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Research Article

Existence Results for a $p(x)$-Kirchhoff-Type Equation without Ambrosetti-Rabinowitz Condition

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We consider the existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type equations without Ambrosetti-Rabinowitz condition. Using the Mountain Pass Lemma, the Fountain Theorem, and its dual, the existence of solutions and infinitely many solutions were obtained, respectively.

1. Introduction

The Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right| dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

was introduced by Kirchhoff [1] in the study of oscillations of stretched strings and plates, where ρ, ρ_0, h, E, and L are constants. The stationary analogue of the Kirchhoff equation, that is, (1), is as follows:

$$-\left(a + b \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u). \tag{2}$$

After the excellent work of Lions [2], problem (2) has received more attention; see [3–10] and references therein.

The $p(x)$-Laplace operator arises from various phenomena, for instance, the image restoration [11], the electro-rheological fluids [12], and the thermoconvective flows of non-Newtonian fluids [13, 14]. The study of the $p(x)$-Laplace operator is based on the theory of the generalized Lebesgue space $L^{p(x)}(\Omega)$ and the Sobolev space $W^{m,p(x)}(\Omega)$, which we called variable exponent Lebesgue and Sobolev space. We refer the reader to [15–19] for an overview on the variable exponent Sobo-lev space, and to [20–29] for the study of the $p(x)$-Laplacian-type equations.

Recently, there has been an increasing interest in studying the Kirchhoff equation involving the $p(x)$-Laplace operator. Autuori et al. [30, 31] have dealt with the nonstationary Kirchhoff-type equation involving the $p(x)$-Laplacian of the form

$$\begin{aligned} & u_{tt} - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u \\ & \qquad + Q(t, x, u, u_t) + f(x, u) = 0, \\ & u_{tt} - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u \\ & \qquad + \mu |\nabla u|^{p(x)-2} u + Q(t, x, u, u_t) = f(t, x, u). \end{aligned} \tag{3}$$

In [32–35], applying variational method and Ambrosetti-Rabinowitz (AR) condition, Guowei Dai has studied the existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type equations with Dirichlet or Neumann boundary condition. In [36], by using (S_+) mapping theory and the Mountain Pass Lemma, Fan has discussed the nonlocal $p(x)$-Laplacian Dirichlet problem with the nonvariational form

$$\begin{aligned} -A(u) \Delta_{p(x)} u &= B(u) f(x, u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

and the $p(x)$-Kirchhoff-type equation with the variational form

$$-a\left(\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = b\left(\int_{\Omega}F(x,u)\,dx\right)f(x,u), \quad \text{in } \Omega, \qquad u=0, \quad \text{on } \partial\Omega, \tag{5}$$

under (AR) condition, where A, B are two functionals defined on $W_0^{1,p(x)}(\Omega)$, and $F(x,t)=\int_0^t f(x,s)ds$.

Motivated by the above works, the purpose of this paper is to study the $p(x)$-Kirchhoff-type equation

$$-\left(a+b\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = f(x,u), \quad \text{in } \Omega, \qquad u=0, \quad \text{on } \partial\Omega, \tag{6}$$

without (AR) condition, where Ω is a smooth bounded domain in $\mathbb{R}^N$, a, b are two positive constants, $\Delta_{p(x)}u = \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x))$, $p \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$, and

$$1 < p^- := \inf_{\Omega} p(x) \le p^+ := \sup_{\Omega} p(x) < +\infty. \tag{7}$$

By taking the famous Mountain Pass Lemma, the Fountain Theorem, and its dual, we obtain the existence of solutions and infinitely many solutions for the $p(x)$-Kirchhoff-type equation (6) under no (AR) condition.

2. Preliminary

We recall in this section some definitions and properties of variable exponent Lebesgue-Sobolev space. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{u : u:\Omega \to \mathbb{R} \text{ is measurable}, \int_{\Omega}|u|^{p(x)}dx < \infty\right\} \tag{8}$$

with the norm

$$|u|_{L^{p(x)}} = |u|_{p(x)} = \inf\left\{\sigma > 0 : \int_{\Omega}\left|\frac{u}{\sigma}\right|^{p(x)}dx \le 1\right\}. \tag{9}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\} \tag{10}$$

with the norm

$$\|u\|_{W^{1,p(x)}} = \|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}. \tag{11}$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$. $|\nabla u|_{p(x)}$ is an equivalent norm on $W_0^{1,p(x)}(\Omega)$. In this paper we use the notation $\|u\| = |\nabla u|_{p(x)}$ for $u \in W_0^{1,p(x)}(\Omega)$. Define

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases} \tag{12}$$

We refer the reader to [36–38] for the elementary properties of the space $W^{1,p(x)}(\Omega)$.

Proposition 1 (see [38]). *Set $\rho(u) = \int_{\Omega}|u(x)|^{p(x)}dx$. For any $u \in L^{p(x)}(\Omega)$, then the following are given:*

(1) $|u|_{p(x)} = \lambda \Leftrightarrow \rho(u/\lambda) = 1$ *if* $u \neq 0$;

(2) $|u|_{p(x)} < 1 (=1; >1) \Leftrightarrow \rho(u) < 1 (=1; >1)$;

(3) $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$ *if* $|u|_{p(x)} > 1$;

(4) $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$ *if* $|u|_{p(x)} < 1$;

(5) $\lim_{k\to+\infty}|u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k\to+\infty}\rho(u_k) = 0$;

(6) $\lim_{k\to+\infty}|u_k|_{p(x)} = +\infty \Leftrightarrow \lim_{k\to+\infty}\rho(u_k) = +\infty$.

3. Positive Energy Solution

In this section we discuss the existence of weak solutions of (6). For simplicity we write $X = W_0^{1,p(x)}(\Omega)$.

First, we state the assumptions on f as follows.

(f_0) Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and there exist positive constants c_1, c_2 such that

$$|f(x,t)| \le c_1 + c_2|t|^{\alpha(x)-1}, \tag{13}$$

where $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

(f_0') Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and there exist positive constants c_1, c_2 such that

$$|f(x,t)| \le c_1 + c_2|t|^{\alpha(x)-1}, \tag{14}$$

where $\alpha \in C(\overline{\Omega})$ and $p^+ < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$; $tf(x,t) \ge 0$ for all $t > 0$.

(f_1) Let $\lim_{t\to+\infty}(F(x,t)/|t|^{2p^+}) = +\infty$, uniformly for $x \in \overline{\Omega}$, where $F(x,t) = \int_0^t f(x,s)ds$.

(f_2) There exists $\theta \ge 1$ such that $\theta G(x,t) \ge G(x,st)$ for $(x,t) \in \Omega \times \mathbb{R}$ and $s \in [0,1]$, where

$$G(x,t) = tf(x,t) - 2p^+F(x,t). \tag{15}$$

(f_3) Let $\lim_{t\to 0}(F(x,t)/|t|^{p^+}) = 0$, uniformly on $x \in \Omega$.

(f_3') There exists $\delta > 0$, such that $F(x,t) \le 0$ for $x \in \overline{\Omega}$, $|t| < \delta$.

(f_4) Let $f(x,t) = -f(x,-t)$ for $x \in \Omega$ and $t \in \mathbb{R}$.

(f_5) Let $\lim_{t\to 0}(F(x,t)/|t|^{q^+}) = 0$, uniformly on $x \in \overline{\Omega}$, where $q \in C(\overline{\Omega})$ satisfies $1 < q(x) < p(x)$ for $x \in \overline{\Omega}$.

Remark 2. Condition (f_2) was first introduced by Jeanjean [39] for the case $p(x) = 2$. Let $f(x,t) = 2p^+|t|^{2p^+-2}t\ln|t|$, then

$$F(x,t) = |t|^{2p^+}\ln|t| - \frac{1}{2p^+}|t|^{2p^+}, \qquad G(x,t) = |t|^{2p^+}. \tag{16}$$

It is easy to see that the function f does not satisfy (AR) condition, but it satisfies (f_1)–(f_5) and (f_3').

Define $I(u) = J(u) - \Phi(u)$, where

$$J(u) = \left(a + \frac{b}{2}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx,$$
$$\Phi(u) = \int_\Omega F(x,u)\,du. \tag{17}$$

Then $I \in C^1(X,\mathbb{R})$.

Proposition 3 (see [38]). *Assume that (f_0) hold, then the functional $J : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous, $\Phi : X \to \mathbb{R}$ is sequentially weakly continuous, and I is sequentially weakly lower semicontinuous.*

Proposition 4 (see [37]). *Assume that (f_0) hold, and let $u_0 \in W_0^{1,p(x)}(\Omega)$ be a local minimizer (resp., a strictly local minimizer) of I in the $C^1(\overline{\Omega})$ topology. Then u_0 is a local minimizer (resp., a strictly local minimizer) of I in the $W_0^{1,p(x)}(\Omega)$ topology.*

Definition 5. We say that $u \in X$ is a weak solution of (6), if

$$\left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx = \int_\Omega f(x,u)\,v\,dx \tag{18}$$

for any $v \in X$.

Definition 6. Let X be a Banach space and $I \in C^1(X,\mathbb{R})$. Given $c \in \mathbb{R}$. we say that I satisfies the Cerami c condition (we denote by $(C)_c$ the condition), if

(i) any bounded sequence $\{u_n\} \subset X$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ has a convergent subsequence;

(ii) there exist constants $\delta, R, \beta > 0$ such that

$$\|u\|\left\|I'(u)\right\| \geq \beta, \quad \forall u \in I^{-1}[c-\delta, c+\delta], \quad \|u\| \geq R. \tag{19}$$

If $I \in C^1(X,\mathbb{R})$ satisfies $(C)_c$ condition for every $c \in \mathbb{R}$, then we say that I satisfies (C) condition.

Remark 7. Although (PS) condition is stronger than (C) condition, the Deformation Theorem is still valid under (C) condition; we see that the Mountain Pass Lemma, the Fountain Theorem, and its dual are true under (C) condition.

Lemma 8. *Assume that conditions (f_0)–(f_2) hold. Then I satisfies (C) condition.*

Proof. From [36, Proposition 3.1], I satisfies (i) of (C) condition.

Now we check that I satisfies (ii) of (C) condition. Arguing by contradiction, we may assume that, for some $c \in \mathbb{R}$,

$$I(u_n) \longrightarrow c, \qquad \|u_n\| \longrightarrow \infty, \qquad \|u_n\|\left\|I'(u_n)\right\| \longrightarrow 0. \tag{20}$$

Then we have

$$\begin{aligned}\lim_{n\to\infty}\Big\{&a\int_\Omega\left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)|\nabla u|^{p(x)}dx \\ &+ \frac{b}{2}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ &\times \int_\Omega\left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla u|^{p(x)}dx \\ &+ \frac{1}{2p^+}\int_\Omega G(x,u)\,dx\Big\} \\ &= \lim_{n\to\infty}\left\{I(u_n) - \frac{1}{2p^+}\left\langle I'(u_n), u_n\right\rangle\right\} = c.\end{aligned} \tag{21}$$

Let $v_n = u_n/\|u_n\|$, then up to a subsequence we may assume that

$$\begin{gathered}v_n \rightharpoonup v \quad \text{in } X, \\ v_n \longrightarrow v \quad \text{in } L^{\alpha(x)}(\Omega), \\ v_n(x) \longrightarrow v(x) \quad \text{a.e. } x \in \Omega.\end{gathered} \tag{22}$$

If $v = 0$, inspired by [13, 14], then we define

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n). \tag{23}$$

For any $m > 1/2p^+$, let $w_n = (2mp^+)^{1/p^-} v_n$. Since $w_n \to 0$ in $L^{\alpha(x)}(\Omega)$ and

$$|F(x,t)| \leq c_5 + c_6|t|^{\alpha(x)}, \tag{24}$$

by the continuity of $F(x,\cdot)$, $F(\cdot,w_n) \to 0$ in $L^1(\Omega)$, thus,

$$\lim_{n\to 0}\int_\Omega F(\cdot,w_n)\,dx = 0. \tag{25}$$

Then for n large enough, $(2mp^+)^{1/p^-}/\|u_n\| \in (0,1)$ and

$$\begin{aligned}
I(t_n u_n) &\geq I(w_n) \\
&= a\int_\Omega \frac{1}{p(x)}|\nabla w_n|^{p(x)}dx \\
&\quad + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla w_n|^{p(x)}dx\right)^2 - \int_\Omega F(x,w_n)\,dx \\
&= a\int_\Omega \frac{1}{p(x)}\left(\left(2mp^+\right)^{1/p^-}|\nabla v_n|\right)^{p(x)}dx \\
&\quad + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}\left(\left(2mp^+\right)^{1/p^-}|\nabla v_n|\right)^{p(x)}dx\right)^2 \\
&\quad - \int_\Omega F(x,w_n)\,dx \\
&\geq \frac{2ma}{p^+}\int_\Omega |\nabla v_n|^{p(x)}dx \\
&\quad + \frac{2m^2b}{(p^+)^2}\left(\int_\Omega |\nabla v_n|^{p(x)}dx\right)^2 - \int_\Omega F(x,w_n)\,dx \\
&\geq \frac{2ma}{p^+} + \frac{2m^2b}{(p^+)^2} - \int_\Omega F(x,w_n)\,dx.
\end{aligned} \tag{26}$$

That is, $I(t_n u_n) \to \infty$. From $I(0) = 0$ and $I(u_n) \to c$, we know that $t_n \in (0,1)$ and

$$\begin{aligned}
a&\int_\Omega |\nabla t_n u_n|^{p(x)}dx \\
&\quad + b\left(\int_\Omega \frac{1}{p(x)}|\nabla t_n u_n|^{p(x)}dx\right)\int_\Omega |\nabla t_n u_n|^{p(x)}dx \\
&\quad - \int_\Omega f(x,t_n u_n)\,u_n dx \\
&= \left\langle I'(t_n u_n), t_n u_n\right\rangle = t_n \frac{d}{dt}\bigg|_{t=t_n} I(tu_n) = 0.
\end{aligned} \tag{27}$$

Therefore, from (f_2), we have

$$\begin{aligned}
a&\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)|\nabla u_n|^{p(x)}dx \\
&\quad + \frac{b}{2}\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \\
&\quad \times \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla u_n|^{p(x)}dx \\
&\quad + \frac{1}{2p^+}\int_\Omega G(x,u_n)\,dx \\
&\geq a\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)|\nabla u_n|^{p(x)}dx \\
&\quad + \frac{b}{2}\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \\
&\quad \times \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla u_n|^{p(x)}dx \\
&\quad + \frac{1}{2p^+}\int_\Omega \frac{G(x,t_n u_n)}{\theta}dx \\
&\geq \frac{a}{\theta}\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{2p^+}\right)t_n^{p(x)}|\nabla u_n|^{p(x)}dx \\
&\quad + \frac{b}{2\theta}\int_\Omega \frac{1}{p(x)}t_n^{p(x)}|\nabla u_n|^{p(x)}dx \\
&\quad \times \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)t_n^{p(x)}|\nabla u_n|^{p(x)}dx \\
&\quad + \frac{1}{2p^+}\int_\Omega \frac{G(x,t_n u_n)}{\theta}dx \\
&= \frac{1}{\theta}\left(I(t_n u_n) - \frac{1}{2p^+}\left\langle I'(t_n u_n), t_n u_n\right\rangle\right) \\
&= \frac{1}{\theta}I(t_n u_n) \to \infty.
\end{aligned} \tag{28}$$

This contradicts (21).

If $v \neq 0$, from (20), when $\|u_n\| \geq 1$,

$$\frac{a}{p^-}\|u_n\|^{p^+} + \frac{b}{2(p^-)^2}\|u_n\|^{2p^+} - (c + o(1)) \geq \int_\Omega F(x,u_n)\,dx. \tag{29}$$

Then from (f_1) we have

$$\begin{aligned}
&\frac{a}{p^-}\frac{1}{\|u_n\|^{p^+}} + \frac{b}{2(p^-)^2} - \frac{c+o(1)}{\|u_n\|^{2p^+}} \\
&\quad \geq \int_\Omega \frac{F(x,u_n)}{\|u_n\|^{2p^+}}dx \\
&\quad = \left(\int_{v_n \neq 0} + \int_{v_n = 0}\right)\frac{F(x,u_n)}{|u_n|^{2p^+}}|v_n|^{2p^+}dx \\
&\quad = \int_{v_n \neq 0}\frac{F(x,u_n)}{|u_n|^{2p^+}}|v_n|^{2p^+}dx.
\end{aligned} \tag{30}$$

For $x \in \Theta := \{x \in \Omega : v(x) \neq 0\}$, $|u_n(x)| \to +\infty$. By (f_1) we have

$$\frac{F(x,u_n)}{|u_n|^{p^+}}|v_n|^{p^+} \longrightarrow +\infty. \tag{31}$$

Note that the Lebesgue measure of Θ is positive; using the Fatou Lemma, we have

$$\int_{v_n \neq 0} \frac{F(x,u_n)}{|u_n|^{2p^+}} |v_n|^{2p^+} dx \longrightarrow +\infty. \tag{32}$$

This contradicts (30).

The technique used in this lemma was first introduced by [39, 40]. □

Theorem 9. *Assume that conditions (f_0)–(f_2) and (f_3) (or (f_3')) hold. Then (6) has a nontrivial solution with positive energy.*

Proof. From Lemma 8, I satisfies (C) condition. Let us show that the functional I has a Mountain-Pass-type geometry.

Note that $I(0) = 0$. By (f_3), there exists $\delta > 0$, and for any $u \in X$ with $|u|_{L^\infty(\Omega)} < \delta$,

$$\begin{aligned} I(u) &= a \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &\quad + \frac{b}{2} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_\Omega F(x,u)\, dx \\ &\geq \frac{a}{p^+} \|u\|^{p^+} + \frac{b}{(p^+)^2} \|u\|^{2p^+} - \int_\Omega F(x,u)\, dx > 0. \end{aligned} \tag{33}$$

This shows that 0 is a strictly local minimizer of I in the $C(\overline{\Omega})$ topology, and hence 0 is a strictly local minimizer of I in the $C^1(\overline{\Omega})$ topology. By [37, Theorem 1.1], 0 is a strictly local minimizer of I in the $W_0^{1,p(x)}(\Omega)$ topology. Thus there exists $r > 0$ such that $I(u) > 0$ for every $u \in X \setminus \{0\}$ with $\|u\| \leq r$.

We claim that $\inf_{\|u\|=r} I(u) > 0$. To prove this claim, arguing by contradiction, assume that there exists a sequence $\{u_n\} \subset X$ with $\|u_n\| = r$ such that $I(u_n) \to 0$ as $n \to \infty$. We may assume that $u_n \rightharpoonup u_0$ in X. Since I is sequentially weakly lower semicontinuous, we have that $I(u_0) = 0$, and hence $u_0 = 0$. Since Φ is sequentially weakly continuous, then we have that $\Phi(u_n) \to \Phi(0) = 0$, and hence $J(u_n) = I(u_n) + \Phi(u_n) \to 0$. It follows from this that $u_n \to 0$ in X which contradicts with $\|u_n\| = r$.

Let $y \in X$ with $y > 0$ in Ω and $\|y\| = 1$. By (f_0) and (f_1), for $s \geq 1$ we have

$$\begin{aligned} I(sy) &= a \int_\Omega \frac{1}{p(x)} |\nabla sy|^{p(x)} dx \\ &\quad + b \left(\int_\Omega \frac{1}{p(x)} |\nabla sy|^{p(x)} dx \right)^2 - \int_\Omega F(x,sy)\, dx \\ &\leq \frac{a}{p^-} s^{p^+} + \frac{b}{(p^-)^2} s^{2p^+} \\ &\quad - c_1 s^{2p^+} \int_\Omega |y|^{2p^+} dx + c_2 \longrightarrow -\infty \quad \text{as } s \longrightarrow +\infty. \end{aligned} \tag{34}$$

We set $e = sy$. Then for s large, we obtain

$$\|e\| > r, \qquad I(e) < 0. \tag{35}$$

Hence by the famous Mountain Pass Lemma, problem (6) has a nontrivial weak solution with positive energy. □

4. Infinitely Many Solutions

Since X is a reflexive and separable Banach space, then there exists $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$\begin{gathered} X = \overline{\operatorname{span} \{e_j : j = 1,2,\ldots\}}, \\ X^* = \overline{\operatorname{span} \{e_j^* : j = 1,2,\ldots\}}, \\ \langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \end{gathered} \tag{36}$$

For convenience, we write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$.

Lemma 10 (see [21]). *If $\alpha \in C(\overline{\Omega})$, $1 < \alpha(x) < p^*$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k = \sup \{ |u|_{\alpha(x)} : \|u\| = 1, u \in Z_k \}. \tag{37}$$

Then $\lim_{k \to +\infty} \beta_k = 0$.

Proposition 11 (Fountain Theorem). *Assume that $I \in C^1(X, \mathbb{R})$ is an even functional. If, for any $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that*

(A_1) $a_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0$,

(A_2) $b_k = \inf_{u \in Z_k, \|u\| = r_k} I(u) \to +\infty$ *as* $k \to \infty$,

(A_3) *I satisfies $(C)_c$ condition for every $c > 0$, then I has an unbounded sequence of critical values.*

Proposition 12 (Dual Fountain Theorem). *Assume that $I \in C^1(X, \mathbb{R})$ is an even functional. If, for any $k \geq k_0$, there exists $\rho_k > r_k > 0$ such that*

(B_1) $a_k = \inf_{u \in Z_k, \|u\| = \rho_k} I(u) \geq 0$,

(B_2) $b_k = \max_{u \in Y_k, \|u\| = r_k} I(u) < 0$,

(B_3) $d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} I(u) \to 0$ *as* $k \to \infty$,

(B_4) *I satisfies $(c)_c^*$ condition for every $c \in [d_{k_0}, 0]$, then I has a sequence of negative critical values converging to 0.*

Theorem 13. *Assume that the conditions (f_0'), (f_1)–(f_4) hold. Then (6) has infinitely many solutions $\{u_k\}$ such that $I(u_k) \to \infty$ as $k \to \infty$.*

Proof. By conditions (f_0'), (f_1), and (f_3), for any $\varepsilon > 0$, there exists C_ε such that

$$F(x,u) \geq C_\varepsilon |u|^{2p^+} - \varepsilon |u|^{p^+}, \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{38}$$

For $u \in Y_k$, when $\|u\| > 1$,

$$\begin{aligned} I(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ &\quad + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega F(x,u)\,dx \\ &\le \frac{a}{p^-}\|u\|^{p^+} + \frac{b}{2(p^-)^2}\|u\|^{2p^+} \\ &\quad - C_\varepsilon |u|_{2p^+}^{2p^+} + \varepsilon|u|_{p^+}^{p^+} \longrightarrow -\infty \quad \text{as } \|u\| \longrightarrow +\infty. \end{aligned} \tag{39}$$

Then for some $\rho_k > 0$ large enough,

$$a_k := \max_{u\in Y_k, \|u\|=\rho_k} I(u) \le 0. \tag{40}$$

On the other hand, by (f_0') and (f_3), there exists $C_\varepsilon > 0$ such that

$$|F(x,u)| \le \varepsilon|u|^{p^+} + C_\varepsilon|u|^{\alpha(x)}, \quad \forall (x,u) \in \Omega\times\mathbb{R}. \tag{41}$$

Let $\beta_k := \sup_{u\in Z_k,\|u\|=\rho_k}|u|_{\alpha^-}$. From Lemma 10, $\beta_k \to 0$ as $k \to \infty$. For $u \in Z_k$, when $\|u\| \le 1$ and ε small enough,

$$\begin{aligned} I(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ &\quad + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega F(x,u)\,dx \\ &\ge \frac{a}{p^+}\|u\|^{p^-} + \frac{b}{2(p^+)^2}\|u\|^{2p^-} - C_\varepsilon|u|_{\alpha^-}^{\alpha^-} - \varepsilon|u|_{p^+}^{p^+} \\ &\ge \frac{a}{2p^+}\|u\|^{p^-} - c|u|_{\alpha^-}^{\alpha^-} \\ &\ge \frac{a}{2p^+}\|u\|^{p^-} - c\beta_k\|u\|^{\alpha^-}. \end{aligned} \tag{42}$$

If we choose $r_k := (a/4cp^+\beta_k^{\alpha-})^{1/(\alpha^- - p^-)} \to \infty$ as $k \to \infty$, then, for $u \in Z_k$ with $\|u\| = r_k$,

$$I(u) \ge \frac{a}{4p^+}\left(\frac{a}{4cp^+\beta_k^{\alpha^-}}\right)^{p^-/(\alpha^- - p^-)} := \overline{b}_k, \tag{43}$$

which implies that $b_k := \inf_{u\in Z_k,\|u\|=r_k} I(u) \ge \overline{b}_k \to +\infty$ as $k \to +\infty$. □

Theorem 14. *Assume that conditions (f_0'), (f_1), (f_2), (f_4), and (f_5) hold. Then* (6) *has infinitely many solutions $\{u_k\}$ such that $I(u_k) \to 0$ as $k \to \infty$.*

Proof. By conditions (f_0'), (f_1), and (f_5), for any $\varepsilon > 0$, there exists C_ε such that

$$F(x,u) \ge C_\varepsilon|u|^{2p^+} - \varepsilon|u|^{q^+}, \quad \forall (x,u) \in \Omega\times\mathbb{R}. \tag{44}$$

For $u \in Y_k$, when $\|u\|$ is large enough,

$$\begin{aligned} I(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ &\quad + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega F(x,u)\,dx \\ &\le \frac{a}{p^-}\|u\|^{p^+} + \frac{b}{(p^-)^2}\|u\|^{2p^+} \\ &\quad - C_\varepsilon|u|_{2p^+}^{2p^+} + \varepsilon|u|_{q^-}^{q^-} \longrightarrow -\infty \\ &\quad \text{as } \|u\| \longrightarrow +\infty. \end{aligned} \tag{45}$$

Then for some $r_k > 0$ large enough,

$$b_k := \max_{u\in Y_k,\|u\|=r_k} I(u) < 0. \tag{46}$$

On the other hand, by (f_5), there exists $C_\varepsilon > 0$ such that

$$|F(x,u)| \le \varepsilon|u|^{q^-} + C_\varepsilon|u|^{\alpha(x)}, \quad \forall (x,u) \in \Omega\times\mathbb{R}. \tag{47}$$

Let $\beta_k := \sup_{u\in Z_k,\|u\|=r_k}|u|_{q^-}$, then $\beta_k \to 0$ as $k \to \infty$. For $u \in Z_k$, when $\|u\|$ and ε small enough,

$$\begin{aligned} I(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ &\quad + \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega F(x,u)\,dx \\ &\ge \frac{a}{p^+}\|u\|^{p^+} - cC_\varepsilon|u|_{\alpha^-}^{\alpha^-} - c\varepsilon|u|_{q^+}^{q^+} \\ &\ge \frac{a}{2p^+}\|u\|^{p^+} - c|u|_{q^+}^{q^+} \\ &\ge \frac{a}{2p^+}\|u\|^{p^+} - c\beta_k^{q^+}\|u\|^{q^+}. \end{aligned} \tag{48}$$

If we choose $\rho_k := (4cp^+\beta_k^{q^+}/a)^{1/(p^+ - q^+)} \to 0$ as $k \to \infty$, then, for $u \in Z_k$ with $\|u\| = \rho_k$,

$$I(u) \ge c\beta_k^{q^+}\left(\frac{4cp^+\beta_k^{q^+}}{a}\right)^{q^+/(p^+ - q^+)} := \overline{a}_k, \tag{49}$$

which implies that $a_k := \inf_{u\in Z_k,\|u\|=\rho_k} I(u) \ge \overline{a}_k \to 0$ as $k \to +\infty$.

Furthermore, if $u \in Z_k$ with $\|u\| \le \rho_k$, then

$$I(u) \ge -c\beta_k^{q^-}\rho_k^{q^-} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{50}$$

which implies that $d_k = \inf_{u\in Z_k,\|u\|\le\rho_k} I(u) \to 0$ as $k \to \infty$. □

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