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Review Article

Nonlinear Random Stability via Fixed-Point Method

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We prove the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation $f(x+2y)+f(x-2y) = 4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$ in various complete random normed spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Cholewa [6] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Czerwik [7] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [8–12]).

In [13], Jun and Kim consider the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.2}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.2), which is called a *cubic functional equation*, and every solution of the cubic functional equation is said to be a *cubic mapping*.

Considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.3}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation, which is called a *quartic functional equation*, and every solution of the quartic functional equation is said to be a *quartic mapping*. One can easily show that an odd mapping $f : \mathrm{X} \to Y$ satisfies the additive-quadratic-cubic-quadratic functional equation

$$\begin{aligned} f(x+2y) + f(x-2y) = {} & 4f(x+y) + 4f(x-y) - 6f(x) \\ & + f(2y) + f(-2y) - 4f(y) - 4f(-y) \end{aligned} \tag{1.4}$$

if and only if it is an additive-cubic mapping, that is,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x). \tag{1.5}$$

It was shown in Lemma 2.2 of [14] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = (1/6)g(x) - (1/6)h(x)$.

One can easily show that an even mapping $f : \mathrm{X} \to Y$ satisfies (1.4) if and only if it is a quadratic-quartic mapping, that is,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + 2f(2y) - 8f(y). \tag{1.6}$$

Also $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and $f(x) = (1/12)g(x) - (1/12)h(x)$.

For a given mapping $f : \mathrm{X} \to Y$, we define

$$\begin{aligned} Df(x,y) := {} & f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 6f(x) \\ & - f(2y) - f(-2y) + 4f(y) + 4f(-y) \end{aligned} \tag{1.7}$$

for all $x, y \in \mathrm{X}$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the fixed-point alternative of Diaz and Margolis.

Theorem 1.1 (see [15, 16]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$, then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.8}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$,

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J,

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$,

(4) $d(y, y^*) \leq (1/(1 - L)) d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [18–21]).

2. Preliminaries

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [22–26]. Throughout this paper, Δ^+ is the space of all probability distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$, such taht F is left continuous, nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $\{F(+\infty) = 1\}$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{2.1}$$

A *triangular norm* (shortly *t-norm*) is a binary operation on the unit interval $[0, 1]$, that is, a function $T : [0, 1] \times [0, 1] \to [0, 1]$, such that for all $a, b, c \in [0, 1]$ the following four axioms satisfied:

(T1) $T(a, b) = T(b, a)$ (commutativity),

(T2) $T(a, (T(b, c))) = T(T(a, b), c)$ (associativity), (T3) $T(a,1) = a$ (boundary condition),

(T4) $T(a,b) \leq T(a,c)$ whenever $b \leq c$ (monotonicity).

Basic examples are the Łukasiewicz t-norm $T_L, T_L(a,b) = \max(a+b-1,0)$ for all $a,b \in [0,1]$ and the t-norms T_P, T_M, T_D, where $T_P(a,b) := ab$, $T_M(a,b) := \min\{a,b\}$,

$$T_D(a,b) := \begin{cases} \min(a,b), & \text{if } \max(a,b) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

If T is a t-norm, then $x_T^{(n)}$ is defined for every $x \in [0,1]$ and $n \in N \cup \{0\}$ by 1, if $n = 0$ and $T(x_T^{(n-1)}, x)$ if $n \geq 1$. A t-norm T is said to be *of Hadžić type* (we denote by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n\in N}$ is equicontinuous at $x = 1$ (cf. [27]).

Other important triangular norms are the following (see [28]):

(1) The *Sugeno-Weber family* $\{T_\lambda^{\mathrm{SW}}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{\mathrm{SW}} = T_D$, $T_\infty^{\mathrm{SW}} = T_P$ and

$$T_\lambda^{\mathrm{SW}}(x,y) = \max\left(0, \frac{x+y-1+\lambda xy}{1+\lambda}\right) \tag{2.3}$$

if $\lambda \in (-1,\infty)$.

(2) The *Domby family* $\{T_\lambda^{D}\}_{\lambda\in[0,\infty]}$ is defined by T_D if $\lambda = 0$, T_M if $\lambda = \infty$, and

$$T_\lambda^{D}(x,y) = \frac{1}{1+\left(((1-x)/x)^\lambda + ((1-y)/y)^\lambda\right)^{1/\lambda}} \tag{2.4}$$

if $\lambda \in (0,\infty)$.

(3) The *Aczel-Alsina family* $\{T_\lambda^{\mathrm{AA}}\}_{\lambda\in[0,\infty]}$ is defined by T_D if $\lambda = 0$, T_M if $\lambda = \infty$ and

$$T_\lambda^{\mathrm{AA}}(x,y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}} \tag{2.5}$$

if $\lambda \in (0,\infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1,\ldots,x_n) \in [0,1]^n$ the value $T(x_1,\ldots,x_n)$ defined by

$$T_{i=1}^{0} x_i = 1, \qquad T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) = T(x_1,\ldots,x_n). \tag{2.6}$$

T can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in $[0,1]$ the value

$$T_{i=1}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{n} x_i. \tag{2.7}$$

The limit on the right side of (6.4) exists since the sequence $(T_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

Proposition 2.1 (see [28]). *We have the following.*

(1) *For $T \geq T_L$, the following implication holds:*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty. \tag{2.8}$$

(2) *If T is of Hadžić type, then*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \tag{2.9}$$

for every sequence $(x_n)_{n\in N}$ in $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$.

(3) *If $T \in \{T_\lambda^{\mathrm{AA}}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n)^{\alpha} < \infty. \tag{2.10}$$

(4) *If $T \in \{T_\lambda^{\mathrm{SW}}\}_{\lambda\in[-1,\infty)}$, then*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty. \tag{2.11}$$

Definition 2.2 (see [26]). A *Random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in \mathrm{X}$, and $\alpha \neq 0$,

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in \mathrm{X}$ and $t, s \geq 0$.

Definition 2.3. Let (X,μ,T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x}(\epsilon) > 1-\lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1-\lambda$ whenever $n \geq m \geq N$.

(3) An RN-space (X,μ,T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X. A complete RN-space is said to be random Banach space.

Theorem 2.4 (see [25]). *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ *almost everywhere.*

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us with the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces, and fuzzy normed spaces has been recently studied [20, 24, 29–39].

3. Non-Archimedean Random Normed Space

By a *non-Archimedean field,* we mean a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r+s| \le \max\{|r|,|s|\}$ for all $r,s \in \mathcal{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$. By the *trivial valuation*, we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. Let X be a vector space over a field $\mathcal{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(NAN1) $\|x\| = 0$ if and only if $x = 0$,

(NAN2) for any $r \in \mathcal{K}$ and $x \in X$, $\|rx\| = |r|\|x\|$,

(NAN3) the strong triangle inequality (ultrametric), namely,

$$\|x+y\| \le \max\{\|x\|, \|y\|\} \quad (x, y \in X), \tag{3.1}$$

then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*. Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} \quad (n > m), \tag{3.2}$$

a sequence $\{x_n\}$ is a Cauchy sequence if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [40] discovered the p-adic numbers of as a number theoretical analogues of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*.

Throughout the paper, we assume that X is a vector space and Y is a complete non-Archimedean normed space.

Definition 3.1. A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple (X, μ, T), where X is a linear space over a non-Archimedean field $\mathcal{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(NA-RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(NA-RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $t > 0$, and $\alpha \ne 0$,

(NA-RN3) $\mu_{x+y}(\max\{t,s\}) \ge T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \ge 0$.

It is easy to see that if (NA-RN3) holds, then so is

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$.

As a classical example, if $(X, \|.\|)$ is a non-Archimedean normed linear space, then the triple (X, μ, T_M), where

$$\mu_x(t) = \begin{cases} 0, & t \leq \|x\|, \\ 1, & t > \|x\|, \end{cases} \tag{3.3}$$

is a non-Archimedean RN-space.

Example 3.2. Let $(X, \|\cdot\|)$ be a non-Archimedean normed linear space. Define

$$\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X,\ t > 0), \tag{3.4}$$

then (X, μ, T_M) is a non-Archimedean RN-space.

Definition 3.3. Let (X, μ, T) be a non-Archimedean RN-space. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that

$$\lim_{n \to \infty} \mu_{x_n - x}(t) = 1 \tag{3.5}$$

for all $t > 0$. In that case, x is called the *limit* of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $\mu_{x_{n+p}-x_n}(t) > 1 - \varepsilon$.

If each Cauchy sequence is convergent, then the random norm is said to be *complete* and the non-Archimedean RN-space is called a non-Archimedean *random Banach space*.

Remark 3.4 (see [41]). Let (X, μ, T_M) be a non-Archimedean RN-space, then

$$\mu_{x_{n+p}-x_n}(t) \geq \min\left\{\mu_{x_{n+j+1}-x_{n+j}}(t) : j = 0, 1, 2, \ldots, p-1\right\}. \tag{3.6}$$

So, the sequence $\{x_n\}$ is a Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$ there exists n_0 such that for all $n \geq n_0$,

$$\mu_{x_{n+1}-x_n}(t) > 1 - \varepsilon. \tag{3.7}$$

4. Generalized Ulam-Hyers Stability for a Quartic Functional Equation in Non-Archimedean RN-Spaces of Functional Equation (1.4): An Odd Case

Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$, and let (Y, μ, T) be a non-Archimedean random Banach space over $\mathcal{K}$.

Next, we define a random approximately AQCQ mapping. Let Ψ be a distribution function on $X \times X \times [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing and

$$\Psi(cx, cx, t) \geq \Psi\left(x, x, \frac{t}{|c|}\right) \quad (x \in X,\ c \neq 0). \tag{4.1}$$

Definition 4.1. A mapping $f : X \to Y$ is said to be Ψ-*approximately AQCQ* if

$$\mu_{Df(x,y)}(t) \geq \Psi(x, y, t) \quad (x, y \in X,\ t > 0). \tag{4.2}$$

In this section, we assume that $2 \neq 0$ in $\mathcal{K}$ (i.e., characteristic of $\mathcal{K}$ is not 2). Our main result, in this section, is the following.

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean random spaces, an odd case.

Theorem 4.2. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$ and let (Y, μ, T) be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : X \to Y$ be an odd mapping and Ψ-approximately AQCQ mapping. If for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 3$ with $|2^k| < \alpha$,*

$$\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X,\ t > 0), \tag{4.3}$$

$$\lim_{n \to \infty} T_{j=n}^{\infty} M\left(2x, \frac{\alpha^j t}{|8|^{kj}}\right) = 1 \quad (x \in X,\ t > 0), \tag{4.4}$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|8|^{ki}}\right) \tag{4.5}$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) := T^{k-1}\left[\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{|4|}\right), \Psi\left(x, \frac{x}{2}, t\right), \ldots, \Psi\left(\frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, \frac{t}{|4|}\right), \Psi\left(2^{k-1}x, \frac{2^{k-1}x}{2}, t\right)\right] \quad (x \in X,\ t > 0). \tag{4.6}$$

Proof. Letting $x = y$ in (4.2), we get

$$\mu_{f(3y)-4f(2y)+5f(y)}(t) \geq \Psi(y, y, t) \tag{4.7}$$

for all $y \in X$ and $t > 0$. Replacing x by $2y$ in (4.2), we get

$$\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq \Psi(2y, y, t) \tag{4.8}$$

for all $y \in X$ and $t > 0$. By (4.7) and (4.8), we have

$$\begin{aligned}\mu_{f(4y)-10f(2y)+16f(y)}(t) &\geq T\left(\mu_{4(f(3y)-4f(2y)+5f(y))}(t), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right)\\ &= T\left(\mu_{f(3y)-4f(2y)+5f(y)}\left(\frac{t}{|4|}\right), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right)\\ &\geq T\left(\Psi\left(y, y, \frac{t}{|4|}\right), \Psi(2y, y, t)\right)\end{aligned} \tag{4.9}$$

for all $y \in X$ and $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$ in (4.9), we get

$$\mu_{g(x)-8g(x/2)}(t) \geq T\left(\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{|4|}\right), \Psi\left(x, \frac{x}{2}, t\right)\right) \tag{4.10}$$

for all $x \in X$ and $t > 0$. Now, we show by induction on j that for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\begin{aligned}&\mu_{g(2^{j-1}x)-8^j g(x/2)}(t)\\ &\quad\geq M_j(x,t)\\ &\quad:= T^{2j-1}\left[\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{|4|}\right), \Psi\left(x, \frac{x}{2}, t\right), \ldots, \Psi\left(\frac{2^{j-1}x}{2}, \frac{2^{j-1}x}{2}, \frac{t}{|4|}\right), \Psi\left(2^{j-1}x, \frac{2^{j-1}x}{2}, t\right)\right].\end{aligned} \tag{4.11}$$

Putting $j = 1$ in (4.11), we obtain (4.10). Assume that (4.11) holds for some $j \geq 1$. Replacing x by $2^j x$ in (4.10), we get

$$\mu_{g(2^j x)-8g(2^{j-1}x)}(t) \geq T\left(\Psi\left(2^{j-1}x, 2^{j-1}x, \frac{t}{|4|}\right), \Psi\left(2^j x, 2^{j-1}x, t\right)\right). \tag{4.12}$$

Since $|8| \leq 1$,

$$\begin{aligned}\mu_{g(2^j x)-8^{j+1}g(x/2)}(t) &\geq T\left(\mu_{g(2^j x)-8g(2^{j-1}x)}(t), \mu_{8g(2^{j-1}x)-8^{j+1}g(x/2)}(t)\right)\\ &= T\left(\mu_{g(2^j x)-8g(2^{j-1}x)}(t), \mu_{g(2^{j-1}x)-8^j g(x/2)}\left(\frac{t}{|8|}\right)\right)\\ &\geq T^2\left(\Psi\left(2^{j-1}x, 2^{j-1}x, \frac{t}{|4|}\right), \Psi\left(2^j x, 2^{j-1}x, t\right), M_j(x,t)\right)\\ &= M_{j+1}(x,t)\end{aligned} \tag{4.13}$$

for all $x \in X$ and $t > 0$. Thus, (4.11) holds for all $j \geq 2$. In particular,

$$\mu_{g(2^{k-1}x)-8^k g(x/2)}(t) \geq M(x,t) \quad (x \in X,\ t > 0). \tag{4.14}$$

Replacing x by $2^{-(kn+k-1)}x$ in (4.14) and using inequality (4.3), we obtain

$$\mu_{g(x/2^{kn})-8^k g(x/2^{k(n+1)})}(t) \geq M\left(\frac{2x}{2^{k(n+1)}}, t\right) \quad (x \in X,\ t > 0,\ n = 0,1,2,\ldots). \tag{4.15}$$

Then

$$\mu_{8^{kn}g(x/2^{kn})-8^{k(n+1)}g(x/2^{k(n+1)})}(t) \geq M\left(2x, \frac{\alpha^{n+1}}{\left|8^{k(n+1)}\right|}t\right) \quad (x \in X,\ t > 0,\ n = 0,1,2,\ldots). \tag{4.16}$$

Hence

$$\begin{aligned}
\mu_{8^{kn}g(x/2^{kn})-8^{k(n+p)}g(x/2^{k(n+p)})}(t) &\geq T_{j=n}^{n+p}\left(\mu_{8^{kj}g(x/2^{kj})-8^{k(j+p)}g(x/2^{k(j+p)})}(t)\right) \\
&\geq T_{j=n}^{n+p} M\left(2x, \frac{\alpha^{j+1}}{\left|\left(8^k\right)^{j+1}\right|}t\right) \\
&\geq T_{j=n}^{n+p} M\left(2x, \frac{\alpha^{j+1}}{\left|\left(8^k\right)^{j+1}\right|}t\right) \quad (x \in X,\ t > 0,\ n = 0,1,2,\ldots).
\end{aligned} \tag{4.17}$$

Since

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(2x, \frac{\alpha^{j+1}}{\left|\left(8^k\right)^{j+1}\right|}t\right) = 1 \quad (x \in X,\ t > 0), \tag{4.18}$$

then

$$\left\{8^{kn} g\left(\frac{x}{2^{kn}}\right)\right\}_{n\in\mathbb{N}} \tag{4.19}$$

is a Cauchy sequence in the non-Archimedean random Banach space (Y, μ, T). Hence we can define a mapping $C : X \to Y$ such that

$$\lim_{n\to\infty} \mu_{(8^{8k})^n g(x/2^{kn})-C(x)}(t) = 1 \quad (x \in X,\ t > 0). \tag{4.20}$$

Next for each $n \geq 1$, $x \in X$ and $t > 0$,

$$\begin{aligned}
\mu_{g(x)-(8^{8k})^n g(x/2^{kn})}(t) &= \mu_{\sum_{i=0}^{n-1}(8^{8k})^i g(x/2^{ki})-(8^{8k})^{i+1}g(x/2^{k(i+1)})}(t) \\
&\geq T_{i=0}^{n-1}\left(\mu_{(8^{8k})^i g(x/2^{ki})-(8^{8k})^{i+1}g(x/2^{k(i+1)})}(t)\right) \\
&\geq T_{i=0}^{n-1} M\left(2x, \frac{\alpha^{i+1}t}{\left|8^k\right|^{i+1}}\right).
\end{aligned} \tag{4.21}$$

Therefore,

$$\begin{aligned}\mu_{g(x)-C(x)}(t) &\geq T\left(\mu_{g(x)-(8^{8k})^n g(x/2^{kn})}(t), \mu_{(8^{8k})^n g(x/2^{kn})-C(x)}(t)\right)\\ &\geq T\left(T_{i=0}^{n-1} M\left(2x, \frac{\alpha^{i+1}t}{|8^k|^{i+1}}\right), \mu_{(8^{8k})^n g(x/2^{kn})-C(x)}(t)\right).\end{aligned} \tag{4.22}$$

By letting $n \to \infty$, we obtain

$$\mu_{g(x)-C(x)}(t) \geq T_{i=1}^{\infty} M\left(2x, \frac{\alpha^{i+1}t}{|8^k|^{i+1}}\right). \tag{4.23}$$

So,

$$\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|8^k|^{i+1}}\right). \tag{4.24}$$

This proves (4.5). From $Dg(x,y) = Df(2x,2y) - 2Df(x,y)$, by (4.2), we deduce that

$$\begin{aligned}\mu_{Df(2x,2y)}(t) &\geq \Psi(2x,2y,t),\\ \mu_{-2Df(x,y)}(t) = \mu_{Df(x,y)}\left(\frac{t}{|2|}\right) &\geq \mu_{Df(x,y)}(t) \geq \Psi(x,y,t),\end{aligned} \tag{4.25}$$

and so, by (NA-RN3) and (4.2), we obtain

$$\mu_{Dg(x,y)}(t) \geq T\left(\mu_{Df(2x,2y)}(t), \mu_{-2Df(x,y)}(t)\right) \geq T\left(\Psi(2x,2y,t), \Psi(x,y,t)\right) := N(x,y,t). \tag{4.26}$$

It follows that

$$\begin{aligned}\mu_{8^{kn}Dg(x/2^{kn},y/2^{kn})}(t) &= \mu_{Dg(x/2^{kn},y/2^{kn})}\left(\frac{t}{|8|^{kn}}\right)\\ &\geq N\left(\frac{x}{2^{kn}}, \frac{y}{2^{kn}}, \frac{t}{|8|^{kn}}\right) \geq \cdots \geq N\left(x, y, \frac{\alpha^{n-1}t}{|8|^{k(n-1)}}\right)\end{aligned} \tag{4.27}$$

for all $x, y \in X$, $t > 0$, and $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} N\left(x, y, \frac{\alpha^{n-1}t}{|8|^{k(n-1)}}\right) = 1 \tag{4.28}$$

for all $x, y \in X$ and $t > 0$, by Theorem 2.4, we deduce that

$$\mu_{DC(x,y)}(t) = 1 \tag{4.29}$$

for all $x, y \in X$ and $t > 0$. Thus, the mapping $C : X \to Y$ satisfies (1.4).

Now, we have

$$\begin{aligned} C(2x) - 8C(x) &= \lim_{n\to\infty}\left[8^n g\left(\frac{x}{2^{n-1}}\right) - 8^{n+1} g\left(\frac{x}{2^n}\right)\right] \\ &= 8\lim_{n\to\infty}\left[8^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 8^n g\left(\frac{x}{2^n}\right)\right] = 0 \end{aligned} \tag{4.30}$$

for all $x \in X$. Since the mapping $x \to C(2x) - 2C(x)$ is cubic (see Lemma 2.2 of [14]), from the equality $C(2x) = 8C(x)$, we deduce that the mapping $C : X \to Y$ is cubic. □

Corollary 4.3. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$, and let (Y, μ, T) be a non-Archimedean random Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be an odd and Ψ-approximately AQCQ mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 3$, with $|2^k| < \alpha$,*

$$\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X,\ t > 0), \tag{4.31}$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|8|^{ki}}\right) \tag{4.32}$$

for all $x \in X$ and $t > 0$.

Proof. Since

$$\lim_{n\to\infty} M\left(x, \frac{\alpha^j t}{|8|^{kj}}\right) = 1 \quad (x \in X,\ t > 0) \tag{4.33}$$

and T is of Hadžić type, from Proposition 2.1, it follows that

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|8|^{kj}}\right) = 1 \quad (x \in X,\ t > 0). \tag{4.34}$$

Now, we can apply Theorem 4.2 to obtain the result. □

Example 4.4. Let (X, μ, T_M) be non-Archimedean random normed space in which

$$\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X,\ t > 0). \tag{4.35}$$

And let (Y, μ, T_M) be a complete non-Archimedean random normed space (see Example 3.2). Define

$$\Psi(x, y, t) = \frac{t}{1+t}. \tag{4.36}$$

It is easy to see that (4.3) holds for $\alpha = 1$. Also, since

$$M(x, t) = \frac{t}{1+t}, \tag{4.37}$$

we have

$$\begin{aligned} \lim_{n\to\infty} T^{\infty}_{M,j=n} M\left(x, \frac{\alpha^j t}{|8|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} T^{m}_{M,j=n} M\left(x, \frac{t}{|8|^{kj}}\right)\right) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t+\left|8^k\right|^n}\right) \\ &= 1 \quad (x \in X,\ t > 0). \end{aligned} \tag{4.38}$$

Let $f : X \to Y$ be an odd and Ψ-approximately AQCQ mapping. Thus, all the conditions of Theorem 4.2 hold, and so there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq \frac{t}{t+\left|8^k\right|}. \tag{4.39}$$

Theorem 4.5. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$, and let (Y, μ, T) be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : X \to Y$ be an odd mapping and Ψ-approximately AQCQ mapping. If for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 1$ with $|2^k| < \alpha$,*

$$\begin{aligned} &\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X,\ t > 0), \\ &\lim_{n\to\infty} T^{\infty}_{j=n} M\left(2x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1 \quad (x \in X,\ t > 0), \end{aligned} \tag{4.40}$$

then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-8f(x/2)-A(x/2)}(t) \geq T^{\infty}_{i=1} M\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right) \tag{4.41}$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := T^{k-1}\left[\Psi\left(\frac{x}{2},\frac{x}{2},\frac{t}{|4|}\right),\Psi\left(x,\frac{x}{2},t\right),\ldots,\Psi\left(\frac{2^{k-1}x}{2},\frac{2^{k-1}x}{2},\frac{t}{|4|}\right),\Psi\left(2^{k-1}x,\frac{2^{k-1}x}{2},t\right)\right] \quad (x \in X,\ t > 0) \tag{4.42}$$

Proof. Letting $y := x/2$ and $g(x) := f(2x) - 8f(x)$ for all $x \in X$ in (4.9), we get

$$\mu_{g(x)-2g(x/2)}(t) \geq T\left(\Psi\left(\frac{x}{2},\frac{x}{2},\frac{t}{|4|}\right),\Psi\left(x,\frac{x}{2},t\right)\right) \tag{4.43}$$

for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 4.2. □

Corollary 4.6. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$, and let (Y,μ,T) be a non-Archimedean random Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be an odd and Ψ-approximately AQCQ mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 1$, with $|2^k| < \alpha$,*

$$\Psi\left(2^{-k}x,2^{-k}y,t\right) \geq \Psi(x,y,\alpha t) \quad (x \in X,\ t > 0), \tag{4.44}$$

then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-8f(x/2)-A(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x,\frac{\alpha^{i+1}t}{|2|^{ki}}\right) \tag{4.45}$$

for all $x \in X$ and $t > 0$.

Proof. Since

$$\lim_{n\to\infty} M\left(x,\frac{\alpha^{j}t}{|2|^{kj}}\right) = 1 \quad (x \in X,\ t > 0) \tag{4.46}$$

and T is of Hadžić type, from Proposition 2.1, it follows that

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x,\frac{\alpha^{j}t}{|2|^{kj}}\right) = 1 \quad (x \in X,\ t > 0). \tag{4.47}$$

Now, we can apply Theorem 4.5 to obtain the result. □

Example 4.7. Let (X, μ, T_M) non-Archimedean random normed space in which

$$\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0), \tag{4.48}$$

and let (Y, μ, T_M) be a complete non-Archimedean random normed space (see Example 3.2). Define

$$\Psi(x, y, t) = \frac{t}{1 + t}. \tag{4.49}$$

It is easy to see that (4.3) holds for $\alpha = 1$. Also, since

$$M(x, t) = \frac{t}{1 + t}, \tag{4.50}$$

we have

$$\begin{aligned} \lim_{n\to\infty} T^{\infty}_{M,j=n} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} T^{m}_{M,j=n} M\left(x, \frac{t}{|2|^{kj}}\right)\right) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t + \left|2^k\right|^n}\right) \\ &= 1 \quad (x \in X, \ t > 0). \end{aligned} \tag{4.51}$$

Let $f : X \to Y$ be an odd and Ψ-approximately AQCQ mapping. Thus, all the conditions of Theorem 4.2 hold, and so there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-8f(x/2)-A(x/2)}(t) \geq \frac{t}{t + \left|2^k\right|}. \tag{4.52}$$

5. Generalized Hyers-Ulam Stability of the Functional Equation (1.4) in Non-Archimedean Random Normed Spaces: An Even Case

Now, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean Banach spaces, an even case.

Theorem 5.1. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$, and let (Y, μ, T) be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : X \to Y$ be an even mapping, $f(0) = 0$, and Ψ-approximately AQCQ mapping. If for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 4$ with $|2^k| < \alpha$,*

$$\begin{gathered} \Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0), \\ \lim_{n\to\infty} T^{\infty}_{j=n} M\left(2x, \frac{\alpha^j t}{|16|^{kj}}\right) = 1 \quad (x \in X, \ t > 0), \end{gathered} \tag{5.1}$$

then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-4f(x/2)-Q(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|16|^{ki}}\right) \tag{5.2}$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := T^{k-1}\left[\Psi\left(\frac{x}{2},\frac{x}{2},\frac{t}{|4|}\right),\Psi\left(x,\frac{x}{2},t\right),\ldots,\Psi\left(\frac{2^{k-1}x}{2},\frac{2^{k-1}x}{2},\frac{t}{|4|}\right),\Psi\left(2^{k-1}x,\frac{2^{k-1}x}{2},t\right)\right]$$
$$(x \in X,\ t > 0). \tag{5.3}$$

Proof. Letting $x = y$ in (4.2), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq \Psi(y,y,t) \tag{5.4}$$

for all $y \in X$ and $t > 0$. Replacing x by $2y$ in (4.2), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq \Psi(2y,y,t) \tag{5.5}$$

for all $y \in X$ and $t > 0$. By (5.4) and (5.5), we have

$$\begin{aligned}\mu_{f(4y)-20f(2y)+64f(y)}(t) &\geq T\left(\mu_{4(f(3y)-4f(2y)+5f(y))}(t),\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right)\\ &= T\left(\mu_{f(3y)-4f(2y)+5f(y)}\left(\frac{t}{|4|}\right),\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right)\\ &\geq T\left(\Psi\left(y,y,\frac{t}{|4|}\right),\Psi(2y,y,t)\right)\end{aligned} \tag{5.6}$$

for all $y \in X$ and $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 4f(x)$ for all $x \in X$ in (5.6), we get

$$\mu_{g(x)-16g(x/2)}(t) \geq T\left(\Psi\left(\frac{x}{2},\frac{x}{2},\frac{t}{|4|}\right),\Psi\left(x,\frac{x}{2},t\right)\right) \tag{5.7}$$

for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 4.2. □

Corollary 5.2. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$, and let (Y,μ,T) be a non-Archimedean random Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be an even, $f(0) = 0$, and Ψ-approximately AQCQ mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 4$, with $|2^k| < \alpha$,*

$$\Psi\left(2^{-k}x,2^{-k}y,t\right) \geq \Psi(x,y,\alpha t) \quad (x \in X,\ t > 0), \tag{5.8}$$

then there exists a unique quartic mapping $Q : X \to Y$ *such that*

$$\mu_{f(x)-4f(x/2)-Q(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|16|^{ki}}\right) \tag{5.9}$$

for all $x \in X$ *and* $t > 0$.

Proof. Since

$$\lim_{n\to\infty} M\left(x, \frac{\alpha^j t}{|16|^{kj}}\right) = 1 \quad (x \in X,\ t > 0) \tag{5.10}$$

and T is of Hadžić type, from Proposition 2.1, it follows that

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|16|^{kj}}\right) = 1 \quad (x \in X,\ t > 0). \tag{5.11}$$

Now, we can apply Theorem 5.1 to obtain the result. □

Example 5.3. Let (X, μ, T_M) be non-Archimedean random normed space in which

$$\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X,\ t > 0). \tag{5.12}$$

And let (Y, μ, T_M) be a complete non-Archimedean random normed space (see Example 3.2). Define

$$\Psi(x, y, t) = \frac{t}{1+t}. \tag{5.13}$$

It is easy to see that (4.3) holds for $\alpha = 1$. Also, since

$$M(x, t) = \frac{t}{1+t}, \tag{5.14}$$

we have

$$\begin{aligned} \lim_{n\to\infty} T_{M,j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|16|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} T_{M,j=n}^{m} M\left(x, \frac{t}{|16|^{kj}}\right)\right) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t + \left|16^k\right|^n}\right) \\ &= 1 \quad (x \in X,\ t > 0). \end{aligned} \tag{5.15}$$

Let $f : X \to Y$ be an even, $f(0) = 0$, and Ψ-approximately AQCQ mapping. Thus all the conditions of Theorem 5.1 hold, and so there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-4f(x/2)-Q(x/2)}(t) \geq \frac{t}{t+|16^k|}. \tag{5.16}$$

Theorem 5.4. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$ and let (Y, μ, T) be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : X \to Y$ be an even mapping, $f(0) = 0$ and Ψ-approximately AQCQ mapping. If for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 2$ with $|2^k| < \alpha$,*

$$\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X,\ t > 0),$$
$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(2x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1 \quad (x \in X,\ t > 0), \tag{5.17}$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-16f(x/2)-Q(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|4|^{ki}}\right) \tag{5.18}$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := T^{k-1}\left[\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{|4|}\right), \Psi\left(x, \frac{x}{2}, t\right), \ldots, \Psi\left(\frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, \frac{t}{|4|}\right), \Psi\left(2^{k-1}x, \frac{2^{k-1}x}{2}, t\right)\right] \quad (x \in X,\ t > 0). \tag{5.19}$$

Proof. Letting $y := x/2$ and $g(x) := f(2x) - 16f(x)$ for all $x \in X$ in (5.6), we get

$$\mu_{g(x)-4g(x/2)}(t) \geq T\left(\Psi\left(\frac{x}{2}, \frac{x}{2}, \frac{t}{|4|}\right), \Psi\left(x, \frac{x}{2}, t\right)\right) \tag{5.20}$$

for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 5.1. □

Corollary 5.5. *Let $\mathcal{K}$ be a non-Archimedean field, let X be a vector space over $\mathcal{K}$, and let (Y, μ, T) be a non-Archimedean random Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be an even, $f(0) = 0$, and Ψ-approximately AQCQ mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer k, $k > 2$, with $|2^k| < \alpha$,*

$$\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X,\ t > 0), \tag{5.21}$$

then there exists a unique quadratic mapping $Q : X \to Y$ *such that*

$$\mu_{f(x)-16f(x/2)-Q(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}t}{|4|^{ki}}\right) \quad (5.22)$$

for all $x \in X$ *and* $t > 0$*.*

Proof. Since

$$\lim_{n\to\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1 \quad (x \in X,\ t > 0) \quad (5.23)$$

and T is of Hadžić type, from Proposition 2.1, it follows that

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) = 1 \quad (x \in X,\ t > 0). \quad (5.24)$$

Now, we can apply Theorem 5.4 to obtain the result. □

Example 5.6. Let (X, μ, T_M) be a non-Archimedean random normed space in which

$$\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X,\ t > 0). \quad (5.25)$$

And let (Y, μ, T_M) be a complete non-Archimedean random normed space (see Example 3.2). Define

$$\Psi(x, y, t) = \frac{t}{1+t}. \quad (5.26)$$

It is easy to see that (4.3) holds for $\alpha = 1$. Also, since

$$M(x, t) = \frac{t}{1+t}, \quad (5.27)$$

we have

$$\begin{aligned}\lim_{n\to\infty} T_{M,j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|4|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} T_{M,j=n}^{m} M\left(x, \frac{t}{|4|^{kj}}\right)\right)\\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t+\left|4^k\right|^n}\right)\\ &= 1 \quad (x \in X,\ t > 0).\end{aligned} \quad (5.28)$$

Let $f : X \to Y$ be an even, $f(0) = 0$, and Ψ-approximately AQCQ mapping. Thus, all the conditions of Theorem 5.4 hold, and so there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-16f(x/2)-Q(x/2)}(t) \geq \frac{t}{t+|4^k|}. \tag{5.29}$$

6. Latticetic Random Normed Space

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, that is, a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$. The space of latticetic random distribution functions, denoted by Δ_L^+, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ such that F is left continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0_{\mathcal{L}}$, $F(+\infty) = 1_{\mathcal{L}}$.

$D_L^+ \subseteq \Delta_L^+$ is defined as $D_L^+ = \{F \in \Delta_L^+ : l^- F(+\infty) = 1_{\mathcal{L}}\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ_L^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ_L^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathcal{L}}, & \text{if } t \leq 0, \\ 1_{\mathcal{L}}, & \text{if } t > 0. \end{cases} \tag{6.1}$$

In Section 2, we defined t-norms on $[0,1]$, and now we extend t-norms on a complete lattice.

Definition 6.1 (see [42]). A *triangular norm* (t-norm) on L is a mapping $\mathcal{T} : (L)^2 \to L$ satisfying the following conditions:

(a) (for all $x \in L$)($\mathcal{T}(x, 1_{\mathcal{L}}) = x$) (boundary condition);

(b) (for all $(x, y) \in (L)^2$)($\mathcal{T}(x, y) = \mathcal{T}(y, x)$) (commutativity);

(c) (for all $(x, y, z) \in (L)^3$)($\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$) (associativity);

(d) (for all $(x, x', y, y') \in (L)^4$)($x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y')$) (monotonicity).

Let $\{x_n\}$ be a sequence in L converges to $x \in L$ (equipped order topology). The t-norm $\mathcal{T}$ is said to be a *continuous t-norm* if

$$\lim_{n \to \infty} \mathcal{T}(x_n, y) = \mathcal{T}(x, y) \tag{6.2}$$

for all $y \in L$.

A t-norm $\mathcal{T}$ can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1, \ldots, x_n) \in L^n$ the value $\mathcal{T}(x_1, \ldots, x_n)$ defined by

$$\mathcal{T}_{i=1}^0 x_i = 1, \qquad \mathcal{T}_{i=1}^n x_i = \mathcal{T}\left(\mathcal{T}_{i=1}^{n-1} x_i, x_n\right) = \mathcal{T}(x_1, \ldots, x_n). \tag{6.3}$$

$\mathcal{T}$ can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in L the value

$$\mathcal{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathcal{T}_{i=1}^{n} x_i. \tag{6.4}$$

The limit on the right side of (6.4) exists since the sequence $(\mathcal{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

Note that we put $\mathcal{T} = T$ whenever $L = [0,1]$. If T is a t-norm, then $x_T^{(n)}$ is defined for every $x \in [0,1]$ and $n \in N \cup \{0\}$ by 1 if $n = 0$ and $T(x_T^{(n-1)}, x)$ if $n \geq 1$. A t-norm T is said to be *of Hadžić type*, (we denote by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n\in N}$ is equicontinuous at $x = 1$ (cf. [27]).

Definition 6.2 (see [42]). A continuous t-norm $\mathcal{T}$ on $L = [0,1]^2$ is said to be *continuous t–representable* if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L$,

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2). \tag{6.5}$$

For example,

$$\begin{aligned} \mathcal{T}(a,b) &= (a_1 b_1, \min\{a_2 + b_2, 1\}), \\ \mathbf{M}(a,b) &= (\min\{a_1, b_1\}, \max\{a_2, b_2\}) \end{aligned} \tag{6.6}$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0,1]^2$ are continuous t-representable. Define the mapping $\mathcal{T}_\wedge$ from L^2 to L by

$$\mathcal{T}_\wedge(x,y) = \begin{cases} x, & \text{if } y \geq_L x, \\ y, & \text{if } x \geq_L y. \end{cases} \tag{6.7}$$

Recall (see [27, 28]) that if $\{x_n\}$ is a given sequence in L, $(\mathcal{T}_\wedge)_{i=1}^{n} x_i$ is defined recurrently by $(\mathcal{T}_\wedge)_{i=1}^{1} x_i = x_1$ and $(\mathcal{T}_\wedge)_{i=1}^{n} x_i = \mathcal{T}_\wedge((\mathcal{T}_\wedge)_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$.

A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an *involutive negation*. In the following, $\mathcal{L}$ is endowed with a (fixed) negation $\mathcal{N}$.

Definition 6.3. A *latticetic random normed space* (in short LRN-space) is a triple $(X, \mu, \mathcal{T}_\wedge)$, where X is a vector space and μ is a mapping from X into D_L^+ such that the following conditions hold:

(LRN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(LRN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all x in X, $\alpha \neq 0$ and $t \geq 0$,

(LRN3) $\mu_{x+y}(t+s) \geq_L \mathcal{T}_\wedge(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

We note that from (LPN2) it follows that $\mu_{-x}(t) = \mu_x(t)$ for all $x \in X$ and $t \geq 0$.

Example 6.4. Let $L = [0,1] \times [0,1]$ and operation $\leq_L$ be defined by

$$L = \{(a_1, a_2) : (a_1, a_2) \in [0,1] \times [0,1], \quad a_1 + a_2 \leq 1\},$$
$$(a_1, a_2) \leq_L (b_1, b_2) \Longleftrightarrow a_1 \leq b_1,\ a_2 \geq b_2, \quad \forall a = (a_1, a_2),\ b = (b_1, b_2) \in L. \tag{6.8}$$

then $(L, \leq_L)$ is a complete lattice (see [42]). In this complete lattice, we denote its units by $0_L = (0,1)$ and $1_L = (1,0)$. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0,1] \times [0,1]$ and μ be a mapping defined by

$$\mu_x(t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right) \quad (t \in \mathbb{R}^+), \tag{6.9}$$

then $(X, \mu, \mathcal{T})$ is a latticetic random normed spaces.

If $(X, \mu, \mathcal{T}_\wedge)$ is a latticetic random normed space, then

$$\mathcal{V} = \{V(\varepsilon, \lambda) : \varepsilon >_L 0_{\mathcal{L}},\ \lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}\}, \quad V(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) >_L \mathcal{N}(\lambda)\}, \tag{6.10}$$

is a complete system of neighborhoods of null vector for a linear topology on X generated by the norm F.

Definition 6.5. Let $(X, \mu, \mathcal{T}_\wedge)$ be a latticetic random normed spaces.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer N such that $\mu_{x_n - x}(t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq m \geq N$.

(3) A latticetic random normed spaces $(X, \mu, \mathcal{T}_\wedge)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 6.6. *If $(X, \mu, \mathcal{T}_\wedge)$ is a latticetic random normed space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Proof. The proof is the same as classical random normed spaces, see [25]. □

7. Generalized Hyers-Ulam Stability of the Functional Equation (1.4): An Odd Case via Fixed-Point Method

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in random Banach spaces: an odd case.

Theorem 7.1. *Let X be a linear space, let $(Y, \mu, \mathcal{T}_\wedge)$ be a complete LRN-space, and Φ let be a mapping from X^2 to D_L^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/8$,*

$$\Phi_{2x,2y}(t) \leq_L \Phi_{x,y}(\alpha t) \quad (x, y \in X,\ t > 0). \tag{7.1}$$

Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{Df(x,y)}(t) \geq_L \Phi_{x,y}(t) \tag{7.2}$$

for all $x, y \in X$ and $t > 0$. Then

$$C(x) := \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right) \tag{7.3}$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{x,x}\left(\frac{1-8\alpha}{5\alpha}t\right), \Phi_{2x,x}\left(\frac{1-8\alpha}{5\alpha}t\right)\right) \tag{7.4}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $x = y$ in (7.2), we get

$$\mu_{f(3y)-4f(2y)+5f(y)}(t) \geq_L \Phi_{y,y}(t) \tag{7.5}$$

for all $y \in X$ and $t > 0$. Replacing x by $2y$ in (7.2), we get

$$\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq_L \Phi_{2y,y}(t) \tag{7.6}$$

for all $y \in X$ and $t > 0$. By (7.5) and (7.6),

$$\begin{aligned}\mu_{f(4y)-10f(2y)+16f(y)}(5t) &\geq_L \mathcal{T}_\wedge\left(\mu_{4(f(3y)-4f(2y)+5f(y))}(4t), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right)\\ &= \mathcal{T}_\wedge\left(\mu_{f(3y)-4f(2y)+5f(y)}(t), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right)\\ &\geq_L \mathcal{T}_\wedge\left(\Phi_{y,y}(t), \Phi_{2y,y}(t)\right)\end{aligned} \tag{7.7}$$

for all $y \in X$ and $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\mu_{g(x)-8g(x/2)}(5t) \geq_L \mathcal{T}_\wedge(\Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t)) \tag{7.8}$$

for all $x \in X$ and $t > 0$.

Consider the set

$$S := \{h : X \longrightarrow Y,\ h(0) = 0\} \tag{7.9}$$

and introduce the generalized metric on S:

$$d(h,k) = \inf\left\{u \in \mathbb{R}^+ : \mu_{h(x)-k(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)),\ \forall x \in X,\ \forall t > 0\right\} \tag{7.10}$$

where, as usual, $\inf\emptyset = +\infty$. It is easy to show that (S, d) is complete (see the proof of Lemma 2.1 of [24]).

Now, we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 8h\left(\frac{x}{2}\right) \tag{7.11}$$

for all $x \in X$, and we prove that J is a strictly contractive mapping with the Lipschitz constant 8α.

Let $h, k \in S$ be given such that $d(h, k) < \varepsilon$. Then

$$\mu_{h(x)-k(x)}(\varepsilon t) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.12}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}\mu_{Jh(x)-Jk(x)}(8\alpha\varepsilon t) &= \mu_{8h(x/2)-8k(x/2)}(8\alpha\varepsilon t)\\ &= \mu_{h(x/2)-k(x/2)}(\alpha\varepsilon t)\\ &\geq \mathcal{T}_\wedge(\Phi_{x/2,x/2}(\alpha t), \Phi_{x,x/2}(\alpha t))\\ &\geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t))\end{aligned} \tag{7.13}$$

for all $x \in X$ and $t > 0$. So, $d(h, k) < \varepsilon$ implies that

$$d(Jh, Jk) \leq \frac{\alpha}{8}\varepsilon. \tag{7.14}$$

This means that

$$d(Jh, Jk) \leq \frac{\alpha}{8} d(h, k) \tag{7.15}$$

for all $h, k \in S$. It follows from (7.8) that

$$\mu_{g(x)-8g(x/2)}(5\alpha t) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.16}$$

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5\alpha \leq 5/8$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, that is,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{7.17}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{h \in S : d(h, g) < \infty\}. \tag{7.18}$$

This implies that C is a unique mapping satisfying (7.17) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-C(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.19}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x) \tag{7.20}$$

for all $x \in X$.

(3) $d(h, C) \leq (1/(1-8\alpha))d(h, Jh)$ with $h \in M$, which implies the inequality

$$d(g, C) \leq \frac{5\alpha}{1-8\alpha}, \tag{7.21}$$

from which it follows that

$$\mu_{g(x)-C(x)}\left(\frac{5\alpha}{1-8\alpha}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)). \tag{7.22}$$

This implies that the inequality (7.4) holds. From $Dg(x, y) = Df(2x, 2y) - 2Df(x, y)$, by (7.2), we deduce that

$$\begin{gathered} \mu_{Df(2x,2y)}(t) \geq_L \Phi_{2x,2y}(t), \\ \mu_{-2Df(x,y)}(t) = \mu_{Df(x,y)}\left(\frac{t}{2}\right) \geq_L \Phi_{x,y}\left(\frac{t}{2}\right) \end{gathered} \tag{7.23}$$

and so, by (LRN3) and (7.1), we obtain

$$\begin{aligned} \mu_{Dg(x,y)}(3t) &\geq_L \mathcal{T}_\wedge(\mu_{Df2x,2y}(t), \mu_{-2Df(x,y)}(2t)) \\ &\geq_L \mathcal{T}_\wedge(\Phi_{2x,2y}(t), \Phi_{x,y}(t)) \geq_L \Phi_{2x,2y}(t). \end{aligned} \tag{7.24}$$

It follows that

$$\begin{aligned} \mu_{8^n Dg(x/2^n, y/2^n)}(3t) &= \mu_{Dg(x/2^n, y/2^n)}\left(3\frac{t}{8^n}\right) \\ &\geq \Phi_{x/2^{n-1}, y/2^{n-1}}\left(\frac{t}{8^n}\right) \geq_L \cdots \geq_L \Phi_{x,y}\left(\frac{1}{8}\frac{t}{(8\alpha)^{n-1}}\right) \end{aligned} \tag{7.25}$$

for all $x, y \in X$, $t > 0$ and $n \in \mathbb{N}$.

Since $\lim_{n\to\infty}\Phi_{x,y}((3/8)(t/(8\alpha)^{n-1}))=1$ for all $x,y\in X$ and $t>0$, by Theorem 2.4, we deduce that

$$\mu_{DC(x,y)}(3t)=1_{\mathcal{L}} \tag{7.26}$$

for all $x,y\in X$ and $t>0$. Thus the mapping $C:X\to Y$ satisfies (1.4).

Now, we have

$$\begin{aligned} C(2x)-8C(x)&=\lim_{n\to\infty}\left[8^n g\left(\frac{x}{2^{n-1}}\right)-8^{n+1}g\left(\frac{x}{2^n}\right)\right]\\ &=8\lim_{n\to\infty}\left[8^{n-1}g\left(\frac{x}{2^{n-1}}\right)-8^n g\left(\frac{x}{2^n}\right)\right]=0 \end{aligned} \tag{7.27}$$

for all $x\in X$. Since the mapping $x\to C(2x)-2C(x)$ is cubic (see Lemma 2.2 of [14]), from the equality $C(2x)=8C(x)$, we deduce that the mapping $C:X\to Y$ is cubic. □

Corollary 7.2. *Let $\theta\geq 0$ and let p be a real number with $p>3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t)\geq\frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{7.28}$$

for all $x,y\in X$ and $t>0$. Note that (X,μ,T_M) is a complete LRN-space, in which $L=[0,1]$, then

$$C(x):=\lim_{n\to\infty}8^n\left(f\left(\frac{x}{2^{n-1}}\right)-2f\left(\frac{x}{2^n}\right)\right) \tag{7.29}$$

exists for each $x\in X$ and defines a cubic mapping $C:X\to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t)\geq\frac{(2^p-8)t}{(2^p-8)t+5(1+2^p)\theta\|x\|^p} \tag{7.30}$$

for all $x\in X$ and $t>0$.

Proof. The proof follows from Theorem 7.1 by taking

$$\Phi_{x,y}(t):=\frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{7.31}$$

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^{-p}$, and we get

$$\begin{aligned}\mu_{f(2x)-2f(x)-C(x)}(t) &\geq \min\left(\frac{(1-2^{3-p})t}{(1-2^{3-p})t + 5\cdot 2^{-p}\theta(2\|x\|^p)}, \frac{(1-2^{3-p})t}{(1-2^{3-p})t + 5\cdot 2^{-p}\theta(\|2x\|^p + \|x\|^p)}\right)\\ &\geq \frac{(1-2^{3-p})t}{(1-2^{3-p})t + 5\cdot 2^{-p}\theta(\|2x\|^p + \|x\|^p)}\\ &= \frac{(2^p-8)t}{(2^p-8)t + 5\cdot(2^p+1)\theta\|x\|^p},\end{aligned} \tag{7.32}$$

which is the desired result. □

Theorem 7.3. *Let X be a linear space, let $(Y, \mu, \mathcal{T}_\wedge)$ be a complete LRN-space, and let Φ be a mapping from X^2 to D_L^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 8$,*

$$\Phi_{x/2,y/2}(t) \leq_L \Phi_{x,y}(\alpha t) \quad (x, y \in X,\ t > 0). \tag{7.33}$$

Let $f : X \to Y$ be an odd mapping satisfying (7.2)*, then*

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right) \tag{7.34}$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \ \geq_L \mathcal{T}_\wedge\left(\Phi_{x,x}\left(\frac{8-\alpha}{5}t\right), \Phi_{2x,x}\left(\frac{8-\alpha}{5}t\right)\right) \tag{7.35}$$

for all $x \in X$ and $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 7.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{8}h(2x) \tag{7.36}$$

for all $x \in X$, and we prove that J is a strictly contractive mapping with the Lipschitz constant $\alpha/8$.

Let $h, k \in S$ be given such that $d(h, k) < \varepsilon$, then

$$\mu_{h(x)-k(x)}(\varepsilon t) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.37}$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned}\mu_{Jh(x)-Jk(x)}\left(\frac{\alpha}{8}\varepsilon t\right) &= \mu_{(1/8)h(2x)-(1/8)k(2x)}\left(\frac{\alpha}{8}\varepsilon t\right)\\ &= \mu_{h(2x)-k(2x)}(\alpha\varepsilon t)\\ &\geq_L \mathcal{T}_\wedge(\Phi_{2x,2x}(\alpha t), \Phi_{4x,2x}(\alpha t))\\ &\geq \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t))\end{aligned} \tag{7.38}$$

for all $x \in X$ and $t > 0$. So, $d(h,k) < \varepsilon$ implies that

$$d(Jh, Jk) \leq \frac{\alpha}{8}\varepsilon. \tag{7.39}$$

This means that

$$d(Jh, Jk) \leq \frac{\alpha}{8}d(h,k) \tag{7.40}$$

for all $g, h \in S$. Letting $g(x) := f(2x) - 2f(x)$ for all $x \in X$, from (7.8), we get that

$$\mu_{g(x)-(1/8)g(2x)}\left(\frac{5}{8}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.41}$$

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5/8$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, that is,

$$C(2x) = 8C(x) \tag{7.42}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{h \in S : d(h,g) < \infty\}. \tag{7.43}$$

This implies that C is a unique mapping satisfying (7.42) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-C(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.44}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equalit

$$\lim_{n\to\infty} \frac{1}{8^n} g(2^n x) = C(x) \tag{7.45}$$

for all $x \in X$.

(3) $d(h, C) \leq (1/(1-\alpha/8))d(h, Jh)$ for every $h \in M$, which implies the inequality

$$d(g, C) \leq \frac{5}{8-\alpha}, \tag{7.46}$$

from which it follows that

$$\mu_{g(x)-C(x)}\left(\frac{5}{8-\alpha}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.47}$$

for all $x \in X$ and $t > 0$. This implies that the inequality (7.35) holds.

From

$$\mu_{Dg(x,y)}(3t) \geq_L \mathcal{T}_\wedge(\Phi_{2x,2y}(t), \Phi_{x,y}(t)) \geq_L \mathcal{T}_\wedge\left(\Phi_{2x,2y}(t), \Phi_{x,y}\left(\frac{t}{8}\right)\right), \tag{7.48}$$

by (7.33), we deduce that

$$\mu_{8^{-n}Dg(2^n x, 2^n y)}(3t) = \mu_{Dg(2^n x, 2^n y)}(3 \cdot 8^n t) \geq_L \Phi_{2^n x, 2^n y}\left(8^{n-1} t\right) \geq_L \cdots \geq \Phi_{x,y}\left(\left(\frac{8}{\alpha}\right)^{n-1} \frac{t}{\alpha}\right) \tag{7.49}$$

for all $x, y \in X$, $t > 0$, and $n \in \mathbb{N}$. As $n \to \infty$, we deduce that

$$\mu_{DC(x,y)}(3t) = 1_{\mathcal{L}} \tag{7.50}$$

for all $x, y \in X$ and $t > 0$. Thus the mapping $C : X \to Y$ satisfies (1.4).

Now, we have

$$\begin{aligned} C(2x) - 8C(x) &= \lim_{n\to\infty}\left[\frac{1}{8^n} g\left(2^{n+1} x\right) - \frac{1}{8^{n-1}} g(2^n x)\right] \\ &= 8 \lim_{n\to\infty}\left[\frac{1}{8^{n+1}} g\left(2^{n+1} x\right) - \frac{1}{8^n} g(2^n x)\right] = 0 \end{aligned} \tag{7.51}$$

for all $x \in X$. Since the mapping $x \to C(2x) - 2C(x)$ is cubic (see Lemma 2.2 of [14]), from the equality $C(2x) = 8C(x)$, we deduce that the mapping $C : X \to Y$ is cubic. □

Corollary 7.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (7.28)*, then*

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1} x\right) - 2f(2^n x)\right) \tag{7.52}$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(8-2^p)t}{(8-2^p)t + 5(1+2^p)\theta\|x\|^p} \tag{7.53}$$

for all $x \in X$ and $t > 0$. Note that (X, μ, T_M) is a complete LRN-space, in which $L = [0,1]$.

Proof. The proof follows from Theorem 7.3 by taking

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{7.54}$$

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^p$, and we get the desired result. □

Theorem 7.5. *Let X be a linear space, let $(Y, \mu, \mathcal{T}_\wedge)$ be a complete LRN-space, and let Φ be a mapping from X^2 to D_L^+ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/2$,*

$$\Phi_{2x,2y}(t) \leq_L \Phi_{x,y}(\alpha t) \quad (x, y \in X,\ t > 0). \tag{7.55}$$

Let $f : X \to Y$ be an odd mapping satisfying (7.2), then

$$A(x) := \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right) \tag{7.56}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{x,x}\left(\frac{1-2\alpha}{5\alpha}t\right), \Phi_{2x,x}\left(\frac{1-2\alpha}{5\alpha}t\right)\right) \tag{7.57}$$

for all $x \in X$ and $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 7.1.

Letting $y := x/2$ and $g(x) := f(2x) - 8f(x)$ for all $x \in X$ in (7.7), we get

$$\mu_{g(x)-2g(x/2)}(5t) \geq_L \mathcal{T}_\wedge(\Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t)) \tag{7.58}$$

for all $x \in X$ and $t > 0$.

Now, we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right) \tag{7.59}$$

for all $x \in X$. It is easy to see that J is a strictly contractive self-mapping on S with the Lipschitz constant 2α.

It follows from (7.58) and (7.55) that

$$\mu_{g(x)-2g(x/2)}(5\alpha t) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.60}$$

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5\alpha < \infty$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{7.61}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{h \in S : d(h, g) < \infty\}. \tag{7.62}$$

This implies that A is a unique mapping satisfying (7.61) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-A(x)}(ut) \geq_L \mathcal{T}_{\wedge}(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.63}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 2^n g\left(\frac{x}{2^n}\right) = A(x) \tag{7.64}$$

for all $x \in X$.

(3) $d(h, A) \leq (1/(1-2\alpha))d(h, Jh)$ for each $h \in M$, which implies the inequality

$$d(g, A) \leq \frac{5\alpha}{1-2\alpha}. \tag{7.65}$$

This implies that the inequality (7.57) holds. Since $\mu_{Dg(x,y)}(3t) \geq_L \Phi_{2x,2y}(t)$, it follows that

$$\begin{aligned} \mu_{2^n Dg(x/2^n, y/2^n)}(3t) &= \mu_{Dg(x/2^n, y/2^n)}\left(3\frac{t}{2^n}\right) \\ &\geq \Phi_{x/2^{n-1}, y/2^{n-1}}\left(\frac{t}{2^n}\right) \geq_L \cdots \geq_L \Phi_{x,y}\left(\frac{1}{2}\frac{t}{(2\alpha)^{n-1}}\right) \end{aligned} \tag{7.66}$$

for all $x, y \in X$, $t > 0$, and $n \in \mathbb{N}$. As $n \to \infty$, we deduce that

$$\mu_{DA(x,y)}(3t) = 1_{\mathcal{L}} \tag{7.67}$$

for all $x, y \in X$ and $t > 0$. Thus, the mapping $A : X \to Y$ satisfies (1.4).

Now, we have

$$\begin{aligned} A(2x) - 2A(x) &= \lim_{n\to\infty}\left[2^n g\left(\frac{x}{2^{n-1}}\right) - 2^{n+1} g\left(\frac{x}{2^n}\right)\right] \\ &= 2\lim_{n\to\infty}\left[2^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 2^n g\left(\frac{x}{2^n}\right)\right] = 0 \end{aligned} \tag{7.68}$$

for all $x \in X$. Since the mapping $x \to A(2x) - 8A(x)$ is additive (see Lemma 2.2 of [14]), from the equality $A(2x) = 2A(x)$, we deduce that the mapping $A : X \to Y$ is additive. □

Corollary 7.6. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (7.28)*, then*

$$A(x) := \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right) \tag{7.69}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2^p-2)t}{(2^p-2)t + 5(1+2^p)\theta\|x\|^p} \tag{7.70}$$

for all $x \in X$ and $t > 0$, where (X, μ, T_M) is a complete LRN-space in which $L = [0,1]$.

Proof. The proof follows from Theorem 7.5 by taking

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{7.71}$$

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^{-p}$, and we get the desired result. □

Theorem 7.7. *Let X be a linear space, let $(Y, \mu, \mathcal{T}_\wedge)$ be a complete LRN-space, and let Φ be a mapping from X^2 to D_L^+ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 2$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{x/2,y/2}(t) \quad (x, y \in X,\ t > 0). \tag{7.72}$$

Let $f : X \to Y$ be an odd mapping satisfying (7.2)*, then*

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right) \tag{7.73}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{x,x}\left(\frac{2-\alpha}{5\alpha}t\right), \Phi_{2x,x}\left(\frac{2-\alpha}{5\alpha}t\right)\right) \tag{7.74}$$

for all $x \in X$ and $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 7.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x) \tag{7.75}$$

for all $x \in X$. It is easy to see that J is a strictly contractive self-mapping on S with the Lipschitz constant $\alpha/2$. Let $g(x) = f(2x) - 8f(x)$, from (7.58), it follows that

$$\mu_{g(x)-1/2g(2x)}\left(\frac{5}{2}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.76}$$

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5/2$. By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A(2x) = 2A(x) \tag{7.77}$$

for all $x \in X$. Since $h : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{h \in S : d(h, g) < \infty\}. \tag{7.78}$$

This implies that A is a unique mapping satisfying (7.77) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-A(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{7.79}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} g(2^n x) = A(x) \tag{7.80}$$

for all $x \in X$.

(3) $d(h, A) \le (1/(1-\alpha/2))d(h, Jh)$, which implies the inequality

$$d(g, A) \le \frac{5}{2-\alpha}. \tag{7.81}$$

This implies that the inequality (7.74) holds.

Proceeding as in the proof of Theorem 7.5, we obtain that the mapping $A : X \to Y$ satisfies (1.4). Now, we have

$$\begin{aligned} A(2x) - 2A(x) &= \lim_{n\to\infty}\left[\frac{1}{2^n} g\left(2^{n+1}x\right) - \frac{1}{2^{n-1}} g(2^n x)\right] \\ &= 2\lim_{n\to\infty}\left[\frac{1}{2^{n+1}} g\left(2^{n+1}x\right) - \frac{1}{2^n} g(2^n x)\right] = 0 \end{aligned} \tag{7.82}$$

for all $x \in X$. Since the mapping $x \to A(2x) - 8A(x)$ is additive (see Lemma 2.2 of [14]), from the equality $A(2x) = 2A(x)$, we deduce that the mapping $A : X \to Y$ is additive. □

Corollary 7.8. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (7.28), then*

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right) \tag{7.83}$$

exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \ge \frac{(2-2^p)t}{(2-2^p)t + 5(1+2^p)\theta\|x\|^p} \tag{7.84}$$

for all $x \in X$ and $t > 0$, where (X, μ, T_M) is a complete LRN-space in which $L = [0,1]$.

Proof. The proof follows from Theorem 7.7 by taking

$$\mu_{Df(x,y)}(t) \ge \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{7.85}$$

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^p$, and we get the desired result. □

8. Generalized Hyers-Ulam Stability of the Functional Equation (1.4): An Even Case via Fixed-Point Method

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in random Banach spaces, an even case.

Theorem 8.1. *Let X be a linear space, let $(\Upsilon, \mu, \mathcal{T}_\wedge)$ be a complete LRN-space, and let Φ be a mapping from X^2 to D_L^+ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/16$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{2x,2y}(t) \quad (x, y \in X,\ t > 0). \tag{8.1}$$

Let $f : X \to \Upsilon$ be an even mapping satisfying $f(0) = 0$ and (7.2), then

$$Q(x) := \lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right) \tag{8.2}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to \Upsilon$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{x,x}\left(\frac{1-16\alpha}{5\alpha}t\right), \Phi_{2x,x}\left(\frac{1-16\alpha}{5\alpha}t\right)\right) \tag{8.3}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $x = y$ in (7.2), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq_L \Phi_{y,y}(t) \tag{8.4}$$

for all $y \in X$ and $t > 0$. Replacing x by $2y$ in (7.2), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq_L \Phi_{2y,y}(t) \tag{8.5}$$

for all $y \in X$ and $t > 0$. By (8.4) and (8.5),

$$\begin{aligned} \mu_{f(4x)-20f(2x)+64f(x)}(5t) &\geq_L \mathcal{T}_\wedge\left(\mu_{4(f(3x)-6f(2x)+15f(x))}(4t), \mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t)\right) \\ &\geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \end{aligned} \tag{8.6}$$

for all $x \in X$ and $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\mu_{g(x)-16g(x/2)}(5t) \geq_L \mathcal{T}_\wedge(\Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t)) \tag{8.7}$$

for all $x \in X$ and $t > 0$. Let (S, d) be the generalized metric space defined in the proof of Theorem 7.1.

Now we consider the linear mapping $J : S \to S$ such that $Jh(x) := 16h(x/2)$ for all $x \in X$. It is easy to see that J is a strictly contractive self-mapping on S with the Lipschitz constant 16α. It follows from (8.7) that

$$\mu_{g(x)-16g(x/2)}(5\alpha t) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.8}$$

for all $x \in X$ and $t > 0$. So,

$$d(g, Jg) \leq 5\alpha \leq \frac{5}{16} < \infty. \tag{8.9}$$

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{8.10}$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $Q : X \to Y$ is an even mapping with $Q(0) = 0$. The mapping Q is a unique fixed point of J in the set

$$M = \{h \in S : d(h, g) < \infty\}. \tag{8.11}$$

This implies that Q is a unique mapping satisfying (8.10) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-Q(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.12}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x) \tag{8.13}$$

for all $x \in X$.

(3) $d(h, Q) \leq (1/(1-16\alpha))d(h, Jh)$ for every $h \in M$, which implies the inequality

$$d(g, Q) \leq \frac{5\alpha}{1-16\alpha}. \tag{8.14}$$

This implies that the inequality (8.3) holds.

Proceeding as in the proof of Theorem 7.1, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$\begin{aligned} Q(2x) - 16Q(x) &= \lim_{n\to\infty}\left[16^n g\left(\frac{x}{2^{n-1}}\right) - 16^{n+1} g\left(\frac{x}{2^n}\right)\right] \\ &= 16\lim_{n\to\infty}\left[16^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 16^n g\left(\frac{x}{2^n}\right)\right] = 0 \end{aligned} \tag{8.15}$$

for all $x \in X$. Since the mapping $x \to Q(2x) - 4Q(x)$ is quartic, we get that the mapping $Q : X \to Y$ is quartic. □

Corollary 8.2. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (7.28)*, then*

$$Q(x) := \lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right) \tag{8.16}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(2^p - 16)t}{(2^p - 16)t + 5(1 + 2^p)\theta\|x\|^p} \tag{8.17}$$

for all $x \in X$ and $t > 0$, where (X, μ, T_M) is a complete LRN-space in which $L = [0, 1]$.

Proof. The proof follows from Theorem 8.1 by taking

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{8.18}$$

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^{-p}$, and we get the desired result. □

Theorem 8.3. *Let X be a linear space, let $(Y, \mu, \mathcal{T}_\wedge)$ be a complete LRN-space, and let Φ be a mapping from X^2 to D_L^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 16$,*

$$\Phi_{x,y}(\alpha t) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X,\ t > 0). \tag{8.19}$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.2), then

$$Q(x) := \lim_{n\to\infty} \frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right) \tag{8.20}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{x,x}\left(\frac{16-\alpha}{5}t\right), \Phi_{2x,x}\left(\frac{16-\alpha}{5}t\right)\right) \tag{8.21}$$

for all $x \in X$ and $t > 0$.

Proof. In the generalized metric space (S, d) defined in the proof of Theorem 7.1, we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{16}h(2x) \tag{8.22}$$

for all $x \in X$. It is easy to see that J is a strictly contractive self-mapping on S with the Lipschitz constant $\alpha/16$.

Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, by (8.7), we get

$$\mu_{g(x)-(1/16)g(2x)}\left(\frac{5}{16}t\right) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.23}$$

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5/16$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q(2x) = 16Q(x) \tag{8.24}$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $Q : X \to Y$ is an even mapping with $Q(0) = 0$. The mapping Q is a unique fixed point of J in the set

$$M = \{h \in S : d(h,g) < \infty\}. \tag{8.25}$$

This implies that Q is a unique mapping satisfying (8.24) such that there exists a $u \in (0,\infty)$ satisfying

$$\mu_{g(x)-Q(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.26}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{16^n} g(2^n x) = Q(x) \tag{8.27}$$

for all $x \in X$.

(3) $d(g,Q) \leq (16/(16-\alpha))d(g,Jg)$ for each $h \in M$, which implies the inequality

$$d(g,Q) \leq 5/(16-\alpha). \tag{8.28}$$

This implies that the inequality (8.21) holds.

Proceeding as in the proof of Theorem 7.3, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$\begin{aligned} Q(2x) - 16Q(x) &= \lim_{n\to\infty}\left[\frac{1}{16^n} g\left(2^{n+1}x\right) - \frac{1}{16^{n-1}} g(2^n x)\right] \\ &= 16 \lim_{n\to\infty}\left[\frac{1}{16^{n+1}} g\left(2^{n+1}x\right) - \frac{1}{16^n} g(2^n x)\right] = 0 \end{aligned} \tag{8.29}$$

for all $x \in X$. Since the mapping $x \to Q(2x) - 4Q(x)$ is quartic, we get that the mapping $Q : X \to Y$ is quartic. □

Corollary 8.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28), then*

$$Q(x) := \lim_{n\to\infty} \frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right) \tag{8.30}$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(16-2^p)t}{(16-2^p)t+5(1+2^p)\theta\|x\|^p} \tag{8.31}$$

for all $x \in X$ and $t > 0$, where (X, μ, T_M) is a complete LRN-space in which $L = [0,1]$.

Proof. The proof follows from Theorem 8.3 by taking

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{8.32}$$

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^p$, and we get the desired result. □

Theorem 8.5. *Let X be a linear space, let $(Y, \mu, \mathcal{T}_\wedge)$ be a complete LRN-space, and let Φ be a mapping from X^2 to D_L^+ ($\Phi(x,y)$ is by denoted $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/4$,*

$$\Phi_{x,y}(\alpha t) \geq_L \Phi_{2x,2y}(t) \quad (x, y \in X,\ t > 0). \tag{8.33}$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.2), then

$$T(x) := \lim_{n\to\infty} 4^n\left(f\left(\frac{x}{2^{n-1}}\right) - 16f\left(\frac{x}{2^n}\right)\right) \tag{8.34}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq_L \mathcal{T}_\wedge\left(\Phi_{x,x}\left(\frac{1-4\alpha}{5\alpha}t\right), \Phi_{2x,x}\left(\frac{1-4\alpha}{5\alpha}t\right)\right) \tag{8.35}$$

for all $x \in X$ and $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 7.1.

Letting $g(x) := f(2x) - 16f(x)$ for all $x \in X$ in (8.6), we get

$$\mu_{g(x)-4g(x/2)}(5t) \geq_L \mathcal{T}_\wedge(\Phi_{x/2,x/2}(t), \Phi_{x,x/2}(t)) \tag{8.36}$$

for all $x \in X$ and $t > 0$. It is easy to see that the linear mapping $J : S \to S$ such that

$$Jh(x) := 4h\left(\frac{x}{2}\right) \tag{8.37}$$

for all $x \in X$, is a strictly contractive self-mapping with the Lipschitz constant 4α.

It follows from (8.36) that

$$\mu_{g(x)-4g(x/2)}(5\alpha t) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.38}$$

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5\alpha < \infty$.

By Theorem 1.1, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, that is,

$$T\left(\frac{x}{2}\right) = \frac{1}{4}T(x) \tag{8.39}$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $T : X \to Y$ is an even mapping with $T(0) = 0$. The mapping T is a unique fixed point of J in the set $M = \{h \in S : d(h, g) < \infty\}$. This implies that T is a unique mapping satisfying (8.39) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-T(x)}(ut) \geq_L \mathcal{T}_\wedge(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.40}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n g\left(\frac{x}{2^n}\right) = T(x) \tag{8.41}$$

for all $x \in X$.

(3) $d(h, T) \leq (1/(1-4\alpha))d(h, Jh)$ for each $h \in M$, which implies the inequality

$$d(g, T) \leq \frac{5\alpha}{1-4\alpha}. \tag{8.42}$$

This implies that the inequality (8.35) holds.

Proceeding as in the proof of Theorem 7.1, we obtain that the mapping $T : X \to Y$ satisfies (1.4). Now, we have

$$\begin{aligned} T(2x) - 4T(x) &= \lim_{n\to\infty}\left[4^n g\left(\frac{x}{2^{n-1}}\right) - 4^{n+1} g\left(\frac{x}{2^n}\right)\right] \\ &= 4\lim_{n\to\infty}\left[4^{n-1} g\left(\frac{x}{2^{n-1}}\right) - 4^n g\left(\frac{x}{2^n}\right)\right] = 0 \end{aligned} \tag{8.43}$$

for all $x \in X$. Since the mapping $x \to T(2x) - 16T(x)$ is quadratic, we get that the mapping $T : X \to Y$ is quadratic. □

Corollary 8.6. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28), then*

$$T(x) := \lim_{n\to\infty} 4^n\left(f\left(\frac{x}{2^{n-1}}\right) - 16 f\left(\frac{x}{2^n}\right)\right) \tag{8.44}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(2^p-4)t}{(2^p-4)t+5(1+2^p)\theta\|x\|^p} \tag{8.45}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 8.5 by taking

$$\Phi_{x,y}(t) := \frac{t}{t+\theta(\|x\|^p+\|y\|^p)} \tag{8.46}$$

for all $x, y \in X$. Then we can choose $\alpha = 2^{-p}$, and we get the desired result. □

Theorem 8.7. *Let X be a linear space, let (Y, μ, T_M) be a complete RN-space, and let Φ be a mapping from X^2 to D^+ ($\Phi(x,y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 4$,*

$$\Phi_{x,y}(\alpha t) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X,\ t > 0). \tag{8.47}$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.2), then

$$T(x) := \lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right) \tag{8.48}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq T_M\left(\Phi_{x,x}\left(\frac{4-\alpha}{5}t\right), \Phi_{2x,x}\left(\frac{4-\alpha}{5}t\right)\right) \tag{8.49}$$

for all $x \in X$ and $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 7.1.

It is easy to see that the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{4}h(2x) \tag{8.50}$$

for all $x \in X$ is a strictly contractive self-mapping with the Lipschitz constant $\alpha/4$.

Letting $g(x) := f(2x) - 16f(x)$ for all $x \in X$, from (8.36), we get

$$\mu_{g(x)-1/4g(2x)}\left(\frac{5}{4}t\right) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.51}$$

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5/4$.

By Theorem 1.1, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, that is,

$$T(2x) = 4T(x) \tag{8.52}$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0, T : X \to Y$ is an even mapping with $T(0) = 0$. The mapping T is a unique fixed point of J in the set

$$M = \{h \in S : d(h, g) < \infty\}. \tag{8.53}$$

This implies that T is a unique mapping satisfying (8.52) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-T(x)}(ut) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.54}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{4^n} g(2^n x) = T(x) \tag{8.55}$$

for all $x \in X$.

(3) $d(h, T) \leq (1/(1 - \alpha/4))d(h, Jh)$ for each $h \in M$, which implies the inequality

$$d(g, T) \leq 5/(4 - \alpha). \tag{8.56}$$

This implies that the inequality (8.49) holds.

Proceeding as in the proof of Theorem 2.3, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$\begin{aligned} T(2x) - 4T(x) &= \lim_{n\to\infty} \left[\frac{1}{4^n} g\left(2^{n+1}x\right) - \frac{1}{4^{n-1}} g(2^n x)\right] \\ &= 4 \lim_{n\to\infty} \left[\frac{1}{4^{n+1}} g\left(2^{n+1}x\right) - \frac{1}{4^n} g(2^n x)\right] = 0 \end{aligned} \tag{8.57}$$

for all $x \in X$. Since the mapping $x \to T(2x) - 16T(x)$ is quadratic, we get that the mapping $T : X \to Y$ is quadratic. □

Corollary 8.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28). Then*

$$T(x) := \lim_{n\to\infty} \frac{1}{4^n} \left(f\left(2^{n+1}x\right) - 16f(2^n x)\right) \tag{8.58}$$

exists for each $x \in X$ and defines a quadratic mapping $T : X \rightarrow Y$ such that

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(4-2^p)t}{(4-2^p)t + 5(1+2^p)\theta\|x\|^p} \tag{8.59}$$

for all $x \in X$ and $t > 0$, where (X, μ, T_M) is a complete LRN-space in which $L = [0,1]$.

Proof. The proof follows from Theorem 8.5 by taking

$$\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{8.60}$$

for all $x, y \in X$ and $t > 0$. Then we can choose $\alpha = 2^p$, and we get the desired result. □

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