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Research Article

Superconvergence Analysis of a Multiscale Finite Element Method for Elliptic Problems with Rapidly Oscillating Coefficients

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A new multiscale finite element method is presented for solving the elliptic equations with rapidly oscillating coefficients. The proposed method is based on asymptotic analysis and careful numerical treatments for the boundary corrector terms by virtue of the recovery technique. Under the assumption that the oscillating coefficient is periodic, some superconvergence results are derived, which seem to be never discovered in the previous literature. Finally, some numerical experiments are carried out to demonstrate the efficiency and accuracy of this method, and it is seen that they agree very well with the analytical result.

1. Introduction

In this paper, we consider the following elliptic boundary value problem with rapidly oscillatory coefficients:

$$
\begin{gathered}
L_\varepsilon u^\varepsilon \equiv \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x_j}\right) = f(x), \quad \text{in } \Omega, \\
u^\varepsilon = g(x), \quad \text{on } \partial\Omega,
\end{gathered} \tag{1}
$$

where $\Omega \subset \mathfrak{R}^2$ is a smooth-bounded domain, $a_{ij}(\xi) : \mathfrak{R}^2 \to \mathfrak{R}$ is symmetric and satisfies

$$
\begin{aligned}
&(1)\ \lambda|\xi|^2 \le a_{ij}(\xi)\,\xi_i\xi_j \le \lambda^{-1}|\xi|^2, \quad \forall\xi \in \mathfrak{R}^2,\ \exists\lambda \in (0,1], \\
&(2)\ a_{ij}(\xi + \xi') = a_{ij}(\xi), \quad \forall\xi \in \mathfrak{R}^2,\ \exists\xi' \in Z^2, \\
&\qquad\qquad\qquad\qquad\qquad 1 \le i,\ j \le n, \\
&(3)\ \left\|a_{ij}\right\|_{H^1(\mathfrak{R}^2)} \le C, \quad \exists C > 0,
\end{aligned} \tag{2}
$$

where $\xi = x/\varepsilon$, ε is a small scale parameter. This kind of equation has widely been applied in many areas, such as the behavior of flow in porous media or the thermal and mechanical behavior of composite material structure. In practice, the oscillatory coefficients may span many scales to a great extent. In such cases, the direct accurate numerical computation of the solution becomes difficult because it would require a very fine mesh, and it can easily exceed the limit of today's computer resources because of the requirement of tremendous amount of computer memory and CPU time. Meanwhile, it is desirable to have a numerical method that can solve this equation on a large-scale mesh with capturing the effect of small scales details. Thus, various methods of upscaling or homogenization have been developed.

Based on the homogenization method, there are many discussions [1–4] about the numerical methods of (1). A large amount of examples and applications can also be found in the classical books [5–8], where the formal asymptotic expansions for the limit solution are deduced when ε is small enough. In these books, the first-order approximation of these expansions is justified by proving sharp error estimates, from which a general method that allowed us to treat some structures with rapidly oscillatory coefficients is also developed. However, the general method cannot effectively compute the boundary corrector on boundary layer. It should be noted that the boundary corrector is the important source of error estimates. In [9], He and Cui present a novel finite element method to solve (1) which can effectively compute the boundary corrector even if the boundary layer is very small. The crucial idea is to combine the numerical approximation of the first-order terms of asymptotic expansions with the numerical approximation of the boundary corrector from different meshes exploiting the need for different levels of resolution. The following result (Theorem 2.13 in [9]) can be obtained.

Lemma 1. *Assume that u^ε is the solution of* (1) *and $\widehat{u}^{h_0,h_1,h}$ is the finite element solution [9]. For all $p, 1 < p < +\infty$, there exists a constant C such that*

$$\begin{aligned}&\left\| \frac{\nabla\left(u^\varepsilon - \widehat{u}^{h_0,h_1,h}\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}} \right\|_{L^2(\Omega)} \\ &\quad \leq C\left[\left(h_1+h_0+h+\varepsilon\right)|\ln\varepsilon|^{1/2}+\varepsilon^{(2p-1)/2p}\right] \\ &\quad\quad \times\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^3(\Omega)}\right),\end{aligned} \tag{3}$$

where u^0 is the homogenization solution of (1)*, and* $\operatorname{dist}(x,\partial\Omega)$ *is the distance between the point x and the boundary $\partial\Omega$.*

Unfortunately, the needed CPU time of the method presented in [9] is $O(\varepsilon^{1-n}h^{-n})$. In this paper, a high-effective finite element method to compute boundary corrector by virtue of the recovery technique is proposed, and some superconvergence results for the multiscale finite element approximation of (1) are obtained. The rest of this paper is organized as follows. In the next section, we present a multiscale finite element method to compute $u^\varepsilon(x)$. Its convergence analysis are shown in Section 3. Finally, some numerical results conforming our analytical estimates are given in Section 4.

Notation. Before closing this section, we would like to fix some notations. First, the Einstein summation is used. Let $Q = \{\xi \mid 0 < \xi_i < 1,\ i = 1,2\}$, and the capital letter C (with or without subscripts) denotes a positive constant, which is independent of the small parameter ε and the mesh size h (with or without subscripts).

2. An Improved Multiscale Finite Element Method

Firstly, let us simply recall the homogenization method described in [5].

2.1. Homogenization Method. Let $N_k(\xi)\,(k = 1,2)$ be a 1-periodic function, which satisfies

$$\begin{gathered}\frac{\partial}{\partial\xi_i}\left(a_{ij}(\xi)\frac{\partial N_k}{\partial\xi_j}\right) = -\frac{\partial}{\partial\xi_i}a_{ik}, \quad \text{in } \mathfrak{R}^2, \\ \int_Q N_k(\xi)\,d\xi = 0.\end{gathered} \tag{4}$$

Then, the matrix $\widehat{a} = (\widehat{a}_{ij})_{2\times 2}$ can be obtained by

$$\widehat{a}_{ij} = \int_Q\left(a_{ij} + a_{ik}\frac{\partial N_j}{\partial\xi_k}\right)d\xi. \tag{5}$$

The first-order approximation of $u^\varepsilon(x)$ can be written as

$$\widetilde{u}(x) = u^0(x) + \varepsilon N_k(\xi)\frac{\partial u^0(x)}{\partial x_k}, \tag{6}$$

where $u^0(x)$ satisfies the homogenization problem

$$\begin{gathered}L_0u^0(x) := \frac{\partial}{\partial x_i}\left(\widehat{a}_{ij}\frac{\partial u^0}{\partial x_j}\right) = f(x), \quad \text{in } \Omega, \\ u^0(x) = g(x), \quad \text{on } \partial\Omega.\end{gathered} \tag{7}$$

The boundary corrector term of the homogenization method θ_ε is defined by

$$\begin{gathered}L_\varepsilon\theta_\varepsilon = 0, \quad \text{in } \Omega, \\ \theta_\varepsilon = -\varepsilon N_k\frac{\partial u^0}{\partial x_k}, \quad \text{on } \partial\Omega.\end{gathered} \tag{8}$$

In the next two subsections, we will compute numerically the first-order approximation $\widetilde{u}$ and the boundary corrector term θ_ε, respectively, and furthermore give the multiscale finite element solution of (1).

2.2. Finite Element Approximation of $\widetilde{u}$. Let $\mathcal{T}_{h_0}$ be a quasi-uniform triangular partition of Q with the mesh size h_0. S^{h_0} denotes the conforming P_1 finite element spaces with respect to $\mathcal{T}_{h_0}$, and $S_0^{h_0} = S^{h_0} \cap H_0^1(Q)$. The finite element scheme of (4) is to find $N_k^{h_0} \in S^{h_0}$ such that

$$\begin{gathered}\int_Q a_{ij}(\xi)\frac{\partial N_k^{h_0}(\xi)}{\partial\xi_j}\frac{\partial v(\xi)}{\partial\xi_i}d\xi \\ = -\int_Q a_{ik}(\xi)\frac{\partial v(\xi)}{\partial\xi_i}d\xi, \quad \forall v \in S_0^{h_0}(Q), \\ N_k^{h_0}(\xi) \text{ is a 1-periodic function.}\end{gathered} \tag{9}$$

Then, the numerical approximation $\widehat{a}_{ij}^{h_0}$ of $\widehat{a}_{ij}$ can be calculated by

$$\widehat{a}_{ij}^{h_0} = \int_Q\left(a_{ij}(\xi) + a_{ik}(\xi)\frac{\partial N_j^{h_0}(\xi)}{\partial\xi_k}\right)d\xi. \tag{10}$$

Let $\mathcal{T}_{h_1}$ be a quasiuniform triangular partition of Ω with the mesh size h_1 and satisfy

$$\min_{e_k\in\mathcal{T}_{h_1}} S_{e_k} \geq Ch_1^2, \tag{11}$$

where S_{e_k} is the area of the triangular element e_k. $S_g^{h_1}$ denotes the corresponding conforming P_1 finite element spaces, and $S_0^{h_1} = S^{h_1} \cap H_0^1(Q)$.

The finite element approximation $u_0^{h_0,h_1}$ of the homogenization problem (7) is to find $u_0^{h_0,h_1} \in S_g^{h_1}$ such that

$$\int_\Omega \widehat{a}_{ij}^{h_0} \frac{\partial u_0^{h_0,h_1}}{\partial x_i} \frac{\partial v_h}{\partial x_j} dx = \int_\Omega f(x) v_h(x)\, dx, \quad \forall v_h \in S_0^{h_1}. \tag{12}$$

Furthermore, we turn to the computation of $\partial u_0^{h_0,h_1}(x)/\partial x_k$ and $\widetilde{u}(x)$. Let Σ^{h_1} be the set of all nodal points of the mesh $\mathcal{T}_{h_1}$. Define $u_k^{h_0,h_1}$ $(k = 1, 2)$ by the following:

(A_1) for all $x \in \Sigma^{h_1}$, the value of $u_k^{h_0,h_1}$ at the nodal point x is the average of $\partial u_0^{h_0,h_1}(x)/\partial x_k$ in all elements including x,

(A_2) $u_k^{h_0,h_1}(x)$ is a piecewise linear function in every element.

Therefore, we have a numerical approximation $\widetilde{u}^{h_0,h_1}(x)$ of $\widetilde{u}(x)$ which is defined by

$$\widetilde{u}^{h_0,h_1}(x) = u_0^{h_0,h_1}(x) + \varepsilon N_k^{h_0}(\xi)\, u_k^{h_0,h_1}(x). \tag{13}$$

2.3. Finite Element Approximation of $\theta_\varepsilon(x)$. Let m be a positive integer satisfying

$$2^{m-1}\varepsilon < \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) \le 2^m \varepsilon. \tag{14}$$

Then, the domain Ω can be divided by $\Omega = \bigcup_{i=0}^m \Omega_i$ and

$$\begin{gathered} \Omega_i = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \le \varepsilon\}, \quad \text{if } i = 0, \\ \Omega_i = \left\{x \in \Omega \mid 2^{i-1}\varepsilon < \operatorname{dist}(x, \partial\Omega) \le 2^i \varepsilon\right\}, \quad \text{if } 1 \le i \le m. \end{gathered} \tag{15}$$

Let $\mathcal{T}_h$ be the regular triangular partition of Ω with the mesh size h and satisfy

$$\begin{gathered} (B_1)\ 2^{i/2}\lambda\varepsilon h \le \sqrt{S_{e_i}} \le 2^{i/2}\lambda^{-1}\varepsilon h, \quad \forall e_i \in \mathcal{T}_h, \\ e_i \subset \Omega_i \cup \Omega_{i+1}, \end{gathered} \tag{16}$$

$$(B_2)\ \left|l_i - l_j\right| \le C h l_i, \quad \forall e_i \in \mathcal{T}_h, \tag{17}$$

where λ and C are independent of i and e_i, S_{e_i} denotes the area of e_i, and l_i, l_j are the length of two edges of e_i. Let Σ^h be the set of all nodal points in $\mathcal{T}_h$, $\Sigma_B^h = \Sigma^h \cap \partial\Omega$, and let S^h be the conforming P_1 finite element spaces with respect to $\mathcal{T}_h$; we define

$$\begin{gathered} S_1^h = \left\{v \in S^h \mid v(x) = -\varepsilon N_k^{h_0}\left(\frac{x}{\varepsilon}\right) u_k^{h_0,h_1}(x),\ \forall x \in \Sigma_B^h\right\}, \\ S_0^h = \left\{v \in S^h \mid v(x) = 0,\ \forall x \in \Sigma_B^h\right\}. \end{gathered} \tag{18}$$

Then, the finite element approximation of θ_ε is to find $\widetilde{\theta}_\varepsilon^{h_0,h_1,h} \in S_1^h$ such that

$$\int_\Omega a_{ij}(\xi) \frac{\partial \widetilde{\theta}_\varepsilon^{h_0,h_1,h}}{\partial x_i} \frac{\partial v}{\partial x_j} dx = 0, \quad \forall v \in S_0^h. \tag{19}$$

2.4. Multiscale Finite Element Approximation of u^ε. For any $v \in S^h(\Omega)$, we define the linear operator R_h by

$$\begin{gathered} \frac{\partial R_h v(x)}{\partial x_i} \text{ is a piecewise function on } \mathcal{T}_h, \\ \frac{\partial R_h v(x)}{\partial x_i} = \frac{\partial v(x)}{\partial x_i}, \quad \text{if } x \text{ is the middle point of elements of } \mathcal{T}_h. \end{gathered} \tag{20}$$

Then, the multiscale finite element approximation $\widehat{u}^{h_0,h_1,h}(x)$ of $u^\varepsilon(x)$ can be defined by

$$\widehat{u}^{h_0,h_1,h}(x) = \widetilde{u}^{h_0,h_1}(x) + R_h \widetilde{\theta}_\varepsilon^{h_0,h_1,h}(x). \tag{21}$$

3. Superconvergence Result of $\widehat{u}^{h_0,h_1,h}$

Firstly, we have the following assumption.

Assumption C1. The functions $a_{ij} \in W^{1,\infty}(Q) \cap H^2(Q)$, and the homogenization solution $u^0 \in W^{2,\infty}(\Omega) \cap H^4(\Omega)$.

Then, we introduce the following lemma.

Lemma 2 (see [1, 5]). *Let $\Omega_r = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \ge r\}$. Assuming that (C1) holds, then there exists C, which is independent of ε and r such that*

$$\begin{gathered} \left\|\nabla\theta_\varepsilon\right\|_{L^2(\Omega_r)} \le C r^{-1/2} \varepsilon \left\|u^0\right\|_{H^2(\Omega)}, \\ \left\|\theta_\varepsilon\right\|_{H^2(\Omega)} \le C \varepsilon^{-1/2} \left\|u^0\right\|_{H^3(\Omega)}, \\ \left\|\theta_\varepsilon\right\|_{H^1(\Omega_r)} \le C r^{-1/2} \varepsilon \left\|u^0\right\|_{W^{1,\infty}(\partial\Omega)}, \\ \left\|\theta_\varepsilon\right\|_{H^2(\Omega_r)} \le C r^{-1/2} \left\|u^0\right\|_{W^{1,\infty}(\partial\Omega)}. \end{gathered} \tag{22}$$

From Lemma 2, one can easily deduce.

Lemma 3. *Assuming that (C1) holds, then there exists C, which is independent of ε such that*

$$\left\|\theta_\varepsilon\right\|_{H^3(\Omega)} \le C \varepsilon^{-3/2} \left\|u^0\right\|_{H^4(\Omega)}, \tag{23}$$

$$\left\|\theta_\varepsilon\right\|_{H^3(\Omega_r)} \le C r^{-1/2} \varepsilon^{-1} \left(\left\|u^0\right\|_{W^{2,\infty}(\partial\Omega)} + \left\|u^0\right\|_{H^4(\Omega)}\right). \tag{24}$$

Proof. For $k = 1, 2$, we define

$$v_k^\varepsilon(x) = \frac{\partial \theta_\varepsilon(x)}{\partial x_k}. \tag{25}$$

Then, the upper bound of $\|v_k^\varepsilon\|_{H^2(\Omega)}$ and $\|v_k^\varepsilon\|_{H^1(\Omega_r)}$ can be estimated, respectively.

Using the result from (8) and $\xi = x/\varepsilon$, we have

$$\begin{gathered} L_\varepsilon v_k^\varepsilon = -\varepsilon^{-1} \frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}(\xi)}{\partial \xi_k} \frac{\partial \theta_\varepsilon}{\partial x_j}\right), \quad x \in \Omega, \\ v_k^\varepsilon(x) = -\varepsilon \frac{\partial\left(N_l(\xi)\left(\partial u^0(x)/\partial x_l\right)\right)}{\partial x_k}, \quad x \in \partial\Omega. \end{gathered} \tag{26}$$

Then, we estimate $\|v_k^\varepsilon\|_{H^2(\Omega)}$. Obviously, v_k^ε can be divided into

$$v_k^\varepsilon = v_{k,1}^\varepsilon + v_{k,2}^\varepsilon, \tag{27}$$

where $v_{k,1}^\varepsilon$ satisfies

$$\begin{gathered} L_\varepsilon v_{k,1}^\varepsilon = -\varepsilon^{-1}\frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}(\xi)}{\partial \xi_k}\frac{\partial \theta_\varepsilon}{\partial x_j}\right), \quad x \in \Omega, \\ v_{k,1}^\varepsilon = 0, \quad x \in \partial\Omega, \end{gathered} \tag{28}$$

and $v_{k,2}^\varepsilon$ satisfies

$$\begin{gathered} L_\varepsilon v_{k,2}^\varepsilon = 0, \quad x \in \Omega, \\ v_{k,2}^\varepsilon = -\varepsilon\frac{\partial\left(N_l(x/\varepsilon)\left(\partial u^0/\partial x_l\right)\right)}{\partial x_k}, \quad x \in \partial\Omega. \end{gathered} \tag{29}$$

Using the result from Lemma 2 and (28), we have

$$\left\|v_{k,1}^\varepsilon\right\|_{H^2(\Omega)} \le C\varepsilon^{-1}\|\theta_\varepsilon\|_{H^2(\Omega)} \le C\varepsilon^{-3/2}\left\|u^0\right\|_{H^3(\Omega)}. \tag{30}$$

Let $v_{k,2,k'}^\varepsilon = \partial v_{k,2}^\varepsilon/\partial x_{k'}$, and using the result from (29), we have

$$\begin{gathered} L_\varepsilon v_{k,2,k'}^\varepsilon = 0, \quad x \in \Omega, \\ v_{k,2,k'}^\varepsilon = -\varepsilon\frac{\partial^2\left(N_l(x/\varepsilon)\left(\partial u^0/\partial x_l\right)\right)}{\partial x_k\partial x_{k'}}, \quad x \in \partial\Omega. \end{gathered} \tag{31}$$

Following the same line of [5] (1992, Theorem 1.2, pages 124–128), we have

$$\left\|v_{k,2,k'}\right\|_{H^1(\Omega)} \le C\varepsilon^{-3/2}\left\|u^0\right\|_{H^4(\Omega)} \le C\varepsilon^{-3/2}\left\|u^0\right\|_{H^4(\Omega)} \tag{32}$$

which indicates

$$\left\|v_{k,2}^\varepsilon\right\|_{H^2(\Omega)} \le C\varepsilon^{-3/2}\left\|u^0\right\|_{H^4(\Omega)}. \tag{33}$$

Combining (30) with (33), we can derive (24) immediately.

Considering the proof of (23), v_k^ε can be divided into

$$v_k^\varepsilon = \widehat{v}_{k,1}^\varepsilon + \widehat{v}_{k,2}^\varepsilon, \tag{34}$$

where $\widehat{v}_{k,1}^\varepsilon$ satisfies

$$\begin{gathered} L_\varepsilon \widehat{v}_{k,1}^\varepsilon = -\varepsilon^{-1}\frac{\partial}{\partial x_i}\left(\frac{\partial a_{ij}(\xi)}{\partial \xi_k}\frac{\partial \theta_\varepsilon}{\partial x_j}\right), \quad x \in \Omega_r, \\ \widehat{v}_{k,1}^\varepsilon = 0, \quad x \in \partial\Omega_r, \end{gathered} \tag{35}$$

and $\widehat{v}_{k,2}^\varepsilon$ satisfies

$$\begin{gathered} L_\varepsilon \widehat{v}_{k,2}^\varepsilon = 0, \quad x \in \Omega_r, \\ \widehat{v}_{k,2}^\varepsilon = \frac{\partial\theta_\varepsilon}{\partial x_k}, \quad x \in \partial\Omega_r. \end{gathered} \tag{36}$$

In view of Lemma 2, we have

$$\begin{aligned} \left\|\widehat{v}_{k,1}^\varepsilon\right\|_{H^2(\Omega_r)} &\le C\varepsilon^{-1}\|\theta_\varepsilon\|_{H^2(\Omega_r)} \\ &\le C\varepsilon^{-1}r^{-1/2}\left\|u^0\right\|_{W^{2,\infty}(\partial\Omega)}. \end{aligned} \tag{37}$$

Following the same line of [5] (1992, Theorem 1.2, pages 124–128), we have

$$\begin{aligned} \left\|\widehat{v}_{k,2}^\varepsilon\right\|_{H^2(\Omega_r)} &\le C\varepsilon^{-1}r^{-1/2}\|\theta\|_{W^{1,\infty}(\Omega_{r/2})} + Cr^{-1/2}\|\theta\|_{W^{2,\infty}(\Omega_{r/2})} \\ &\le C\varepsilon^{-1}r^{-1/2}r^{-1}\varepsilon\left(\left\|u^0\right\|_{W^{2,\infty}(\partial\Omega)} + \left\|u^0\right\|_{H^4(\Omega)}\right) \\ &\quad + Cr^{-1/2}r^{-1}\left(\left\|u^0\right\|_{W^{2,\infty}(\partial\Omega)} + \left\|u^0\right\|_{H^4(\Omega)}\right) \\ &\le C\varepsilon^{-1}r^{-1/2}\left(\left\|u^0\right\|_{W^{2,\infty}(\partial\Omega)} + \left\|u^0\right\|_{H^4(\Omega)}\right). \end{aligned} \tag{38}$$

Combining (37) with (38), we can conclude the result of this lemma. □

Assuming that $\mathscr{T}_h$ is defined as (16), and let θ_ε^h and θ_ε^I be the linear finite element approximation and the linear interpolation of θ_ε with respect to $\mathscr{T}_h$, respectively. Then, we have the following.

Lemma 4. *Assuming that (C1) holds, then there exists C such that*

$$\begin{aligned} &\left\|\frac{\nabla\left(\theta_\varepsilon^I - \theta_\varepsilon^h\right)}{\operatorname{dist}(\cdot,\partial\Omega) + \varepsilon}\right\|_{L^2(\Omega)} \\ &\quad \le C\varepsilon^{-1/2}h^2\left(\left\|u^0\right\|_{W^{1,\infty}(\partial\Omega)} + \left\|u^0\right\|_{H^4(\Omega)}\right). \end{aligned} \tag{39}$$

Proof. Assuming that m is defined as (14) and $\Omega = \bigcup_{i=k+1}^m \Omega_i$ ($k < m, k \in \mathbb{N}$). Considering $a_{ij} \in W^{1,\infty}(\Omega)$ and using the result from Lemma 2, we have

$$\begin{aligned} &\left(\left\|\nabla\left(\theta_\varepsilon^I - \theta_\varepsilon^h\right)\right\|_{L^2(\Omega)}\right)^2 \\ &\le C\left|\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial\left(\theta_{\varepsilon,r}^h - \theta_\varepsilon^I\right)}{\partial x_i}\frac{\partial\left(\theta_\varepsilon^h - \theta_\varepsilon^I\right)}{\partial x_j}dx\right| \\ &\le C\left|\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial\left(\theta_\varepsilon - \theta_\varepsilon^I\right)}{\partial x_i}\frac{\partial\left(\theta_{\varepsilon,r}^h - \theta_\varepsilon^I\right)}{\partial x_j}dx\right| \\ &\le \sum_{i=1}^k\left|\int_{\Omega_i} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial\left(\theta_\varepsilon - \theta_\varepsilon^I\right)}{\partial x_i}\frac{\partial\left(\theta_{\varepsilon,r}^h - \theta_\varepsilon^I\right)}{\partial x_j}dx\right| \\ &\le C\sum_{i=1}^k\left(2^{i/2}\varepsilon h\right)^2\left(\varepsilon^{-1}\|\theta_\varepsilon\|_{H^2(\Omega_i)}\left\|\nabla\left(\theta_\varepsilon^I - \theta_{\varepsilon,r}^h\right)\right\|_{L^2(\Omega_i)}\right. \\ &\qquad \left. + \|\theta_\varepsilon\|_{H^3(\Omega_i)}\left\|\nabla\left(\theta_\varepsilon^I - \theta_\varepsilon^{r,h}\right)\right\|_{L^2(\Omega_i)}\right) \\ &\le C\sum_{i=1}^k 2^i\varepsilon^2h^2\left(\varepsilon^{-1}(2^i\varepsilon)^{-1/2}\left(\left\|u^0\right\|_{W^{1,\infty}(\partial\Omega)} + \left\|u^0\right\|_{H^4(\Omega)}\right)\right. \\ &\qquad \left.\times\left\|\nabla\left(\theta_\varepsilon^I - \theta_{\varepsilon,r}^h\right)\right\|_{L^2(\Omega-\Omega_r)}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq C2^{k/2}\varepsilon^{1/2}h^2\left(\left\|u^0\right\|_{W^{1,\infty}(\partial\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\
&\quad\times\left\|\nabla\left(\theta_\varepsilon^I-\theta_{\varepsilon,r}^h\right)\right\|_{L^2(\Omega)}\\
&\leq Ch^2\left(\left\|u^0\right\|_{W^{1,\infty}(\partial\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\
&\quad\times\left\|\nabla\left(\theta_\varepsilon^I-\theta_{\varepsilon,r}^h\right)\right\|_{L^2(\Omega-\Omega_r)},
\end{aligned}
\tag{40}
$$

which indicates that

$$
\left\|\nabla\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)\right\|_{L^2(\Omega)}\leq Ch^2\left(\left\|u^0\right\|_{W^{1,\infty}(\partial\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right). \tag{41}
$$

Then Lemma 4 can be easily derived. □

Furthermore, we can obtain the following lemma.

Lemma 5. *Assuming that (C1) holds, then there exists C such that*

$$
\begin{aligned}
&\left\|\frac{\nabla\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}\\
&\quad\leq C\left|\ln\varepsilon\right|h^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right).
\end{aligned}
\tag{42}
$$

Proof. Assume that Ω_{r_j} is defined as Lemma 4. Considering $\theta_\varepsilon^h=\theta_{\varepsilon,r_0}^h$ $(\forall x\in\Omega_{r_j})$, and we divide $\theta_\varepsilon^I-\theta_\varepsilon^h$ into

$$
\theta_\varepsilon^I-\theta_\varepsilon^h=\left(\theta_\varepsilon^I-\theta_{\varepsilon,r_j}^h\right)+\sum_{i=0}^{j-1}\left(\theta_{\varepsilon,r_i}^h-\theta_{\varepsilon,r_{i+1}}^h\right). \tag{43}
$$

Then,

$$
\begin{aligned}
\left\|\theta_\varepsilon^I-\theta_\varepsilon^h\right\|_{H^1(\Omega-\Omega_{r_j})}&\leq\left\|\theta_\varepsilon^I-\theta_{\varepsilon,r_{j+1}}^h\right\|_{H^1(\Omega-\Omega_{r_j})}\\
&\quad+\sum_{i=1}^{j-1}\left\|\theta_{\varepsilon,r_i}^I-\theta_{\varepsilon,r_{i+1}}^h\right\|_{H^1(\Omega-\Omega_{r_j})}\\
&\leq cr_j^{1/2}h^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\
&\quad+\sum_{i=1}^{j-1}\left\|\theta_{\varepsilon,r_i}^I-\theta_{\varepsilon,r_{i+1}}^h\right\|_{H^1(\Omega-\Omega_{r_j})}\\
&\leq Cr_j^{1/2}h^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\
&\quad+Cr_jjh^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\
&\leq Cr_j^{1/2}h^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right).
\end{aligned}
\tag{44}
$$

Using the result from (44), we have

$$
\begin{aligned}
&\left\|\frac{\nabla\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}\\
&\quad\leq C\sum_{j=1}^{m}\left\|\frac{\nabla\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega_{r_j}-\Omega_{r_{j-1}})}\\
&\quad\leq\sum_{j=1}^{m}Cr_j^{-1/2}\left\|\nabla\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)\right\|_{L^2(\Omega_{r_j}-\Omega_{r_{j-1}})}\\
&\quad\leq\sum_{j=1}^{m}Cr_j^{-1/2}\left\|\nabla\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)\right\|_{L^2(\Omega-\Omega_{r_{j-1}})}\\
&\quad\leq\sum_{j=1}^{m}Cr_j^{-1/2}r_j^{1/2}h^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\
&\quad\leq Cmh^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\
&\quad\leq C\left|\ln\varepsilon\right|h^2\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right).
\end{aligned}
\tag{45}
$$

□

Based on the previous lemmas, the estimate of $\||\nabla(\theta_\varepsilon^I-\widetilde{\theta}_\varepsilon^{h_0,h_1,h})|/\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}\|_{L^2(\Omega)}$ can be given as follows.

Lemma 6. *Assuming that (C1) holds, then there exists C such that*

$$
\begin{aligned}
\left\|\frac{\left|\nabla\left(\theta_\varepsilon^I-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}\right)\right|}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}&\leq C\left(h_0+h_1+h^2\right)\left|\ln\varepsilon\right|\\
&\quad\times\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right).
\end{aligned}
\tag{46}
$$

Proof. Assuming that $\widetilde{\theta}_\varepsilon^{h_0,h_1}$ satisfies

$$
\begin{aligned}
&L_\varepsilon\widetilde{\theta}_\varepsilon^{h_0,h_1}(x)=0,\quad x\in\Omega,\\
&\widetilde{\theta}_\varepsilon^{h_0,h_1}(x)=-\varepsilon N_k^{h_0}\left(\frac{x}{\varepsilon}\right)u_k^{h_0,h_1}(x),\quad x\in\partial\Omega,
\end{aligned}
\tag{47}
$$

and $\widetilde{\theta}_\varepsilon^{I,h_0,h_1}$ is the linear interpolation of $\widetilde{\theta}_\varepsilon^{h_0,h_1}$ on $\mathcal{T}_h$.

Then, we divide $\theta_\varepsilon^I-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}$ into

$$
\theta_\varepsilon^I-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}=\left(\theta_\varepsilon^I-\widetilde{\theta}_\varepsilon^{I,h_0,h_1}\right)+\left(\widetilde{\theta}_\varepsilon^{I,h_0,h_1}-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}\right). \tag{48}
$$

Firstly, considering the first item of the right-hand side of (48) and assuming that $\theta_\varepsilon^{h_0,h_1}(x)$ satisfies the problem

$$
\begin{aligned}
&L_\varepsilon\theta_\varepsilon^{h_0,h_1}(x)=0,\quad x\in\Omega,\\
&\theta_\varepsilon^{h_0,h_1}(x)=-\varepsilon N_k^{h_0}\left(\frac{x}{\varepsilon}\right)u_k^{h_0,h_1}(x),\quad x\in\partial\Omega,
\end{aligned}
\tag{49}
$$

we have

$$\begin{aligned}&\left\|\nabla\left(\theta_\varepsilon^I-\widetilde{\theta}_\varepsilon^{I,h_0,h_1}\right)\right\|_{L^2(\Omega)}\\&\quad\le C\left\|\nabla\left(\theta_\varepsilon-\widetilde{\theta}_\varepsilon^{h_0,h_1}\right)\right\|_{L^2(\Omega)}\\&\quad\le C\left(\left\|\nabla\left(\theta_\varepsilon-\theta_\varepsilon^{h_0,h_1}\right)\right\|_{L^2(\Omega)}+\left\|\nabla\left(\widetilde{\theta}_\varepsilon^{h_0,h_1}-\theta_\varepsilon^{h_0,h_1}\right)\right\|_{L^2(\Omega)}\right).\end{aligned}\tag{50}$$

Using the same method of Lemma 2, we have

$$\left\|\theta_\varepsilon-\theta_\varepsilon^{h_0,h_1}\right\|_{L^\infty(\Omega)}\le C\varepsilon\left(h_0+h_1\right),\tag{51}$$

$$\left\|\nabla\left(\theta_\varepsilon-\theta_\varepsilon^{h_0,h_1}\right)\right\|_{L^2(\Omega)}\le C\varepsilon^{1/2}\left(h_0+h_1\right).\tag{52}$$

Then, we have

$$\begin{aligned}&\left|\varepsilon N_k^{h_0}\left(\frac{x}{\varepsilon}\right)u_k^{h_0,h_1}(x)-\varepsilon N_k\left(\frac{x}{\varepsilon}\right)\frac{\partial u^0}{\partial x_k}\right|_{L^\infty(\Omega)}\\&\quad\le C\varepsilon\left(h_0+h_1\right)\left\|u^0\right\|_{W^{2,\infty}(\Omega)},\end{aligned}\tag{53}$$

$$\left\|\nabla\left(\widetilde{\theta}_\varepsilon^{h_0,h_1}-\theta_\varepsilon^{h_0,h_1}\right)\right\|_{L^2(\Omega)}\le C\varepsilon^{1/2}\left(h_0+h_1\right)\left\|u^0\right\|_{W^{2,\infty}(\Omega)}.\tag{54}$$

Combining (52) with (54), we have

$$\left\|\nabla\left(\theta_\varepsilon^I-\widetilde{\theta}_\varepsilon^{I,h_0,h_1}\right)\right\|_{L^2(\Omega)}\le C\varepsilon^{1/2}\left(h_0+h_1\right)\left\|u^0\right\|_{W^{2,\infty}(\Omega)}.\tag{55}$$

Next, considering the second item of the right-hand side of (48), $\widetilde{\theta}_\varepsilon^{I,h_0,h_1}-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}$ can be divided into

$$\begin{aligned}\widetilde{\theta}_\varepsilon^{I,h_0,h_1}-\widetilde{\theta}_\varepsilon^{h,h_0,h_1}=&\left(\widetilde{\theta}_\varepsilon^{I,h_0,h_1}-\theta_\varepsilon^I\right)+\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)\\&+\left(\theta_\varepsilon^h-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}\right).\end{aligned}\tag{56}$$

Similarly, we have

$$\begin{aligned}&\left|\nabla\left(\widetilde{\theta}_\varepsilon^{I,h_0,h_1}-\theta_\varepsilon^I\right)\right|_{L^2(\Omega)}\le C\varepsilon^{1/2}\left(h_0+h_1\right)\left\|u^0\right\|_{W^{2,\infty}(\Omega)},\\&\quad\left|\nabla\left(\theta_\varepsilon^I-\theta_\varepsilon^h\right)\right|_{L^2(\Omega)}\le C\varepsilon^{1/2}\left(h_0+h_1\right)\left\|u^0\right\|_{W^{2,\infty}(\Omega)},\end{aligned}\tag{57}$$

$$\left|\nabla\left(\theta_\varepsilon^h-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}\right)\right|_{L^2(\Omega)}\le C\varepsilon^{1/2}\left(h_0+h_1\right)\left\|u^0\right\|_{W^{2,\infty}(\Omega)}.\tag{58}$$

Combining Lemma 6 with (55)–(58), we have (46). □

Next, using the extrapolation technique [10], we are in a position to estimate

$$\theta_\varepsilon(x)-R_h\frac{4\widetilde{\theta}_\varepsilon^{h_0,h_1,h/2}(x)-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}(x)}{3}\tag{59}$$

instead of

$$\theta_\varepsilon(x)-R_h\widetilde{\theta}_\varepsilon^{h_0,h_1,h}(x).\tag{60}$$

Lemma 7. *Assuming that (C1) holds, then there exists C such that*

$$\begin{aligned}&\left\|\frac{\nabla\left(\theta_\varepsilon-R_h\left(\left(4\widetilde{\theta}_\varepsilon^{h_0,h_1,h/2}-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}\right)/3\right)\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}\\&\quad\le C\left(h_0+h_1+h^2\right)|\ln\varepsilon|\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right).\end{aligned}\tag{61}$$

Proof. Let $\mathcal{T}_h$ and θ_ε^I be defined as above. Assuming that $\theta_\varepsilon^{I,h/2}$ is the linear interpolation of θ_ε on $\mathcal{T}_{h/2}$, we have

$$\begin{aligned}&\left\|\nabla\left(\theta_\varepsilon-R_h\frac{4\widetilde{\theta}_\varepsilon^{h_0,h_1,h/2}-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}}{3}\right)\right\|_{L^2(\Omega)}\\&\quad\le\left\|\nabla\left(\theta_\varepsilon-R_h\frac{4\theta_\varepsilon^{I,h/2}-\theta_\varepsilon^I}{3}\right)\right\|_{L^2(\Omega)}\\&\qquad+\left\|\nabla R_h\left(\frac{4\theta_\varepsilon^{I,h/2}-\theta_\varepsilon^I}{3}-\frac{4\widetilde{\theta}_\varepsilon^{h_0,h_1,h/2}-\widetilde{\theta}_\varepsilon^{h_0,h_1,h}}{3}\right)\right\|_{L^2(\Omega)}\\&\quad\le C\varepsilon^{1/2}h^2|\ln\varepsilon|\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\&\qquad+C\varepsilon^{1/2}\left(h_0+h_1+h^2\right)\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right)\\&\quad\le C\varepsilon^{1/2}\left(h_0+h_1+h^2|\ln\varepsilon|\right)\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right).\end{aligned}\tag{62}$$

Then, (61) can be easily derived. □

Next, we turn to estimate $\|\nabla w^\varepsilon/\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}\|_{L^2(\Omega)}$.

Lemma 8. *Assuming that (C1) holds, then there exists C such that*

$$\left\|\frac{\nabla w^\varepsilon}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}\le C\varepsilon|\ln\varepsilon|\left\|u^0\right\|_{W^{2,\infty}(\Omega)}.\tag{63}$$

Proof. Following the same line of [5] and $\omega^\varepsilon=u^\varepsilon-\widetilde{u}-\theta_\varepsilon$, there exists C such that

$$\begin{aligned}&\left\|\frac{\nabla w^\varepsilon}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}\\&\quad\le C\left\|\nabla w^\varepsilon\right\|_{L^{2|\ln\varepsilon|}(\Omega)}\times\left\|\frac{1}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^{4|\ln\varepsilon|/(2|\ln\varepsilon|-1)}(\Omega)}\\&\quad\le C\varepsilon|\ln\varepsilon|^{1/2}\left\|u^0\right\|_{W^{2,2|\ln\varepsilon|}(\Omega)}|\ln\varepsilon|^{1/2}\\&\quad\le C\varepsilon|\ln\varepsilon|\left\|u^0\right\|_{W^{2,\infty}(\Omega)}.\end{aligned}\tag{64}$$

Table 1: Comparison of computational results with $\varepsilon = 1/30$.

$h_0 \downarrow$	$h_1 \downarrow$	$h \downarrow$	e_0	e_1
1/8	1/8	1/8	0.1856	0.0426
1/32	1/32	1/16	0.1629	0.0178
1/128	1/128	1/32	0.1573	0.0043

Finally, noting the definitions of $\widetilde{u}, \widetilde{u}^{h_0,h_1}, \theta_\varepsilon, R_h\widetilde{\theta}_\varepsilon^{h_0,h_1,h}$, and ω^ε, and combining Lemma 3 with Lemmas 7-8 and Lemma 2.4 in [9], we have

$$\begin{aligned}\left\|\frac{\nabla\left(u^\varepsilon-\widehat{u}^{h_0,h_1,h}\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)} &\le \left\|\frac{\nabla\left(\widetilde{u}-\widetilde{u}^{h_0,h_1}\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)} \\ &\quad+\left\|\frac{\nabla\left(\theta_\varepsilon-R_h\widetilde{\theta}_\varepsilon^{h_0,h_1,h}\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)} \\ &\quad+\left\|\frac{\nabla\omega^\varepsilon}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)} \\ &\le C\left(h_1+h_0+\varepsilon+h^2\right)|\ln\varepsilon| \\ &\quad\times\left(\|u^0\|_{W^{2,\infty}(\Omega)}+\|u^0\|_{H^4(\Omega)}\right).\end{aligned} \tag{65}$$

□

Combining the above lemmas, we can conclude the following result.

Theorem 9. *Assuming that (C1) holds, then there exists C such that*

$$\begin{aligned}\left\|\frac{\nabla\left(u^\varepsilon-\widehat{u}^{h_0,h_1,h}\right)}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)} &\le C\left[\left(h_1+h_0+\varepsilon+h^2\right)|\ln\varepsilon|\right] \\ &\quad\times\left(\left\|u^0\right\|_{W^{2,\infty}(\Omega)}+\left\|u^0\right\|_{H^4(\Omega)}\right).\end{aligned} \tag{66}$$

4. Numerical Example

In this section, some numerical results will be shown. In order to show the numerical accuracy of the method presented in this paper, the exact solution of problem (1) should firstly be obtained. However, it is very difficult to find them out. Then, the exact solution will be replaced by the finite element solution in a fine mesh with the mesh size 1/256.

It should not be confused that u^* denotes the finite element solution of (1) in a fine mesh, and $\widehat{u}^{h_0,h_1,h}$, obtained by the multiscale finite element scheme presented in the above section, is the multiscale finite element solution of problem (1). Some numerical results will be presented by solving the following model problem:

$$\begin{aligned}a_{11} &= 0.3+2x_1\left(1-x_1\right),\\ a_{12}=a_{21} &= x_1\left(1-x_1\right)x_2\left(1-x_2\right),\\ a_{22} &= 0.1+2x_2\left(1-x_2\right),\\ f(x)=e^{x_1+x_2}, &\qquad g(x)=2\sin\left(x_1\right)+4\cos\left(x_2\right),\\ \Omega &= \left\{x \mid \left(x_1-0.5\right)^2+\left(x_1-0.5\right)^2<0.25\right\}.\end{aligned} \tag{67}$$

Table 2: Comparison of error order.

$\left\|\frac{\nabla(u^*-\widetilde{u}^{h_0,h_1})}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}$	$O(1)$
$\left\|\frac{\nabla(u^*-\widehat{u}^{h_0,h_1,h})}{\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}}\right\|_{L^2(\Omega)}$	$O((h_1+h_0+\varepsilon+h^2)\|\ln\varepsilon\|)$

Moreover, let

$$\begin{aligned}e_0 &= \frac{\left\|\nabla\left(u^*-\widetilde{u}^{h_0,h_1}\right)/\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}\right\|_{L^2(\Omega)}}{\left\|\nabla u^*/\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}\right\|_{L^2(\Omega)}},\\ e_1 &= \frac{\left\|\nabla\left(u^*-\widehat{u}^{h_0,h_1,h}\right)/\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}\right\|_{L^2(\Omega)}}{\left\|\nabla u^*/\sqrt{\operatorname{dist}(\cdot,\partial\Omega)+\varepsilon}\right\|_{L^2(\Omega)}}.\end{aligned} \tag{68}$$

In Table 1, the numerical results of the multiscale method for e_0 and e_1 are given. It can be seen that the improvement obtained in the final approximation by considering the numerical approximation for the boundary corrector, and the numerical result agree well with the theoretical result from Theorem 9.

According to Table 2, it can be seen that $\nabla u^\varepsilon(x)$ can effectively be computed for problem (1) by using the above method, even if $\operatorname{dist}(x,\Omega)$ is very small. If we only need to get a good numerical solution for problem (1) in Sobolev space $H_1(\Omega)$, the boundary corrector needs not to be computed. However, the boundary corrector is a very important part of error estimate in the real applications. It can be concluded that this method is an exceedingly important and effective finite element algorithm.

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