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Research Article

Superconvergence Analysis of Finite Element Method for a Second-Type Variational Inequality

Dongyang Shi,[1] Hongbo Guan,[1,2] and Xiaofei Guan[3]

[1] *Department of Mathematics, Zhengzhou University, Zhengzhou 450001, China*
[2] *Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China*
[3] *Department of Mathematics, Tongji University, Shanghai 200092, China*

Correspondence should be addressed to Xiaofei Guan, guanxf@tongji.edu.cn

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This paper studies the finite element (FE) approximation to a second-type variational inequality. The supe rclose and superconvergence results are obtained for conforming bilinear FE and nonconforming EQ^{rot} FE schemes under a reasonable regularity of the exact solution $u \in H^{5/2}(\Omega)$, which seem to be never discovered in the previous literature. The optimal L^2-norm error estimate is also derived for EQ^{rot} FE. At last, some numerical results are provided to verify the theoretical analysis.

1. Introduction

Variational inequality (VI) theory has been playing an important role in the obstacle problem, contact problem, elasticity problem, and so on [1]. FE methods for solving VI problems have attracted more and more attentions. For example, as regards to the first type-VI case, the authors of [2] used piecewise quadratic FE to approximate the obstacle problem and suggested the error order between the FE solution and the exact solution should be $O(h^{3/2})$. The authors of [3] first obtained the error bound $O(h^{3/2-\varepsilon})$ (for any $\varepsilon > 0$) for the above FE when the obstacle vanished. Then through a detailed analysis, the authors of [4] obtained the same error bound as the ones of [3] under the hypothesis that the free boundary has finite length. Later, the authors of [5] obtained the same error bound as the ones of [3] for the same element without the hypothesis of finite length of the free boundary. Furthermore, [6] investigated the Wilson's element approximation to the obstacle problem and derived the error bound with order $O(h)$. The authors of [7] obtained the same error estimate with order

$O(h)$ on anisotropic meshes by making the full use of the bilinear part of the Wilson element, which relaxed the interpolation restriction and simplified the proofs of [5, 6]. Recently, the authors of [8] proposed a class of nonconforming FE methods for the parabolic obstacle VI problem with moving grids and obtained the optimal error estimates on anisotropic meshes. On the other hand, some studies [9–11] have been devoted to FE approximation to Signorini problem which arises in contact problems and obtained different error estimates under different assumptions. The authors of [12] derived the convergence result of $O(h^{3/4}|\log h|^{1/4})$ if the displacement field is of H^2 regularity and also showed that if stronger but reasonable regularity is available ($u \in W^{2,p}$, $p > 2$), the above result can be improved to optimal order $O(h)$. The authors of [13] applied a class of Crouzeix-Raviart-type FEs to Signorini problem and obtained $O(h)$ order estimate on anisotropic meshes. The authors of [14] used the bilinear FE to approximate the frictionless Signorini problem by virtue of the information on the contact zone and derived a superconvergence rate of $O(h^{3/2})$ when the exact solution $u \in H^{5/2}(\Omega)$. The authors of [15] presented the nonconforming Carey FE approximation to the problem of [14] and obtained the same convergence and superconvergence results are also obtained.

For the second type case, the authors of [16] proposed a Galerkin FE schemes for deriving a posteriori error estimates for a friction problem and a model flow of Bingham fluid. The authors of [17] considered the FE approximation to the plate contact problem and obtained some error estimates by employing the technique of mesh dependent norm.

In this paper, we will consider the following second type-VI problem [18, 19]:

$$\begin{gathered}\text{find} \quad u \in K^*, \quad \text{such that} \\ a(u, v-u) + j(v) - j(u) \geq (f, v-u), \quad \forall v \in K^*,\end{gathered} \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded convex polygonal domain; K^* is defined as follows:

$$K^* = \left\{ v \in H^1(\Omega) \mid v = 0, \text{ on } \Gamma - \Gamma_d;\ v \geq 0,\ \frac{\partial v}{\partial n} \geq 0,\ v\frac{\partial v}{\partial n} = 0, \text{ on } \Gamma_d = \Gamma_d^0 \cup \Gamma_d^+ \right\}, \tag{1.2}$$

in which $\Gamma = \partial\Omega, \Gamma_d \subset \Gamma$ and $\Gamma_d^0 = \{x \in \Gamma_d \mid v(x) = 0\}$, $\Gamma_d^+ = \{x \in \Gamma_d \mid v(x) > 0\}$. $a(u,v) = \int_\Omega (\nabla u \nabla v + \mu uv)\,dx\,dy$, μ is a positive constant, $(f,v) = \int_\Omega fv\,dx\,dy$, $j(v) = \int_{\Gamma_d} \psi(v)ds$, and

$$\psi(t) = \int_0^t \varphi(\tau)d\tau, \qquad \varphi(\tau) = \begin{cases} g, & \tau \geq kg, \\ \dfrac{\tau}{k}, & |\tau| \leq kg, \\ -g & \tau \leq -kg, \end{cases} \tag{1.3}$$

and g and k are positive constants. (1.1) may describe many practical engineering problems and attracts many scholars' interests. For instance, the authors of [20] obtained the $O(h^{1/2-\varepsilon})$ error estimate of energy norm for linear FE; the authors of [21] got the $O(h^{1/2})$ error estimate in energy norm by improving the result of [20] for $u \in H^{3/2}(\Omega)$; the authors of [22] derived the optimal $O(h^2)$ error estimate of L^2 norm and $O(h)$ error estimate of energy norm when $u \in H^2(\Omega)$. But all the above studies mentioned above only paid attention to the convergence analysis for the conforming FE with no consideration on the superconvergence property, although it is surely an interesting and useful phenomenon in scientific computing of industrial problems [23].

In this paper, as a first attempt, we try to investigate the superconvergence of conforming and nonconforming FE schemes for problem (1.1) with a reasonable assumption of $u \in H^{5/2}(\Omega)$. The rest of this paper is organized as follows. In the next section, we give the equivalent form of (1.1) and the conforming bilinear FE (see [14]) approximation of (1.1). Moreover, superclose result of $O(h^{3/2})$ is derived under the broken energy norm. In Section 3, the nonconforming $\mathrm{EQ}^{\mathrm{rot}}$ FE (see [26]) approximation is used, and the same superclose result is obtained under the energy norm; the optimal error estimate of L^2-norm is also derived when $u \in H^2(\Omega)$. In Section 4, we construct a postprocessing interpolation operator to obtain the superconvergence properties. In Section 5, we present some numerical results to verify the theoretical analysis.

2. The Equivalent Form and Conforming FE Scheme

It has been shown in [21, 22] that (1.1) is equivalent to

$$\begin{gathered} \text{find } u \in K^*, \quad \text{such that} \\ a(u,v) + \int_{\Gamma_d} \varphi(u) v \, ds = (f, v), \quad \forall v \in K^*, \end{gathered} \tag{2.1}$$

and (2.1) has the unique solution u in K^*. It can be verified that $\varphi(t)$ satisfies the following two properties: for all $a, b \in R^1$,

$$\left|\varphi(a) - \varphi(b)\right| \le \frac{1}{k}|a-b|, \tag{2.2}$$

$$\left(\varphi(a) - \varphi(b)\right)(a-b) \ge 0. \tag{2.3}$$

Let T_h be a rectangular partition with a maximum size h in (x, y) plane, $K \in T_h$ a general element; V_h^1 and V_h^2 are the conforming bilinear FE space and the nonconforming $\mathrm{EQ}^{\mathrm{rot}}$ FE space. We denote by Π_h^1 and Π_h^2 the associated interpolation operators on V_h^1 and V_h^2, respectively. In the meantime, we denote K_h^i by a convex set associated with K^* in V_h^i $(i = 1, 2)$ as follows:

$$\begin{gathered} K_h^1 = \left\{ v_h \in V_h^1 \mid v_h = 0 \quad \text{on } \Gamma - \Gamma_d \right\}, \\ K_h^2 = \left\{ v_h \in V_h^2 \mid \int_F v_h \, ds = 0, F \subset \Gamma - \Gamma_d, \int_F v_h \, ds \ge 0, F \subset \Gamma_d \right\}, \end{gathered} \tag{2.4}$$

where F is an edge of K. The following two lemmas will play an important role in the FE analysis, which can be found in [14, 24], respectively.

Lemma 2.1. *For all $u \in H^2(\Omega), F \subset \partial K$, there holds $\|u - \Pi_h^i u\|_{0,F} \leq Ch^{3/2}|u|_{2,K}$.*

Lemma 2.2. *Let $u \in H^{5/2}(\Omega)$, then for $v_h \in K_h^1$, there holds*

$$\left(\nabla\left(u - \Pi_h^1 u\right), v_h\right) = O\left(h^{3/2}\right)|u|_{5/2}|v_h|_1, \tag{2.5}$$

where $|u|_{5/2} = \sum_{|\alpha|=2} \iint_\Omega |u^{(\alpha)}(\vartheta) - u^{(\alpha)}(\theta)|^2/|\vartheta - \theta|^3 d\vartheta\, d\theta$.

The corresponding conforming FE approximation version of (2.1) reads as

$$\begin{gathered}\text{find} \quad u \in K_h^1, \quad \text{such that} \\ a(u_h, v_h) + \int_{\Gamma_d} \varphi(u_h) v_h ds = (f, v_h), \quad \forall v_h \in K_h^1.\end{gathered} \tag{2.6}$$

Theorem 2.3. *Let $u \in H^{5/2}(\Omega)$ be the exact solution of (1.1) and $u_h \in K_h^1$ the bilinear FE solution of (2.6), then there holds*

$$\left|\Pi_h^1 u - u_h\right|_1 \leq ch^{3/2}|u|_{5/2}, \tag{2.7}$$

here and later, c is a generic positive constant, which is independent of h, K, and u.

Proof. Subtracting (2.1) from (2.6), then taking $v = v_h$ in it, one can get

$$a(u - u_h, v_h) + \int_{\Gamma_d} (\varphi(u) - \varphi(u_h)) v_h ds = 0. \tag{2.8}$$

Let $\xi = \Pi_h^1 u - u_h$ and $\eta = u - \Pi_h^1 u$. Taking $v_h = \xi$ in the above equation, there yields

$$a(u - u_h, \xi) + \int_{\Gamma_d} (\varphi(u) - \varphi(u_h)) \xi ds = 0. \tag{2.9}$$

By the definition of $a(v, v)$, we have

$$\begin{aligned}|\xi|_1^2 &\leq a(\xi, \xi) = a(u - u_h, \xi) - a(\eta, \xi) \\ &= -\int_{\Gamma_d} (\varphi(u) - \varphi(u_h)) \xi ds - a(\eta, \xi) \\ &= -\int_{\Gamma_d} \left(\varphi(u) - \varphi\left(\Pi_h^1 u\right)\right) \xi ds - \int_{\Gamma_d} \left(\varphi\left(\Pi_h^1 u\right) - \varphi(u_h)\right) \xi ds - (\nabla \eta, \nabla \xi) - \mu(\eta, \xi).\end{aligned} \tag{2.10}$$

Noticing (2.3), we have $-\int_{\Gamma_d} (\varphi(\Pi_h^1 u) - \varphi(u_h)) \xi ds \leq 0$; thus

$$|\xi|_1^2 \leq I_1 + I_2, \tag{2.11}$$

in which $I_1 = -\int_{\Gamma_d} (\varphi(u) - \varphi(\Pi_h^1 u)) \xi ds, I_2 = -(\nabla \eta, \nabla \xi) - \mu(\eta, \xi)$.

From (2.2) and Lemma 2.1, I_1 can be estimated as

$$|I_1| \leq \frac{c}{k}\int_{\Gamma_d} |\eta||\xi| ds \leq \|\eta\|_{0,\Gamma_d}\|\xi\|_{0,\Gamma_d} \leq ch^{3/2}|u|_2|\xi|_1. \tag{2.12}$$

Applying the interpolation theory and Lemma 2.2, we get

$$|I_2| \leq ch^{3/2}|u|_{5/2}|\xi|_1. \tag{2.13}$$

The desired result follows directly from the combination of (2.12) and (2.13). □

3. The Nonconforming FE Scheme

The corresponding nonconforming FE approximation scheme of (2.1) reads as

$$\begin{gathered} \text{find} \quad u \in K_h^2, \quad \text{such that} \\ a_h(u_h, v_h) + \int_{\Gamma_d} \varphi(u_h) v_h ds = (f, v_h), \quad \forall v_h \in K_h^2, \end{gathered} \tag{3.1}$$

where $a_h(u,v) = \sum_K \int_K (\nabla u \nabla v + \mu uv) dx\, dy$.

First, we introduce the following Lemma 3.1, which can be found in [25].

Lemma 3.1 (see [25]). *If $u \in H^2(\Omega), v_h \in K_h^2$, one has*

$$\left(\nabla\left(u - \Pi_h^2 u\right), \nabla v_h\right) = 0. \tag{3.2}$$

By using the similar technique in [26], one now states and proves the following important conclusion.

Lemma 3.2. *For all $u \in H^{5/2}(\Omega), v_h \in K_h^2$, there holds*

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h ds \leq ch^{3/2}|u|_{5/2}\|v_h\|_h, \tag{3.3}$$

where $\|v_h\|_h = (\sum_{K\in T_h} |v_h|_{1,K}^2)^{1/2}$.

Proof. Let $Z_1 = (x_0 - h_x, y_0 - h_y), Z_2 = (x_0 + h_x, y_0 - h_y), Z_3 = (x_0 + h_x, y_0 + h_y)$, and $Z_4 = (x_0 - h_x, y_0 + h_y)$ be the four vertices of K, $F_i = \overline{Z_i Z_{i+1}}$ $(i = 1, 2, 3, 4, \mod 4)$. We define operators P_0 and P_{0i} as

$$P_0 v = \frac{1}{|K|}\int_K v\, dx, \qquad P_{0i}\omega = \frac{1}{|F_i|}\int_{F_i} \omega\, ds, \tag{3.4}$$

respectively, where $|K|$ and $|F_i|$ denote the measures of K and F_i, respectively.

It can be checked that

$$\begin{aligned}\sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h ds = \sum_K \Bigg[&- \int_{F_1} \frac{\partial u}{\partial y}(v_h - P_{01}v_h)dx + \int_{F_2} \frac{\partial u}{\partial x}(v_h - P_{02}v_h)dy \\ &+ \int_{F_3} \frac{\partial u}{\partial y}(v_h - P_{03}v_h)dx - \int_{F_4} \frac{\partial u}{\partial x}(v_h - P_{04}v_h)dy \Bigg] + \sum_{F\subset\Gamma_d} \int_F \frac{\partial u}{\partial n} v_h ds \\ &\doteq \sum_K \sum_{i=1}^{4} M_i + M.\end{aligned} \tag{3.5}$$

By the definition of P_{01}, we get

$$\begin{aligned}&\int_K (v_h(x, y_0 - h_y) - P_{01}v_h(x, y_0 - h_y))dx\,dy \\ &\quad = 2h_y \int_{F_1} v_h(x, y_0 - h_y)dx - \frac{4h_x h_y}{|F_1|} \int_{F_1} v_h(x, y_0 - h_y)dx = 0.\end{aligned} \tag{3.6}$$

Noticing that $(v_h - P_{01}v_h)|_{F_1}$ equals $(v_h - P_{03}v_h)|_{F_3}$ and $\partial v_h/\partial x$ is only dependent on x, we can derive that

$$\begin{aligned}M_1 + M_3 &= \int_{x_0-h_x}^{x_0+h_x} \left[\frac{\partial u}{\partial y}(x, y_0 + h_y) - \frac{\partial u}{\partial y}(x, y_0 - h_y)\right](v_h - P_{01}v_h)dx \\ &= \int_{x_0-h_x}^{x_0+h_x} \left[\int_{y_0-h_y}^{y_0+h_y} \frac{\partial^2 u}{\partial y^2}(x, y)dy\right](v_h - P_{01}v_h)dx \\ &= \int_{x_0-h_x}^{x_0+h_x} \int_{y_0-h_y}^{y_0+h_y} \left(\frac{\partial^2 u}{\partial y^2} - P_0 \frac{\partial^2 u}{\partial y^2}\right)(v_h - P_{01}v_h)dy\,dx \\ &= \left\|\frac{\partial^2 u}{\partial y^2} - P_0 \frac{\partial^2 u}{\partial y^2}\right\|_{0,K} \|v_h - P_{01}v_h\|_{0,K} \\ &\leq ch^{3/2}|u|_{5/2,K}|v_h|_{1,K}.\end{aligned} \tag{3.7}$$

Similarly, $M_2 + M_4 \leq ch^{3/2}|u|_{5/2,K}|v_h|_{1,K}$. By using the same technique as [14, 15], M can be estimated as

$$|M| \leq ch^3|u|_{5/2}\|v_h\|_h. \tag{3.8}$$

Thus the desired result follows. □

Theorem 3.3. *Let $u \in H^{5/2}(\Omega)$ be the exact solution of* (1.1) *and $u_h \in K_h^2$ the nonconforming FE solution of* (3.1). *Then one has*

$$\left\|\Pi_h^2 u - u_h\right\|_h \leq C h^{3/2} |u|_{5/2}. \tag{3.9}$$

Proof. Subtracting (2.1) from (3.1) gives

$$a_h(u - u_h, v_h) + \int_{\Gamma_d} (\varphi(u) - \varphi(u_h)) v_h ds = \sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h ds. \tag{3.10}$$

For convenience, we still denote $\xi = \Pi_h^2 u - u_h$ and $\eta = u - \Pi_h^2 u$. Taking $v_h = \Pi_h^2 u - u_h$ in (3.10) yields

$$a_h(u - u_h, \xi) + \int_{\Gamma_d} (\varphi(u) - \varphi(u_h)) \xi ds = \sum_K \int_{\partial K} \frac{\partial u}{\partial n} \xi ds. \tag{3.11}$$

By Lemma 3.1, we can derive that

$$\begin{aligned} \|\xi\|_h^2 &\leq a_h(\xi, \xi) = a_h(u - u_h, \xi) - a_h(\eta, \xi) \\ &= -\int_{\Gamma_d} (\varphi(u) - \varphi(u_h)) \xi ds - a_h(\eta, \xi) + \sum_K \int_{\partial K} \frac{\partial u}{\partial n} \xi ds \\ &= -\int_{\Gamma_d} \left(\varphi(u) - \varphi\left(\Pi_h^2 u\right)\right) \xi ds - \int_{\Gamma_d} \left(\varphi\left(\Pi_h^2 u\right) - \varphi(u_h)\right) \xi ds - \mu(\eta, \xi) + \sum_K \int_{\partial K} \frac{\partial u}{\partial n} \xi ds. \end{aligned} \tag{3.12}$$

Noticing Lemma 3.2 and using the analysis technique of Theorem 2.3, one can immediately get the desired result. □

Remark 3.4. As a by-product, if we assume $u \in H^2(\Omega)$ instead of $u \in H^{5/2}(\Omega)$, the consistency error can be estimated as

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h ds \leq c h |u|_2 \|v_h\|_h, \tag{3.13}$$

which can be found in [26]. Then we can derive the following optimal error estimate:

$$\|u - u_h\|_h \leq C h |u|_2. \tag{3.14}$$

Now we start to give the L^2-norm estimate through a duality argument.

Theorem 3.5. *Let $u \in K^2(\Omega)$ and $u_h \in V_h^2$ be the solutions of* (1.1) *and* (3.1), *respectively, there holds*

$$\|u - u_h\|_0 \leq C h^2 |u|_2. \tag{3.15}$$

Proof. Let $\omega \in H^2(\Omega)$ be the solution of the following auxiliary elliptic problem:

$$\begin{aligned} -\triangle w + \mu w &= u - u_h, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \Gamma - \Gamma_d, \\ \frac{\partial w}{\partial n} &= -\beta(x) w, \quad \text{on } \Gamma_d, \end{aligned} \tag{3.16}$$

in which $\beta(x) = (\varphi(u) - \varphi(u_h))/(u - u_h)$, then

$$\|w\|_2 \le c\|u - u_h\|_0. \tag{3.17}$$

By (3.16) and Lemma 3.1, we can derive that

$$\begin{aligned} \|u - u_h\|_0^2 = (u - u_h, u - u_h) &= a_h(u - u_h, w) \\ &\quad + \int_{\Gamma_d} \beta w(u - u_h)ds + \sum_K \int_{\partial K} \frac{\partial w}{\partial n}(u - u_h)ds \\ &= a_h\left(u - u_h, w - \Pi_h^2 w\right) + a_h\left(u - u_h, \Pi_h^2 w\right) \\ &\quad + \int_{\Gamma_d} \beta w(u - u_h)ds + \sum_K \int_{\partial K} \frac{\partial w}{\partial n}(u - u_h)ds \\ &= a_h\left(u - u_h, w - \Pi_h^2 w\right) - \int_{\Gamma_d} (\varphi(u) - \varphi(u_h))\Pi_h^2 w\, ds + \int_{\Gamma_d} \beta w(u - u_h)ds \\ &\quad + \sum_K \int_{\partial K} \frac{\partial w}{\partial n}(u - u_h)ds + \sum_K \int_{\partial K} \frac{\partial u}{\partial n}\Pi_h^2 w\, ds \\ &= a_h\left(u - u_h, w - \Pi_h^2 w\right) + \frac{1}{k}\int_{\Gamma_d} (u - u_h)\left(w - \Pi_h^2 w\right)ds \\ &\quad + \sum_K \int_{\partial K} \frac{\partial w}{\partial n}(u - u_h)ds + \sum_K \int_{\partial K} \frac{\partial u}{\partial n}\left(w - \Pi_h^2 w\right)ds \\ &= J_1 + J_2 + J_3, \end{aligned} \tag{3.18}$$

where $J_1 = a_h(u-u_h, w-\Pi_h^2 w)$, $J_2 = 1/k \int_{\Gamma_d}(u-u_h)(w-\Pi_h^2 w)ds$, and $J_3 = \sum_K \int_{\partial K} \partial w/\partial n(u-u_h)ds + \sum_K \int_{\partial K} \partial u/\partial n(w-\Pi_h^2 w)ds$. These three terms can be estimated one by one as follows. By (3.14), (3.17), and the interpolation theory, J_1 can be estimated as

$$\begin{aligned} J_1 &= \left(\nabla(u - u_h), \nabla\left(w - \Pi_h^2 w\right)\right) + \mu\left(u - u_h, w - \Pi_h^2 w\right) \\ &\le ch^2|u|_2|w|_2 + ch^2\|u - u_h\|_0|w|_2 \\ &\le ch^2|u|_2\|u - u_h\|_0 + ch^2\|u - u_h\|_0^2. \end{aligned} \tag{3.19}$$

By the trace theorem, (3.17), and Lemma 2.1, one gets

$$J_2 \le \frac{1}{k}\|u-u_h\|_{0,\Gamma_d}\left\|u-\Pi_h^2 u\right\|_{0,\Gamma_d} \le ch^{5/2}|u|_2|w|_2 \le ch^{5/2}|u|_2\|u-u_h\|_0. \tag{3.20}$$

By (3.13), (3.14), and (3.17), we have

$$J_3 \le ch|u|_2\left\|w-\Pi_h^2 w\right\|_h + ch|w|_2\|u-u_h\|_h \le ch^2|u|_2|w|_2 \le ch^2|u|_2\|u-u_h\|_0. \tag{3.21}$$

The desired result follows the combination of the above estimates of J_1, J_2, and J_3. □

Remark 3.6. As to the L^2-norm error estimate of bilinear FE scheme, the readers may refer to [21, 22].

4. The Global Superconvergence Result

In order to obtain the global superconvergence, we combine the four neighbouring elements $K_1, K_2, K_3, K_4 \in T_h$ into one new rectangular element K_0, whose four edges are L_1, L_2, L_3, and L_4. T_{2h} represents the corresponding new partition. For the conforming FE scheme, we construct the postprocessing operator $\Pi_{2h}^1 u|_{K_0} : C(K_0) \to P_2(K_0)$ as follows:

$$\Pi_{2h}^1 u(Z_j) = u(Z_j), \quad j = 1, 2, \ldots, 8, \tag{4.1}$$

in which Z_j is the four vertices and four mid point of edges of K_0. For the nonconforming FE scheme, we construct the postprocessing Π_{2h}^2 operator as

$$\begin{gathered} \Pi_{2h}^2 u|_{K_0} \in P_2(K_0), \quad \forall K_0 \in T_{2h}, \\ \int_{L_j}\left(\Pi_{2h}^2 u - u\right)ds = 0, \quad j = 1, 2, 3, 4, \\ \int_{K_1\cup K_3}\left(\Pi_{2h}^2 u - u\right)dx = 0, \quad \int_{K_2\cup K_4}\left(\Pi_{2h}^2 u - u\right)dx = 0, \quad \forall K_0 \in T_{2h}. \end{gathered} \tag{4.2}$$

It is easy to validate that the interpolation operator is well posed and has the following properties [23]:

$$\begin{gathered} \Pi_{2h}^i \Pi_h^i u = \Pi_{2h}^i u, \quad \forall u \in H^2(\Omega), \\ \left\|\Pi_{2h}^i u - u\right\|_h \le ch^r|u|_{r+1}, \quad \forall u \in H^{r+1}(\Omega), 0 \le r \le 2, \\ \left\|\Pi_{2h}^i v_h\right\|_h \le c\|v_h\|_h, \quad \forall v_h \in K_h^i. \end{gathered} \tag{4.3}$$

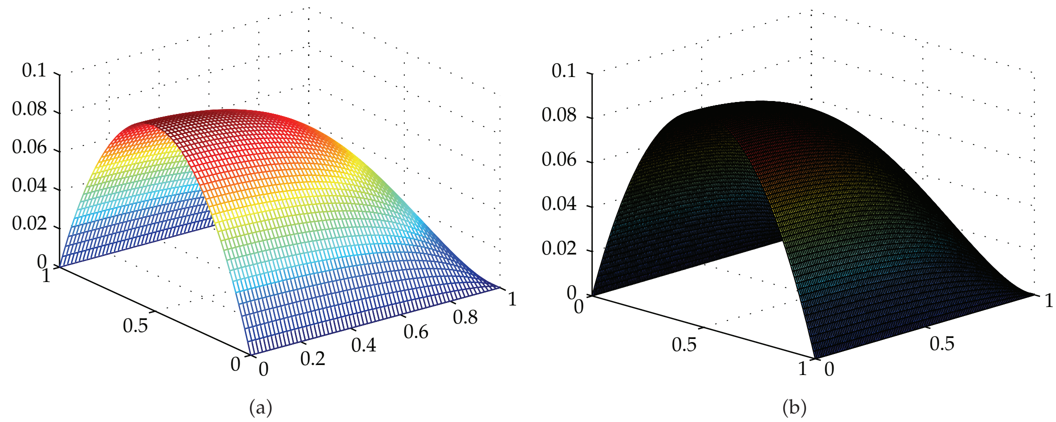

Figure 1: The conforming FE solution (a) and the nonconforming FE solution (b) on the 64 × 64 mesh.

Theorem 4.1. *If $u \in H^{5/2}(\Omega)$ is the exact solution of* (1.1), u_h *is the conforming or nonconforming FE solution. The following superconvergence result*

$$\left\| u - \Pi_{2h}^i u_h \right\|_h \leq ch^{3/2} |u|_{5/2} \tag{4.4}$$

holds.

Proof. By (4.3), one gets

$$\begin{aligned} \left\| \Pi_{2h}^i \Pi_h^i u - \Pi_{2h}^i u_h \right\|_h &= \left\| \Pi_{2h}^i (\Pi_h^i u - u_h) \right\|_h \leq c \left\| \Pi_h^i u - u_h \right\|_h \leq ch^{3/2} |u|_{5/2}, \\ \left\| \Pi_{2h}^i \Pi_h^i u - u \right\|_h &= \left\| \Pi_{2h}^i u - u \right\|_h \leq ch^{3/2} |u|_{5/2}. \end{aligned} \tag{4.5}$$

Noticing $\Pi_{2h}^i u_h - u = \Pi_{2h}^i u_h - \Pi_{2h}^i \Pi_h^i u + \Pi_{2h}^i \Pi_h^i u - u$, the proof is completed. □

5. Numerical Results

In this section, we will present an example to confirm the correctness of our theoretical analysis. In (1.1), we choose $\Omega = [0,1] \times [0,1]$ with boundary $\partial\Omega = \Gamma, \mu = 1, \varphi(u) = u, \Gamma_d = \{0\} \times [0,1], u|_{\Gamma_d} = (x_1 - 1/2)^2 - 1/4, u|_{\Gamma - \Gamma_d} = 0$. The right hand term $f = 1$. Since there may be no exact solution to the above problem, we use the conforming FE solution on a sufficient refined mesh $h = 1/256$ as the reference solution. Then we compare the conforming and nonconforming FE solutions (see Figure 1) on the coarser meshes ($h = 1/2, 1/4, 1/8, 1/16, 1/32, 1/64$) with the reference one in Tables 1 and 2.

From the above tables, we can see that the conforming and nonconforming FE solutions both converge. At the same time, the superconvergence results in our experiments are a little better than the theoretical ones. We may explain this phenomenon with some special properties of this nonconforming FE that we have not discovered.

Table 1: The error estimates for conforming FE scheme.

h	1/2	1/4	1/8	1/16	1/32	1/64
$\Vert \Pi_h^1 u - u_h \Vert_h$	2.1780E−02	6.6596E−03	1.8851E−03	5.1760E−04	1.3861E−04	3.5405E−05
order	/	1.8084	1.8796	1.9084	1.9324	1.9786
$\Vert u - \Pi_{2h}^1 u_h \Vert_h$	5.3923E−03	1.5926E−03	4.1215E−04	1.0371E−04	2.5691E−05	6.1214E−06
order	/	1.8401	1.9657	1.9935	2.0092	2.0486

Table 2: The error estimates for nonconforming FE scheme.

h	1/2	1/4	1/8	1/16	1/32	1/64
$\Vert \Pi_h^2 u - u_h \Vert_h$	1.1264E−01	4.6572E−02	1.8679E−02	8.0004E−03	3.1977E−03	1.3906E−03
order	/	1.5552	1.5790	1.5280	1.5817	1.5164
$\Vert u - \Pi_{2h}^2 u_h \Vert_h$	8.1523E−02	3.1059E−02	1.0260E−02	3.1725E−03	9.4359E−04	2.7340E−04
order	/	1.6201	1.7399	1.7983	1.8336	1.8578
$\Vert u - u_h \Vert_0$	1.0995E−02	2.7109E−03	6.4263E−04	1.5941E−04	3.9258E−05	9.3430E−06
order	/	2.0139	2.0539	2.0078	2.0151	2.0498

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