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Research Article

Fractional Variational Iteration Method versus Adomian's Decomposition Method in Some Fractional Partial Differential Equations

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A comparative study is presented about the Adomian's decomposition method (ADM), variational iteration method (VIM), and fractional variational iteration method (FVIM) in dealing with fractional partial differential equations (FPDEs). The study outlines the significant features of the ADM and FVIM methods. It is found that FVIM is identical to ADM in certain scenarios. Numerical results from three examples demonstrate that FVIM has similar efficiency, convenience, and accuracy like ADM. Moreover, the approximate series are also part of the exact solution while not requiring the evaluation of the Adomian's polynomials.

1. Introduction

Fractional differential equations (FDEs), as a generalization of ordinary differential equations to an arbitrary (noninteger) order, have been proved to be a valuable tool in modelling many phenomena in the fields of physics, chemistry, engineering, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth [1–9]. The reasons are that fractional derivatives provide an excellent instrument for description of memory and hereditary properties of various materials and processes.

Considerable attention has been paid to developing accurate and efficient methods for solving fractional partial differential equations (FPDEs). Most of the nonlinear fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. Recently, some approximate methods such as Adomian's decomposition method (ADM) [10–13], homotopy perturbation method (HPM) [14–16], variational iteration method (VIM) [17–22], homotopy analysis method (HAM) [23–26], fractional complex transform (FCT) [27–31], and wavelets method [32–34] have been given to find an analytical approximation to FDEs.

The variational iteration method (VIM), which was first proposed by He et al. [17–22] and has been shown to be very efficient for handling a wide class of physical problems. As early as 1998, the variational iteration method was shown to be an effective tool for factional calculus [35]; hereafter, the method has been routinely used to solve various fractional differential equations [11, 36–41] for many years. In order to improve the accuracy and efficiency of the VIM for factional calculus, a modification called fractional variational iteration method (FVIM) [42, 43] was proposed and some successes [44, 45] have been achieved. In the field of fractional differential equations, the main difference between VIM and FVIM is the evaluation of Lagrange multipliers: VIM usually get Lagrange multipliers by some approximations.

The three methods (i.e., VIM, FVIM, and ADM) are relatively new and effective approaches to find the approximate solution of PDEs, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear PDEs without linearization or discretization. The prior work [46–49] has performed a comparative study of ADM and VIM and got two useful conclusions: on the one hand, ADM needs specific algorithms to evaluate the Adomian's polynomials, while VIM handles linear and nonlinear problems in a similar manner without any additional requirement or restriction; on the other hand, Adomian's decomposition method provides the components of the exact solution. However, it has to be validated whether these conclusions are also true for the scenario of FPDEs.

In this paper, we consider the following fractional initial value problem:

$$D_t^{\alpha} u(x,t) + N[u(x,t)] + L[u(x,t)] = g(x,t), \quad t > 0, \tag{1}$$

where L is a linear operator, N is a nonlinear operator in x, t, and D^{α} is the modified Riemann-Liouville derivative of order α, subject to the initial conditions

$$\begin{gathered} u^{(k)}(x,0) = c_k(x), \\ k = 0, 1, 2, \cdots, m-1, \; m-1 < \alpha \le m. \end{gathered} \tag{2}$$

We will provide a comparative study of ADM and FVIM in dealing with the above FPDEs. The remainder of the paper is organized as follows. We begin by introducing some necessary definitions and mathematical preliminaries for the fractional calculus theory in Section 2. We present the VIM/FVIM method and the ADM in Sections 3 and 4, respectively. In Section 5, three examples are given to demonstrate our conclusions. Finally, a brief summary is presented.

2. Preliminaries and Notations

In this section, we describe some necessary definitions and mathematical preliminaries of the fractional calculus theory.

Definition 1. A real function $h(t)$, $t > 0$, is said to be in the space C_{μ}, $\mu \in R$, if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0,\infty)$, and it is said to be in the space C_{μ}^n if and only if $h^{(n)} \in C_{\mu}$, $n \in N$.

Definition 2. Riemann-Liouville fractional integral operator (J^{α}) of order $\alpha \ge 0$, of a function $f \in C_{\mu}$, $\mu \ge -1$ is defined as

$$\begin{gathered} J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > 0, \\ J^0 f(t) = f(t). \end{gathered} \tag{3}$$

$\Gamma(\alpha)$ is the well-known gamma function. Some properties of the operator J^{α} can be found in [4, 8, 9, 50]. We only recall the following ones:

$$\begin{gathered} J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \\ J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t), \\ J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}, \end{gathered} \tag{4}$$

for $f \in C_{\mu}$, $\mu \ge -1$, α, $\beta \ge 0$, and $\gamma > -1$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with FDEs. Therefore, we will introduce a modified fractional differential operator D_x^{α} proposed by Caputo [51].

Definition 3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$\begin{aligned} &\left(D_x^{\alpha} f\right)(x) \\ &= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi, & (\alpha > 0, m-1 < \alpha < m) \\ \dfrac{\partial^m f(x)}{\partial x^m}, & \alpha = m, \end{cases} \end{aligned} \tag{5}$$

where $f: R \to R$, $x \to f(x)$ denotes a continuous (but not necessarily differentiable) function.

Some useful formulas and results of modified Riemann-Liouville derivative, which we need here, are listed as follows:

$$\begin{gathered} D_x^{\alpha} c = 0, \quad \alpha > 0, \; c = \text{constant}, \\ D_x^{\alpha} [c f(x)] = c D_x^{\alpha} f(x), \quad \alpha > 0, \; c = \text{constant}, \\ D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > \alpha > 0, \\ D_x^{\alpha} [f(x) g(x)] = [D_x^{\alpha} f(x)] g(x) + f(x) [D_x^{\alpha} g(x)], \\ D_x^{\alpha} [f(x(t))] = f_x'(x) x^{(\alpha)}(t). \end{gathered} \tag{6}$$

Lemma 4. *Let $n-1 < \alpha \le n$, $n \in N$, $t > 0$, $h \in C_{\mu}^n$, $\mu \ge -1$. Then*

$$\left(J^{\alpha} D^{\alpha}\right) h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^{+}) \frac{t^k}{k!}. \tag{7}$$

3. VIM and FVIM

3.1. Variational Iteration Method. In this section, the basic ideas of variational iteration method (VIM) are introduced. Here a description of the method (please refer to publications [17–19] for more details) is given to handle the general nonlinear problem as

$$L(u) + N(u) = g(t), \tag{8}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytic function.

According to He's variational iteration method [17–22], we can construct a correction functional as follows:

$$\begin{aligned} u_{n+1}(t) &= u_n(t) \\ &\quad + \int_0^t \lambda(\tau) \{L(u_n(\tau)) + N(\tilde{u}_n(\tau)) - g(\tau)\}, \quad n \ge 0, \end{aligned} \tag{9}$$

where λ is a general Lagrange multiplier which can be optimally identified via variational theory and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Therefore, the Lagrange multiplier λ should be first determined via integration by parts. The successive approximation $u_n(t)$ $(n \geq 0)$ of the solution $u(t)$ will be readily obtained by using the obtained Lagrange multiplier and any selective function u_0. The zeroth approximation u_0 may be selected by any function that just meets, at least, the initial and boundary conditions. With λ determined, then several approximations $u_n(t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained as

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{10}$$

The VIM depends on the proper selection of the initial approximation $u_0(t)$. Finally, we approximate the solution of the initial value problem (1) by the nth-order term $u_n(t)$. It has been validated that VIM is capable of effectively, easily, and accurately solving a large class of nonlinear problems.

3.2. Fractional Variational Iteration Method. We can construct a correction functional for (1) as follows:

$$\begin{aligned} u_{k+1}(x,t) &= u_k(x,t) + \int_0^t \lambda(t,\tau)\left(D_\tau^\alpha u_k(x,\tau) + N\left[\tilde{u}_k(x,\tau)\right]\right. \\ &\qquad \left. + L\left[\tilde{u}_k(x,\tau)\right] - g(x,\tau)\right) d\tau, \end{aligned} \tag{11}$$

where $\tilde{u}(x,t)$ is a restricted variation.

Taking Laplace transform to both sides of (11) as

$$\begin{aligned} \overline{u}_{k+1}(x,t) &= \overline{u}_k(x,t) \\ &\quad + \overline{L}\left[\int_0^t \lambda(t,\tau)\left(D_t^\alpha u_k(x,\tau) + N\left[\tilde{u}_k(x,\tau)\right]\right.\right. \\ &\qquad \left.\left. + L\left[\tilde{u}_k(x,\tau)\right] - g(x,\tau)\right) d\tau\right], \end{aligned} \tag{12}$$

where $\overline{u}_k(x,t)$ is Laplace transform of $u_k(x,t)$ with respect to t and $\overline{L}$ is operator of Laplace transform.

By assuming that the Lagrange multiplier has the form as $\lambda(t,\tau) = \lambda(t-\tau)$, so that $\overline{L}[J_\tau^\alpha \lambda(D_\tau^\alpha u_k(x,\tau) + N[\tilde{u}_k(x,\tau)] + L[\tilde{u}_k(x,\tau)] - g(x,\tau))]$ is the convolution of the function $\lambda(t)$ and $D_t^\alpha u_k(x,t) + N[\tilde{u}_k(x,t)] + L[\tilde{u}_k(x,t)] - g(x,t)$.

Because $\tilde{u}(x,t)$ is a restricted variation, we have

$$\delta\overline{L}\left[J_t^\alpha \lambda\left(N\left[\tilde{u}_k(x,t)\right] + L\left[\tilde{u}_k(x,t)\right] - g(x,t)\right)\right] = 0. \tag{13}$$

Taking the variation derivative δ on the both sides of (12), we can derive

$$\begin{aligned} \delta\overline{u}_{k+1}(x,t) &= \delta\overline{u}_k(x,t) + \delta\overline{L}\left[J_t^\alpha \lambda\left(D_t^\alpha u_k(x,t) + N\left[\tilde{u}_k(x,t)\right]\right.\right. \\ &\qquad \left.\left. + L\left[\tilde{u}_k(x,t)\right] - g(x,t)\right)\right] \\ &= \left(1 + \overline{\lambda}(s)s^\alpha\right)\delta\overline{u}_k(x,s). \end{aligned} \tag{14}$$

If setting the coefficient of $\delta\overline{u}_k(x,s)$ to zero, we can get

$$\overline{\lambda}(s) = -\frac{1}{s^\alpha}, \tag{15}$$

and the Lagrange multiplier can be identified by using the inverse Laplace transform

$$\lambda(t,\tau) = -\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} = \frac{(-1)^\alpha(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}. \tag{16}$$

Substituting (16) into (12) and using the definition of Riemann-Liouville fractional integral operator, we get the iteration formula as follows:

$$\begin{aligned} u_{k+1}(x,t) = u_k(x,t) - J_t^\alpha\left(D_t^\alpha u_k(x,t) + N\left[u_k(x,t)\right]\right. \\ \left. + L\left[u_k(x,t)\right] - g(x,t)\right). \end{aligned} \tag{17}$$

4. Adomian's Decomposition Method

Applying the operator J^α and the inverse of the operator D_t^α to both sides of (1) yields

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^\alpha g(x,t) \\ &\quad - J^\alpha\left[Lu(x,t) + Nu(x,t)\right]. \end{aligned} \tag{18}$$

The Adomian's decomposition method [52–55] suggests that the solution $u(x,t)$ should be decomposed into the infinite series of components as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{19}$$

and the nonlinear function in (1) be decomposed as follows:

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{20}$$

where A_n are the so-called the Adomian's polynomials.

Substituting the decomposition series equations (19) and (20) into both sides of (18) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^\alpha g(x,t) \\ &\quad - J^\alpha\left[L\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right]. \end{aligned} \tag{21}$$

From this equation, the iterates are determined by the following recursive way:

$$
\begin{aligned}
u_0(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^\alpha g(x,t),\\
u_1(x,t) &= -J^\alpha(Lu_0 + A_0),\\
u_2(x,t) &= -J^\alpha(Lu_1 + A_1),\\
&\vdots\\
u_{n+1}(x,t) &= -J^\alpha(Lu_n + A_n).
\end{aligned} \tag{22}
$$

The Adomian's polynomial A_n can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [54]. The Adomian polynomials can be easily calculated by the homotopy perturbation method (for more details see [56]). The general formulation for an Adomian's polynomials is

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}. \tag{23}$$

This formula is easy to compute by using mathematical software or by writing a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. Finally, we approximate the solution $u(x,t)$ by the truncated series as

$$\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t), \qquad \lim_{N\to\infty}\phi_N(x,t) = u(x,t). \tag{24}$$

5. Applications and Results

From (17) and according to Lemma 4, we could get an approximate solution as

$$
\begin{aligned}
u(x,t) &= \lim_{k\to\infty} u_k(x,t)\\
&= \sum_{i=0}^{m-1}\frac{\partial^i u}{\partial t^i}(x,0^+)\frac{t^i}{i!}\\
&\quad - J_t^\alpha(N[u(x,t)] + L[u(x,t)]) + J_t^\alpha g(x,t).
\end{aligned} \tag{25}
$$

From (21), we could get an approximate solution as

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty}\widehat{u}_k(x,t)\\
&= \sum_{i=0}^{m-1}\frac{\partial^i u}{\partial t^i}(x,0^+)\frac{t^i}{i!}\\
&\quad - J_t^\alpha\left[\sum_{k=0}^{\infty}A_k + L\left(\sum_{k=0}^{\infty}\widehat{u}_k(x,t)\right)\right] + J_t^\alpha g(x,t),
\end{aligned} \tag{26}
$$

where $u(x,t) = \sum_{k=0}^{\infty}\widehat{u}_k(x,t)$ and $N[u(x,t)] = \sum_{n=0}^{\infty}A_n$.

When $\widehat{u}_k(x,t) = u_k(x,t) - u_{k-1}(x,t)$ and $A_k = N(\widehat{u}_k(x,t)) = N(u_k(x,t) - u_k(x,t))$, we could find that (25) and (26) are identical.

In this section, we will provide three examples for performing comparative studies. The exact solutions of these examples are known for the special cases $\alpha = 1$ or 2 and have been solved in [11, 12, 57] by using the VIM, HPM, ADM, and some other methods. It is to be noted that Lagrange multiplier of VIM in [11, 12] is an approximation.

Example 5. Consider the following linear time-fractional diffusion equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x \in R,\ 0 < \alpha \le 1, \tag{27}$$

subject to the initial condition

$$u(x,0) = \sin x. \tag{28}$$

Momani and Odibat [11, 12, 57] have made a study about this equation by using the VIM, HPM, and the ADM and drew a conclusion that using the modified HPM is the same as the fourth-order term of the VIM solution. When $\alpha = 1$, the VIM solution and the decomposition solution are identical.

By using the VIM described in [6], the iteration formula for (27) is given by

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi) - \frac{\partial^2}{\partial x^2}u_k(x,\xi)\right)d\xi. \tag{29}$$

By employing the above variational iteration formula and beginning with $u_0 = \sin x$, we can obtain the following approximations:

$$
\begin{aligned}
u_0(x,t) &= \sin(x),\\
u_1(x,t) &= (1-t)\sin(x),\\
u_2(x,t) &= \left(1 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2t + \frac{1}{2}t^2\right)\sin(x),\\
u_3(x,t) &= \left(1 - \frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{3t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{2t^{3-\alpha}}{\Gamma(4-\alpha)}\right.\\
&\qquad \left. -3t + \frac{3}{2}t^2 - \frac{1}{6}t^3\right)\sin(x),\\
&\vdots
\end{aligned} \tag{30}
$$

and so on. The rest components of the iteration formula (29) can be obtained in the same manner.

To solve the problem with the ADM, the recurrence relation is obtained as follows:

$$
\begin{aligned}
u_0(x,t) &= u(x,0) = \sin(x),\\
u_{j+1}(x,t) &= J^\alpha\left(L_{2x}u_j(x,t)\right), \quad j \ge 0.
\end{aligned} \tag{31}
$$

Table 1: Numerical values when $\alpha = 0.5, 0.75$, and 1.0 for (27).

t	x	$\alpha = 0.50$		$\alpha = 0.75$		$\alpha = 1.0$		
		FVIM	VIM	FVIM	VIM	FVIM	VIM	Exact
0.25	0.25	0.14640842	0.14180807	0.17120511	0.16836254	0.19264006	0.19264006	0.19267840
	0.50	0.28371388	0.27479919	0.33176551	0.32625710	0.37330270	0.37330270	0.37337698
	0.75	0.40337938	0.39070464	0.47169834	0.46386658	0.53075518	0.53075518	0.53086080
	1.00	0.49796471	0.48231796	0.58230325	0.57263509	0.65520788	0.65520788	0.65533826
0.50	0.25	0.10790621	0.10407459	0.13273945	0.13274483	0.14947323	0.14947323	0.15005809
	0.50	0.20910333	0.20167832	0.25722580	0.25723624	0.28965293	0.28965293	0.29078629
	0.75	0.29729942	0.28674267	0.36571910	0.36573394	0.41182342	0.41182342	0.41343481
	1.00	0.36701087	0.35397875	0.45147375	0.45149207	0.50838872	0.50838872	0.51037795
0.75	0.25	0.07031034	0.10263146	0.10053521	0.11682841	0.11403776	0.11403776	0.11686536
	0.50	0.13624913	0.19888179	0.19481962	0.22639299	0.22098521	0.22098521	0.22646459
	0.75	0.19371660	0.28276661	0.27699110	0.32188155	0.31419287	0.31419287	0.32198335
	1.00	0.23913971	0.34907038	0.34194061	0.39735708	0.38786553	0.38786553	0.39748275

In view of (31), the first few components of the decomposition series are derived as follows:

$$\begin{aligned}
u_0(x,t) &= \sin(x),\\
u_1(x,t) &= J^\alpha\left(L_{2x}u_0(x,t)\right) = \frac{-t^\alpha}{\Gamma(\alpha+1)}\sin(x),\\
u_2(x,t) &= J^\alpha\left(L_{2x}u_1(x,t)\right) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x),\\
u_3(x,t) &= J^\alpha\left(L_{2x}u_2(x,t)\right) = \frac{-t^{3\alpha}}{\Gamma(3\alpha+1)}\sin(x),\\
&\vdots
\end{aligned} \tag{32}$$

and so on. The rest of components of the decomposition series can be obtained in this manner.

The solution in series form is given by

$$\begin{aligned}
u(x,t) = {} & \sin(x) - \frac{t^\alpha}{\Gamma(\alpha+1)}\sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(x)\\
& + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)}\sin(x).
\end{aligned} \tag{33}$$

To solve (27) by means of FVIM, we construct a correctional functional that reads as

$$u_{k+1} = u_k(x,0) + J^\alpha\frac{\partial^2 u_k}{\partial x^2}. \tag{34}$$

From the initial value, we can derive

$$\begin{aligned}
u_0(x,t) &= \sin(x),\\
u_1(x,t) &= \sin(x)\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)}\right),\\
u_2(x,t) &= \sin(x)\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right),\\
u_3(x,t) &= \sin(x)\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right),\\
&\vdots
\end{aligned} \tag{35}$$

Consequently, the exact solution can be obtained as

$$u(t) = \lim_{n\to\infty}u_n(t) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{n\alpha}}{\Gamma(n\alpha+1)}\sin(x), \tag{36}$$

which is the same as that obtained by ADM.

Table 1 shows the approximate solutions for (27) obtained for different values of α using methods VIM, ADM, and FVIM. The values of $\alpha = 1$ are the only case for which we know the exact solution. From (33) and (36), it is obvious that the solution of (27) obtained using the FVIM is the same as the ADM. Moreover, when α is a positive integer, the Lagrange multiplier of FVIM is identical to that of VIM, so the solutions obtained by the two methods are the same. It should be noted that only the fourth-order term of the VIM and FVIM is used in evaluating the approximate solutions.

Example 6. We next consider the following linear time-fractional wave equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x \in R,\ 1 < \alpha \le 2, \tag{37}$$

Table 2: Numerical values when $\alpha = 1.5$, 1.75, and 2.0 for (37).

t	x	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2.0$		
		FVIM	VIM	FVIM	VIM	FVIM	VIM	Exact
0.25	0.25	0.26622298	0.26599883	0.26593959	0.26590628	0.26578827	0.26578827	0.26578827
	0.50	0.56489190	0.56399533	0.56375836	0.56362512	0.56315308	0.56315308	0.56315308
	0.75	0.89600678	0.89398950	0.89345630	0.89315652	0.89209443	0.89209443	0.89209443
	1.00	1.25956762	1.25598133	1.25503343	1.25450047	1.25261232	1.25261232	1.25261232
0.50	0.25	0.28474208	0.28393355	0.28340402	0.28328354	0.28256846	0.28256846	0.28256846
	0.50	0.63896831	0.63573419	0.63361610	0.63313417	0.63027383	0.63027383	0.63027383
	0.75	1.06267869	1.05540192	1.05063622	1.04955189	1.04311611	1.04311611	1.04311611
	1.00	1.55587323	1.54293675	1.53446439	1.53253670	1.52109530	1.52109530	1.52109531
0.75	0.25	0.30690489	0.30527637	0.30361709	0.30335993	0.30139478	0.30139478	0.30139480
	0.50	0.72761955	0.72110549	0.71446834	0.71343972	0.70557913	0.70557913	0.70557918
	0.75	1.26214400	1.24748736	1.23255378	1.23023936	1.21255304	1.21255304	1.21255316
	1.00	1.91047821	1.88442198	1.85787338	1.85375886	1.82231652	1.82231652	1.82231673

subject to the initial conditions

$$u(x,0) = x, \qquad \frac{\partial u(x,0)}{\partial t} = x^2. \tag{38}$$

By using the VIM described in [6], the iteration formula for (37) is given by

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^\alpha}{\partial \xi^\alpha} u_k(x,\xi) - \frac{1}{2} x^2 \frac{\partial^2}{\partial x^2} u_k(x,\xi) \right) d\xi. \tag{39}$$

By using variational iteration formula and beginning with $u_0 = x + x^2 t$, we can obtain the following approximations:

$$\begin{aligned}
u_1 &= x + x^2 \left(t + \frac{t^3}{3!} \right), \\
u_2 &= x + x^2 \left(t + \frac{t^3}{3} + \frac{t^5}{5!} - \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} \right), \\
u_3 &= x + x^2 \left(t + \frac{t^3}{2} + \frac{t^5}{40} + \frac{t^7}{7!} - \frac{3t^{5-\alpha}}{\Gamma(6-\alpha)} - \frac{2t^{7-\alpha}}{\Gamma(8-\alpha)} \right. \\
&\qquad \left. + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)} \right), \\
&\vdots
\end{aligned} \tag{40}$$

To solve the problem by using the decomposition method, we substitute (37) and the initial conditions equation (38) into (22), and we obtain the following recurrence relation:

$$\begin{aligned}
u_0(x,t) &= u(x,0) = x + x^2 t, \\
u_{j+1}(x,t) &= \frac{1}{2} J^\alpha \left(x^2 L_{2x} u_j(x,t) \right), \quad j \geq 0.
\end{aligned} \tag{41}$$

In view of (22), the first few components of the decomposition series are derived as follows:

$$\begin{aligned}
u_0(x,t) &= x + x^2 t, \\
u_1(x,t) &= \frac{1}{2} J^\alpha \left(x^2 L_{2x} u_0(x,t) \right) = x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \\
u_2(x,t) &= \frac{1}{2} J^\alpha \left(x^2 L_{2x} u_1(x,t) \right) = x^2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}, \\
u_3(x,t) &= \frac{1}{2} J^\alpha \left(x^2 L_{2x} u_2(x,t) \right) = x^2 \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}.
\end{aligned} \tag{42}$$

By using fractional variational iteration formula (17) and beginning with $u_0 = x + x^2 t$, we can obtain the following approximations:

$$\begin{aligned}
u_0 &= x + x^2 t, \\
u_1 &= x + x^2 t + \frac{1}{\Gamma(2+\alpha)} x^2 t^{1+\alpha}, \\
u_2 &= x + x^2 \left[t + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)} \right], \\
u_3 &= x + x^2 \left[t + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)} + \frac{t^{1+3\alpha}}{\Gamma(2+3\alpha)} \right], \\
u_4 &= x + x^2 \\
&\quad \times \left[t + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)} + \frac{t^{1+3\alpha}}{\Gamma(2+3\alpha)} + \frac{t^{1+4\alpha}}{\Gamma(2+4\alpha)} \right], \\
&\vdots
\end{aligned} \tag{43}$$

Table 2 shows the approximate solutions for (37) obtained for different values of α using methods VIM, ADM, and FVIM.

The values of $\alpha = 2$ are the only case for which we know the exact solution $u(x,t) = x + x^2 \sinh(t)$. From (42) and (43), it is obvious that the solution of (37) obtained by using the FVIM is the same as the ADM. As the previous example, the fourth-order term of the VIM/FVIM is utilized in evaluating the approximate solutions.

Example 7. Consider the following nonlinear time-fractional advection partial differential equation [57]:

$$D_t^{\alpha} u(x,t) + u(x,t)\, u_x(x,t) = x + xt^2, \qquad t > 0, \quad x \in R, \quad 0 < \alpha \le 1, \tag{44}$$

subject to the initial conditions

$$u(x,0) = 0. \tag{45}$$

By using the VIM described in [6], the iteration formula for (44) is given by

$$u_{k+1}(x,t) = u_k(x,t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} u_k(x,\xi) + u_k(x,\xi) \frac{\partial u_k(x,\xi)}{\partial x} - \left(x + x\xi^2\right) \right) d\xi. \tag{46}$$

By the variational iteration method, starting with $u_0(x,t) = 0$, we can obtain the following approximations:

$$\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= x\left(t + \frac{t^3}{3}\right),\\
u_2(x,t) &= x\left(2t + \frac{t^3}{3} - \frac{2t^5}{15} - \frac{t^7}{63} - \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{\Gamma(4)\, t^{4-\alpha}}{3\Gamma(5-\alpha)}\right),\\
&\ \vdots
\end{aligned} \tag{47}$$

In the same manner, the rest of components of the iteration formula (46) can be obtained by using the Mathematica package.

To solve the problem using the decomposition method, we obtain the following recurrence relation:

$$\begin{aligned}
u_0(x,t) &= u(x,0) + J^{\alpha}\left(x + xt^2\right) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),\\
u_{j+1}(x,t) &= -J^{\alpha}\left(A_j\right), \quad j \ge 0,
\end{aligned} \tag{48}$$

where A_j are the Adomian's polynomials for the nonlinear function $N = uu_x$. In view of (22), the first few components of the decomposition series are derived as follows:

$$\begin{aligned}
u_0(x,t) &= x\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\right),\\
u_1(x,t) &= -x\left(\frac{\Gamma(1+2\alpha)\, t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{4\Gamma(3+2\alpha)\, t^{3\alpha+2}}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)} + \frac{4\Gamma(5+2\alpha)\, t^{3\alpha+4}}{\Gamma(3+\alpha)^2\Gamma(5+3\alpha)}\right),\\
u_2(x,t) &= 2x\left(\frac{\Gamma(1+2\alpha)\Gamma(1+4\alpha)\, t^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} + \frac{8\Gamma(5+2\alpha)^2\Gamma(8+6\alpha)\, t^{5\alpha+6}}{\Gamma(3+\alpha)^3\Gamma(5+3\alpha)\Gamma(7+5\alpha)} + \cdots\right),\\
&\ \vdots
\end{aligned} \tag{49}$$

The first three terms of the decomposition series (48) are given by

$$u(x,t) = x\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{\Gamma(1+2\alpha)\, t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{4\Gamma(3+2\alpha)\, t^{3\alpha+2}}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)} + \cdots\right). \tag{50}$$

By the FVIM and beginning with $u_0(x,t) = 0$, we can obtain the following approximations:

$$u_{k+1}(x,t) = u_k(x,0) + J^{\alpha}\left(x + xt^2\right) - J^{\alpha}\left(u_k \frac{\partial u_k}{\partial x}\right), \tag{51}$$

$$\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= x\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}\right),\\
u_2(x,t) &= x\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{\Gamma(1+2\alpha)\, t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{4\Gamma(3+2\alpha)\, t^{3\alpha+2}}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)} - \frac{4\Gamma(5+2\alpha)\, t^{3\alpha+4}}{\Gamma(3+\alpha)^2\Gamma(5+3\alpha)}\right),
\end{aligned}$$

Table 3: Numerical values when $\alpha = 0.5$, 0.75, and 1.0 for (44).

t	x	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
		FVIM	VIM	FVIM	VIM	FVIM	VIM	Exact
0.25	0.25	0.12422501	0.12306887	0.09230374	0.09291265	0.06250058	0.06250058	0.062500
	0.50	0.24845002	0.24613773	0.18460748	0.18582531	0.12500117	0.12500117	0.125000
	0.75	0.37267504	0.36920660	0.27691122	0.27873796	0.18750175	0.18750175	0.187500
	1.00	0.49690005	0.49227547	0.36921496	0.37165062	0.25000234	0.25000234	0.250000
0.50	0.25	0.18377520	0.19472445	0.15148283	0.15611713	0.12507592	0.12507592	0.125000
	0.50	0.36755040	0.38944890	0.30296566	0.31223426	0.25015184	0.25015184	0.250000
	0.75	0.55132559	0.58417334	0.45444848	0.46835139	0.37522776	0.37522776	0.375000
	1.00	0.73510079	0.77889779	0.60593131	0.62446853	0.50030368	0.50030368	0.500000
0.75	0.25	0.27227270	0.22829012	0.21407798	0.20170432	0.18881843	0.18881843	0.187500
	0.50	0.54454540	0.45658025	0.42815596	0.40340864	0.37763687	0.37763687	0.375000
	0.75	0.81681810	0.68487037	0.64223394	0.60511296	0.56645530	0.56645530	0.562500
	1.00	1.08909080	0.91316050	0.85631192	0.80681728	0.75527373	0.75527373	0.750000

$$\begin{aligned} u_3(x,t) &= x\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{2t^{2+\alpha}}{\Gamma(3+\alpha)}-\frac{\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right.\\ &\quad -\frac{4\Gamma(3+2\alpha)t^{3\alpha+2}}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)}\\ &\quad -\frac{4\Gamma(5+2\alpha)t^{3\alpha+4}}{\Gamma(3+\alpha)^2\Gamma(5+3\alpha)}\\ &\quad +2\left(\frac{\Gamma(1+2\alpha)\Gamma(1+4\alpha)t^{5\alpha}}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)}\right.\\ &\quad\left.\left. +\frac{8\Gamma(5+2\alpha)^2\Gamma(8+6\alpha)t^{5\alpha+6}}{\Gamma(3+\alpha)^3\Gamma(5+3\alpha)\Gamma(7+5\alpha)}+\cdots\right)\right). \end{aligned} \tag{52}$$

Table 3 shows the approximate solutions for (44) obtained for different values of α using methods VIM, ADM, and FVIM. The values of $\alpha = 1$ is the only case for which we know the exact solution $u(x,t) = xt$. From (49) and (52), it is obvious that the solution of (44) obtained using the FVIM is the same as that of ADM. As the previous examples, the fourth-order term of the VIM solution and four terms of the FVIM are used in evaluating the approximate solutions for Table 3.

6. Conclusion

The main goal of this work is to conduct a comparative study between fractional variational iteration method and the Adomian's decomposition method. The two methods are powerful and effective tools for the solution of fractional partial differential equations, and both give approximations of higher accuracy and closed form solutions if existing. There are three important points to make here. First, FVIM is identical to the decomposition method in some sense. Second, FVIM reduces the computational workload by avoiding the evaluation of Adomian's polynomials, hence the iteration is straightforward. Third, FVIM provides the components of the exact solution like the ADM, but there is no need to add successive components to get the series solution. So, FVIM is more effective than ADM in solving the fractional partial differential equations.

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