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*Research Article*

# An Implicit Iteration Process for Common Fixed Points of Two Infinite Families of Asymptotically Nonexpansive Mappings in Banach Spaces

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Let $K$ be a nonempty, closed, and convex subset of a real uniformly convex Banach space $E$. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$ be two infinite families of asymptotically nonexpansive mappings from $K$ to itself with $F := \{x \in K : T_\lambda x = x = S_\lambda x, \lambda \in \Lambda\} \neq \emptyset$. For an arbitrary initial point $x_0 \in K$, $\{x_n\}$ is defined as follows: $x_n = \alpha_n x_{n-1} + \beta_n (T^*_{n-1})^{m_{n-1}} x_{n-1} + \gamma_n (T^*_n)^{m_n} y_n$, $y_n = \alpha'_n x_n + \beta'_n (S^*_{n-1})^{m_{n-1}} x_{n-1} + \gamma'_n (S^*_n)^{m_n} x_n$, $n = 1, 2, 3, \ldots$, where $T^*_n = T_{\lambda_{i_n}}$ and $S^*_n = S_{\lambda_{i_n}}$ with $i_n$ and $m_n$ satisfying the positive integer equation: $n = i + (m-1)m/2$, $m \geq i$; $\{T_{\lambda_i}\}_{i=1}^{\infty}$ and $\{S_{\lambda_i}\}_{i=1}^{\infty}$ are two countable subsets of $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$, respectively; $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ are sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $\alpha'_n + \beta'_n + \gamma'_n = 1$. Under some suitable conditions, a strong convergence theorem for common fixed points of the mappings $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$ is obtained. The results extend those of the authors whose related researches are restricted to the situation of finite families of asymptotically nonexpansive mappings.

## 1. Introduction

Let $K$ be a nonempty, closed, and convex subset of a real uniformly convex Banach space $E$. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. $T$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ $(n \to \infty)$ such that

$$\left\|T^n x - T^n y\right\| \leq k_n \left\|x - y\right\|, \quad \forall x, y \in K, \ n = 1, 2, 3, \ldots. \tag{1}$$

It is obvious that a nonexpansive mapping is an asymptotically nonexpansive one, but the converse is not true. Denote by $F(T)$ the set of fixed points of $T$, that is, $F(T) = \{x \in K : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \emptyset$. As an important generalization of nonexpansive mappings, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972, who proved that if $K$ is a nonempty, closed, and convex subset of a real uniformly convex Banach space and $T : K \to K$ is an asymptotically nonexpansive mapping, then $T$ has a fixed point.

Since then, iterative techniques for approximating fixed points of asymptotically nonexpansive mappings have been studied by various authors (see, e.g., [2–9]). However, these researches are all restricted to the situation of at most finite families of asymptotically nonexpansive mappings. For the extension of finite families to infinite ones, we develop an original method, namely, a specific way of choosing the indexes, for the iterative approximation of common fixed points of the involved mappings.

We now cite an announced result as the object of our extension. In 2010, Wang et al. [10] constructed the following iteration process for two asymptotically nonexpansive mappings and obtained some strong convergence theorems for common fixed points of the given mappings in Banach spaces. For an arbitrary initial point $x_0 \in K$, $\{x_n\}, \{y_n\}$ are defined as follows:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T^{n-1} x_{n-1} + \gamma_n T^n y_n, \\ y_n &= \alpha'_n x_n + \beta'_n S^{n-1} x_{n-1} + \gamma'_n S^n x_n, \\ &\qquad n = 1, 2, 3, \ldots, \end{aligned} \tag{2}$$

where $T, S : K \to K$ are two asymptotically nonexpansive mappings; $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, and $\{\gamma_n'\}$ are real sequences in $[0,1)$ satisfying $\alpha_n+\beta_n+\gamma_n = 1$ and $\alpha_n'+\beta_n'+\gamma_n' = 1$.

In this paper, a modified iteration scheme of (2) is used for approximating common fixed points of two infinite families of asymptotically nonexpansive mappings; a strong convergence theorem is established in the framework of uniformly convex Banach spaces. The results show the feasibility of the newly developed technique and extend those of the authors whose related researches are restricted to the situation of finite families of such mappings.

## 2. Preliminaries

Throughout this paper, we use $F$ to denote the set of common fixed points of two infinite families of asymptotically nonexpansive mappings $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$, that is, $F := \{x \in K : T_\lambda x = x = S_\lambda, \lambda \in \Lambda\}$.

Let $K$ be a nonempty, closed, and convex subset of a real Banach space $E$. Let $\{T_{\lambda_i}\}_{i=1}^{\infty}$ and $\{S_{\lambda_i}\}_{i=1}^{\infty}$ be two countable subsets of $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$, respectively. In order to approximate some member of $F$, we define, from an arbitrary $x_0 \in K$, the following implicit iteration scheme:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n (T_{n-1}^*)^{m_{n-1}} x_{n-1} + \gamma_n (T_n^*)^{m_n} y_n, \\ y_n &= \alpha_n' x_n + \beta_n' (S_{n-1}^*)^{m_{n-1}} x_{n-1} + \gamma_n' (S_n^*)^{m_n} x_n, \\ &\qquad n = 1,2,3,\ldots, \end{aligned} \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, and $\{\gamma_n'\}$ are sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$, satisfying $\alpha_n+\beta_n+\gamma_n = 1$ and $\alpha_n'+\beta_n'+\gamma_n' = 1$; $T_n^* = T_{\lambda_{i_n}}$ and $S_n^* = S_{\lambda_{i_n}}$ with $i_n$ and $m_n$ being the solutions to the positive integer equation: $n = i + (m-1)m/2$ $(m \ge i, n = 1,2,3,\ldots)$, that is, for each $n \ge 1$, there exist unique $i_n$ and $m_n$ such that

$$\begin{aligned} &i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \\ &i_4 = 1, \quad i_5 = 2, \quad i_6 = 3, \\ &i_7 = 1, \quad i_8 = 2, \ldots, \\ &m_1 = 1, \quad m_2 = 2, \quad m_3 = 2, \\ &m_4 = 3, \quad m_5 = 3, \quad m_6 = 3, \\ &m_7 = 4, \quad m_8 = 4, \ldots. \end{aligned} \tag{4}$$

For convenience, we restate the following concepts and results.

A Banach space $E$ is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in $E$, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \tag{5}$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to $x$.

A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be demiclosed at $p$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to $p$, then $Tx^* = p$.

We now need the following lemmas for our main results.

**Lemma 1** (see [11]). *Let $\{a_n\}$, $\{\delta_n\}$, and $\{b_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1+\delta_n) a_n + b_n, \quad n = 1,2,3,\ldots. \tag{6}$$

*If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

**Lemma 2** (see [6]). *Let $E$ be a real uniformly convex Banach space, and let $a$, and $b$ be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a,b]$ is a real sequence and $\{x_n\}$ and $\{y_n\}$ are two sequences in $E$. Then, the conditions*

$$\begin{gathered} \lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = d, \\ \limsup_{n\to\infty} \|x_n\| \le d, \qquad \limsup_{n\to\infty} \|y_n\| \le d \end{gathered} \tag{7}$$

*imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \ge 0$ is a constant.*

**Lemma 3** (see [2]). *Let $E$ be a real uniformly convex Banach space, $K$ a nonempty, closed, convex subset of $E$, and let $T : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$ as $n \to \infty$. Then, $I - T$ is demiclosed at zero.*

**Lemma 4.** *The unique solutions to the positive integer equation*

$$n = i + \frac{(m-1)m}{2}, \qquad m \ge i, \ n = 1,2,3,\ldots \tag{8}$$

*are*

$$\begin{gathered} i = n - \frac{(m-1)m}{2}, \qquad m = -\left[\frac{1}{2} - \sqrt{2n + \frac{1}{4}}\right], \\ n = 1,2,3,\ldots, \end{gathered} \tag{9}$$

*where $[x]$ denotes the maximal integer that is not larger than $x$.*

*Proof.* It follows from (8) that

$$i = n - \frac{(m-1)m}{2}, \quad i \le m, \ n = 1,2,3,\ldots, \tag{10}$$

and hence

$$1 \le i = n - \frac{(m-1)m}{2} \le m, \quad n = 1,2,3,\ldots, \tag{11}$$

that is,

$$\frac{(m-1)m}{2} + 1 \le n \le \frac{(m+1)m}{2}, \quad n = 1,2,3,\ldots, \tag{12}$$

which implies that

$$\begin{gathered} \left(m - \frac{1}{2}\right)^2 \le 2n - \frac{7}{4}, \qquad \left(m + \frac{1}{2}\right)^2 \ge 2n + \frac{1}{4}, \\ n = 1,2,3,\ldots. \end{gathered} \tag{13}$$

Thus,

$$\sqrt{2n+\frac{1}{4}}-\frac{1}{2}\le m\le\frac{1}{2}+\sqrt{2n-\frac{7}{4}},\quad n=1,2,3\ldots, \tag{14}$$

that is,

$$-\sqrt{2n-\frac{7}{4}}-\frac{1}{2}\le -m\le\frac{1}{2}-\sqrt{2n+\frac{1}{4}},\quad n=1,2,3,\ldots, \tag{15}$$

while the difference of the two sides of the inequality above is

$$\begin{aligned}1&-\left(\sqrt{2n+\frac{1}{4}}-\sqrt{2n-\frac{7}{4}}\right)\\&=1-\frac{2}{\sqrt{2n+1/4}+\sqrt{2n-7/4}}\in[0,1),\\&\qquad n=1,2,3,\ldots.\end{aligned} \tag{16}$$

Then, it follows from (15) that (9) holds obviously. □

## 3. Main Results

**Lemma 5.** *Let $K$ be a nonempty, closed, and convex subset of a real uniformly convex Banach space $E$, and let $\{T_{\lambda_i}\}_{i=1}^{\infty}$ and $\{S_{\lambda_i}\}_{i=1}^{\infty}$ be two countable subsets of the asymptotically nonexpansive mappings $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$ from $K$ to itself, respectively, with corresponding sequences $\{k_n^{(i)}\}\subset[1,\infty)$ and $\{r_n^{(i)}\}\subset[1,\infty)$ such that $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(r_n^{(i)}-1)<\infty$. Suppose that $\{x_n\}$ is generated by (3), where $\gamma_n k_n^{(i)}(\alpha_n'+\gamma_n' r_n^{(i)})<1$ for all $i\ge 1$. If $F\ne\emptyset$, then $\lim_{n\to\infty}\|x_n-q\|$ exists for each $q\in F$.*

*Proof.* Set $u_{m_n}^{(i_n)}=k_{m_n}^{(i_n)}-1$ and $v_{m_n}^{(i_n)}=r_{m_n}^{(i_n)}-1$ for each positive integer $n\ge 1$, where $i_n$ and $m_n$ satisfy the positive integer equation: $n=i+(m-1)m/2(m\ge i, m\in\mathbb{N})$. For any $q\in F$, it follows from (3) that

$$\begin{aligned}\|y_n-q\|&\le\alpha_n'\|x_n-q\|+\beta_n' r_{m_{n-1}}^{(i_{n-1})}\|x_{n-1}-q\|\\&\quad+\gamma_n' r_{m_n}^{(i_n)}\|x_{n-1}-q\|=\left(\alpha_n'+\gamma_n' r_{m_n}^{(i_n)}\right)\|x_n-q\|\\&\quad+\beta_n' r_{m_{n-1}}^{(i_{n-1})}\|x_{n-1}-q\|,\end{aligned} \tag{17}$$

$$\begin{aligned}\|x_n-q\|&\le\alpha_n\|x_{n-1}-q\|+\beta_n k_{m_{n-1}}^{(i_{n-1})}\|x_{n-1}-q\|\\&\quad+\gamma_n k_{m_n}^{(i_n)}\|y_n-q\|=\left(\alpha_n+\beta_n k_{m_{n-1}}^{(i_{n-1})}\right)\|x_{n-1}-q\|\\&\quad+\gamma_n k_{m_n}^{(i_n)}\|y_n-q\|.\end{aligned} \tag{18}$$

Substituting (17) into (18) yields that

$$\begin{aligned}&\left[1-\gamma_n k_{m_n}^{(i_n)}\left(\alpha_n'+\gamma_n' r_{m_n}^{(i_n)}\right)\right]\|x_n-q\|\\&\quad\le\left(\alpha_n+\beta_n k_{m_{n-1}}^{(i_{n-1})}+\gamma_n k_{m_n}^{(i_n)}\beta_n' r_{m_{n-1}}^{(i_{n-1})}\right)\|x_{n-1}-q\|.\end{aligned} \tag{19}$$

Note that $a_n:=1-\gamma_n k_{m_n}^{(i_n)}(\alpha_n'+\gamma_n' r_{m_n}^{(i_n)})>0$. We have

$$\|x_n-q\|\le\frac{\alpha_n+\beta_n k_{m_{n-1}}^{(i_{n-1})}+\gamma_n k_{m_n}^{(i_n)}\beta_n' r_{m_{n-1}}^{(i_{n-1})}}{a_n}\|x_{n-1}-q\|, \tag{20}$$

which implies that $\|x_n-q\|\le(1+b_n)\|x_{n-1}-q\|$, where

$$\begin{aligned}b_n=&\left(\gamma_n\gamma_n' v_{m_n}^{(i_n)}+\gamma_n u_{m_n}^{(i_n)}+\gamma_n\gamma_n' u_{m_n}^{(i_n)}v_{m_n}^{(i_n)}+\beta_n u_{m_{n-1}}^{(i_{n-1})}\right.\\&\left.+\gamma_n\beta_n' v_{m_{n-1}}^{(i_{n-1})}+\gamma_n\beta_n' u_{m_n}^{(i_n)}v_{m_{n-1}}^{(i_{n-1})}\right)\\&\times\left(1-\gamma_n\left(1-\beta_n'\right)-\gamma_n\gamma_n' v_{m_n}^{(i_n)}\right.\\&\left.-\gamma_n u_{m_n}^{(i_n)}\left(1-\beta_n'\right)-\gamma_n\gamma_n' u_{m_n}^{(i_n)}v_{m_n}^{(i_n)}\right)^{-1}.\end{aligned} \tag{21}$$

Note that $\sum_{n=1}^{\infty}u_{m_n}^{(i_n)}=\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}(k_n^{(i)}-1)\le\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}v_{m_n}^{(i_n)}=\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}(r_n^{(i)}-1)\le\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(r_n^{(i)}-1)<\infty$, which implies that $\lim_{n\to\infty}u_{m_n}^{(i_n)}=\lim_{n\to\infty}v_{m_n}^{(i_n)}=0$. Then, for a given $\epsilon_0\in(0,\delta)$, there exists a positive $n_0$ such that

$$\begin{aligned}&\gamma_n\left(1-\beta_n'\right)+\gamma_n\gamma_n' v_{m_n}^{(i_n)}+\gamma_n u_{m_n}^{(i_n)}\left(1-\beta_n'\right)\\&\quad+\gamma_n\gamma_n' u_{m_n}^{(i_n)}v_{m_n}^{(i_n)}<1-\epsilon_0,\end{aligned} \tag{22}$$

as $n\ge n_0$. Then, it follows from (20) and (22) that

$$\|x_n-q\|\le\left(1+c_n\right)\|x_{n-1}-q\|, \tag{23}$$

where $c_n=(1/\epsilon_0)[v_{m_n}^{(i_n)}+u_{m_n}^{(i_n)}+u_{m_n}^{(i_n)}v_{m_n}^{(i_n)}+u_{m_{n-1}}^{(i_{n-1})}+v_{m_{n-1}}^{(i_{n-1})}+u_{m_n}^{(i_n)}v_{m_{n-1}}^{(i_{n-1})}]$, and so $\sum_{n=1}^{\infty}c_n<\infty$. Hence, it follows from (23) and Lemma 1 that $\lim_{n\to\infty}\|x_n-q\|$ exists for each $q\in F$. The proof is completed. □

*Remark 6.* Because of the importance of the condition that $F$ is nonempty, we now give an example satisfying the lemma with the set of common fixed points of $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\lambda\}_{\lambda\in\Lambda}$ being a non single point set. Let $E:=\mathbb{R}^1$, and let $K:=[-1,1]$. Define an infinite family of mappings $\{T_\lambda\}_{\lambda\ge 1}:K\to K$ by

$$T_\lambda(x)=\begin{cases}\dfrac{1}{\lambda}x^\lambda, & x\in[0,1],\\ x, & x\in[-1,0),\end{cases} \tag{24}$$

and an infinite family of mappings $\{S_\lambda\}_{\lambda\ge 1}:K\to K$ by

$$S_\lambda(x)=\begin{cases}\dfrac{1}{\lambda}\sin x^\lambda, & x\in[0,1],\\ x, & x\in[-1,0).\end{cases} \tag{25}$$

Then, clearly, $\{T_\lambda\}_{\lambda\ge 1}$ and $\{S_\lambda\}_{\lambda\ge 1}$ are two infinite families of asymptotically nonexpansive mappings with $F=[-1,0]$.

**Lemma 7.** *Let $K$, $E$, $\{T_{\lambda_i}\}_{i=1}^{\infty}$, $\{S_{\lambda_i}\}_{i=1}^{\infty}$, and $\{x_n\}$ be the same as those in Lemma 5. If $F\ne\emptyset$, then for each $i\ge 1$, there exists a subsequence $\{x_n^{(i)}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty}\|x_n^{(i)}-T_{\lambda_i}x_n^{(i)}\|=\lim_{n\to\infty}\|x_n^{(i)}-S_{\lambda_i}x_n^{(i)}\|=0$.*

*Proof.* By Lemma 5, we may assume that $\lim_{n\to\infty}\|x_n-q\| = d$ for a given $q \in F$, that is,

$$\lim_{n\to\infty} d_n = d, \tag{26}$$

where $d_n = \|(1-\gamma_n)[\alpha_n(x_{n-1}-q)/(1-\gamma_n)+\beta_n((T_{n-1}^*)^{m_{n-1}}x_{n-1}-q)/(1-\gamma_n)] + \gamma_n((T_n^*)^{m_n}y_n - q)\|$. It follows from (17) that

$$\begin{aligned}
&\left\|(T_n^*)^{m_n}y_n - q\right\| \\
&\quad\le k_{m_n}^{(i_n)}\left\|y_n - q\right\| \\
&\quad\le k_{m_n}^{(i_n)}\left[\left(\alpha_n' + \gamma_n' r_{m_n}^{(i_n)}\right)\left\|x_n - q\right\| + \beta_n' r_{m_{n-1}}^{(i_{n-1})}\left\|x_{n-1} - q\right\|\right] \\
&\quad= k_{m_n}^{(i_n)}\left[\left\|x_n - q\right\| + \beta_n'\left(\left\|x_n - q\right\| - \left\|x_{n-1} - q\right\|\right)\right. \\
&\qquad\left. + \gamma_n' v_{m_n}^{(i_n)}\left\|x_n - q\right\| + \beta_n' v_{m_{n-1}}^{(i_{n-1})}\left\|x_{n-1} - q\right\|\right].
\end{aligned} \tag{27}$$

Taking lim sup on both sides in (27) yields that

$$\limsup_{n\to\infty}\left\|(T_n^*)^{m_n}y_n - q\right\| \le \limsup_{n\to\infty}\left\|y_n - q\right\| \le d. \tag{28}$$

Next, it follows from (26) that

$$\begin{aligned}
\limsup_{n\to\infty} e_n &\le \limsup_{n\to\infty}\left(\frac{\alpha_n}{1-\gamma_n}\left\|x_{n-1} - q\right\|\right. \\
&\qquad\left. + \frac{\beta_n}{1-\gamma_n}k_{m_{n-1}}^{(i_{n-1})}\left\|x_{n-1} - q\right\|\right) \\
&= \limsup_{n\to\infty}\left(1 + \frac{\beta_n}{1-\gamma_n}u_{m_{n-1}}^{(i_{n-1})}\right)\left\|x_{n-1} - q\right\| = d,
\end{aligned} \tag{29}$$

where $e_n = \|\alpha_n(x_{n-1} - q) + \beta_n((T_{n-1}^*)^{m_{n-1}}x_{n-1} - q)\|/(1 - \gamma_n)$. It then follows from (26), (28), (29), and Lemma 2 that

$$\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}x_{n-1} + \frac{\beta_n}{1-\gamma_n}(T_{n-1}^*)^{m_{n-1}}x_{n-1} - (T_n^*)^{m_n}y_n\right\| = 0, \tag{30}$$

which, in addition to (3), implies that

$$\lim_{n\to\infty}\left\|x_n - (T_n^*)^{m_n}y_n\right\| = 0. \tag{31}$$

Now, we show that $\|y_n - q\| \to d$ as $n \to \infty$. It follows from (18) that

$$\begin{aligned}
&\frac{\alpha_n + \beta_n}{\gamma_n}\left(\left\|x_n - q\right\| - \left\|x_{n-1} - q\right\|\right) + \left\|x_n - q\right\| \\
&\quad - \frac{\beta_n}{\gamma_n}u_{m_{n-1}}^{(i_{n-1})}\left\|x_{n-1} - q\right\| \le k_{m_n}^{(i_n)}\left\|y_n - q\right\|.
\end{aligned} \tag{32}$$

Taking lim inf on both sides in the inequality above yields that

$$d = \liminf_{n\to\infty}\left\|x_n - q\right\| \le \liminf_{n\to\infty}\left\|y_n - q\right\|. \tag{33}$$

Combining (28) with (33), we have $\lim_{n\to\infty}\|y_n - q\| = d$. Then, by ways similar to the preceding ones, it is easily shown that

$$\lim_{n\to\infty}\left\|\frac{\beta_n'}{1-\alpha_n'}(S_{n-1}^*)^{m_{n-1}}x_{n-1} + \frac{\gamma_n'}{1-\alpha_n'}(S_n^*)^{m_n}x_n - x_n\right\| = 0, \tag{34}$$

which means that

$$\lim_{n\to\infty}\left\|y_n - x_n\right\| = 0. \tag{35}$$

Set $d_n' = \|(1-\beta_n)[\alpha_n(x_{n-1}-q)/(1-\beta_n)+\gamma_n((T_n^*)^{m_n}y_n-q)/(1-\beta_n)] + \beta_n((T_{n-1}^*)^{m_{n-1}}x_{n-1} - q)\|$. Since

$$\lim_{n\to\infty}\left\|x_n - q\right\| = \lim_{n\to\infty} d_n' = d, \tag{36}$$

then

$$\begin{aligned}
&\limsup_{n\to\infty}\left\|(T_{n-1}^*)^{m_{n-1}}x_{n-1} - q\right\| \\
&\quad\le \limsup_{n\to\infty} k_{m_{n-1}}^{(i_{n-1})}\left\|x_{n-1} - q\right\| \le d,
\end{aligned} \tag{37}$$

$$\begin{aligned}
\limsup_{n\to\infty} e_n' &\le \limsup_{n\to\infty}\left(\frac{\alpha_n}{1-\beta_n}\left\|x_{n-1} - q\right\|\right. \\
&\qquad\left. + \frac{\gamma_n}{1-\beta_n}k_{m_n}^{(i_n)}\left\|y_n - q\right\|\right) = d,
\end{aligned} \tag{38}$$

where $e_n' = \|\alpha_n(x_{n-1} - q) + \gamma_n((T_n^*)^{m_n}y_n - q)\|/(1 - \beta_n)$. It then follows from (36)–(38) and Lemma 2 that

$$\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\beta_n}x_{n-1} + \frac{\gamma_n}{1-\beta_n}(T_n^*)^{m_n}y_n - (T_{n-1}^*)^{m_{n-1}}x_{n-1}\right\| = 0, \tag{39}$$

which implies that

$$\lim_{n\to\infty}\left\|x_n - (T_{n-1}^*)^{m_{n-1}}x_{n-1}\right\| = 0. \tag{40}$$

Then, it follows from (31), (35), and (40) that

$$\lim_{n\to\infty}\left\|x_n - x_{n-1}\right\| = 0. \tag{41}$$

Similarly, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left\|y_n - (S_n^*)^{m_n}x_n\right\| = 0, \\
&\lim_{n\to\infty}\left\|y_n - (S_{n-1}^*)^{m_{n-1}}x_{n-1}\right\| = 0.
\end{aligned} \tag{42}$$

Next, for any $i \ge 1$, we consider the corresponding subsequence $\{x_l^{(i)}\}_{l\in\Gamma_i}$ of $\{x_n\}$, where $l \in \Gamma_i := \{l : l = i + (j-1)j/2, j \ge i, j \in \mathbb{N}\}$. For example, by the definition of $\Gamma_1$, we have $\Gamma_1 = \{1, 2, 4, 7, 11, 16, \ldots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. For simplicity, $\{x_l^{(i)}\}_{l\in\Gamma_i}$, $\{y_l^{(i)}\}_{l\in\Gamma_i}$, $\{T_l^{*(i)}\}_{l\in\Gamma_i}$, and $\{j_l^{(i)}\}_{l\in\Gamma_i}$ are written as $\{x_n'\}$, $\{y_n'\}$, $\{T_n'\}$ and $\{m_n\}$, respectively. Note that $\{m_n\}_{n\in\Gamma_i} = \{i, i+1, i+2, \ldots\}$, that is,

$k_1^{(i_l)} = k_1^{(i)}, m_n - 1 = m_{n-1}$, and $T_n' = T_{\lambda_i} = T_{n-1}'$ whenever $l \in \Gamma_i$. Then, we have

$$\begin{aligned}
&\left\|x_n' - T_n' x_n\right\| \\
&\quad \le \left\|x_n' - \left(T_n'\right)^{m_n} y_n'\right\| \\
&\qquad + \left\|\left(T_n'\right)^{m_n} y_n' - \left(T_n'\right)^{m_n} x_n'\right\| + \left\|\left(T_n'\right)^{m_n} x_n' - T_n' x_n'\right\| \\
&\quad \le \left\|x_n' - \left(T_n'\right)^{m_n} y_n'\right\| + k_{m_n}^{(i)} \left\|y_n' - x_n'\right\| \\
&\qquad + k_1^{(i)} \left\|\left(T_n'\right)^{m_n - 1} x_n' - x_n'\right\| \le \left\|x_n' - \left(T_n'\right)^{m_n} y_n'\right\| \\
&\qquad + k_{m_n}^{(i)} \left\|y_n' - x_n'\right\| + k_1^{(i)} \\
&\qquad \times \left[\left\|\left(T_n'\right)^{m_{n}-1} x_n' - \left(T_{n-1}'\right)^{m_{n-1}} x_{n-1}'\right\| \right. \\
&\qquad\quad \left. + \left\|\left(T_{n-1}'\right)^{m_{n-1}} x_{n-1}' - x_n'\right\|\right] \\
&\quad \le \left\|x_n' - \left(T_n'\right)^{m_n} y_n'\right\| + k_{m_n}^{(i)} \left\|y_n' - x_n'\right\| \\
&\qquad + k_1^{(i)} \left[k_{m_{n-1}}^{(i)} \left\|x_n' - x_{n-1}'\right\| + \left\|\left(T_{n-1}'\right)^{m_{n-1}} x_{n-1}' - x_n'\right\|\right].
\end{aligned} \tag{43}$$

It hence follows from (31), (35), (40), and (41) that $\lim_{n\to\infty}\|x_n' - T_n' x_n'\| = 0$. That is, for each $i \ge 1$, there exists a subsequence $\{x_n^{(i)}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty}\|x_n^{(i)} - T_n^{*(i)} x_n^{(i)}\| = 0$. Since $T_n^{*(i)} = T_{\lambda_i}$, we have, for each $i \ge 1$,

$$\lim_{n\to\infty} \left\|x_n^{(i)} - T_{\lambda_i} x_n^{(i)}\right\| = 0. \tag{44}$$

Similarly, it can be shown that, for each $i \ge 1$,

$$\lim_{n\to\infty} \left\|y_n^{(i)} - S_{\lambda_i} y_n^{(i)}\right\| = 0. \tag{45}$$

This completes the proof. □

*Remark 8.* The key point of the proof of Lemma 7 lies in the use of a specific way of choosing the indexes of the involved mappings, which makes the generalization of finite families of nonlinear mappings to infinite ones possible.

**Theorem 9.** *Let $K$, $E$, $\{T_{\lambda_i}\}_{i=1}^{\infty}$, $\{S_{\lambda_i}\}_{i=1}^{\infty}$, and $\{x_n\}$ be the same as those in Lemma 5. If $F \ne \emptyset$ and there exist a $T_{\lambda_{i_0}} \in \{T_{\lambda_i}\}_{i=1}^{\infty}$ or an $S_{\lambda_{i_0}} \in \{S_{\lambda_i}\}_{i=1}^{\infty}$ and a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $f(d(x_n, F)) \le \|x_n - T_{\lambda_{i_0}} x_n\|$ or $f(d(x_n, F)) \le \|x_n - S_{\lambda_{i_0}} x_n\|$ for all $n \ge 1$, then $\{x_n\}$ converges strongly to some point of $F$.*

*Proof.* By Lemma 7, there exists a subsequence $\{x_n^{(i_0)}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty}\|x_n^{(i_0)} - T_{\lambda_{i_0}} x_n^{(i_0)}\| = \lim_{n\to\infty}\|x_n^{(i_0)} - S_{\lambda_{i_0}} x_n^{(i_0)}\| = 0$. Since

$$f\left(d\left(x_n^{(i_0)}, F\right)\right) \le \left\|x_n^{(i_0)} - T_{\lambda_{i_0}} x_n^{(i_0)}\right\|, \tag{46}$$

or

$$f\left(d\left(x_n^{(i_0)}, F\right)\right) \le \left\|x_n^{(i_0)} - S_{\lambda_{i_0}} x_n^{(i_0)}\right\|, \tag{47}$$

by taking lim sup as $n \to \infty$ on both sides in the inequality above, we have

$$\lim_{n\to\infty} f\left(d\left(x_n^{(i_0)}, F\right)\right) = 0, \tag{48}$$

which implies $\lim_{n\to\infty} d(x_n^{(i_0)}, F) = 0$ by the definition of the function $f$.

Now, we will show that $\{x_n^{(i_0)}\}$ is a Cauchy sequence. By Lemma 5, there exists a constant $M > 0$ such that $\|x_n - q\| \le M\|x_m - q\|$ as $n > m$. And for any $\epsilon > 0$, there exists a positive integer $N$ such that $d(x_n^{(i_0)}, F) < \epsilon/2M$ for all $n \ge N$. Then, for any $q \in F$ and $n,\ m \ge N$, we have

$$\left\|x_n^{(i_0)} - x_m^{(i_0)}\right\| \le \left\|x_n^{(i_0)} - q\right\| + \left\|x_m^{(i_0)} - q\right\| \le 2M \left\|x_N^{(i_0)} - q\right\|. \tag{49}$$

Taking the infimum in the above inequalities for all $q \in F$ yields that

$$\left\|x_n^{(i_0)} - x_m^{(i_0)}\right\| \le 2Md\left(x_N^{(i_0)}, F\right) < \epsilon, \tag{50}$$

which implies that $\{x_n^{(i_0)}\}$ is a Cauchy sequence. Therefore, there exists a $p \in K$ such that $x_n^{(i_0)} \to p$ as $n \to \infty$ since $E$ is complete. Furthermore, $\lim_{n\to\infty} d(x_n^{(i_0)}, F) = 0$ shows that $d(p, F) = 0$, which implies that $p \in F$ since $F$ is closed. It follows from the existence of $\lim_{n\to\infty}\|x_n - p\|$ that $x_n \to p$ as $n \to \infty$. This completes the proof. □

*Remark 10.* The result of Theorem 9 extends that of Wang et al. [10] whose related research is restricted to the situation of two asymptotically nonexpansive mappings.

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