \times X \to E$ by

$$\begin{gathered} d(\alpha, \beta) = d(\beta, \alpha) = (0.9, 0.9), \\ d(\alpha, \gamma) = d(\gamma, \alpha) = (0.5, 3), \\ d(\alpha, \delta) = d(\delta, \alpha) = (1, 2.2), \\ d(\beta, \gamma) = d(\gamma, \beta) = (0.5, 3), \\ d(\beta, \delta) = d(\delta, \beta) = (1, 2.5), \\ d(\gamma, \delta) = d(\delta, \gamma) = (1, 3), \\ d(\alpha, \alpha) = d(\beta, \beta) = d(\gamma, \gamma) = d(\delta, \delta) = (0, 0), \end{gathered} \tag{36}$$

then (X, d) is a cone metric space. Let f, g, and $h : X \to X$ be defined, respectively, as follows:

$$\begin{array}{llll} f\alpha = \beta, & f\beta = \beta, & f\gamma = \alpha, & f\delta = \beta, \\ g\alpha = \beta, & g\beta = \beta, & g\gamma = \delta, & g\delta = \beta, \\ h\alpha = \delta, & h\beta = \beta, & h\gamma = \gamma, & h\delta = \alpha. \end{array} \tag{37}$$

Then f, g, and h have the properties mentioned in Theorem 10, and also f, g, and h satisfy the inequality (1).

Hence the conditions of Theorem 10 are satisfied. Therefore we conclude that f, g, and h have unique point of coincidence and also unique common fixed point.

Here it is seen that β is unique point of coincidence and also the unique common fixed point of f, g, and h.

Remark 15. Example 14 does not satisfy the conditions (29) and (30) at the points $x = \gamma$, $y = \gamma$ and $x = \beta$, $y = \gamma$, respectively. Therefore, we can say that inequalities of Theorems 2.1 and 2.3 of [4] fail at the points $x = \gamma$, $y = \gamma$ and $x = \beta$, $y = \gamma$, respectively. Hence, Theorem 2.1 and Theorem 2.3 of [4] cannot apply to Example 14.

Example 16. Let $E = R$, with the norm $\|x\| = |x|$, be a real Banach space and let $P = \{x \in E : x \geq 0\}$. Let $X = \{0, 1, 2\}$ and also define $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$.

Then (X, d) is a cone metric space. Let $f, h : X \to X$ be defined, respectively, as follows:

$$fx = \begin{cases} 2 - x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{38}$$

Also

$$hx = x, \quad \text{for } x \in X. \tag{39}$$

Then f and h have the properties mentioned in Corollary 12, and also f and h satisfy the inequality (32).

Hence the conditions of Corollary 12 are satisfied. Therefore we conclude that f and h have unique point of coincidence and also unique common fixed point.

Here it is seen that 0 is unique point of coincidence and also the unique common fixed point of f and h.

Remark 17. Example 16 does not satisfy the conditions ((33), (35)), and (34) at the points $x = 1$, $y = 2$ and $x = 2$, $y = 0$, respectively. Therefore, we can say that inequalities ((2.4), (2.6)) and (2.5) of [3] fail at the points $x = 1$, $y = 2$ and $x = 2$, $y = 0$, respectively. Hence, Theorem 2.3 of [3] cannot apply to Example 16.

Remark 18. Example 14 does not satisfy the inequality 2.8 of [5] at the point $x = \alpha$, $y = \gamma$. Therefore, it is clear that Corollary 2.10 of [5] cannot apply to Example 14. Hence Theorem 10 is more general than Corollary 2.10 of [5].

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