## Research Article
# Fuzzy Bases of Fuzzy Domains

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This paper is an attempt to develop quantitative domain theory over frames. Firstly, we propose the notion of a fuzzy basis, and several equivalent characterizations of fuzzy bases are obtained. Furthermore, the concept of a fuzzy algebraic domain is introduced, and a relationship between fuzzy algebraic domains and fuzzy domains is discussed from the viewpoint of fuzzy basis. We finally give an application of fuzzy bases, where the image of a fuzzy domain can be preserved under some special kinds of fuzzy Galois connections.

## 1. Introduction

Since the pioneering work of Scott [1, 2], domain and its generalization have attracted more and more attention. Domain provides models for various types of programming languages that include imperative, functional, nondeterministic, and probabilistic languages. When domains appear in theoretical computer science, one typically wants them to be objects suitable for computation. In particular, one is motivated to find a suitable notion of a recursive or recursively enumerable domain. This leads to the notion of a basis (cf. [3]).

Quantitative domain theory has been developed to supply models for concurrent systems. Now it forms a new focus on domain theory and has undergone active research. Rutten's generalized (ultra)metric spaces [4], Flagg's continuity spaces [5], and Wagner's $\Omega$-categories [6] are good examples, which consist of basic frameworks of quantitative domain theory (cf. [7]).

Recently, based on complete residuated lattices, Yao and Shi [8, 9] investigated quantitative domains via fuzzy set theory. They defined a fuzzy way-below relation via fuzzy ideals to examine the continuity of fuzzy domains and later discussed fuzzy Scott topology over fuzzy dcpos. Zhang and Fan [7] studied quantitative domains over frames. From the very beginning, they defined a fuzzy partial order which is really a degree function on a nonempty set. After that, they defined and studied fuzzy dcpos and fuzzy domains. Roughly

speaking, the definition of a fuzzy directed subset in [7] which is based on a kind of special relations looks relatively complex. Furthermore, from the viewpoint of category, Hofmann and Waszkiewicz [10–12], Lai and Zhang [13, 14], and Stubbe [15, 16] studied quantitative domain theory.

It is well known that the notion of a basis plays an important role in domain theory. The results not only are handy in establishing certain equivalent characterizations for domains but also are critical to study some properties of domains. Then, how can we describe a fuzzy basis in a fuzzy dcpo? And what is the role of it in fuzzy ordered set theory? For this purpose, we are motivated to introduce the notion of a fuzzy basis as a new approach to study fuzzy domains. From the viewpoint of fuzzy basis, we try to build a relationship between fuzzy domains and fuzzy algebraic domains. Moreover, we investigate some applications of fuzzy bases to examine the relationships of the definitions.

The contents of this paper are organized as follows. In Section 2, some preliminary concepts and properties are recalled. In Section 3, the concept of a fuzzy basis is proposed, and an equivalent characterization of fuzzy bases is obtained. Furthermore, the notion of a fuzzy algebraic domain is proposed; it is proved that a fuzzy dcpo is a fuzzy algebraic if and only if it is a fuzzy domain and the fuzzy basis satisfies some special interpolation property. In Section 4, an application of fuzzy bases is given, where we investigate some special kinds of fuzzy Galois connections, under which the

image of a fuzzy domain is also a fuzzy domain. Conclusions are settled in the last section.

## 2. Preliminary

A frame will be used as the structures of truth values in this paper. Throughout this paper, unless otherwise stated, $L$ always denotes a frame. For more properties about frames, we refer to [3, 17, 18].

Let $X$ be a nonempty set, an $L$-subset on $X$ is a mapping from $X$ to $L$, and the family of all $L$-subsets on $X$ will be denoted by $L^X$. We denote the constant $L$-subsets on $X$ taking the values 0 and 1 by $0_X$ and $1_X$, respectively. Let $A, B \in L^X$. The equality of $A$ and $B$ is defined as the usual equality of mappings; that is, $A = B \Leftrightarrow A(x) = B(x)$ for any $x \in X$. The inclusion $A \leq B$ is also defined pointwisely: $A \leq B \Leftrightarrow A(x) \leq B(x)$ for any $x \in X$.

The following definitions and propositions can be found in [7–9, 14, 19–24].

*Definition 1.* A fuzzy poset is a pair $(X, e)$ such that $X$ is a nonempty set, and $e : X \times X \to L$ is a mapping, called a fuzzy order, that satisfies for any $x, y, z \in X$,

(1) $e(x, x) = 1$,

(2) $e(x, y) \wedge e(y, z) \leq e(x, z)$,

(3) $e(x, y) = e(y, x) = 1$ implies $x = y$.

To study fuzzy relational systems, Bělohlávek [19] defined and studied an $L$-order over complete residuated lattices. It is shown in [25] that the previous notion is equivalent to Bělohlávek's one.

*Definition 2.* Let $(X, e)$ be a fuzzy poset. An element $x_0 \in X$ is called a join (or meet) of a fuzzy subset $A$, in symbols $x_0 = \sqcup A$ (or $x_0 = \sqcap A$) if

(1) for any $x \in X$, $A(x) \leq e(x, x_0)$ (or $A(x) \leq e(x_0, x)$),

(2) for any $y \in X$, $\bigwedge_{x \in X}(A(x) \to e(x, y)) \leq e(x_0, y)$ (or $\bigwedge_{x \in X}(A(x) \to e(y, x)) \leq e(y, x_0)$).

It is easy to check if $x_1, x_2$ are two joins (or meets) of $A$, then $x_1 = x_2$. This means if $A \in L^X$ has a join (or meet), then it is unique.

**Proposition 3.** *Let $(X, e)$ be a fuzzy poset. Then*

(1) $x_0 = \sqcup A$ *if and only if for any $y \in X$, $e(x_0, y) = \bigwedge_{z \in X}(A(z) \to e(z, y))$;*

(2) $x_0 = \sqcap A$ *if and only if for any $y \in X$, $e(y, x_0) = \bigwedge_{z \in X}(A(z) \to e(y, z))$.*

*Example 4.* Given a nonempty set $X$, the subsethood degree mapping $\mathrm{sub}(-, -) : L^X \times L^X \to L$ is defined by for each pair $(A, B) \in L^X \times L^X$, $\mathrm{sub}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then $\mathrm{sub}(-, -)$ is an $L$-partial order on $L^X$. Moreover, if $A \leq B$, then $\mathrm{sub}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x)) = 1$.

*Definition 5.* Let $(X, e)$ be a fuzzy poset. $A \in L^X$ is called a fuzzy upper set (*or a fuzzy lower set*) if for any $x, y \in X$, $A(x) \wedge e(x, y) \leq A(y)$ (*or* $A(x) \wedge e(y, x) \leq A(y)$).

*Definition 6.* Let $(X, e)$ be a fuzzy poset. For $x \in X$, $\downarrow x \in L^X$ (or $\uparrow x \in L^X$) is defined as for any $y \in X$, $\downarrow x(y) = e(y, x)$ (or $\uparrow x(y) = e(x, y)$). And $\downarrow D$ is defined by for any $x \in X$, $\downarrow D(x) = \bigvee_{y \in X} D(y) \wedge e(x, y)$.

Note that $x = \sqcup \downarrow x$. When $A = \downarrow x$, by Proposition 3, we have

$$e(x, y) = \bigwedge_{z \in X}(e(z, x) \longrightarrow e(z, y)). \tag{1}$$

*Definition 7.* Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy posets. A mapping $f : (X, e_X) \to (Y, e_Y)$ is called a fuzzy monotone mapping if for any $x, y \in X$, $e_X(x, y) \leq e_Y(f(x), f(y))$.

*Definition 8.* Let $X, Y$ be two nonempty sets. For each mapping $f : X \to Y$, the $L$-forward powerset operator $f_L^\to : L^X \to L^Y$ is defined by

$$\text{for any } y \in Y, \ A \in L^X, \quad f_L^\to(A)(y) = \bigvee_{f(x) = y} A(x). \tag{2}$$

The $L$-backward powerset operator $f_L^\leftarrow : L^Y \to L^X$ is defined by

$$\text{for any } B \in L^Y, \quad f_L^\leftarrow(B) = B \circ f. \tag{3}$$

Furthermore, $f$ can be always lifted as $\widetilde{f}^\to : L^X \to L^Y$, which is defined by

$$\text{for any } y \in Y, \ A \in L^X,$$
$$\widetilde{f}^\to(A)(y) = \bigvee_{x \in X} A(x) \wedge e(y, f(x)). \tag{4}$$

In the literature one can find several different fuzzy versions of directed subsets. We will focus on one of them, which is introduced in [8, 14].

*Definition 9.* Let $(X, e)$ be a fuzzy poset. $D \in L^X$ is called a fuzzy directed subset if

(1) $\bigvee_{x \in X} D(x) = 1$,

(2) for any $a, b \in X$, $D(a) \wedge D(b) \leq \bigvee_{d \in X} D(d) \wedge e(a, d) \wedge e(b, d)$.

A fuzzy ideal is a fuzzy lower directed subset. We denote the set of all fuzzy directed subsets and all fuzzy ideals on $X$ by $\mathscr{D}_L(X)$ and $\mathscr{I}_L(X)$, respectively. A fuzzy poset is called a fuzzy dcpo if every fuzzy directed subset has a join.

*Definition 10.* Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy dcpos. A fuzzy monotone mapping $f : (X, e_X) \to (Y, e_Y)$ is said to be fuzzy Scott continuous if for any $D \in \mathscr{D}_L(X)$, $f(\sqcup D) = \sqcup \widetilde{f}^\to(D)$.

We now introduce one of the most efficient tools in dealing with fuzzy poset, which were extensively studied in [8, 13, 14, 19, 20, 25]. One reason for this great efficiency is that the pairs of mappings of the kind we are about to single out exist in great profusion.

*Definition 11.* Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy posets, $f : (X, e_X) \rightarrow (Y, e_Y)$ and $g : (Y, e_Y) \rightarrow (X, e_X)$ two fuzzy monotone mappings. The pair $(f, g)$ is called a fuzzy Galois connection between $(X, e_X)$ and $(Y, e_Y)$ provided that

$$\text{for any } x \in X, \ y \in Y, \quad e_Y(y, f(x)) = e_X(g(y), x), \quad (5)$$

where $f$ is called the upper adjoint of $g$ and dually $g$ is called the lower adjoint of $f$.

**Proposition 12.** *Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy posets. $(f, g)$ is a fuzzy Galois connection on $(X, e_X)$ and $(Y, e_Y)$ if and only if both $f$ and $g$ are fuzzy monotone mappings, and $(f, g)$ is a crisp Galois connection on $(X, \leq_{e_X})$ and $(Y, \leq_{e_Y})$, where $\leq_{e_X}$ is defined as follows: $e_X(x_1, x_2) = 1 \Leftrightarrow x_1 \leq_{e_X} x_2$.*

The crisp Galois connection is defined as follows: $y \leq f(x) \Leftrightarrow g(y) \leq x$ for any $x \in X, y \in Y$, and its relative properties can be found in [3].

**Proposition 13.** *Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy posets, $f : (X, e_X) \rightarrow (Y, e_Y)$ and $g : (Y, e_Y) \rightarrow (X, e_X)$ two mappings.*

(1) *If $f$ is a fuzzy monotone mapping and has a lower adjoint, then for any $S \in L^X$ such that $\sqcap S$ exists, $f(\sqcap S) = \sqcap f_L^{\rightarrow}(S)$.*

(2) *If $g$ is a fuzzy monotone mapping and has an upper adjoint, then for any $D \in L^Y$ such that $\sqcup D$ exists, $g(\sqcup D) = \sqcup g_L^{\rightarrow}(D)$.*

The fuzzy visions of way-below relations were extensively studied in [7, 8, 10–14]. Hofmann and Waszkiewicz [11] presented a systematic investigation of such relation in quantale-enriched categories.

*Definition 14.* Let $(X, e)$ be a fuzzy dcpo. For any $x, y \in X$, define $\Downarrow x \in L^X$ by

$$\Downarrow x(y) = \bigwedge_{I \in \mathscr{I}_L(X)} (e(x, \sqcup I) \longrightarrow I(y)). \quad (6)$$

It is a fact that in the crisp setting, the way-below relation can be defined by ideals and directed subsets, respectively. And in this case, the two way-below relations are equivalent. Then, does the equivalence of such relations also hold? Here we present a proof to confirm it.

**Lemma 15.** *Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy posets, $f : (X, e_X) \rightarrow (Y, e_Y)$ a fuzzy monotone mapping. Then for any $D \in \mathscr{D}_L(X)$, $\widetilde{f}^{\rightarrow}(D) \in \mathscr{I}_L(Y)$.*

**Proposition 16.** *Let $(X, e)$ be a fuzzy dcpo. Then for any $x, y \in X$,*

$$\bigwedge_{D \in \mathscr{D}_L(X)} \left( e(x, \sqcup D) \longrightarrow \left( \bigvee_{d \in X} D(d) \wedge e(y, d) \right) \right)$$

$$= \bigwedge_{I \in \mathscr{I}_L(X)} (e(x, \sqcup I) \longrightarrow I(y)). \quad (7)$$

*That is, $\Downarrow x(y) = \bigwedge_{D \in \mathscr{D}_L(X)}(e(x, \sqcup D) \rightarrow (\bigvee_{d \in X} D(d) \wedge e(y, d)))$.*

*Proof.* Obviously, for any $I \in \mathscr{I}_L(X)$, $I \in \mathscr{D}_L(X)$. On the one hand,

$$\bigwedge_{D \in \mathscr{D}_L(X)} \left( e(x, \sqcup D) \longrightarrow \left( \bigvee_{d \in X} D(d) \wedge e(y, d) \right) \right)$$

$$\leq \bigwedge_{I \in \mathscr{I}_L(X)} \left( e(x, \sqcup I) \longrightarrow \left( \bigvee_{d \in X} I(d) \wedge e(y, d) \right) \right)$$

$$\leq \bigwedge_{I \in \mathscr{I}_L(X)} \left( e(x, \sqcup I) \longrightarrow \bigvee_{d \in X} I(y) \right)$$

$$= \bigwedge_{I \in \mathscr{I}_L(X)} (e(x, \sqcup I) \longrightarrow I(y)). \quad (8)$$

On the other hand, for any $D \in \mathscr{D}_L(X)$, it is routine to check that $\downarrow D \in \mathscr{I}_L(X)$. Then

$$\bigwedge_{I \in \mathscr{I}_L(X)} (e(x, \sqcup I) \longrightarrow I(y))$$

$$\leq \bigwedge_{D \in \mathscr{D}_L(X)} (e(x, \sqcup \downarrow D) \longrightarrow \downarrow D(y))$$

$$= \bigwedge_{D \in \mathscr{D}_L(X)} \left( e(x, \sqcup D) \longrightarrow \left( \bigvee_{d \in X} D(d) \wedge e(y, d) \right) \right). \quad (9)$$

$\square$

By Proposition 16, for all statements, it is valid for the fuzzy way-below relation over fuzzy directed subsets if and only if it holds for the one over fuzzy ideals.

Some basic properties of the fuzzy relation are listed in the following proposition.

**Proposition 17.** *Let $(X, e)$ be a fuzzy dcpo. For any $x, y, u, v \in X$, then*

(1) $\Downarrow x \leq \downarrow x$,

(2) $e(u, x) \wedge \Downarrow y(x) \wedge e(y, v) \leq \Downarrow v(u)$.

*Definition 18.* A fuzzy dcpo $(X, e)$ is called a fuzzy domain or continuous if for any $x \in X$, $\Downarrow x \in \mathscr{D}_L(X)$ (or $\Downarrow x \in \mathscr{I}_L(X)$) and $x = \sqcup \Downarrow x$.

The following theorem exhibits an important property of the fuzzy way-below relation on fuzzy domains, the interpolation property. It has been widely discussed in [7, 8, 11, 13].

**Theorem 19.** *If $(X, e)$ is a fuzzy domain, then for any $x, y \in X$, $\Downarrow y(x) = \bigvee_{z \in X} \Downarrow y(z) \wedge \Downarrow z(x)$.*

## 3. Fuzzy Bases and Fuzzy Algebraic Domains

In this section, we define a fuzzy basis in a fuzzy dcpo, and we obtain some equivalent characterizations of fuzzy bases. Moreover, we also study fuzzy algebraic domains from the viewpoint of fuzzy basis.

*Definition 20.* Let $(X, e)$ be a fuzzy dcpo. $B \in L^X$ is called a fuzzy basis of $X$ if

(1) for any $x \in X$, $B \wedge \Downarrow x$ is a fuzzy directed subset of $X$, and

(2) for any $x \in X$, $x = \sqcup(B \wedge \Downarrow x)$.

Obviously, the previous definition is really a generation of the notion of a basis in [3].

**Proposition 21.** *Let $(X, e)$ be a fuzzy dcpo. For any $x \in X$, if there exists a fuzzy directed subset $A$ such that $x = \sqcup A$ and $A \leq \Downarrow x$, then $\Downarrow x$ is a fuzzy directed subset with $x = \sqcup \Downarrow x$.*

*Proof.* For any $y \in X$, we firstly show that $\Downarrow x(y) \leq \bigvee_{d \in X} A(d) \wedge e(y, d)$. Indeed,

$$
\begin{aligned}
\Downarrow x(y) &= \bigwedge_{D \in \mathscr{D}_L(X)} \left( e(x, \sqcup D) \longrightarrow \left( \bigvee_{d \in X} D(d) \wedge e(y, d) \right) \right) \\
&\leq e(x, \sqcup A) \longrightarrow \left( \bigvee_{d \in X} A(d) \wedge e(y, d) \right) \\
&= 1 \longrightarrow \left( \bigvee_{d \in X} A(d) \wedge e(y, d) \right) \\
&= \bigvee_{d \in X} A(d) \wedge e(y, d).
\end{aligned}
\tag{10}
$$

Then for any $a, b \in X$, we have

$$
\begin{aligned}
\Downarrow x(a) &\wedge \Downarrow x(b) \\
&\leq \bigvee_{d_1, d_2 \in X} A(d_1) \wedge A(d_2) \wedge e(a, d_1) \wedge e(b, d_2) \\
&\leq \bigvee_{d_1, d_2 \in X} \bigvee_{d \in X} A(d) \wedge e(d_1, d) \wedge e(d_2, d) \\
&\qquad \wedge e(a, d_1) \wedge e(b, d_2) \\
&\leq \bigvee_{d_1, d_2 \in X} \bigvee_{d \in X} A(d) \wedge e(a, d) \wedge e(b, d)
\end{aligned}
$$

$$
\begin{aligned}
&= \bigvee_{d \in X} A(d) \wedge e(a, d) \wedge e(b, d) \\
&\leq \bigvee_{d \in X} \Downarrow x(d) \wedge e(a, d) \wedge e(b, d).
\end{aligned}
\tag{11}
$$

Moreover, $\bigvee_{y \in X} \Downarrow x(y) = 1$ follows from $1 = \bigvee_{y \in X} A(y) \leq \bigvee_{y \in X} \Downarrow x(y)$. Hence $\Downarrow x$ is fuzzy directed.

It is easy to verify that $\sqcup$ is fuzzy monotone. Since $A \leq \Downarrow x$, then $1 = \mathrm{sub}(A, \Downarrow x) \leq e(\sqcup A, \sqcup \Downarrow x) = e(x, \sqcup \Downarrow x)$. Meanwhile, $1 = \mathrm{sub}(\Downarrow x, \downarrow x) \leq e(\sqcup \Downarrow x, \sqcup \downarrow x) = e(\sqcup \Downarrow x, x)$. Therefore, $x = \sqcup \Downarrow x$. $\qquad\square$

**Theorem 22.** *A fuzzy dcpo has a fuzzy basis if and only if it is a fuzzy domain.*

*Proof. Necessity.* Suppose that $B$ is a fuzzy basis of $X$, then for any $x \in X$, $B \wedge \Downarrow x \in \mathscr{D}_L(X)$ with $x = \sqcup(B \wedge \Downarrow x)$. It is clear that $B \wedge \Downarrow x \leq \Downarrow x$. Thus $\Downarrow x \in \mathscr{D}_L(X)$ with $x = \sqcup \Downarrow x$ follows from Proposition 21. Therefore, $(X, e)$ is a fuzzy domain.

*Sufficiency.* It is easy to check that $1_X$ is a fuzzy basis of $X$. $\qquad\square$

**Proposition 23.** *Let $(X, e)$ be a fuzzy domain. For any $z \in X$, if $z = \sqcup D$ for some $D \in \mathscr{D}_L(X)$, then $\Downarrow z = \bigvee_{d \in X} D(d) \wedge \Downarrow d$.*

*Proof.* For any $x \in X$, denote that $A \in L^X$ as $A(x) = \bigvee_{d \in X} D(d) \wedge \Downarrow d(x)$.

(a) $A$ is a fuzzy directed subset as follows:

$$
\begin{aligned}
\bigvee_{x \in X} A(x) &= \bigvee_{x \in X} \bigvee_{d \in X} D(d) \wedge \Downarrow d(x) \\
&= \bigvee_{d \in X} D(d) \wedge \left( \bigvee_{x \in X} \Downarrow d(x) \right) = 1.
\end{aligned}
\tag{12}
$$

Furthermore, for any $a, b \in X$,

$$
\begin{aligned}
A(a) &\wedge A(b) \\
&= \bigvee_{d_1, d_2 \in X} D(d_1) \wedge \Downarrow d_1(a) \wedge D(d_2) \wedge \Downarrow d_2(b) \\
&\leq \bigvee_{d_1, d_2 \in X} \bigvee_{d \in X} D(d) \wedge e(d_1, d) \wedge e(d_2, d) \\
&\qquad \wedge \Downarrow d_1(a) \wedge \Downarrow d_2(b) \\
&\leq \bigvee_{d_1, d_2 \in X} \bigvee_{d \in X} D(d) \wedge \Downarrow d(a) \wedge \Downarrow d(b) \\
&= \bigvee_{d \in X} D(d) \wedge \Downarrow d(a) \wedge \Downarrow d(b) \\
&\leq \bigvee_{d \in X} D(d) \wedge \left( \bigvee_{c \in X} \Downarrow d(c) \wedge e(a, c) \wedge e(b, c) \right)
\end{aligned}
$$

$$= \bigvee_{c \in X} e(a,c) \wedge e(b,c) \wedge \left( \bigvee_{d \in X} D(d) \wedge \Downarrow d(c) \right)$$

$$= \bigvee_{c \in X} A(c) \wedge e(a,c) \wedge e(b,c). \tag{13}$$

(b) $\sqcup A = z$. For any $y \in X$, we have

$$\bigwedge_{x \in X} (A(x) \longrightarrow e(x,y))$$

$$= \bigwedge_{x \in X} \left( \left( \bigvee_{d \in X} D(d) \wedge \Downarrow d(x) \right) \longrightarrow e(x,y) \right)$$

$$= \bigwedge_{x \in X} \bigwedge_{d \in X} (D(d) \longrightarrow (\Downarrow d(x) \longrightarrow e(x,y)))$$

$$= \bigwedge_{d \in X} \left( D(d) \longrightarrow \bigwedge_{x \in X} (\Downarrow d(x) \longrightarrow e(x,y)) \right) \tag{14}$$

$$= \bigwedge_{d \in X} (D(d) \longrightarrow e(\sqcup \Downarrow d, y))$$

$$= \bigwedge_{d \in X} (D(d) \longrightarrow e(d,y))$$

$$= e(\sqcup D, y).$$

Hence $\sqcup D = \sqcup A = z$.

(c) $\Downarrow z = \bigvee_{d \in X} D(d) \wedge \Downarrow d$. On one hand, for any $x \in X$,

$$\bigvee_{d \in X} D(d) \wedge \Downarrow d(x) \leq \bigvee_{d \in X} e(d, \sqcup D) \wedge \Downarrow d(x)$$

$$\leq \bigvee_{d \in X} \Downarrow z(x) = \Downarrow z(x). \tag{15}$$

On the other hand,

$$\Downarrow z(x) = \bigwedge_{S \in \mathscr{D}_L(X)} \left( e(z, \sqcup S) \longrightarrow \left( \bigvee_{b \in X} S(b) \wedge e(x,b) \right) \right)$$

$$\leq e\left( z, \sqcup \left( \bigvee_{d \in X} D(d) \wedge \Downarrow d \right) \right)$$

$$\longrightarrow \left( \bigvee_{b \in X} \bigvee_{d \in X} D(d) \wedge \Downarrow d(b) \wedge e(x,b) \right)$$

$$\leq e(z, \sqcup A) \longrightarrow \left( \bigvee_{b \in X} \bigvee_{d \in X} D(d) \wedge \Downarrow d(x) \right)$$

$$= \bigvee_{d \in X} D(d) \wedge \Downarrow d(x). \tag{16}$$

Therefore, $\Downarrow z = \bigvee_{d \in X} D(d) \wedge \Downarrow d$. $\qquad \square$

**Theorem 24.** *Let $(X,e)$ be a fuzzy domain. For $B \in L^X$, the following are equivalent:*

(1) *$B$ is a fuzzy basis of $X$;*

(2) *for any $x \in X$, there exists a fuzzy directed subset $D \leq B \wedge \Downarrow x$ such that $x = \sqcup D$;*

(3) *for any $x, y \in X$, $\Downarrow y(x) = \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \Downarrow b(x)$;*

(4) *for any $x, y \in X$, $\Downarrow y(x) = \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \downarrow b(x)$.*

*Proof.* (1) implies (2). It is evident from Definition 20.

(2) implies (3). Since $(X,e)$ is a fuzzy domain, by Proposition 23, for any $z \in X$, if $z = \sqcup D$ for some $D \in \mathscr{D}_L(X)$, then $\Downarrow z = \bigvee_{b \in X} D(b) \wedge \Downarrow b$. Indeed, for any $x, y \in X$,

$$\Downarrow y(x) = \bigvee_{z \in X} \Downarrow y(z) \wedge \Downarrow z(x)$$

$$= \bigvee_{z \in X} \Downarrow y(z) \wedge \left( \bigvee_{b \in X} D(b) \wedge \Downarrow b(x) \right)$$

$$\leq \bigvee_{z \in X} \Downarrow y(z) \wedge \left( \bigvee_{b \in X} (B \wedge \Downarrow z)(b) \wedge \Downarrow b(x) \right) \tag{17}$$

$$= \bigvee_{z \in X} \bigvee_{b \in X} B(b) \wedge \Downarrow y(z) \wedge \Downarrow z(b) \wedge \Downarrow b(x)$$

$$\leq \bigvee_{z \in X} \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \Downarrow b(x)$$

$$= \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \Downarrow b(x).$$

(3) implies (4). For any $x, y \in X$, we have

$$\Downarrow y(x) = \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \Downarrow b(x)$$

$$\leq \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \downarrow b(x)$$

$$\leq \bigvee_{b \in X} B(b) \wedge \Downarrow y(x) \tag{18}$$

$$\leq \Downarrow y(x).$$

Hence $\Downarrow y(x) = \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \downarrow b(x)$.

(4) implies (1). Assuming (4), we next show that $B$ is a fuzzy basis of $X$. In fact, for any $y \in X$,

(a) $y = \sqcup(B \wedge \Downarrow y)$. Since $y = \sqcup \Downarrow y$, then for any $u \in X$,

$$
\begin{aligned}
e(y,u) \\
&= \bigwedge_{x \in X} (\Downarrow y(x) \longrightarrow e(x,u)) \\
&= \bigwedge_{x \in X} \left( \left( \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \downarrow b(x) \right) \longrightarrow e(x,u) \right) \\
&= \bigwedge_{x \in X} \bigwedge_{b \in X} ((B(b) \wedge \Downarrow y(b) \wedge \downarrow b(x)) \longrightarrow e(x,u)) \\
&= \bigwedge_{x \in X} \bigwedge_{b \in X} ((B(b) \wedge \Downarrow y(b)) \longrightarrow (e(x,b) \longrightarrow e(x,u))) \\
&= \bigwedge_{b \in X} \left( (B(b) \wedge \Downarrow y(b)) \longrightarrow \bigwedge_{x \in X} (e(x,b) \longrightarrow e(x,u)) \right) \\
&= \bigwedge_{b \in X} ((B \wedge \Downarrow y)(b) \longrightarrow e(b,u)) \\
&= e(\sqcup(B \wedge \Downarrow y),u).
\end{aligned}
$$
(19)

Hence $y = \sqcup(B \wedge \Downarrow y)$.

(b) $B \wedge \Downarrow y$ is a fuzzy directed subset. Firstly, for any $a, b \in X$,

$$
\begin{aligned}
(B \wedge \Downarrow y)(a) &\wedge (B \wedge \Downarrow y)(b) \\
&\leq \Downarrow y(a) \wedge \Downarrow y(b) \\
&\leq \bigvee_{c \in X} \Downarrow y(c) \wedge e(a,c) \wedge e(b,c) \\
&= \bigvee_{c \in X} \bigvee_{d \in X} B(d) \wedge \Downarrow y(d) \wedge \downarrow d(c) \wedge e(a,c) \wedge e(b,c) \\
&\leq \bigvee_{c \in X} \bigvee_{d \in X} B(d) \wedge \Downarrow y(d) \wedge e(a,d) \wedge e(b,d) \\
&= \bigvee_{d \in X} (B \wedge \Downarrow y)(d) \wedge e(a,d) \wedge e(b,d).
\end{aligned}
$$
(20)

Moreover, note that $\bigvee_{x \in X} \Downarrow y(x) = 1$ and $\Downarrow y(x) = \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \downarrow b(x)$. Then $\bigvee_{x \in X} \Downarrow y(x) = \bigvee_{b \in X} B(b) \wedge \Downarrow y(b) \wedge \bigvee_{x \in X} \downarrow b(x)$. Hence $\bigvee_{b \in X} (B \wedge \Downarrow y)(b) = 1$. By Definition 20, (1) holds. $\square$

**Proposition 25.** *If $B$ is a fuzzy basis of $X$, then so is $\downarrow B$.*

*Proof.* In fact, for any $x \in X$,

(a) $x = \sqcup(\downarrow B \wedge \Downarrow x)$. It is clear that $B \wedge \Downarrow x \leq \downarrow B \wedge \Downarrow x$. Then

$$
\begin{aligned}
1 &= \mathrm{sub}(B \wedge \Downarrow x, \downarrow B \wedge \Downarrow x) \\
&\leq e(\sqcup(B \wedge \Downarrow x), \sqcup(\downarrow B \wedge \Downarrow x)) \\
&= e(x, \sqcup(\downarrow B \wedge \Downarrow x)).
\end{aligned}
$$
(21)

Note that $\downarrow B \wedge \Downarrow x \leq \Downarrow x \leq \downarrow x$. Then

$$
\begin{aligned}
1 &= \mathrm{sub}(\downarrow B \wedge \Downarrow x, \downarrow x) \\
&\leq e(\sqcup(\downarrow B \wedge \Downarrow x), \sqcup \downarrow x) \\
&= e(\sqcup(\downarrow B \wedge \Downarrow x), x).
\end{aligned}
$$
(22)

Hence $x = \sqcup(\downarrow B \wedge \Downarrow x)$.

(b) $\downarrow B \wedge \Downarrow x$ is a fuzzy directed subset. Since $B$ is a fuzzy basis of $X$, by Theorems 24, for any $c \in X$, we have $\Downarrow x(c) = \bigvee_{d \in X} B(d) \wedge \Downarrow x(d) \wedge \downarrow d(c)$. Then for any $a, b \in X$,

$$
\begin{aligned}
(\downarrow B \wedge \Downarrow x)(a) &\wedge (\downarrow B \wedge \Downarrow x)(b) \\
&\leq \Downarrow x(a) \wedge \Downarrow x(b) \\
&\leq \bigvee_{c \in X} \Downarrow x(c) \wedge e(a,c) \wedge e(b,c) \\
&= \bigvee_{c \in X} \bigvee_{d \in X} B(d) \wedge \Downarrow x(d) \wedge \downarrow d(c) \wedge e(a,c) \wedge e(b,c) \\
&\leq \bigvee_{c \in X} \bigvee_{d \in X} B(d) \wedge \Downarrow x(d) \wedge e(a,d) \wedge e(b,d) \\
&= \bigvee_{d \in X} (B \wedge \Downarrow x)(d) \wedge e(a,d) \wedge e(b,d) \\
&\leq \bigvee_{d \in X} (\downarrow B \wedge \Downarrow x)(d) \wedge e(a,d) \wedge e(b,d).
\end{aligned}
$$
(23)

Furthermore, $\bigvee_{d \in X} (\downarrow B \wedge \Downarrow x)(d) = 1$ follows from $1 = \bigvee_{d \in X} (B \wedge \Downarrow x)(d) \leq \bigvee_{d \in X} (\downarrow B \wedge \Downarrow x)(d)$. Therefore, $\downarrow B$ is a fuzzy basis of $X$. $\square$

Since for any $D \in L^X$, $\downarrow D$ is a fuzzy lower set. Then we can deduce the following.

**Corollary 26.** *If $X$ has a fuzzy basis, then there exists a fuzzy lower one.*

Although the definition of fuzzy algebraic domain was introduced by compact elements in [8], we next introduce the notion of a fuzzy algebraic domain and discuss the relationships between fuzzy algebraic domains and fuzzy domains from the viewpoint of fuzzy basis.

*Definition 27.* A fuzzy dcpo $(X, e)$ is called a fuzzy algebraic domain if

(1) for any $x \in X$, $K \wedge \downarrow x$ is a fuzzy directed subset of $X$, and

(2) for any $x \in X$, $x = \sqcup(K \wedge \downarrow x)$,

where $K \in L^X$ is defined as follows: for any $y \in X$, $K(y) = \Downarrow y(y)$. If no confusion arises, $K$ always denotes the previous definition in the sequel.

**Theorem 28.** *Let $(X, e)$ be a fuzzy dcpo. $(X, e)$ is a fuzzy algebraic domain if and only if $K$ is precisely a fuzzy basis of $X$.*

*Proof.* By Definitions 20 and 27, it suffices to show that for any $x \in X$, $K \wedge \Downarrow x = K \wedge \downarrow x$.

It is clear that $K \wedge \Downarrow x \leq K \wedge \downarrow x$. Conversely, for any $y \in X$, obviously, $(K \wedge \downarrow x)(y) \leq K(y)$. Meanwhile, $(K \wedge \downarrow x)(y) = \Downarrow y(y) \wedge \downarrow x(y) \leq \Downarrow x(y)$. Thus $(K \wedge \downarrow x)(y) \leq K(y) \wedge \Downarrow x(y) = (K \wedge \Downarrow x)(y)$. By the arbitrariness of $y$, $K \wedge \downarrow x \leq K \wedge \Downarrow x$. Therefore, $K \wedge \Downarrow x = K \wedge \downarrow x$. $\qquad\square$

**Theorem 29.** *Let $(X, e)$ be a fuzzy dcpo. Then $(X, e)$ is a fuzzy algebraic domain if and only if*

(1) *$(X, e)$ is a fuzzy domain, and*

(2) *for any $x, y \in X$, $\Downarrow y(x) = \bigvee_{z \in X} K(z) \wedge \downarrow y(z) \wedge \downarrow z(x)$.*

*Proof. Necessity.* Suppose that $(X, e)$ is a fuzzy algebraic domain, by Theorem 28, and $K$ is a fuzzy basis of $X$. Thus $(X, e)$ is a fuzzy domain follows from Theorem 22. It remains to show that for any $x, y \in X$, $\Downarrow y(x) = \bigvee_{z \in X} K(z) \wedge \downarrow y(z) \wedge \downarrow z(x)$. On one hand,

$$\bigvee_{z \in X} K(z) \wedge \downarrow y(z) \wedge \downarrow z(x)$$
$$= \bigvee_{z \in X} \Downarrow z(z) \wedge \downarrow y(z) \wedge \downarrow z(x)$$
$$\leq \bigvee_{z \in X} \Downarrow y(z) \wedge \downarrow z(x) \tag{24}$$
$$\leq \Downarrow y(x).$$

On the other hand,

$$\Downarrow y(x)$$
$$= \bigwedge_{D \in \mathscr{D}_L(X)} \left( e(y, \sqcup D) \longrightarrow \left( \bigvee_{z \in X} D(z) \wedge e(x, z) \right) \right)$$
$$\leq e(y, \sqcup(K \wedge \downarrow y)) \longrightarrow \left( \bigvee_{z \in X} (K \wedge \downarrow y)(z) \wedge e(x, z) \right)$$
$$= \bigvee_{z \in X} K(z) \wedge \downarrow y(z) \wedge \downarrow z(x). \tag{25}$$

Therefore, $\Downarrow y(x) = \bigvee_{z \in X} K(z) \wedge \downarrow y(z) \wedge \downarrow z(x)$.

*Sufficiency.* In fact, for any $y \in X$,
(a) $y = \sqcup(K \wedge \downarrow y)$. Since $y = \sqcup \Downarrow y$, then for any $u \in X$,

$$e(y, u)$$
$$= \bigwedge_{x \in X} (\Downarrow y(x) \longrightarrow e(x, u))$$
$$= \bigwedge_{x \in X} \left( \left( \bigvee_{z \in X} K(z) \wedge \downarrow y(z) \wedge \downarrow z(x) \right) \longrightarrow e(x, u) \right)$$
$$= \bigwedge_{x \in X} \bigwedge_{z \in X} ((K(z) \wedge \downarrow y(z)) \longrightarrow (e(x, z) \longrightarrow e(x, u)))$$
$$= \bigwedge_{z \in X} \left( (K(z) \wedge \downarrow y(z)) \longrightarrow \bigwedge_{x \in X} (e(x, z) \longrightarrow e(x, u)) \right)$$
$$= \bigwedge_{z \in X} ((K \wedge \downarrow y)(z) \longrightarrow e(z, u))$$
$$= e(\sqcup(K \wedge \downarrow y), u). \tag{26}$$

Hence $y = \sqcup(K \wedge \downarrow y)$.

(b) $K \wedge \downarrow y$ is fuzzy directed. For any $x \in X$, $(K \wedge \downarrow y)(x) \leq \Downarrow y(x)$ and $\Downarrow y(x) = \bigvee_{d \in X} K(d) \wedge \downarrow y(d) \wedge \downarrow d(x)$. Then for any $a, b \in X$,

$$(K \wedge \downarrow y)(a) \wedge (K \wedge \downarrow y)(b)$$
$$\leq \Downarrow y(a) \wedge \Downarrow y(b)$$
$$\leq \bigvee_{c \in X} \Downarrow y(c) \wedge e(a, c) \wedge e(b, c)$$
$$= \bigvee_{c \in X} \bigvee_{d \in X} K(d) \wedge \downarrow y(d) \wedge \downarrow d(c) \wedge e(a, c) \wedge e(b, c)$$
$$\leq \bigvee_{c \in X} \bigvee_{d \in X} K(d) \wedge \downarrow y(d) \wedge e(a, d) \wedge e(b, d)$$
$$= \bigvee_{d \in X} (K \wedge \downarrow y)(d) \wedge e(a, d) \wedge e(b, d). \tag{27}$$

Furthermore, since $\bigvee_{x \in X} \Downarrow y(x) = 1$ and $\Downarrow y(x) = \bigvee_{d \in X} K(d) \wedge \downarrow y(d) \wedge \downarrow d(x)$, then

$$\bigvee_{x \in X} \Downarrow y(x) = \bigvee_{d \in X} K(d) \wedge \downarrow y(d) \wedge \bigvee_{x \in X} \downarrow d(x). \tag{28}$$

Thus $\bigvee_{d \in X} (K \wedge \downarrow y)(d) = 1$.

Therefore, $(X, e)$ is a fuzzy algebraic domain. $\qquad\square$

*Remark 30.* The main results of Theorems 24, 28, and 29 indicate that the definitions of the fuzzy basis and the fuzzy algebraic domain are reasonable.

## 4. An Application of Fuzzy Bases

This section is mainly devoted to giving an application of fuzzy bases. Our aim is to investigate some special kinds of

fuzzy Galois connections, under which the image of a fuzzy domain can be preserved.

*Definition 31.* Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy dcpos. A fuzzy monotone mapping $f : (X, e_X) \rightarrow (Y, e_Y)$ is said to preserve fuzzy way-below relation if for any $x, y \in X$, $\Downarrow x(y) \leq \Downarrow f(x)(f(y))$.

**Proposition 32.** *Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy dcpos, $(f, g)$ a fuzzy Galois connection from $(X, e_X)$ to $(Y, e_Y)$.*

(1) *If $f$ is fuzzy Scott continuous, then $g$ preserves fuzzy way-below relation.*

(2) *If $f$ is surjective, then for any $x, y \in Y$, $\Downarrow g(x)(g(y)) \leq \Downarrow x(y)$.*

*Proof.* (1) For any $D \in \mathscr{D}_L(X)$, by Lemma 15, $\widetilde{f}^{\rightarrow}(D) \in \mathscr{I}_L(Y)$. Then for any $x, y \in Y$,

$$
\Downarrow x(y)
$$

$$
= \bigwedge_{S \in \mathscr{D}_L(Y)} \left( e_Y(x, \sqcup S) \rightarrow \left( \bigvee_{d \in Y} S(d) \wedge e_Y(y, d) \right) \right)
$$

$$
\leq \bigwedge_{D \in \mathscr{D}_L(X)} \left( e_Y\left(x, \sqcup \widetilde{f}^{\rightarrow}(D)\right) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{d \in Y} \widetilde{f}^{\rightarrow}(D)(d) \wedge e_Y(y, d) \right) \right)
$$

$$
= \bigwedge_{D \in \mathscr{D}_L(X)} \left( e_Y(x, f(\sqcup D)) \rightarrow \widetilde{f}^{\rightarrow}(D)(y) \right)
$$

$$
\tag{29}
$$

$$
= \bigwedge_{D \in \mathscr{D}_L(X)} \left( e_X(g(x), \sqcup D) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{b \in X} D(b) \wedge e_Y(y, f(b)) \right) \right)
$$

$$
= \bigwedge_{D \in \mathscr{D}_L(X)} \left( e_X(g(x), \sqcup D) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{b \in X} D(b) \wedge e_X(g(y), b) \right) \right)
$$

$$
= \Downarrow g(x)(g(y)).
$$

(2) Since $f$ is surjective, $fg = id_Y$. Note that for any $S \in \mathscr{D}_L(Y)$, $\widetilde{g}^{\rightarrow}(S) \in \mathscr{I}_L(X)$, and $g$ is a lower adjoint of $f$. Then for any $x, y \in Y$, by Proposition 13, we have

$$
\Downarrow g(x)(g(y))
$$

$$
= \bigwedge_{D \in \mathscr{D}_L(X)} \left( e_X(g(x), \sqcup D) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{b \in X} D(b) \wedge e_X(g(y), b) \right) \right)
$$

$$
\leq \bigwedge_{S \in \mathscr{D}_L(Y)} \left( e_X\left(g(x), \sqcup \widetilde{g}^{\rightarrow}(S)\right) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{b \in X} \widetilde{g}^{\rightarrow}(S)(b) \wedge e_X(g(y), b) \right) \right)
$$

$$
= \bigwedge_{S \in \mathscr{D}_L(Y)} \left( e_X(g(x), g(\sqcup S)) \rightarrow \widetilde{g}^{\rightarrow}(S)(g(y)) \right)
$$

$$
= \bigwedge_{S \in \mathscr{D}_L(Y)} \left( e_Y(x, fg(\sqcup S)) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{d \in Y} S(d) \wedge e_X(g(y), g(d)) \right) \right)
$$

$$
= \bigwedge_{S \in \mathscr{D}_L(Y)} \left( e_Y(x, \sqcup S) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{d \in Y} S(d) \wedge e_X(g(y), g(d)) \right) \right)
$$

$$
\leq \bigwedge_{S \in \mathscr{D}_L(Y)} \left( e_Y(x, \sqcup S) \right.
$$

$$
\left. \rightarrow \left( \bigvee_{d \in Y} S(d) \wedge e_Y(fg(y), fg(d)) \right) \right)
$$

$$
= \bigwedge_{S \in \mathscr{D}_L(Y)} \left( e_Y(x, \sqcup S) \rightarrow \left( \bigvee_{d \in Y} S(d) \wedge e_Y(y, d) \right) \right)
$$

$$
= \Downarrow x(y).
$$

$$
\tag{30}
$$

$\square$

**Proposition 33.** *Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy dcpos, $(f, g)$ a fuzzy Galois connection from $(X, e_X)$ to $(Y, e_Y)$. If $(Y, e_Y)$ is a fuzzy domain and $g$ preserves fuzzy way-below relation, then $f$ is fuzzy Scott continuous.*

*Proof.* For any $D \in \mathscr{D}_L(X)$, we need to show that $f(\sqcup D) = \sqcup \widetilde{f}^{\rightarrow}(D)$. Indeed,

$$
e_Y\left(\sqcup \widetilde{f}^{\rightarrow}(D), f(\sqcup D)\right)
$$

$$
= \bigwedge_{y \in Y}\left(\widetilde{f}^{\rightarrow}(D)(y) \longrightarrow e_Y(y, f(\sqcup D))\right)
$$

$$
= \bigwedge_{y \in Y}\left(\left(\bigvee_{x \in X} D(x) \wedge e_Y(y, f(x))\right) \longrightarrow e_Y(y, f(\sqcup D))\right)
$$

$$
= \bigwedge_{y \in Y}\bigwedge_{x \in X}\left(D(x) \longrightarrow\left(e_Y(y, f(x)) \longrightarrow e_Y(y, f(\sqcup D))\right)\right)
$$

$$
= \bigwedge_{x \in X}\left(D(x) \longrightarrow \bigwedge_{y \in Y}\left(e_Y(y, f(x)) \longrightarrow e_Y(y, f(\sqcup D))\right)\right)
$$

$$
= \bigwedge_{x \in X}\left(D(x) \longrightarrow e_Y(f(x), f(\sqcup D))\right)
$$

$$
\geq \bigwedge_{x \in X}\left(D(x) \longrightarrow e_X(x, \sqcup D)\right)
$$

$$
\geq \bigwedge_{x \in X}\left(D(x) \longrightarrow D(x)\right) = 1.
$$

$$(31)$$

For the converse, let $u = \sqcup D$. Since $(Y, e_Y)$ is a fuzzy domain, $f(\sqcup D) = f(u) = \sqcup \Downarrow f(u)$. Note that $e_X(gf(u), u) = 1$. Then for any $v \in Y$,

$$
\Downarrow f(u)(v) \leq \Downarrow gf(u)(g(v))
$$

$$
= \Downarrow gf(u)(g(v)) \wedge e_X(gf(u), u)
$$

$$
\leq \bigwedge_{S \in \mathscr{D}_L(X)}\left(e_X(u, \sqcup S) \longrightarrow\left(\bigvee_{d \in X} S(d) \wedge e_X(g(v), d)\right)\right)
$$

$$
\leq e_X(u, \sqcup D) \longrightarrow\left(\bigvee_{d \in X} D(d) \wedge e_X(g(v), d)\right)
$$

$$
= 1 \longrightarrow\left(\bigvee_{d \in X} D(d) \wedge e_X(g(v), d)\right)
$$

$$
= \bigvee_{d \in X} D(d) \wedge e_X(g(v), d)
$$

$$
= \bigvee_{d \in X} D(d) \wedge e_Y(v, f(d))
$$

$$
= \widetilde{f}^{\rightarrow}(D)(v).
$$

$$(32)$$

Thus $1 = \mathrm{sub}(\Downarrow f(u), \widetilde{f}^{\rightarrow}(D)) \leq e_Y(\sqcup \Downarrow f(u), \sqcup \widetilde{f}^{\rightarrow}(D)) = e_Y(f(\sqcup D), \sqcup \widetilde{f}^{\rightarrow}(D))$. Therefore, $f(\sqcup D) = \sqcup \widetilde{f}^{\rightarrow}(D)$. □

*Definition 34.* Let $(X, e_X)$ and $(Y, e_Y)$ be two fuzzy dcpos. $f : (X, e_X) \rightarrow (Y, e_Y)$ is called a fuzzy morphism if $f$ is a fuzzy Scott continuous upper adjoint.

**Theorem 35.** *Let $(X, e_X)$ be a fuzzy domain, and let $(Y, e_Y)$ be a fuzzy dcpo. If $f : (X, e_X) \rightarrow (Y, e_Y)$ is a surjective fuzzy morphism, then $(Y, e_Y)$ is a fuzzy domain.*

*Proof.* By Theorem 22 and Corollary 26, there exists a fuzzy lower basis $B$ of $X$. Now we show that $\widetilde{f}^{\rightarrow}(B)$ is a fuzzy basis of $Y$.

Since $f$ is a surjective morphism, then for any $y \in Y$, there exists a lower adjoint $g$ of $f$ and an $x$ in $X$ such that $y = f(x)$. For any $u \in Y$, by Proposition 32 (2), we have

$$
\widetilde{f}^{\rightarrow}(\Downarrow gf(x))(u)
$$

$$
= \bigvee_{v \in X} \Downarrow gf(x)(v) \wedge e_Y(u, f(v))
$$

$$
= \bigvee_{v \in X} \Downarrow gf(x)(v) \wedge e_X(g(u), v) \qquad (33)
$$

$$
\leq \bigvee_{v \in X} \Downarrow gf(x)(g(u))
$$

$$
\leq \Downarrow f(x)(u).
$$

Obviously, $\widetilde{f}^{\rightarrow}(B \wedge \Downarrow gf(x)) \leq \widetilde{f}^{\rightarrow}(B)$.
Hence

$$
\widetilde{f}^{\rightarrow}(B \wedge \Downarrow gf(x)) \leq \widetilde{f}^{\rightarrow}(B) \wedge \widetilde{f}^{\rightarrow}(\Downarrow gf(x))
$$

$$
\leq \widetilde{f}^{\rightarrow}(B) \wedge \Downarrow f(x). \qquad (34)
$$

Furthermore, since $B$ is a fuzzy lower set, then by Proposition 32 (1),

$$
\left(\widetilde{f}^{\rightarrow}(B) \wedge \Downarrow f(x)\right)(u)
$$

$$
= \left(\bigvee_{v \in X} B(v) \wedge e_Y(u, f(v))\right) \wedge \Downarrow f(x)(u)
$$

$$
\leq \left(\bigvee_{v \in X} B(v) \wedge e_X(g(u), v)\right) \wedge \Downarrow gf(x)(g(u))
$$

$$
= B(g(u)) \wedge \Downarrow gf(x)(g(u)) \qquad (35)
$$

$$
\leq \bigvee_{v \in X} B(v) \wedge \Downarrow gf(x)(v) \wedge e_X(g(u), v)
$$

$$
= \bigvee_{v \in X} B(v) \wedge \Downarrow gf(x)(v) \wedge e_Y(u, f(v))
$$

$$
= \widetilde{f}^{\rightarrow}(B \wedge \Downarrow gf(x))(u).
$$

Therefore, $\widetilde{f}^{\rightarrow}(B \wedge \Downarrow gf(x)) = \widetilde{f}^{\rightarrow}(B) \wedge \Downarrow f(x)$. Note that $f$ is fuzzy Scott continuous and $B \wedge \Downarrow gf(x) \in \mathscr{D}_L(X)$, then

$$
\begin{aligned}
\sqcup \left( \widetilde{f}^{\rightarrow}(B) \wedge \Downarrow y \right) &= \sqcup \left( \widetilde{f}^{\rightarrow}(B) \wedge \Downarrow f(x) \right) \\
&= \sqcup \widetilde{f}^{\rightarrow}(B \wedge \Downarrow gf(x)) \\
&= f(\sqcup(B \wedge \Downarrow gf(x))) \qquad (36) \\
&= f(gf(x)) \\
&= f(x) = y.
\end{aligned}
$$

It follows from Lemma 15 that $\widetilde{f}^{\rightarrow}(B) \wedge \Downarrow y = \widetilde{f}^{\rightarrow}(B) \wedge \Downarrow f(x) = \widetilde{f}^{\rightarrow}(B \wedge \Downarrow gf(x)) \in \mathscr{D}_L(Y)$. Thus $\widetilde{f}^{\rightarrow}(B)$ is a fuzzy basis of $Y$. Therefore, by Theorem 22, $(Y, e_Y)$ is a fuzzy domain. □

## 5. Conclusion

In this paper, we propose the notion of a fuzzy basis in a fuzzy dcpo, which generalizes the concept of an ordinary basis. It provides a new approach to explore fuzzy domains. We can extend this approach further; for example, we can define a fuzzy complete basis on a fuzzy complete lattice [24] to investigate fuzzy completely distributive lattices introduced in [8, 13]. Moreover, in crisp setting, the definition of a wight is in close touch with the notion of a basis, and fuzzy Scott topology on fuzzy directed complete posets was given in [9]. As a followup of this paper, we can further give a fuzzy vision of a weight on fuzzy Scott topology and study its relative properties.

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