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Research Article

Calculation of the Reproducing Kernel on the Reproducing Kernel Space with Weighted Integral

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We provide a new definition for reproducing kernel space with weighted integral and present a method to construct and calculate the reproducing kernel for the space. The new reproducing kernel space is an enlarged reproducing kernel space, which contains the traditional reproducing kernel space. The proposed method of this paper is a universal method and is suitable for the case of that the weight is variable. Obviously, this new method will generalize a number of applications of reproducing kernel theory to many areas.

1. Introduction

A reproducing kernel is a basic tool for studying the spline interpolation of differential operators and is also the base of the reproducing kernel method, which were widely used in numerical analysis, genetic models, pattern analysis, and so forth. The concept of reproducing kernel is derived from the study of the integration equation, and paper [1] studied specially the reproducing kernels and presented its primary theory. From then on, the reproducing kernel theory and the reproducing kernel method have been studied by many authors [2–7].

Let $W_2^m[0,T]$ denote the function space on a finite interval $[0,T]$, $W_2^m[0,T] = \{f(t), t \in [0,T], f^{(m-1)}(t) \in L^2[0,T]\}$, and this space becomes a reproducing kernel Hilbert space (RKHS) if we endow it with some inner product. This kind of the reproducing kernel space is the most popular space for solving the boundary value problems using reproducing kernel method. But in paper [8], the author firstly considered the reproducing kernel space with weighted integral $W_{2,\rho}^2[0,T] = \{u(t), t \in [0,T], u'(t)$ is an absolute continuous real-valued function on $[0,T]$, $\int_0^T \sqrt{t}(u''(t))^2 dt < +\infty\}$ and used it solving Volterra integral equation with weakly singular kernel. It is obvious that $W_2^2[0,T] \subseteq W_{2,\rho}^2[0,T]$ and $W_{2,\rho}^2[0,T]$ will be more widely applied.

In this paper, we are concerned with the reproducing kernel space with weighted integral $W_{2,\alpha}^m[0,T] = \{u(t), t \in [0,T], u'(t), \dots, u^{(m-1)}$ are absolute continuous real-valued functions on $[0,T], \int_0^T t^\alpha (u^{(m)}(t))^2\, dt < +\infty\}$, where α is a constant and satisfies $0 \le \alpha < 1$ (when $\alpha = 0$, $W_{2,\alpha}^m[0,T]$ is $W_2^m[0,T]$). The method for computing the corresponding reproducing kernel is given.

2. Preliminaries

In order to get the main results of the paper, we introduce the method of Zhang for calculating the reproducing kernel of $W_2^m[a,b]$ in a nutshell in this section.

The method of Zhang has very powerful system modeling capability. The idea is coming from the relationship between the Green function with reproducing kernel.

Set $L = D^m + a_{m-1}D^{m-1} + \cdots + a_1 D + a_0(t)$, $t \in [a,b]$, where $a_j(t) \in C^j[a,b]$ and $\operatorname{Ker} L = \{f \in W_2^m[a,b] : Lf = 0\}$.

Definition 2.1. $\varphi_1(t), \dots, \varphi_m(t)$ are the basis in $\operatorname{Ker} L$. The i-th row of Wronskian matrix is $(\varphi_1^{(i-1)}(t), \dots, \varphi_m^{(i-1)}(t))$, and the last line of its inverse matrix is $(\widetilde{\varphi}_1(t), \dots, \widetilde{\varphi}_m(t))$. Call $\widetilde{\varphi}_1(t), \dots, \widetilde{\varphi}_m(t)$ are the adjunct functions of $\varphi_1(t), \dots, \varphi_m(t)$.

Lemma 2.2. *Assume*

$$g(t,\tau) = \sum_{i=1}^{m} e_i(t)\widetilde{e}_i(\tau)(t-\tau)_+^0 \tag{2.1}$$

and $\gamma_1, \dots, \gamma_m$ is a system of linear independent functions in Ker L *and satisfies*

$$\gamma_k \int_a^b g(\cdot,\tau)u(\tau)d\tau = \int_a^b \gamma_k g(\cdot,\tau)u(\tau)d\tau, \tag{2.2}$$

where $e_1(t), \dots, e_m(t)$ are the dual basis of Ker L *relative to $\gamma_1, \dots, \gamma_m$, and $\widetilde{e}_1(t), \dots, \widetilde{e}_m(t)$ are the adjunct functions of $e_1(t), \dots, e_m(t)$. Then for any functions $f \in W_2^m[a,b]$, they satisfy the form*

$$f(t) = \sum_{i=1}^{m} (\gamma_i f) e_i(t) + \int_a^b G(t,\tau) \cdot Lf(\tau)d\tau, \tag{2.3}$$

where $G(t,\tau)$ is defined below

$$G(t,\tau) = g(t,\tau) - \sum_{i=1}^{m} (\gamma_i g(\cdot,\tau)) e_i(t) \tag{2.4}$$

and the expression is exclusive.

Lemma 2.3. *L is a linear differential operator. Assume $\gamma_1, \ldots, \gamma_m$ are linear independent functions in* $\operatorname{Ker} L$*, satisfying* (2.2)*. Then $W_2^m[a,b]$ is a Hilbert space if the inner product is defined by the following form:*

$$(f,h) = \sum_{i=1}^{m} (\gamma_i f)\overline{(\gamma_i h)} + \int_a^b Lf(t)\overline{Lh(t)}\, dt, \quad f,h \in W_2^m[a,b]. \tag{2.5}$$

Lemma 2.4. *Under the assumptions of* Lemma 2.2 *and the inner product* (2.5)*, the Hilbert space $W_2^m[a,b]$ is reproducing kernel Hilbert space with the reproducing kernel can be denoted by*

$$K(t,\tau) = \sum_{i=1}^{m} e_i(t)\overline{e_i(\tau)} + \int_a^b G(t,x)\overline{G(\tau,x)}\, dx. \tag{2.6}$$

3. The New Method for Computing the Reproducing Kernel

It is known that the reproducing kernel of a reproducing kernel Hilbert space is existence and uniqueness. The reproducing kernel K of a Hilbert space H completely determines the space H.

This section discusses the method of calculating reproducing kernels for the following two cases. The first case is when the weight is constant. In the second case we deal with the general space $W_{2,\alpha}^m[0,T]$, where $0 \le \alpha < 1$, and the result of this part is the main result of this paper.

3.1. Case 1: The Weight Is Constant

For general space $W_2^m[0,T]$, let $L = D^m$ be the linear differential operator of order m, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the linear independent functions on $\operatorname{Ker} L$, where $\operatorname{Ker} L$ is defined by $\operatorname{Ker} L = \{f \in W_2^m[0,T], Lf = 0\}$. Let $e_1, e_2, \ldots, e_m$ be the dual basis of $\operatorname{Ker} L$ relative to $\lambda_1, \lambda_2, \ldots, \lambda_m$. That means

$$Le_i = 0, \quad \lambda_i e_j = \delta_{ij}, \quad i,j = 1,2,\ldots,m. \tag{3.1}$$

Let G be the Green's function of L and satisfy

$$L_t G(t,s) = \delta(t-s), \quad \lambda_i G(\cdot,s) = 0, \quad i = 1,\ldots,m. \tag{3.2}$$

By the Lemma 2.4, $W_2^m[0,T]$ is a reproducing kernel Hilbert space if the inner product is defined by the following form:

$$(f,h)_1 = \sum_{i=1}^{m} (\lambda_i f)\overline{(\lambda_i h)} + \int_0^T Lf(t)\overline{Lh(t)} dt, \quad f,h \in W_2^m[0,T] \tag{3.3}$$

and the reproducing kernel is

$$K_1(t,\tau) = \sum_{i=1}^{m} e_i(t)\overline{e_i(\tau)} + \int_0^T G(t,x)\overline{G(\tau,x)}dx. \tag{3.4}$$

Let

$$(f,h)_2 = \sum_{i=1}^{m} a_i(\lambda_i f)\overline{(\lambda_i h)} + \int_0^T bLf(t)\overline{Lh(t)}dt, \quad f,h \in W_2^m[0,T], \tag{3.5}$$

where both $a_1, a_2, \ldots, a_m$ and b are positive real numbers. The following proposition holds.

Theorem 3.1. *Using the above hypothesis, $W_2^m[0,T]$ is a reproducing kernel Hilbert space if it has been endowed with the inner product* (3.5) *and the reproducing kernel is*

$$K_2(t,\tau) = \sum_{i=1}^{m} \frac{e_i(t)\overline{e_i(\tau)}}{a_i} + \int_0^T \frac{1}{b}G(t,x)\overline{G(\tau,x)}\ dx. \tag{3.6}$$

Proof. Let $\widetilde{L} = \sqrt{b}L = \sqrt{b}D^m$, and $\widetilde{\lambda}_i f = \sqrt{a_i}\lambda_i f$.

It is obvious that $\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_m$ are also the linear independent functions on Ker $\widetilde{L}$. From Lemma 2.3, we have that $W_2^m[0,T]$ is a Hilbert space if the inner product is defined by

$$\begin{aligned}(f,h)_{\widetilde{L}} &= \sum_{i=1}^{m} \left(\widetilde{\lambda}_i f\right)\overline{\left(\widetilde{\lambda}_i h\right)} + \int_0^T \widetilde{L}f(t)\overline{\widetilde{L}h(t)}dt \\ &= \sum_{i=1}^{m} a_i(\lambda_i f)\overline{(\lambda_i h)} + \int_0^T bLf(t)\overline{Lh(t)}dt = (f,h)_2.\end{aligned} \tag{3.7}$$

Next, we will proof $K_2(t,s)$ is the reproducing kernel of the space $W_2^m[0,T]$ with the inner product $(\cdot,\cdot)_2$.

$K_1(t,s)$ is the reproducing kernel of the space $W_2^m[0,T]$ with the inner product $(\cdot,\cdot)_1$. In particular, $K_1(t,s)$ is contained in $W_2^m[0,T]$. So $K_2(t,s)$ is also contained in $W_2^m[0,T]$.

For any $f \in W_2^m[0,T]$,

$$(f(s), K_2(t,s))_2 = \sum_{i=1}^{m} a_i(\lambda_i f)\overline{(\lambda_i K_2(t,\cdot))} + \int_0^T bf^{(m)}(s)\overline{\frac{\partial^m}{\partial s^m}K_2(t,s)}ds. \tag{3.8}$$

From (3.1) and (3.2), we have

$$
\begin{aligned}
&\left(f(\tau), K_2(t,\tau)\right)_2 \\
&= \sum_{i=1}^{m} a_i(\lambda_i f)\overline{\lambda_i\left(\sum_{i=1}^{m}\frac{e_i(t)\overline{e_i(\tau)}}{a_i}\right)} + \int_0^T bf^{(m)}(\tau)\overline{\frac{\partial^m}{\partial\tau^m}\int_0^T \frac{1}{b}G(t,x)\overline{G(\tau,x)}dx\,d\tau} \\
&= \sum_{i=1}^{m}(\lambda_i f)\overline{\lambda_i\left(\sum_{i=1}^{m} e_i(t)\overline{e_i(\tau)}\right)} + \int_0^T f^{(m)}(\tau)\overline{\frac{\partial^m}{\partial\tau^m}\int_0^T G(t,x)\overline{G(\tau,x)}dx\,d\tau}.
\end{aligned}
\tag{3.9}
$$

Similarly, from (3.1) and (3.2), we obtain

$$
\begin{aligned}
f(t) &= \left(f(\tau), K_1(t,\tau)\right)_1 \\
&= \sum_{i=1}^{m}(\lambda_i f)\overline{\lambda_i\left(\sum_{i=1}^{m} e_i(t)\overline{e_i(\tau)}\right)} + \int_0^T f^{(m)}(s)\overline{\frac{\partial^m}{\partial\tau^m}\int_0^T G(t,x)\overline{G(\tau,x)}dx\,d\tau}.
\end{aligned}
\tag{3.10}
$$

So $f(t) = (f(\tau), K_2(t,\tau))_2$ holds.

The proof is complete. □

3.2. Case 2: The Weight Is Variable

In this case, we construct the inner product of the space $W^m_{2,\alpha}[0,T]$, and calculate the corresponding reproducing kernel.

Define $L_1 = t^{\alpha/2}D^m$. From the definition of the space $W^m_{2,\alpha}[0,T]$, we know that $L_1 : W^m_{2,\alpha}[0,T] \to L^2[0,T]$ (mapping the space $W^m_{2,\alpha}[0,T]$ to the square integrable space on $[0,T]$).

Under the hypothesis of Case 1, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are also the linear independent functions on $\operatorname{Ker} L_1$, where $\operatorname{Ker} L_1$ is defined by $\operatorname{Ker} L_1 = \{f \in W^m_{2,\alpha}[0,T], L_1 f = 0\}$, and $e_1, e_2, \ldots, e_m$ is also the dual basis of $\operatorname{Ker} L_1$ relative to $\lambda_1, \lambda_2, \ldots, \lambda_m$. $\tilde{e}_1(t), \ldots, \tilde{e}_m(t)$ are the adjunct functions of $e_1(t), \ldots, e_m(t)$.

Similar to Case 1, define an algorithm $(\cdot,\cdot)_3$ as the following form

$$
(f,h)_3 = \sum_{i=1}^{m}(\lambda_i f)\overline{(\lambda_i h)} + \int_0^T L_1 f(t)\overline{L_1 h(t)}dt, \quad f,h \in W^m_{2,\alpha}[0,T]
\tag{3.11}
$$

Theorem 3.2. *Under the above assumption, $(\cdot,\cdot)_3$ is the inner product of the space $W^m_{2,\alpha}[0,T]$.*

If act in accordance with the four basic rules of the inner product, the proof of this proposition is easy. So one overlaps the proof.

Divide the space $W^m_{2,\alpha}[0,T]$ into two parts $\operatorname{Ker} L_1$ and $(\operatorname{Ker} L_1)^{\perp}$, where $\operatorname{Ker} L_1$ is the linear space of order m. From the results in [9], one has the following proposition.

Proposition 3.3. *Under the above assumption,* $\operatorname{Ker} L_1$ *is the reproducing kernel Hilbert space with the inner product below*

$$(f, h)_4 = \sum_{i=1}^{m} (\lambda_i f)\overline{(\lambda_i h)} \quad f, h \in \operatorname{Ker} L_1 \tag{3.12}$$

and the corresponding reproducing kernel is

$$K_4(t, \tau) = \sum_{i=1}^{m} e_i(t)\overline{e}_i(\tau). \tag{3.13}$$

Let

$$\begin{gathered} g_1(t, \tau) = \frac{1}{\tau^{\alpha/2}} \sum_{i=1}^{m} e_i(t)\widetilde{e}_i(\tau)(t-\tau)_+^0 = \begin{cases} \dfrac{1}{\tau^{\alpha/2}} \displaystyle\sum_{i=1}^{m} e_i(t)\widetilde{e}_i(\tau), & t \geq \tau, \\ 0, & t < \tau, \end{cases} \\ G_1(t, \tau) = g_1(t, \tau) - \sum_{i=1}^{m} (\lambda_i g_1(\cdot, \tau)) e_i(t). \end{gathered} \tag{3.14}$$

It is obvious that $G_1(t, \tau) = (1/\tau^{\alpha/2})\ G(t, \tau)$.
The following theorem holds.

Theorem 3.4. G_1 *is the Green's function of* L_1 *and for any* $f \in W_2^m[0, T]$, $u(t) = L_1 f(t)$, *satisfies*

$$\begin{gathered} L_{1t} \int_0^T G_1(t, \tau) u(\tau) d\tau = u(t), \\ \lambda_i G_1(\cdot, \tau) = 0, \quad i = 1, 2, \ldots, m. \end{gathered} \tag{3.15}$$

Proof. For any $i = 1, 2, \cdot, m$, we have

$$\lambda_i G_1(\cdot, \tau) = \tau^{\alpha/2} \lambda_i G(\cdot, \tau). \tag{3.16}$$

From (3.2),

$$\lambda_i G_1(\cdot, \tau) = \tau^{\alpha/2} \lambda_i G(\cdot, \tau) = 0, \quad i = 1, 2, \cdot, m. \tag{3.17}$$

Then from the results in [9, 10], for any $f \in W_2^m[0, T]$, we obtain

$$L_t \int_0^T G(t, \tau) L f(\tau) d\tau = L f. \tag{3.18}$$

So

$$L_{1t}\int_0^T G_1(t,\tau)u(\tau)d\tau = t^{\alpha/2}L_t\int_0^T \frac{1}{\tau^{\alpha/2}}G(t,\tau)\tau^{\alpha/2}Lf(\tau)d\tau$$
$$= t^{\alpha/2}Lf(t) = L_1 f(t) = u(t). \tag{3.19}$$

The proof is complete. □

Remark 3.5. If acting in accordance with the process of the paper [9], we have

$$K_3(t,\tau) = \sum_{i=1}^{m} e_i(t)\overline{e_i(\tau)} + \int_0^T G_1(t,x)\overline{G_1(\tau,x)}dx. \tag{3.20}$$

But $K_3(t,\tau)$ is not the reproducing kernel of $W_2^m[0,T]$, since $K_3(t,\tau) \notin W_2^m[0,T]$.

Now, we will give an important property of the arbitrary element of $W_{2,\alpha}^m[0,T]$.

Theorem 3.6. *For any $f \in W_{2,\alpha}^m[0,T]$, $L_1 f(t) = u(t)$. Then there are some real constant c_1, $c_2,\dots,c_m$, satisfying*

$$f(t) = \sum_{i=1}^{m} c_i e_i(t) + \int_0^T \left(\sum_{i=1}^{m} e_i(t)\tilde{e}_i(\tau)\right)u(\tau)d\tau, \tag{3.21}$$

and the expression is exclusive.

Proof. $L_1 = t^{\alpha/2}D^m$ is a linear mapping, and $L_1 : W_{2,\alpha}^m[0,T] \to L^2[0,T]$ is a homomorphic mapping. For any $u(t) \in L^2[0,T]$, we have a function $h(x)$ satisfies

$$h(t) = \int_0^T \left(\sum_{i=1}^{m} e_i(t)\tilde{e}_i(\tau)\right)u(\tau)d\tau. \tag{3.22}$$

Because $L_1 h(t) = u(t) \in L^2[0,T]$, $h(t) \in W_{2,\alpha}^m[0,T]$ holds. So $L_1 : W_{2,\alpha}^m[0,T] \to L^2[0,T]$ is a surjective homomorphism.

$\operatorname{Ker} L_1 = \operatorname{Span}\{e_1, e_2, \dots, e_m\}$ is a linear system and the dimension of the system is m. So from the knowledge of the group homomorphism, we have

$$L_1 :\ W_{2,\alpha}^m\ [0,T]/\operatorname{Ker} L_1 \longrightarrow L^2\ [0,T] \tag{3.23}$$

is isomorphic.

On the one hand for any $h(t) \in W_{2,\alpha}^m[0,T]/\operatorname{Ker} L_1$, there exists the exclusive $u(t)$, satisfying $L_1 h(t) = u(t)$. On the other hand for the $u(t) \in L^2[0,T]$, $h_0(t) =$

$\int_0^T (\sum_{i=1}^m e_i(t)\widetilde{e}_i(\tau))u(\tau)d\tau$ satisfies $h_0(t) \in W_{2,\alpha}^m[0,T]/\operatorname{Ker} L_1$ and $L_1 h_0(t) = u(t)$. So for $h(t) \in W_{2,\alpha}^m[0,T]/\operatorname{Ker} L_1$, that holds

$$h(t) = \int_0^T \left(\sum_{i=1}^m e_i(t)\widetilde{e}_i(\tau)\right) L_1 h(\tau)d\tau. \tag{3.24}$$

For any $f(t) \in W_{2,\alpha}^m[0,T]$, $f(t) = f_0(t) + h(t)$ holds, where $f_0(t) \in \operatorname{Ker} L_1$ and $h(t) \in W_{2,\alpha}^m[0,T]/\operatorname{Ker} L_1$. At the same time, the decomposition is exclusive because of the orthogonality between $\operatorname{Ker} L_1$ and $W_{2,\alpha}^m[0,T]/\operatorname{Ker} L_1$.

Furthermore, $\operatorname{Ker} L_1 = \operatorname{Span}\{e_1, e_2, \ldots, e_m\}$, so $f_0(t) = \sum_{i=1}^m c_i e_i(t)$, where $c_1, c_2, \ldots, c_m$ are real numbers, and the expression is exclusive.

So for any $f(t) \in W_{2,\alpha}^m[0,T]$, we have

$$f(t) = f_0(t) + h(t) = \sum_{i=1}^m c_i e_i(t) + \int_0^T \left(\sum_{i=1}^m e_i(t)\widetilde{e}_i(\tau)\right) u(\tau)d\tau, \tag{3.25}$$

where $c_1, c_2, \ldots, c_m$ are real numbers, and the the expression is exclusive.

The proof is complete. □

Then similar to the Theorem 3.4, we have the following theorem.

Theorem 3.7. *G_1 is the Green's function of L_1 and for any $f \in W_{2,\alpha}^m[0,T]$, $L_1 f(t) = u(t)$, satisfies*

$$L_{1t}\int_0^T G_1(t,\tau)u(\tau)d\tau = u(t). \tag{3.26}$$

Proof. For any $f \in W_{2,\alpha}^m[0,T]$, $L_1 f(t) = u(t)$,

$$\begin{aligned} L_{1t}\int_0^T G_1(t,\tau)u(\tau)d\tau &= t^{\alpha/2} L_t \int_0^T \frac{1}{\tau^{\alpha/2}} G(t,\tau)\tau^{\alpha/2} Lf(\tau)d\tau \\ &= t^{\alpha/2} L_t \int_0^T G(t,\tau)Lf(\tau)d\tau. \end{aligned} \tag{3.27}$$

From Theorem 3.6, we have

$$Lf(t) = L_t \int_0^T G(t,\tau)Lf(\tau)d\tau. \tag{3.28}$$

Thus, we know that (3.26) is true.

The proof is complete. □

Theorem 3.8. *Under the above hypothesis and the inner product $(\cdot,\cdot)_3$, $W_{2,\alpha}^m[0,T]$ is the Hilbert space.*

Proof. The norm of the space is denoted by $\|u\|_3 = \sqrt{(u,u)_3}$, where $u \in W^m_{2,\alpha}[0,T]$.

Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^m_{2,\alpha}[0,T]$, that is,

$$\|f_{n+p} - f_n\|_3^2 \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.29}$$

From Theorems 3.6 and 3.1, we have

$$\|f_{n+p} - f_n\|_3^2 = \sum_{i=1}^{m} \left(\lambda_i f_{n+p} - \lambda_i f_n\right)^2 + \int_0^T \left[L_1\left(f_{n+p}(t) - f_n(t)\right)\right]^2 \quad dt \longrightarrow 0 \ (n \longrightarrow \infty). \tag{3.30}$$

By the completeness of $\operatorname{Ker} L_1$ and $L^2[0,T]$, there exist a real number r_i, $(i = 1,2,\dots,m)$ and a real function $h \in L^2[0,T]$, such that

$$\begin{aligned} &\lim_{n\to\infty} \lambda_i f_n = r_i, \quad i = 1,2,\dots,m, \\ &\lim_{n\to\infty} \int_0^T \left[L_1 f_n(t) - h(t)\right]^2 = 0. \end{aligned} \tag{3.31}$$

Set $f_0(t) = \sum_{i=1}^{m} r_i e_i(t) + \int_0^T G_1(t,\tau)h(\tau)\ d\tau$. It follows that $f_0 \in W^m_{2,\alpha}[0,T]$ and

$$\|f_n - f_0\|_3^2 \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.32}$$

So $W^m_{2,\alpha}[0,T]$ is complete. Namely, $W^m_{2,\alpha}[0,T]$ is Hilbert space.

The proof is complete. □

Theorem 3.9. *Under the above hypothesis and the inner product $(\cdot,\cdot)_3$, $W^m_{2,\alpha}[0,T]$ is the reproducing kernel Hilbert space, and the reproducing kernel is*

$$K_3(t,\tau) = \sum_{i=1}^{m} e_i(t)\overline{e_i(\tau)} + \int_0^T G_1(t,x)\overline{G_1(\tau,x)}dx. \tag{3.33}$$

Proof. From Theorem 3.8 and Proposition 3.3, we only need to demonstrate that

$$K_5(t,\tau) = \int_0^T G_1(t,x)\overline{G_1(\tau,x)}dx. \tag{3.34}$$

is the reproducing kernel of $(\operatorname{Ker} L_1)^{\perp}$, where the inner product is defined by

$$(f,h)_5 = \int_0^T L_1 f(t)\overline{L_1 h(t)}dt, \quad f,h \in (\operatorname{Ker} L_1)^{\perp}. \tag{3.35}$$

From Theorem 3.7, $L_1 K_5(t,\tau) \neq 0$, so $K_5(t,\tau) \in (\operatorname{Ker} L_1)^{\perp}$.

For any $h \in (\operatorname{Ker} L_1)^{\perp}$,

$$(h(\tau), K_5(t,\tau))_5 = \int_0^T L_1 h(\tau) \overline{L_1 \int_0^T G_1(t,x)\overline{G_1(\tau,x)}dx}\, d\tau. \tag{3.36}$$

From Theorem 3.7,

$$(h(\tau), K_5(t,\tau))_5 = \int_0^T L_1 h(\tau)\overline{G_1(t,\tau)}d\tau. \tag{3.37}$$

Furthermore, from the definition of the $(\operatorname{Ker} L_1)^{\perp}$, we have

$$\int_0^T L_1 h(\tau)\overline{G_1(t,\tau)}d\tau = \sum_{i=1}^{m}(\lambda_i h)e_i(t) + \int_0^T L_1 h(\tau)\overline{G_1(t,\tau)}d\tau. \tag{3.38}$$

Finally, from the Theorem 3.6,

$$h(t) = \sum_{i=1}^{m}(\lambda_i h)e_i(t) + \int_0^T L_1 h(\tau)\overline{G_1(t,\tau)}d\tau. \tag{3.39}$$

So

$$(h(\tau), K_5(t,\tau))_5 = h(t). \tag{3.40}$$

The proof is complete. □

4. Example

Example 4.1. We consider the space mentioned in the introduction $W^2_{2,1/2}[0,1] = \{u(t), t \in [0,1], u'(t)$ is an absolute continuous real-valued function on $[0,1]$, $\int_0^1 \sqrt{t}(u''(t))^2 dt < +\infty\}$. Let $L = D^2$, $\lambda_1 u = u(0)$, and $\lambda_2 u = u'(0)$. Using Theorems 3.8 and 3.9, $W^2_{2,1/2}[0,1]$ is endowed with the inner product:

$$(u,v)_3 = u(0)v(0) + u'(0)v'(0) + \int_0^1 \sqrt{x}u''(x)v''(x)dx \tag{4.1}$$

and the corresponding reproducing kernel is

$$K(x,y) = \begin{cases} 1 + xy + \dfrac{4xy^{3/2}}{3} - \dfrac{4y^{5/2}}{15}, & y \le x, \\[2ex] 1 + xy + \dfrac{4x^{3/2}y}{3} - \dfrac{4x^{5/2}}{15}, & y > x. \end{cases} \tag{4.2}$$

This result is in accord with Theorem 2.1 of [8].

If $\lambda_1 u = u(0)$ and $\lambda_2 u = u(1)$ and the inner product of $W^2_{2,1/2}[0,1]$ is given by

$$(u,v)_3 = u(0)v(0) + u(1)v(1) + \int_0^1 \sqrt{x}u''(x)v''(x)dx, \tag{4.3}$$

using the method of this paper, the reproducing kernel of the this space is

$$\begin{aligned} K(x,y) &= 1 - x - y + \frac{46xy}{15} + \frac{4(y-5)xy^{3/2}}{15} + \frac{4(x-5)x^{3/2}y}{15} \\ &\quad - \begin{cases} \dfrac{4y^{3/2}(y-5x)}{15}, & y \le x, \\ \dfrac{4x^{3/2}(x-5y)}{15}, & y > x. \end{cases} \end{aligned} \tag{4.4}$$

Example 4.2. We consider the space $W^4_{2,\alpha}[0,1] = \{u(t), t \in [0,1],\ u^{(3)}(t)$ is an absolute continuous real-valued function on $[0,1]$, $\int_0^1 t^\alpha (u^{(4)}(t))^2\, dt < +\infty\}$. Let $L = D^4$, $\lambda_1 u = u(0)$, $\lambda_2 u = u(1)$, $\lambda_3 u = u'(0)$ and $\lambda_4 u = u'(1)$. The inner product is given by

$$(u,v)_3 = u(0)v(0) + u(1)v(1) + u'(0)v'(0) + u'(1)v'(1) + \int_0^1 x^\alpha u^{(4)}(x)v^{(4)}(x)dx \tag{4.5}$$

Similar to Example 4.1, we can compute the reproducing kernel of the reproducing kernel space $W^4_{2,\alpha}[0,1]$ is

$$\begin{aligned} K(x,y) &= (x-1)^2 x(-1+y)^2 y + x^2(-3+2x)y^2(-3+2y) \\ &\quad + (-1+x)x^2(-1+y)y^2 + (-1+x)^2(1+2x)(-1+y)^2(1+2y) \\ &\quad - \frac{(2(-1+x)x^2(3+3\alpha(-1+y)-8y)y^2)}{(720-1764\alpha+1624\alpha^2-735\alpha^3+175\alpha^4-21\alpha^5+\alpha^6)} \\ &\quad - \frac{(10x^2(-3+2x)(1+a(-1+y)-3y)y^2)}{(-5040+13068\alpha-13132\alpha^2+6769\alpha^3-1960\alpha^4+322\alpha^5-28\alpha^6+\alpha^7)} \\ &\quad - r(x,y) - r(y,x) + \begin{cases} R(x,y), & y \le x, \\ R(y,x), & y > x, \end{cases} \end{aligned} \tag{4.6}$$

where

$$
\begin{aligned}
&r(x,y)\\
&\quad = \frac{x^{(4-\alpha)}y^{2}\left(-3\left(-42+55\alpha-14\alpha^{2}+\alpha^{3}\right)x-3\left(-14+23\alpha-10\alpha^{2}+\alpha^{3}\right)x^{2}(-2+y)\right)}{6(-7+\alpha)(-2+\alpha)(-1+\alpha)\left(360-342\alpha+119\alpha^{2}-18\alpha^{3}+\alpha^{4}\right)}\\
&\qquad + \frac{\left(x^{(4-\alpha)}y^{2}\left(\left(-210+107\alpha-18\alpha^{2}+\alpha^{3}\right)y+\left(-6+11\alpha-6\alpha^{2}+\alpha^{3}\right)x^{3}(-3+2y)\right)\right)}{\left(6(-7+\alpha)(-2+\alpha)(-1+\alpha)\left(360-342\alpha+119\alpha^{2}-18\alpha^{3}+\alpha^{4}\right)\right)},\\
&R(x,y)\\
&\quad = \frac{y^{(4-\alpha)}\left((-7+\alpha)(-6+\alpha)(-5+\alpha)x^{3}-3(-7+\alpha)(-6+\alpha)(-1+\alpha)x^{2}y\right)}{6(-7+\alpha)(-6+\alpha)(-5+\alpha)(-4+\alpha)(-3+\alpha)(-2+\alpha)(-1+\alpha)}\\
&\qquad + \frac{y^{(4-\alpha)}\left(3(-7+\alpha)(-2+\alpha)(-1+\alpha)xy^{2}-(-3+\alpha)(-2+\alpha)(-1+\alpha)y^{3}\right)}{6(-7+\alpha)(-6+\alpha)(-5+\alpha)(-4+\alpha)(-3+\alpha)(-2+\alpha)(-1+\alpha)}.
\end{aligned}
\tag{4.7}
$$

5. Conclusion

In this paper, we have proposed a method to compute the reproducing kernel on the reproducing kernel space with weighted integral. Theorems 3.8 and 3.9 are the most important theorems of the paper. To our best knowledge, Theorem 3.6 is the first results about the component of the space $W_{2,\alpha}^{m}[0,T]$. From the example, we know that the reproducing kernel space of [8] is just one space of the $W_{2,\alpha}^{m}[0,T]$, and the proposed method of this paper is a universal method.

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