THE SECOND-ORDER SELF-ASSOCIATED ORTHOGONAL SEQUENCES

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The aim of this work is to describe the orthogonal polynomials sequences which are identical to their second associated sequence. The resulting polynomials are semiclassical of class $s \leq 3$. The characteristic elements of the structure relation and of the second-order differential equation are given explicitly. Integral representations of the corresponding forms are also given. A striking particular case is the case of the so-called electrospheric polynomials.

1. Introduction

A long time ago [4], Guillet and Aubert wrote a paper on electrospheric polynomials. They are a particular case of orthogonal polynomials which are identical to their second associated sequence. This property has not been noticed. More recently [7], the first author studied the second-order self-associated sequences in the case where they are positive definite.

Here, we will describe all the orthogonal sequences which are identical to their second associated sequence. Such a sequence depends on three parameters $(\tau, \upsilon, \varepsilon)$, where $\tau \in \mathbb{C}$, $\upsilon \in \mathbb{C} - \{-1, 1\}$, and $\varepsilon^2 = 1$.

When $\tau = 0$, we obtain the so-called electrospheric polynomials. When $|\tau| \leq \min(1, |\upsilon|)$, we have the positive definite case.

In Section 2, we deal with general features. Section 3 is devoted to the classification of second-order self-associated sequences. In Section 4, we carry out the quadratic decomposition of second-order self-associated sequences. This section is necessary for determining the useful materials needed in Section 5 in which we establish the structure relation between any second-order self-associated sequence and the differential equation fulfilled by any polynomial of such a sequence. Finally, in Section 6, we give the integral representation and the moments of the corresponding forms.

2. Preliminary results

2.1. Computing forms and Stieltjes function. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$

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on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of u. For any form u and any polynomial h, we let $Du = u'$ and hu be the forms defined by duality:

$$\langle u', f \rangle := -\langle u, f' \rangle, \quad \langle hu, f \rangle := \langle u, hf \rangle, \quad f \in \mathcal{P}. \tag{2.1}$$

We recall the definition of right multiplication of a form by a polynomial:

$$(up)(x) := \left\langle u, \frac{xp(x) - \xi p(\xi)}{x - \xi} \right\rangle, \quad u \in \mathcal{P}', \ p \in \mathcal{P}. \tag{2.2}$$

By duality, we obtain the Cauchy's product of two forms:

$$\langle uv, p \rangle := \langle u, vp \rangle, \quad u, v \in \mathcal{P}', \ p \in \mathcal{P}. \tag{2.3}$$

We define [1] the form $(x-c)^{-1}u$, $c \in \mathbb{C}$, through

$$\langle (x-c)^{-1}u, p \rangle := \langle u, \theta_c p \rangle, \tag{2.4}$$

with

$$(\theta_c p)(x) := \frac{p(x) - p(c)}{x - c}, \quad u \in \mathcal{P}', \ p \in \mathcal{P}. \tag{2.5}$$

From the definitions, we have $(u\theta_0 f)(x) = \langle u, (f(x) - f(\xi))/(x - \xi) \rangle$, $u \in \mathcal{P}'$, $f \in \mathcal{P}$.

Hence, $W_n^{(1)}(x) = (w_0 \theta_0 W_{n+1})(x)$.

We introduce the operator $\sigma : \mathcal{P} \to \mathcal{P}$ defined by $(\sigma f)(x) := f(x^2)$ for all $f \in \mathcal{P}$.

By transposition, we define σu by duality:

$$\langle \sigma u, f \rangle = \langle u, \sigma f \rangle, \quad \forall u \in \mathcal{P}', \ \forall f \in \mathcal{P}. \tag{2.6}$$

Consequently, $(\sigma u)_n = (u)_{2n}$. The following results are fundamental [1, 13].

Lemma 2.1. *For any $f, g \in \mathcal{P}$, $u, v \in \mathcal{P}'$, and $c \in \mathbb{C}$,*

$$f(x)(uv) = \big(f(x)v\big)u + x\big(v\theta_0 f\big)(x)u, \tag{2.7}$$

$$(x-c)^{-1}(fu) = f(c)\big((x-c)^{-1}u\big) + (\theta_c f)u - \langle u, \theta_c f \rangle \delta_c \quad \big(\langle \delta_c, f \rangle = f(c)\big), \tag{2.8}$$

$$f\big((x-c)^{-1}u\big) = f(c)\big((x-c)^{-1}u\big) + (\theta_c f)u, \tag{2.9}$$

$$(fu)' = fu' + f'u, \tag{2.10}$$

$$(u\theta_0 f)(x) = (\theta_0 u f)(x), \tag{2.11}$$

$$f(x)\sigma u = \sigma\big(f(x^2)u\big), \tag{2.12}$$

$$2(\sigma u)' = \sigma\big((x^{-1}u)'\big), \tag{2.13}$$

$$\sigma u' = 2\big(\sigma(xu)\big)'. \tag{2.14}$$

We will also use the so-called formal Stieltjes function associated with $u \in \mathcal{P}'$ and defined by

$$S(u)(z) := -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}. \tag{2.15}$$

Lemma 2.2. *For any $f \in \mathcal{P}$ and $u, v \in \mathcal{P}'$* [13],

$$\begin{aligned} S(fu)(z) &= f(z)S(u)(z) + (u\theta_0 f)(z), \\ S(u')(z) &= S'(u)(z), \\ S(uv)(z) &= -zS(u)(z)S(v)(z), \\ S(u^n)(z) &= (-1)^{n-1}z^{n-1}\big(S(u)(z)\big)^n, \quad n \geq 1, \\ S(x^{-n}u)(z) &= z^{-n}S(u)(z), \quad n \geq 1. \end{aligned} \tag{2.16}$$

2.2. Dual sequences and orthogonal sequences. Let $\{W_n\}_{n\geq 0}$ be a monic polynomials sequence (MPS), $\deg W_n = n$, $n \geq 0$, and let $\{w_n\}_{n\geq 0}$ be its dual sequence, $w_n \in \mathcal{P}'$, defined by $\langle w_n, W_m\rangle := \delta_{n,m}$, $n, m \geq 0$. The sequence $\{W_n^{(1)}\}_{n\geq 0}$ defined by

$$W_n^{(1)}(x) := \left\langle w_0, \frac{W_{n+1}(x) - W_{n+1}(\xi)}{x - \xi} \right\rangle, \quad n \geq 0, \tag{2.17}$$

is called an associated sequence of $\{W_n\}_{n\geq 0}$ (with respect to w_0).Any polynomial $W_n^{(1)}$ is monic and $\deg W_n^{(1)} = n$. We denote by $\{w_n^{(1)}\}_{n\geq 0}$ the dual sequence of $\{W_n^{(1)}\}_{n\geq 0}$.

The dual sequence $\{w_n^{(1)}\}_{n\geq 0}$ is given by [8]

$$w_n^{(1)} = (xw_{n+1})w_0^{-1}, \quad n \geq 0, \tag{2.18}$$

where w^{-1} exists if and only if $(w)_0 \neq 0$ and then $ww^{-1} = \delta$ ($\delta = \delta_0$ is the Dirac measure at origin).

The form w is called regular if we can associate with it an MPS $\{W_n\}_{n\geq 0}$ such that

$$\langle w, W_m W_n\rangle = r_n\delta_{n,m}, \quad n, m \geq 0,\ r_n \neq 0,\ n \geq 0. \tag{2.19}$$

The sequence $\{W_n\}_{n\geq 0}$ is orthogonal with respect to w; it is a monic orthogonal polynomials sequence (MOPS). Necessarily, $w = \lambda w_0$, $\lambda \neq 0$. In this case, we have $w_n = (\langle w_0, W_n^2\rangle)^{-1} W_n w_0$, $n \geq 0$, and $\{W_n\}_{n\geq 0}$ fulfils the following second-order recurrence relation:

$$\begin{aligned} W_0(x) &= 1, \quad W_1(x) = x - \beta_0, \\ W_{n+2}(x) &= (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \quad n \geq 0. \end{aligned} \tag{2.20}$$

Likewise, the sequence $\{W_n^{(1)}\}_{n\geq 0}$ verifies the recurrence relation

$$\begin{gathered} W_0^{(1)}(x)=1, \quad W_1^{(1)}(x)=x-\beta_1, \\ W_{n+2}^{(1)}(x)=(x-\beta_{n+2})W_{n+1}^{(1)}(x)-\gamma_{n+2}W_n^{(1)}(x), \quad n\geq 0, \end{gathered} \tag{2.21}$$

and it is orthogonal with respect to $w_0^{(1)}$, where [10]

$$\gamma_1 w_0^{(1)}=-x^2 w_0^{-1}. \tag{2.22}$$

Through the formal Stieltjes function [16],

$$\gamma_1 S(w_0^{(1)})(z)=-\frac{1}{S(w_0)(z)}-(z-\beta_0). \tag{2.23}$$

The successive associated sequences are defined recursively:

$$W_n^{(r+1)}=(W_n^{(r)})^{(1)}, \quad w_n^{(r+1)}=(w_n^{(r)})^{(1)}, \quad n,r\geq 0. \tag{2.24}$$

The sequence $\{W_n^{(r+1)}\}_{n\geq 0}$ satisfies the recurrence relation

$$\begin{gathered} W_0^{(r+1)}(x)=1, \quad W_1^{(r+1)}(x)=x-\beta_{r+1}, \\ W_{n+2}^{(r+1)}(x)=(x-\beta_{n+r+2})W_{n+1}^{(r+1)}(x)-\gamma_{n+r+2}W_n^{(r+1)}(x), \quad n\geq 0. \end{gathered} \tag{2.25}$$

From (2.23), we have

$$\gamma_{n+r+1}S(w_0^{(n+r+1)})(z)=-\frac{1}{S(w_0^{(n+r)})(z)}-(z-\beta_{n+r}), \quad n,r\geq 0. \tag{2.26}$$

Hence, we get [6, 10, 13]

$$\gamma_{n+r+1}S(w_0^{(n+r+1)})(z)=-\frac{W_n^{(r+1)}(z)+W_{n+1}^{(r)}(z)S(w_0^{(r)})(z)}{W_{n-1}^{(r+1)}(z)+W_n^{(r)}(z)S(w_0^{(r)})(z)}, \quad n,r\geq 0. \tag{2.27}$$

Let $\{W_n\}_{n\geq 0}$ be an MPS. It is always possible to associate with it two MPSs $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$, $\deg P_n=\deg R_n=n$, $n\geq 0$, and two polynomials sequences $\{a_n(x)\}_{n\geq 0}$ and $\{b_n(x)\}_{n\geq 0}$ such that

$$\begin{gathered} W_{2n}(x)=P_n(x^2)+xa_{n-1}(x^2), \\ W_{2n+1}(x)=xR_n(x^2)+b_n(x^2), \quad n\geq 0, \end{gathered} \tag{2.28}$$

where $\deg a_n\leq n$ and $\deg b_n\leq n$.

Since $\deg P_n = \deg R_n = n$, $n \geq 0$, there exist two tables of coefficients (λ_ν^n) and (θ_ν^n), $0 \leq \nu \leq n$, $n \geq 0$, such that

$$a_n(x) = \sum_{\nu=0}^{n} \lambda_\nu^n R_n(x), \quad n \geq 0,$$
$$b_n(x) = \sum_{\nu=0}^{n} \theta_\nu^n P_n(x), \quad n \geq 0. \tag{2.29}$$

2.3. Semiclassical forms. Let Φ (monic) and Ψ be two polynomials ($\deg\Psi = p \geq 1$, $\deg\Phi = t$). A form w is called semiclassical when it is regular and satisfies the equation [8, 11]

$$(\Phi w)' + \Psi w = 0. \tag{2.30}$$

When w is semiclassical, the orthogonal sequence $\{W_n\}_{n\geq 0}$ is also called semiclassical.

The pair (Φ, Ψ) is not unique. Equation (2.30) can be simplified if and only if there exists a root c of Φ such that

$$\Psi(c) + \Phi'(c) = 0, \qquad \langle w, \theta_c\Psi + \theta_c^2\Phi\rangle = 0. \tag{2.31}$$

Then u fulfils the equation $((\theta_c\Phi)w)' + \{\theta_c\Psi + \theta_c^2\Phi\}w = 0$.

We call the class of w the minimum value of the integer $\max(\deg\Phi - 2, \deg\Psi - 1)$ for all pairs satisfying (2.30). Given the pair (Φ_0, Ψ_0), the class $s \geq 0$ is unique. When $s = 0$, the form w is classical (Hermite, Laguerre, Bessel, Jacobi).

When the form w is of class s, the orthogonal sequence associated with respect to w is known to be of class s.

The class of semiclassical forms is s if and only if the following condition is satisfied [11]:

$$\prod_{c\in\Theta} \left(|\Psi(c) + \Phi'(c)| + |\langle w, \theta_c\Psi + \theta_c^2\Phi\rangle| \right) \neq 0, \tag{2.32}$$

where $\Theta = \{c, \phi(c) = 0\}$.

Lemma 2.3. *Let w be a regular semiclassical form verifying (2.30). Let a be a root of Φ such that*

$$|\Psi(a) + \Phi'(a)| + |\langle w, \theta_a\Psi + \theta_a^2\Phi\rangle| = 0, \tag{2.33}$$
$$|\Psi(c) + \Phi'(c)| + |\langle w, \theta_c\Psi + \theta_c^2\Phi\rangle| \neq 0, \tag{2.34}$$

for all c roots of Φ different from a. Then the form w satisfies the equation

$$(\Phi_1 w)' + \Psi_1 w = 0, \tag{2.35}$$

where $\Phi_1 = \theta_a\Phi$ *and* $\Psi_1 = \theta_a\Psi + \theta_a^2\Phi$ *such that*

$$\left|\Psi_1(c)+\Phi_1'(c)\right| + \left|\langle w,\theta_c\Psi_1+\theta_c^2\Phi_1\rangle\right| \neq 0 \tag{2.36}$$

for all c roots of Φ *different from a.*

Proof. We suppose that there exists a root c of Φ different from a verifying

$$\Psi_1(c)+\Phi_1'(c) = 0, \qquad \langle w,\theta_c\Psi_1+\theta_c^2\Phi_1\rangle = 0. \tag{2.37}$$

We have

$$\Phi(x) = (x-a)\Phi_1(x), \qquad (\Psi+\Phi_1)(x) = (x-a)\Psi_1(x); \tag{2.38}$$

then

$$\Psi(c)+\Phi'(c) = (c-a)\left(\Psi_1(c)+\Phi_1'(c)\right), \qquad \theta_c\Psi+\theta_c^2\Phi = \Psi_1-(c-a)\left(\theta_c\Psi_1+\theta_c^2\Phi_1\right). \tag{2.39}$$

On account of $\langle w,\Psi_1\rangle = 0$, we deduce that $\Psi(c)+\Phi'(c) = 0$ and $\langle w,\theta_c\Psi+\theta_c^2\Phi\rangle = 0$.
This contradicts the conditions given in (2.34). □

2.4. Affine transformation. We define the linear operators τ_b and h_a in $\mathcal{P}'$ as follows:

$$\begin{aligned} \langle\tau_b u,p\rangle &:= \langle u,\tau_{-b}p\rangle = \langle u,p(x+b)\rangle, \quad b\in\mathbb{C},\ u\in\mathcal{P}',\ p\in\mathcal{P}, \\ \langle h_a u,p\rangle &:= \langle u,h_a p\rangle = \langle u,p(ax)\rangle, \quad a\in\mathbb{C}-\{0\},\ u\in\mathcal{P}',\ p\in\mathcal{P}. \end{aligned} \tag{2.40}$$

Let $\{W_n\}_{n\geq 0}$ be an MPS with its dual sequence $\{w_n\}_{n\geq 0}$. The dual sequence $\{\tilde{w}_n\}_{n\geq 0}$ of $\{\tilde{W}_n\}_{n\geq 0}$ with $\tilde{W}_n(x) = a^{-n}W_n(ax+b)$, $n\geq 0$, $a\neq 0$, is given by $\tilde{w}_n = a^n(h_{a^{-1}}\circ\tau_{-b})w_n$, $n\geq 0$.

Let $\{W_n\}_{n\geq 0}$ be an MOPS with respect to w. Then $\{\tilde{W}_n\}_{n\geq 0}$ is an MOPS with respect to $\tilde{w} = (h_{a^{-1}}\circ\tau_{-b})w$. We have

$$\tilde{\beta}_n = \frac{\beta_n-b}{a}, \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \quad n\geq 0. \tag{2.41}$$

Lemma 2.4. *For any* $f\in\mathcal{P}$, $u,v\in\mathcal{P}'$, *and* $(a,b)\in\mathbb{C}-\{0\}\times\mathbb{C}$ [8, 13],

$$\tau_b(fu) = (\tau_b f)(\tau_b u), \tag{2.42}$$

$$h_a(fu) = (h_{a^{-1}}f)(h_a u), \tag{2.43}$$

$$\tau_b(uv) = (\tau_b u)(\tau_b v)\delta_b^{-1}, \tag{2.44}$$

$$h_a(uv) = (h_a u)(h_a v). \tag{2.45}$$

As a result, if w is a semiclassical form of class s satisfying (2.30), then the shifted form $\tilde{w} = (h_{a^{-1}} \circ \tau_{-b})w$ is of class s satisfying the equation

$$(\tilde{\Phi}\tilde{w})' + \tilde{\Psi}\tilde{w} = 0, \tag{2.46}$$

where

$$\tilde{\Phi}(x) = a^{-t}\Phi(ax+b), \qquad \tilde{\Psi}(x) = a^{1-t}\Psi(ax+b). \tag{2.47}$$

Lemma 2.5. *Let $\{W_n\}_{n\geq 0}$ be an MPS, $\deg W_n = n$, $n \geq 0$, and let $\{w_n\}_{n\geq 0}$ be its dual sequence. For any $(a,b) \in \mathbb{C} - \{0\} \times \mathbb{C}$,*

$$\tau_b\big(w_n^{(1)}\big) = \big(\tau_b w_n\big)^{(1)}, \tag{2.48}$$

$$h_a\big(w_n^{(1)}\big) = \big(h_a w_n\big)^{(1)}. \tag{2.49}$$

Proof. By multiplying the two sides of (2.18) by the form w_0, we obtain

$$w_n^{(1)} w_0 = x w_{n+1}. \tag{2.50}$$

By introducing the operator τ_b in the last expression, from (2.42) and (2.44), we obtain

$$\big(\tau_b\big(w_n^{(1)}\big)\big)\big(\tau_b w_0\big) = \big((x-b)\big(\tau_b w_{n+1}\big)\big)\delta_b. \tag{2.51}$$

From (2.7),

$$\begin{aligned}\big(\tau_b\big(w_n^{(1)}\big)\big)\big(\tau_b w_0\big) = {} & \big((x-b)\delta_b\big)\big(\tau_b w_{n+1}\big) + x\big(\tau_b w_{n+1}\big) \\ & - x\big(\big(\big(\tau_b w_{n+1}\big)\theta_0(\xi - b)\big)(x)\big)\delta_b.\end{aligned} \tag{2.52}$$

Since

$$(x-b)\delta_b = 0, \qquad \big(\big(\tau_b w_{n+1}\big)\theta_0(\xi-b)\big)(x) = 0, \quad n \geq 0, \tag{2.53}$$

then

$$\big(\tau_b\big(w_n^{(1)}\big)\big)\big(\tau_b w_0\big) = x\big(\tau_b w_{n+1}\big), \quad n \geq 0, \tag{2.54}$$

or

$$\tau_b\big(w_n^{(1)}\big) = \big(x\big(\tau_b w_{n+1}\big)\big)\big(\tau_b w_0\big)^{-1}, \quad n \geq 0. \tag{2.55}$$

From (2.18) and (2.55), we deduce (2.48).

To prove (2.48), we introduce the operator h_a in the expression (2.50). From (2.43) and (2.45), we give

$$(h_a(w_n^{(1)}))(h_a w_0) = a^{-1}x(h_a w_{n+1}), \quad n \geq 0. \tag{2.56}$$

But

$$(a^{-n}h_a w_n)^{(1)} = x(a^{-(n+1)}h_a w_{n+1})(h_a w_0)^{-1}, \quad n \geq 0. \tag{2.57}$$

From (2.18) and (2.57), we deduce (2.49). □

2.5. Second-degree forms. The form w is a second-degree form [13] if it is regular and if there exist polynomials B and C such that

$$B(z)S^2(w)(z) + C(z)S(w)(z) + D(z) = 0, \tag{2.58}$$

where D depends on B, C, and w.

The regularity of w means that we must have

$$B \neq 0, \qquad C^2 - 4BD \neq 0, \qquad D \neq 0. \tag{2.59}$$

The following expressions are equivalent to (2.58), [13]:

$$B(x)w^2 = xC(x)w, \qquad \langle w^2, \theta_0 B\rangle = \langle w, C\rangle. \tag{2.60}$$

In the sequel, we will assume B to be monic and we will be looking for any regular form w verifying $(w)_0 = 1$.

A second-degree form w is a semiclassical form and satisfies (2.30), where [13]

$$\begin{aligned} k\phi(x) &= B(x)(C^2(x) - 4B(x)D(x)), \quad \phi \text{ monic}, \ k \neq 0, \\ k\psi(x) &= -\frac{3}{2}B(x)(C^2(x) - 4B(x)D(x))'. \end{aligned} \tag{2.61}$$

3. The second-order self-associated orthogonal sequences and their classification

In this section, we quote the second-order self-associated sequences following the class of their corresponding canonical forms.

Definition 3.1. Let any integer $m \geq 1$ be fixed. Then the MOPS $\{W_n\}_{n\geq 0}$ is called an m-order self-associated polynomials sequence when it fulfils

$$W_n^{(m)} = W_n, \quad n \geq 0. \tag{3.1}$$

In this case, the form w_0 is also called an m-order self-associated form. See also [14, 15].

Then w_0 satisfies

$$w_0^{(m)} = w_0. \tag{3.2}$$

From (3.1), the coefficients of (2.20) are given by

$$\beta_{n+m} = \beta_n, \quad \gamma_{n+m+1} = \gamma_{n+1}, \quad n \geq 0. \tag{3.3}$$

The case $m = 1$ is well known; w_0 is the Tchebychev form of the second kind.

According to Lemma 2.5, we give the following result.

Proposition 3.2. *Let $\{W_n\}_{n\geq 0}$ be an m-order self-associated MPS,* $\deg W_n = n$, $n \geq 0$, *and let $\{w_n\}_{n\geq 0}$ be its dual sequence. Then the shifted sequence form $\{\tilde{w}_n\}_{n\geq 0}$ fulfils*

$$\tilde{w}_n^{(m)} = \tilde{w}_n, \quad m \in \mathbb{N} - \{0\},\ n \geq 0, \tag{3.4}$$

where

$$\tilde{w}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) w_n, \quad b \in \mathbb{C},\ a \in \mathbb{C} - \{0\},\ n \geq 0. \tag{3.5}$$

The object of this subject is to treat the case where $m = 2$ by describing all the second-order self-associated polynomials sequences and their classification. We denote by $\{Z_n\}_{n\geq 0}$ these polynomials sequences and $\{z_n\}_{n\geq 0}$ their dual sequences. From (3.3), we get

$$\beta_{n+2} = \beta_n, \quad \gamma_{n+3} = \gamma_{n+1}, \quad n \geq 0. \tag{3.6}$$

This implies

$$\begin{aligned} \beta_{2n} = \beta_0, \quad \beta_{2n+1} = \beta_1, \quad n \geq 0, \\ \gamma_{2n+1} = \gamma_1, \quad \gamma_{2n+2} = \gamma_2, \quad n \geq 0. \end{aligned} \tag{3.7}$$

For $\alpha = (1/2)(\beta_0 + \beta_1)$, $\beta = (1/2)(\beta_0 - \beta_1)$, $\lambda = (1/2)(\gamma_2 + \gamma_1)$, $\mu = (1/2)(\gamma_1 - \gamma_2)$, $n \geq 0$, we have

$$\begin{aligned} \beta_n = \alpha + (-1)^n \beta, \quad n \geq 0,\ (\alpha,\beta) \in \mathbb{C}^2, \\ \gamma_{n+1} = \lambda + (-1)^n \mu, \quad n \geq 0,\ (\lambda,\mu) \in \mathbb{C}^2, \lambda^2 \neq \mu^2. \end{aligned} \tag{3.8}$$

By means of (2.23), we have

$$\gamma_2 S(z_0^{(2)})(z) = -\frac{1}{S(z_0^{(1)})(z)} - (z - \beta_1), \tag{3.9}$$

$$\gamma_1 S(z_0^{(1)})(z) = -\frac{1}{S(z_0)(z)} d - (z - \beta_0). \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain

$$\gamma_2 S(z_0^{(2)})(z) = \frac{\gamma_1 S(z_0)(z)}{1+(z-\beta_0)S(z_0)(z)} - (z-\beta_1). \tag{3.11}$$

Since

$$z_0^{(2)} = z_0, \tag{3.12}$$

relation (3.11) becomes

$$(z-\beta_0)S^2(z_0)(z) + \frac{1}{\gamma_2}(\gamma_2-\gamma_1+(z-\beta_0)(z-\beta_1))S(z_0)(z) + \frac{1}{\gamma_2}(z-\beta_1) = 0. \tag{3.13}$$

From (3.8), we get

$$(z-\alpha-\beta)S^2(z_0)(z) + \frac{1}{\lambda-\mu}(z^2-2\alpha z+\alpha^2-\beta^2-2\mu)S(z_0)(z) + \frac{1}{\lambda-\mu}(z-\alpha+\beta) = 0. \tag{3.14}$$

Thus, the form z_0 is a second-degree form [10, 14, 15].

It is also a semiclassical form of class $s \le 3$, satisfying the functional equation (2.30) with

$$\begin{aligned} \Phi(x) &= (x-(\alpha+\beta))\big(((x-\alpha)^2-2\lambda-\beta^2)^2-4(\lambda^2-\mu^2)\big), \\ \Psi(x) &= -6(x-\alpha)(x-(\alpha+\beta))((x-\alpha)^2-2\lambda-\beta^2). \end{aligned} \tag{3.15}$$

Let δ_1, δ_2 be two complex numbers such that

$$\delta_1^2 = 2\lambda+\beta^2+2\sqrt{\lambda^2-\mu^2}, \qquad \delta_2^2 = 2\lambda+\beta^2-2\sqrt{\lambda^2-\mu^2}. \tag{3.16}$$

The polynomial Φ becomes

$$\Phi(x) = (x-\alpha-\beta)(x-\alpha-\delta_1)(x-\alpha+\delta_1)(x-\alpha-\delta_2)(x-\alpha+\delta_2). \tag{3.17}$$

We remark that $\delta_1^2-\delta_2^2 = 4\sqrt{\lambda^2-\mu^2}$. The regularity of z_0 leads to $\lambda^2 \ne \mu^2$. Then $\delta_1^2 \ne \delta_2^2$; so necessarily one of these values is different from zero. We can suppose that $\delta_1 \ne 0$.

We make a suitable shift such that $\alpha = 0$ and $\delta_1 = 1$. With $\beta = \tau$ and $\delta_2 = \upsilon$, from (3.16), we have $\lambda = (1/4)(1-2\tau^2+\upsilon^2)$ and $\mu = (1/2)\varepsilon\varsigma_{\tau,\upsilon}$, $\varepsilon = \pm 1$, where

$$\varsigma_{\tau,\upsilon} = \sqrt{(\tau^2-1)(\tau^2-\upsilon^2)}. \tag{3.18}$$

Therefore, (3.14) becomes

$$(z-\tau)S^2(z_0)(z) + \frac{1}{\gamma_2}(z^2-\tau^2-\varepsilon\varsigma_{\tau,\upsilon})S(z_0)(z) + \frac{1}{\gamma_2}(z+\tau) = 0, \tag{3.19}$$

where

$$\gamma_2 = \frac{1}{4}\left(1-2\tau^2+v^2-2\varepsilon\varsigma_{\tau,v}\right). \tag{3.20}$$

The functional equation fulfilled by the form z_0 becomes

$$\left(\Phi z_0\right)' + \Psi z_0 = 0, \tag{3.21}$$

where

$$\Phi(x) = (x-\tau)\left(x^2-1\right)\left(x^2-v^2\right), \tag{3.22}$$

$$\Psi(x) = -3x(x-\tau)\left(2x^2-1-v^2\right). \tag{3.23}$$

Proposition 3.3. *Let $\{Z_n\}_{n\geq 0}$ be a second-order self-associated polynomials sequence with respect to z_0. Then there exists $(\tau, v) \in \mathbb{C}^2$, $v^2 \neq 1$, such that*

$$Z_0(x) = 1, \quad Z_1(x) = x-\tau,$$
$$Z_{n+2}(x) = \left(x-(-1)^{n+1}\tau\right)Z_{n+1}(x) - \left(\frac{1}{4}\left(1-2\tau^2+v^2\right)+\frac{(-1)^n}{2}\varepsilon\varsigma_{\tau,v}\right)Z_n(x), \quad n \geq 0. \tag{3.24}$$

The form z_0 is a semiclassical form of class $s \leq 3$ and satisfies the functional equation (3.21), with the following initial conditions:

$$\langle z_0, 1\rangle = 1, \qquad \langle z_0, x\rangle = \tau, \qquad \langle z_0, x^2\rangle = \frac{1}{4}\left(1+2\tau^2+v^2\right)+\frac{1}{2}\varepsilon\varsigma_{\tau,v},$$
$$\langle z_0, x^3\rangle = \tau\langle z_0, x^2\rangle. \tag{3.25}$$

Noting that the sequence $\{Z_n^{(1)}\}_{n\geq 0}$ is also a second-order self-associated sequence,

$$\left(Z_n(\tau, v, \varepsilon; x)\right)^{(1)} = Z_n(-\tau, v, -\varepsilon; x), \quad n \geq 0. \tag{3.26}$$

Proof. Let $\{W_n\}_{n\geq 0}$ be an MOPS satisfying (2.20) with respect to w_0. Generally, we have

$$\langle w_0, x\rangle = \beta_0, \qquad \langle w_0, x^2\rangle = \beta_0^2+\gamma_1, \qquad \langle w_0, x^3\rangle = \beta_0^3+2\beta_0\gamma_1+\beta_1\gamma_1. \tag{3.27}$$

By means of relations (3.8), (3.22), and (3.23), we deduce the result. □

In the sequel, we quote all the second-order self-associated MPSs $\{Z_n\}_{n\geq 0}$. For this, we need the following lemma. Let c be a root of Φ. We have $c \in \{-1, 1, \tau, -v, v\}$.

Lemma 3.4. *Let $\{Z_n\}_{<n\geq 0}$ be a second-order self-associated polynomials sequence with respect to z_0. The expressions $\Phi'(c)+\Psi(c)$ and $\langle z_0, \theta_c^2\Phi+\theta_c\Psi\rangle$ are given for all c roots of Φ in Table 3.1.*

Proof. From (3.22) and (3.23), a simple calculation gives us the values of $\Phi'(c)+\Psi(c)$ for all c roots of Φ.

Table 3.1

Roots of Φ	$\Phi'(c)+\Psi(c)$	$\langle z_0,\theta_c^2\Phi+\theta_c\Psi\rangle$
1	$(\tau-1)(1-v^2)$	$2(\tau^2-1-\varepsilon\varsigma_{\tau,v})$
-1	$-(\tau+1)(1-v^2)$	$-2(\tau^2-1-\varepsilon\varsigma_{\tau,v})$
v	$-v(v-\tau)(v^2-1)$	$2v(\tau^2-v^2-\varepsilon\varsigma_{\tau,v})$
$-v$	$-v(v+\tau)(v^2-1)$	$-2v(\tau^2-v^2-\varepsilon\varsigma_{\tau,v})$
τ	$(\tau^2-1)(\tau^2-v^2)$	$-2\tau\varepsilon\sqrt{(\tau^2-1)(\tau^2-v^2)}$

For calculating $\langle z_0,\theta_c^2\Phi+\theta_c\Psi\rangle$, we must initially calculate the polynomials $(\theta_c^2\Phi+\theta_c\Psi)(x)$ explicitly. Through definition (3.1) and (3.22), (3.23), we have

$$\begin{aligned}
(\theta_1^2\Phi+\theta_1\Psi)(x) &= -5x^3+(5\tau-4)x^2+(2v^2+4\tau-1)x+v^2-2v^2\tau+\tau-1,\\
(\theta_{-1}^2\Phi+\theta_{-1}\Psi)(x) &= -5x^3+(5\tau+4)x^2+(2v^2-4\tau-1)x-v^2-2v^2\tau+\tau+1,\\
(\theta_\tau^2\Psi+\theta_\tau\Psi)(x) &= -5x^3+\tau x^2+(2v^2+\tau^2+2)x+\tau^3-\tau v^2-\tau,\\
(\theta_v^2\Phi+\theta_v\Psi)(x) &= -5x^3+(5\tau-4v)x^2+(4\tau v-v^2+2)x+\tau v^2-v^3+v-2\tau,\\
(\theta_{-v}^2\Phi+\theta_{-v}\Psi)(x) &= -5x^3+(5\tau+4v)x^2+(-4\tau v-v^2+2)x+\tau v^2+v^3-v-2\tau.
\end{aligned} \tag{3.28}$$

From the expressions of the moments $(z_0)_k$, $0\le k\le 3$, given by (3.25), and relations (3.28), we deduce the results of Table 3.1. □

Proposition 3.5. *Let $\{Z_n\}_{n\ge 0}$ be a second-order self-associated MPS with respect to z_0 (remember that the regularity of z_0 means $v^2\ne 1$). Denoting by s the class of z_0,*

(a) *if $\tau^2\ne 1$, $\tau^2\ne v^2$, and $v\ne 0$, so $s=3$ and z_0 is given by (3.21), (3.22), (3.23), (3.24), and (3.25);*

(b) *if $v\ne 0$ and $\tau=1$, so $s=2$ and z_0 is given by*

$$\big((x^2-1)(x^2-v^2)z_0\big)'+\big(-5x^3+x^2+(3+2v^2)x-v^2\big)z_0=0, \tag{3.29}$$

where

$$(z_0)_1=1,\qquad (z_0)_2=\frac{1}{4}(v^2+3), \tag{3.30}$$

and

$$\beta_n=(-1)^n,\quad \gamma_{n+1}=\frac{v^2-1}{4},\quad v^2\ne 1,\ v\ne 0,\ n\ge 0; \tag{3.31}$$

(c) *if $v=0$, $\tau^2\ne 1$, and $\tau\ne 0$, so $s=2$ and z_0 is given by*

$$\big(x(x-\tau)(x^2-1)z_0\big)'+(x-\tau)(-5x^2+2)z_0=0, \tag{3.32}$$

where

$$(z_0)_1 = \tau, \qquad (z_0)_2 = \frac{1}{4}(1+2\tau^2) + \frac{1}{2}\varepsilon\tau\sqrt{(\tau^2-1)}, \tag{3.33}$$

and

$$\beta_n = (-1)^n\tau, \quad \gamma_{n+1} = -\frac{1}{4}\left(\tau - (-1)^n\varepsilon\sqrt{\tau^2-1}\right)^2, \quad \tau^2 \neq 1,\ \tau \neq 0,\ n \geq 0; \tag{3.34}$$

(d) *if $v = 0$ and $\tau = 1$, so $s = 1$ and z_0 is given by*

$$\begin{gathered} \left(x(x^2-1)z_0\right)' + (-4x^2+x+2)z_0 = 0, \quad (z_0)_1 = 1, \\ \beta_n = (-1)^n, \quad \gamma_{n+1} = -\frac{1}{4}, \quad n \geq 0; \end{gathered} \tag{3.35}$$

(e) *if $v = 0$ and $\tau = 0$, so $s = 0$ and z_0 is the Tchebychev form of the second kind* [10, 12, 13], *given by*

$$\left((x^2-1)z_0\right)' - 3xz_0 = 0, \tag{3.36}$$

$$\beta_n = 0, \quad \gamma_{n+1} = \frac{1}{4}, \quad n \geq 0. \tag{3.37}$$

Proof. (a) In the case $\tau^2 \neq 1$, $\tau^2 \neq v^2$, and $v \neq 0$ and from Table 3.1, we have

$$\left|\Psi(c) + \Phi'(c)\right| + \left|\langle z_0, \theta_c\Psi + \theta_c^2\Phi\rangle\right| \neq 0 \tag{3.38}$$

for all c roots of Φ. Relation (2.32) is realized. Consequently, (3.21) is not simplified, so the form z_0 is of class $s = 3$.

(b) In the second case, the functional equation of z_0 is given by

$$\left((x-1)(x^2-1)(x^2-v^2)z_0\right)' - 3x(x-1)(2x^2-1-v^2)z_0 = 0. \tag{3.39}$$

From Table 3.1, $\Psi(1)+\Phi'(1) = 0$, $\langle z_0, \theta_1\Psi + \theta_1^2\Phi\rangle = 0$, and $|\Psi(c)+\Phi'(c)| + |\langle z_0, \theta_c\Psi + \theta_c^2\Phi\rangle| \neq 0$ for all $c \in \{-1, v, -v\}$.

Then this equation is simplified by $x-1$, and z_0 fulfils

$$\left(\Phi_1 z_0\right)' + \Psi_1 z_0 = 0, \tag{3.40}$$

where $\Phi_1(x) = (x^2-1)(x^2-v^2)$ and $\Psi_1(x) = -5x^3 + x^2 + (3+2v^2)x - v^2$.

From Lemma 2.3,

$$\left|\Psi_1(c) + \Phi_1'(c)\right| + \left|\langle z_0, \theta_c\Psi_1 + \theta_c^2\Phi_1\rangle\right| \neq 0 \tag{3.41}$$

for all $c \in \{-1, v, -v\}$; and taking into account $\Psi_1(1) + \Phi_1'(1) = (1 - v^2) \neq 0$, we deduce the result.

When $v \neq 0$ and $\tau = -1$, z_0 satisfies the following equation and elements characteristics:

$$\big((x^2-1)(x^2-v^2)z_0\big)' + \big(-5x^3 - x^2 + (3+2v^2)x + v^2\big)z_0 = 0, \tag{3.42}$$

where

$$(z_0)_1 = -1, \qquad (z_0)_2 = \frac{1}{4}(v^2+3), \tag{3.43}$$

and

$$\beta_n = (-1)^{n+1}, \quad \gamma_{n+1} = \frac{v^2-1}{4}, \quad v^2 \neq 1,\ v \neq 0,\ n \geq 0. \tag{3.44}$$

This form is of class $s = 2$. Indeed, through a suitable shifting, we apply the operator h_{-1} in (3.42), (3.43), and (3.44). We obtain the previous case.

Likewise, if $v \neq 0$ and $\tau = v$, z_0 is given by

$$\big((x^2-1)(x^2-v^2)z_0\big)' + \big(-5x^3 + vx^2 + (2+3v^2)x - v\big)z_0 = 0, \tag{3.45}$$

where

$$(z_0)_1 = v, \qquad (z_0)_2 = \frac{1}{4}(3v^2+1), \tag{3.46}$$

and

$$\beta_n = \frac{(-1)^n}{v}, \quad \gamma_{n+1} = \frac{1-v^2}{4}, \quad v^2 \neq 1,\ v \neq 0,\ n \geq 0. \tag{3.47}$$

Applying the operator h_v in (3.45) and (3.47), then while replacing v by v^{-1}, we obtain again case (b).

By a similar calculation, if $v \neq 0$ and $\tau = -v$, then z_0 is given by

$$\big((x^2-1)(x^2-v^2)z_0\big)' + \big(-5x^3 - vx^2 + (2+3v^2)x + v\big)z_0 = 0, \tag{3.48}$$

where

$$(z_0)_1 = -v, \qquad (z_0)_2 = \frac{1}{4}(3v^2+1), \tag{3.49}$$

and

$$\beta_n = (-1)^{n+1}v, \quad \gamma_{n+1} = \frac{1-v^2}{4}, \quad v^2 \neq 1,\ v \neq 0,\ n \geq 0. \tag{3.50}$$

Applying the operator h_{-v} in (3.48) and (3.50), then while replacing v by v^{-1}, we obtain again case (b).

(c) In this case, we have

$$\left(x^2(x-\tau)(x^2-1)z_0\right)' - 3x(x-\tau)(2x^2-1)z_0 = 0. \tag{3.51}$$

From Table 3.1, $\Psi(0)+\Phi'(0)=0$, $\langle z_0, \theta_0\Psi+\theta_0^2\Phi\rangle = 0$, and $|\Psi(c)+\Phi'(c)| + |\langle z_0, \theta_c\Psi + \theta_c^2\Phi\rangle| \neq 0$ for all $c \in \{-1,1,\tau\}$.

Then this equation is simplified by x, and z_0 satisfies $(\Phi_1 z_0)' + \Psi_1 z_0 = 0$, where

$$\Phi_1(x) = x(x-\tau)(x^2-1), \qquad \Psi_1(x) = (x-\tau)(-5x^2+2). \tag{3.52}$$

From Lemma 2.3, $\Psi_1(c)+\Phi_1'(c)| + |\langle z_0, \theta_c\Psi_1 + \theta_c^2\Phi_1\rangle| \neq 0$ for all $c \in \{-1,1,\tau\}$; and taking into account $\Psi_1(0)+\Phi_1'(0) = -\tau \neq 0$, we deduce the result.

(d) From Table 3.1, the equation $(x^2(x-1)(x^2-1)z_0)' - 3x(x-1)(2x^2-1)z_0 = 0$ is simplified twice by x and $x-1$. In the first place, we have

$$\left(x(x-1)(x^2-1)z_0\right)' + (x-1)(-5x^2+2)z_0 = 0. \tag{3.53}$$

Next, we simplify once more by $x-1$, and we have $(\Phi_2 z_0)' + \Psi_2 z_0 = 0$, where

$$\Phi_2(x) = x(x^2-1), \qquad \Psi_2(x) = -4x^2+x+2. \tag{3.54}$$

Then we get $\Psi_2(0)+\Phi_2'(0) = 1 \neq 0$, and according to Lemma 2.3, z_0 is a semiclassical form of class $s=1$, which satisfies (3.35).

If $\upsilon = 0$ and $\tau = -1$, z_0 is given by

$$\begin{gathered}\left(x(x^2-1)z_0\right)' + (-4x^2-x+2)z_0 = 0, \quad (z_0)_1 = -1, \\ \beta_n = (-1)^{n+1}, \quad \gamma_{n+1} = -\frac{1}{4}, \quad n \geq 0.\end{gathered} \tag{3.55}$$

This form is of class $s=1$. In fact, applying the operator h_{-1} in (3.55), we have again case (d).

(e) Similarly, from Table 3.1, it is easy to prove that the equation is simplified by x^3. Therefore, z_0 is a classical form given by (3.36). □

4. Quadratic decomposition of the second-order self-associated orthogonal sequences

In order to build a structure relation and a differential equation related to second-order self-associated sequences, we want their quadratic decomposition given by (2.28). In [9], the first author gave necessary and sufficient conditions for the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ to be orthogonal.

Proposition 4.1. *Let $\{W_n\}_{n\geq 0}$ satisfy the recurrence relation (2.20), where*

$$\beta_n = (-1)^n \beta_0, \quad n \geq 0. \tag{4.1}$$

Then there exist two MOPSs $\{P_n\}_{n\geq 0}$, with respect to u_0, and $\{R_n\}_{n\geq 0}$, with respect to v_0, fulfilling the following relations:

$$\begin{gathered} P_0(x) = 1, \quad P_1(x) = x - \gamma_1 - \beta_0^2, \\ P_{n+2}(x) = (x - \gamma_{2n+2} - \gamma_{2n+3} - \beta_0^2) P_{n+1}(x) - \gamma_{2n+1}\gamma_{2n+2} P_n(x), \quad n \geq 0, \end{gathered} \tag{4.2}$$

$$\begin{gathered} R_0(x) = 1, \quad R_1(x) = x - \gamma_1 - \gamma_2 - \beta_0^2, \\ R_{n+2}(x) = (x - \gamma_{2n+3} - \gamma_{2n+4} - \beta_0^2) R_{n+1}(x) - \gamma_{2n+2}\gamma_{2n+3} R_n(x), \quad n \geq 0, \end{gathered} \tag{4.3}$$

$$P_{n+1}(x) = R_{n+1}(x) + \gamma_{2n+2} R_n(x), \quad n \geq 0, \tag{4.4}$$

$$(x - \beta_0^2) R_n(x) = P_{n+1}(x) + \gamma_{2n+1} P_n(x), \quad n \geq 0, \tag{4.5}$$

since, in (2.28), $a_n(x) = 0$ and $b_n(x) = -\beta_0 R_n(x)$, $n \geq 0$.

Moreover, the forms u_0, v_0, and w_0 satisfy

$$u_0 = \sigma w_0, \tag{4.6}$$

$$\sigma(x w_0) = \beta_0 (\sigma w_0), \tag{4.7}$$

$$v_0 = \frac{1}{\gamma_1}(x - \beta_0^2)(\sigma w_0). \tag{4.8}$$

Now, this result will be applied to $\{Z_n\}_{n\geq 0}$ which, by virtue of (3.24), fulfils (4.1) and

$$Z_{2n}(x) = P_n(x^2), \tag{4.9}$$

$$Z_{2n+1}(x) = (x - \tau) R_n(x^2). \tag{4.10}$$

From (3.24) and (4.2), the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ become

$$\begin{gathered} P_0(x) = 1, \quad P_1(x) = x - \frac{1}{4}(1 + v^2 + 2\tau^2) - \frac{1}{2}\varepsilon\varsigma_{\tau,v}, \\ P_{n+2}(x) = \left(x - \frac{1}{2}(1 + v^2)\right) P_{n+1}(x) - \left(\frac{v^2 - 1}{4}\right)^2 P_n(x), \quad n \geq 0, \end{gathered} \tag{4.11}$$

$$\begin{gathered} R_0(x) = 1, \quad R_1(x) = x - \frac{1}{2}(1 + v^2), \\ R_{n+2}(x) = \left(x - \frac{1}{2}(1 + v^2)\right) R_{n+1}(x) - \left(\frac{v^2 - 1}{4}\right)^2 R_n(x), \quad n \geq 0. \end{gathered} \tag{4.12}$$

We remark that the sequence $\{P_n\}_{n\geq 0}$ is the corecursive sequence of $\{R_n\}_{n\geq 0}$ with the value $-\gamma_2 = -(1/4)(1+v^2-2\tau^2)+(1/2)\varepsilon\varsigma_{\tau,v}$. For the parameter $P_n(x) = R_n(-\gamma_2;x)$, $n \geq 0$, we have

$$P_{n+1} = R_{n+1} + \gamma_2 R_n^{(1)} = R_{n+1} + \gamma_2 R_n, \quad n \geq 0, \tag{4.13}$$

in accordance with (4.4). Moreover, (4.5) becomes

$$(x-\tau^2)R_n(x) = P_{n+1}(x) + \gamma_1 P_n(x), \quad n \geq 0. \tag{4.14}$$

From (4.12), we easily see that

$$R_n(x) = a^n \hat{P}_n^{(1/2,1/2)}\left(a^{-1}(x-b)\right), \quad n \geq 0,\ a = \frac{1}{2}(v^2-1),\ b = \frac{1}{2}(1+v^2), \tag{4.15}$$

where $\{\hat{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ is the monic Jacobi polynomials sequence, orthogonal with respect to the Jacobi form $\mathcal{J}(\alpha,\beta)$, with parameters α, β, see [11, 12], fulfilling the following equation:

$$\left((x^2-1)\mathcal{J}(\alpha,\beta)\right)' + \left(-(\alpha+\beta+2)x+\alpha-\beta\right)\mathcal{J}(\alpha,\beta) = 0, \quad \left(\mathcal{J}(\alpha,\beta)\right)_0 = 1. \tag{4.16}$$

Usually, $\mathcal{J}(1/2,1/2)$ is denoted by $\mathcal{U}$ which fulfils (3.36), and $\{\hat{P}_n^{(1/2,1/2)}(x)\}_{n\geq 0}$ is defined by (3.37).

Since $v_0 = (\tau_b \circ h_a)\mathcal{U}$, we have

$$(\Phi_0 v_0)' + \Psi_0 v_0 = 0, \tag{4.17}$$

where

$$\Phi_0(x) = (x-1)(x-v^2), \qquad \Psi_0(x) = -\frac{3}{2}(2x-1-v^2). \tag{4.18}$$

Likewise, from (4.6) and (4.8), taking (4.17) into account, we obtain

$$\begin{gathered}(\Phi_1 u_0)' + \Psi_1 u_0 = 0,\\ (u_0)_1 = (\sigma z_0)_1 = \tau^2 + \gamma_1 = \frac{1}{4}(1+v^2+2\tau^2) + \frac{1}{2}\varepsilon\varsigma_{\tau,v},\end{gathered} \tag{4.19}$$

where

$$\Phi_1(x) = (x-1)(x-v^2)(x-\tau^2), \qquad \Psi_1(x) = -\frac{3}{2}(2x-1-v^2)(x-\tau^2). \tag{4.20}$$

LEMMA 4.2. *The following cases hold:*

(a) *if* $\tau^2 \neq 1$ *and* $\tau^2 \neq v^2$, *the class of* u_0 *is* $s = 1$;
(b) *if* $\tau^2 = 1$ *and* $\tau^2 \neq v^2$, *the form* u_0 *is classical* $(s = 0)$ *and fulfils the equation*

$$((x-1)(x-v^2)u_0)' - \frac{1}{2}(4x-3-v^2)u_0 = 0, \quad (u_0)_1 = \frac{1}{4}(3+v^2); \tag{4.21}$$

this implies

$$u_0 = (\tau_b \circ h_a)\mathcal{J}\left(-\frac{1}{2}, \frac{1}{2}\right) \tag{4.22}$$

with

$$a = \frac{1}{2}(v^2-1), \qquad b = \frac{1}{2}(1+v^2); \tag{4.23}$$

(c) *if* $\tau^2 = v^2$, *the form* u_0 *is classical and fulfils the equation*

$$((x-1)(x-\tau^2)u_0)' - \frac{1}{2}(4x-1-3\tau^2)u_0 = 0, \quad (u_0)_1 = \frac{1}{4}(1+3\tau^2); \tag{4.24}$$

this implies

$$u_0 = (\tau_b \circ h_a)\mathcal{J}\left(\frac{1}{2}, -\frac{1}{2}\right) \tag{4.25}$$

with

$$a = \frac{1}{2}(\tau^2-1), \qquad b = \frac{1}{2}(1+\tau^2). \tag{4.26}$$

Proof. From (4.20), we have

$$\begin{aligned}
\Phi_1'(1) + \Psi_1(1) &= -\frac{1}{2}(1-v^2)(1-\tau^2), \\
\Phi_1'(v^2) + \Psi_1(v^2) &= -\frac{1}{2}(v^2-1)(\tau^2-v^2), \\
\Phi_1'(\tau^2) + \Psi_1(\tau^2) &= (\tau^2-1)(\tau^2-v^2).
\end{aligned} \tag{4.27}$$

Assertion (a) is evident. When $\tau^2 = 1$ and $\tau^2 \neq v^2$, we have

$$\langle u_0, \theta_1^2\Phi_1 + \theta_1\Psi_1 \rangle = \left\langle u_0, -2x + \frac{1}{2}(3+v^2)\right\rangle = -2(u_0)_1 + \frac{1}{2}(3+v^2) = 0, \tag{4.28}$$

whence (4.21) and (4.22). The same applies to (4.24) and (4.25). $\square$

5. Structure relation and differential equation

It is well known that a semiclassical orthogonal polynomials sequence fulfils a second-order differential equation [3, 5, 10]. In this section, we give the following second-order differential equation fulfilled by $\{Z_n\}_{n\geq 0}$. We have

$$J(x;n)Z''_{n+1}(x)+K(x;n)Z'_{n+1}(x)+L(x;n)Z_{n+1}(x)=0, \quad n\geq 0, \tag{5.1}$$

with

$$\begin{gathered} J(x;n)=\Phi(x)D_{n+1}(x), \quad n\geq 0,\\ K(x;n)=C_0(x)D_{n+1}(x)-\mathrm{W}(\Phi,D_{n+1})(x), \quad n\geq 0,\\ L(x;n)=\mathrm{W}\Big(\frac{1}{2}(C_{n+1}-C_0),D_{n+1}\Big)(x)-D_{n+1}(x)\sum_{\nu=0}^{n}D_\nu(x), \quad n\geq 0, \end{gathered} \tag{5.2}$$

where $\mathrm{W}(f,g)=fg'-gf'$ is the Wronskian of f and g.

The sequences $\{C_n\}_{n\geq 0}$ and $\{D_n\}_{n\geq 0}$ are defined by

$$\Phi(z)S'\big(z_0^{(n)}\big)(z)=B_n(z)S^2\big(z_0^{(n)}\big)(z)+C_n(z)S\big(z_0^{(n)}\big)(z)+D_n(z), \quad n\geq 0, \tag{5.3}$$

and fulfil

$$\begin{gathered} B_0(z)=0,\\ C_0(z)=-\Phi'(z)-\Psi(z),\\ D_0(z)=-\big(z_0\theta_0\Phi\big)'(z)-\big(z_0\theta_0\Psi\big)(z), \end{gathered} \tag{5.4}$$

$$\begin{gathered} B_{n+1}(z)=\gamma_{n+1}D_n(z), \quad n\geq 0,\\ C_{n+1}(z)=-C_n(z)+2(z-\beta_n)D_n(z), \quad \deg C_n\leq 4,\ n\geq 0,\\ \gamma_{n+1}D_{n+1}(z)=-\Phi(z)+B_n(z)-(z-\beta_n)C_n(z)+(z-\beta_n)^2D_n(z), \quad \deg D_n\leq 3,\ n\geq 0. \end{gathered} \tag{5.5}$$

They are involved in the so-called structure relation [3, 10]

$$\Phi(x)Z'_{n+1}(x)=\frac{1}{2}\big(C_{n+1}(x)-C_0(x)\big)Z_{n+1}(x)-\gamma_{n+1}D_{n+1}(x)Z_n(x), \quad n\geq 0. \tag{5.6}$$

Here, from (3.22), (3.23), and (5.4), we have

$$\begin{gathered} \Phi(z)=(z-\tau)(z^2-1)(z^2-\upsilon^2),\\ C_0(z)=z^4-2\tau z^3+\tau(1+\upsilon^2)z-\upsilon^2,\\ D_0(z)=2z\Big(z^2+2\gamma_1-\frac{1}{2}(1+\upsilon^2)\Big)=2z(z^2-\tau^2+\varepsilon\varsigma_{\tau,\upsilon}). \end{gathered} \tag{5.7}$$

Indeed, from (2.2), we have

$$
\begin{aligned}
(z_0\theta_0\Phi)(x) &= \left\langle z_0, \frac{\Phi(x)-\Phi(\xi)}{x-\xi}\right\rangle \\
&= \left\langle z_0, \frac{(x-\tau)(x^4-(1+v^2)x^2+v^2)-(\xi-\tau)(\xi^4-(1+v^2)\xi^2+v^2)}{x-\xi}\right\rangle \\
&= \langle z_0, x^4+(\xi-\tau)x^3+(\xi^2-(1+v^2)\xi-\tau\xi)x^2 \\
&\quad + (\xi^3-(1+v^2)\xi-\tau\xi^2+(1+v^2)\tau)x \\
&\quad + \xi^4-\tau\xi^3-(1+v^2)\xi+\tau(1+v^2)\xi+v^2\rangle \\
&= x^4+((z_0)_1-\tau)x^3+((z_0)_2-(1+v^2)-\tau(z_0)_1)x^2 \\
&\quad + ((z_0)_3-\tau(z_0)_2-(1+v^2)((z_0)_1-\tau))x \\
&\quad + (z_0)_4-\tau(z_0)_3-(1+v^2)(z_0)_1+\tau(1+v^2)(z_0)_1+v^2.
\end{aligned}
\tag{5.8}
$$

Through (3.25), $(z_0)_1=\tau$, $(z_0)_2=\gamma_1+\tau^2$, and $(z_0)_3=\tau(z_0)_2$; so

$$
(z_0\theta_0\Phi)'(x) = 4x^3+2(\gamma_1-(1+v^2))x. \tag{5.9}
$$

In the same way, from (2.2) and (3.23), we get

$$
\begin{aligned}
(z_0\theta_0\Psi)(x) &= \langle z_0, -6x^3+(6\tau-6\xi)x^2+(6\tau\xi-6\xi^2+3(1+v^2))x \\
&\qquad -6\xi^3+6\tau\xi^2+3(1+v^2)(\xi-\tau)\rangle \\
&= -6x^3+(3(1+v^2)-6\gamma_1)x.
\end{aligned}
\tag{5.10}
$$

Thus, we deduce the expression of $D_0(x)$.

Generally, it is difficult to give the sequences $\{C_n\}_{n\geq 0}$ and $\{D_n\}_{n\geq 0}$ explicitly using the recurrence relations (5.5). The quadratic decomposition allows us to do it.

Lemma 5.1. *The following structure relations hold:*

$$
\begin{aligned}
(x-1)(x-v^2)R'_{n+1}(x) &= (n+1)\left(x-\frac{1}{2}(1+v^2)\right)R_{n+1}(x) \\
&\qquad - 2(n+2)\left(\frac{1-v^2}{4}\right)^2 R_n(x), \quad n\geq 0,
\end{aligned}
\tag{5.11}
$$

$$
\Phi_1(x)P'_{n+1}(x) = A(n;x)P_{n+1}(x)-B(n;x)P_n(x), \quad n\geq 0, \tag{5.12}
$$

where

$$\Phi_1(x) = (x-1)(x-v^2)(x-\tau^2), \tag{5.13}$$

$$\begin{aligned} A(n;x) = (n+1)\Big(x+2\gamma_2-\frac{1}{2}(v^2+1)\Big)\Big(x+\gamma_1-\frac{1}{2}(v^2+1)\Big) \\ -(n+2)\gamma_2\Big(x+2\gamma_1-\frac{1}{2}(v^2+1)\Big), \quad n\geq 0, \end{aligned} \tag{5.14}$$

$$\begin{aligned} B(n;x) = \gamma_1\gamma_2\Big\{(n+1)\Big(x+2\gamma_2-\frac{1}{2}(v^2+1)\Big) \\ +(n+2)\Big(x+2\gamma_1-\frac{1}{2}(v^2+1)\Big)\Big\}, \quad n\geq 0. \end{aligned} \tag{5.15}$$

Proof. Since, for the Jacobi sequence, we have [10, 11]

$$\begin{aligned} C_n^{(\alpha,\beta)}(x) &= (2n+\alpha+\beta)x-\frac{\alpha^2-\beta^2}{2n+\alpha+\beta}, \quad n\geq 0, \\ D_n^{(\alpha,\beta)}(x) &= 2n+\alpha+\beta+1, \quad n\geq 0, \end{aligned} \tag{5.16}$$

then, in the case $\alpha=\beta=1/2$, we obtain

$$\begin{aligned} C_n^R(x) &= aC_n^{(1/2,1/2)}\Big(\frac{x-b}{a}\Big) = (2n+1)\Big(x-\frac{1}{2}(1+v^2)\Big), \quad n\geq 0, \\ D_n^R(x) &= D_n^{(1/2,1/2)}\Big(\frac{x-b}{a}\Big) = 2n+2, \quad n\geq 0, \end{aligned} \tag{5.17}$$

where $a=(1/2)(v^2-1)$ and $b=(1/2)(1+v^2)$.

Hence, (5.11) holds.

Next, from (4.4), we have

$$\begin{aligned} \Phi_1(x)P'_{n+1}(x) &= (x-1)(x-v^2)(x-\tau^2)R'_{n+1}(x) \\ &\quad +\gamma_2(x-1)(x-v^2)(x-\tau^2)R'_n(x), \quad n\geq 0. \end{aligned} \tag{5.18}$$

According to (5.11) and taking (4.12) into account, we obtain

$$\begin{aligned} \Phi_1(x)P'_{n+1}(x) &= (n+1)\Big(x+2\gamma_1-\frac{1}{2}(v^2+1)\Big)(x-\tau^2)R_{n+1}(x) \\ &\quad -(n+2)\Big(\gamma_2\Big(x-\frac{1}{2}(v^2+1)\Big)+2\gamma_1\gamma_2\Big)(x-\tau^2)R_n(x), \quad n\geq 0. \end{aligned} \tag{5.19}$$

With (4.5), this yields (5.12), (5.13), (5.14), and (5.15). □

Proposition 5.2. *The sequence* $\{Z_n\}_{n\geq 0}$ *fulfils* (5.6)*, where the sequences* $\{C_n\}_{n\geq 0}$ *and* $\{D_n\}_{n\geq 0}$ *are given by*

$$\begin{aligned} C_{2n}(x) &= (4n+1)x^4 - 2\tau(2n+1)x^3 + 4n\left(\frac{1}{2}(v^2+1) - 2(\gamma_1+\tau^2)\right)x^2 \\ &\quad + \tau\left(8(\tau^2+\gamma_1)n - (2n-1)(1+v^2)\right)x - v^2, \quad n\geq 0, \end{aligned} \tag{5.20}$$

$$D_{2n}(x) = 2x\left((2n+1)x^2 - 2n\tau^2 + 2\gamma_1 - \frac{1}{2}(v^2+1)\right), \quad n\geq 0, \tag{5.21}$$

$$\begin{aligned} C_{2n+1}(x) &= (4n+3)x^4 - 2\tau(2n+1)x^3 + 2(n+1)\left(4\gamma_1 - (v^2+1)\right)x^2 \\ &\quad - 2\tau\left(4\gamma_1(n+1) - \frac{1}{2}(2n+1)(v^2+1)\right)x + v^2, \quad n\geq 0, \end{aligned} \tag{5.22}$$

$$D_{2n+1}(x) = 4(n+1)x(x-\tau)^2, \quad n\geq 0. \tag{5.23}$$

Proof. We start with (5.11), where $x\to x^2$. According to

$$Z'_{2n+3}(x) = R_{n+1}(x^2) + 2x(x-\tau)R'_{n+1}(x^2), \quad n\geq 0, \tag{5.24}$$

obtained by differentiating (4.10), relation (5.11) becomes

$$\begin{aligned} \Phi(x)Z'_{2n+3}(x) &= \left((x^2-1)(x^2-v^2) + 2(n+1)x(x-\tau)\left(x^2 - \frac{1}{2}(v^2+1)\right)\right)Z_{2n+3}(x) \\ &\quad - 4\left(\frac{1-v^2}{4}\right)^2(n+2)x(x-\tau)^2R_n(x^2), \quad n\geq 0. \end{aligned} \tag{5.25}$$

But (4.9) and (4.13) provide

$$\Phi(x)Z'_{2n+3}(x) = E(n;x)Z_{2n+3}(x) - 4\gamma_1(n+2)x(x-\tau)^2Z_{2n+2}(x), \quad n\geq 0, \tag{5.26}$$

where

$$E(n;x) = (x^2-1)(x^2-v^2) + 2x(x-\tau)\left((n+1)\left(x^2-\frac{1}{2}(v^2+1)\right) + 2(n+2)\gamma_1\right). \tag{5.27}$$

Comparing (5.26) with (5.6), where $n\to 2n+2$, leads to

$$\begin{aligned} &\left(E(n;x) - \frac{1}{2}\left(C_{2n+3}(x) - C_0(x)\right)\right)Z_{2n+3}(x) \\ &\quad = \gamma_1\left(4(n+2)x(x-\tau)^2 - D_{2n+3}(x)\right)Z_{2n+2}(x), \quad n\geq 0. \end{aligned} \tag{5.28}$$

This yields

$$\begin{aligned}\frac{1}{2}\big(C_{2n+1}(x)-C_0(x)\big)&=E(n-1;x), \quad n\geq 1,\\ D_{2n+1}(x)&=4(n+1)x(x-\tau)^2, \quad n\geq 1,\end{aligned} \tag{5.29}$$

by virtue of a well-known result on orthogonal sequences. Routine calculation from (5.5) shows that (5.29) is valid for $n\geq 0$, whence (5.22) and (5.23).

Next, from (5.12), where $x\to x^2$, and with (4.9), we obtain

$$(x+\tau)\Phi(x)Z'_{2n+2}(x)=2xA\big(n;x^2\big)Z_{2n+2}(x)-2xB\big(n;x^2\big)Z_{2n}(x). \tag{5.30}$$

But

$$Z_{2n}(x)=\frac{1}{\gamma_1}(x+\tau)Z_{2n+1}(x)-\frac{1}{\gamma_1}Z_{2n+2}(x) \tag{5.31}$$

implies

$$\begin{aligned}(x+\tau)\Phi(x)Z'_{2n+2}(x)&=2x\big(A\big(n;x^2\big)+\gamma_1^{-1}B\big(n;x^2\big)\big)Z_{2n+2}(x)\\ &\quad-2\gamma_1^{-1}x(x+\tau)B\big(n;x^2\big)Z_{2n+1}(x).\end{aligned} \tag{5.32}$$

Taking (5.14) and (5.15) into account, we have

$$A\big(n;x^2\big)+\gamma_1^{-1}B\big(n;x^2\big)=(n+1)\big(x^2-\tau^2\big)\Big(x^2+2\gamma_2-\frac{1}{2}\big(v^2+1\big)\Big). \tag{5.33}$$

This leads to

$$\begin{aligned}\Phi(x)Z'_{2n+2}(x)&=2(n+1)x(x-\tau)\Big(x^2+2\gamma_2-\frac{1}{2}\big(v^2+1\big)\Big)Z_{2n+2}(x)\\ &\quad-2\gamma_2x\Big((n+1)\Big(x^2+2\gamma_2-\frac{1}{2}\big(v^2+1\big)\Big)\\ &\quad+(n+2)\Big(x^2+2\gamma_1-\frac{1}{2}\big(v^2+1\big)\Big)\Big)Z_{2n+1}(x), \quad n\geq 0.\end{aligned} \tag{5.34}$$

As above, we obtain

$$\begin{aligned}C_{2n}(x)&=C_0(x)+4nx(x-\tau)\Big(x^2+2\gamma_2-\frac{1}{2}\big(v^2+1\big)\Big),\\ D_{2n}(x)&=2x\Big(n\Big(x^2+2\gamma_2-\frac{1}{2}\big(v^2+1\big)\Big)+(n+1)\Big(x^2+2\gamma_1-\frac{1}{2}\big(v^2+1\big)\Big)\Big), \quad n\geq 2.\end{aligned} \tag{5.35}$$

In fact, these relations are valid for $n\geq 0$, whence (5.20) and (5.21).

Now, we are able to calculate the coefficients of (5.1) defined by (5.2). $\square$

Proposition 5.3. *The sequence $\{Z_n\}_{n\geq 0}$ fulfils (5.1), where the elements characteristics $J(x;n)$, $K(x;n)$, and $L(x;n)$ are given as follows:*

$$J(x;2n) = 4(n+1)x(x-\tau)^3(x^2-1)(x^2-v^2), \tag{5.36}$$

$$J(x;2n+1) = 2x(x-\tau)(x^2-1)(x^2-v^2)\left\{(2n+3)x^2-2(n+1)\tau^2+2\gamma_1-\frac{1}{2}(v^2+1)\right\}, \tag{5.37}$$

$$K(x;2n) = 4(n+1)(x-\tau)^2\{3x^5-5\tau x^4+2\tau(1+v^2)x^2-3v^2x+\tau v^2\}, \quad n\geq 0, \tag{5.38}$$

$$\begin{aligned} K(x;2n+1) = (x-\tau)\{&3(4n+6)x^6-(20(n+1)\tau^2-5(4\gamma_1-(v^2+1)))x^4\\ &+((1+v^2)(8(n+1)\tau^2-2(4\gamma_1-(v^2+1)))-3(4n+6)v^2)x^2\\ &+(4n+1)\tau^2v^2-v^2(4\gamma_1-(v^2+1))\}, \quad n\geq 0, \end{aligned} \tag{5.39}$$

$$\begin{aligned} L(x;2n) = -4(n+1)(x-\tau)\{&(2n+1)(2n+3)x^5-(8n^2+16n+5)\tau x^4\\ &+4n(n+2)\tau^2x^3+2(1+v^2)\tau x^2\\ &-3v^2x+\tau v^2\}, \quad n\geq 0, \end{aligned} \tag{5.40}$$

$$\begin{aligned} L(x;2n+1) = -4(n+1)(n+2)x^2\{&2(2n+3)x^4-2(2n+3)\tau x^3\\ &+(3(4\gamma_1-(v^2+1))-4n\tau^2)x^2-((4\gamma_1-(v^2+1))\\ &+4(n+2)\tau^2)\tau x\}, \quad n\geq 0. \end{aligned} \tag{5.41}$$

Proof. From (5.2), (5.7), (5.21), and (5.23), it is easy to obtain (5.36) and (5.37). Next, we have

$$\begin{aligned} K(x,2n) &= (C_0(x)+\Phi'(x))D_{2n+1}(x)-\Phi(x)D'_{2n+1}(x),\\ K(x,2n+1) &= (C_0(x)+\Phi'(x))D_{2n+2}(x)-\Phi(x)D'_{2n+2}(x). \end{aligned} \tag{5.42}$$

On account of (5.7), (5.21), and (5.23), we have (5.38) and (5.39).

Finally, from (5.2), we have

$$L(x;2n) = \mathrm{W}\left(\frac{1}{2}(C_{2n+1}-C_0),D_{2n+1}\right)(x)-D_{2n+1}(x)\sum_{\nu=0}^{2n}D_\nu(x), \quad n\geq 0. \tag{5.43}$$

Successively, we get

$$\begin{aligned}\frac{1}{2}\left(C_{2n+1}-C_0\right)(x) &= E(n-1;x)\\ &= \left(x^2-1\right)\left(x^2-v^2\right)\\ &\quad + 2x(x-\tau)\left\{n\left(x^2-\frac{1}{2}\left(v^2+1\right)\right)+2(n+1)\gamma_1\right\},\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\left(C_{2n+1}-C_0\right)(x)D'_{2n+1}(x)\\ &\quad= 4(n+1)(x-\tau)(3x-\tau)\left\{(2n+1)x^4-2n\tau x^3+(n+1)\left(4\gamma_1-\left(v^2+1\right)\right)x^2\right.\\ &\qquad\qquad \left.-\tau\left(4(n+1)\gamma_1-n\left(1+v^2\right)\right)x+v^2\right\}\\ &\quad= 4(n+1)(x-\tau)\left\{3(2n+1)x^5-(8n+1)\tau x^4+\left(3(n+1)\left(4\gamma_1-\left(v^2+1\right)\right)+2n\tau^2\right)x^3\right.\\ &\qquad\qquad -\tau\left(16(n+1)\gamma_1-(4n+1)\left(1+v^2\right)\right)x^2\\ &\qquad\qquad \left.+\left(\tau^2\left(4(n+1)\gamma_1-n\left(1+v^2\right)\right)+3v^2\right)x-\tau v^2\right\}.\end{aligned} \tag{5.44}$$

Next

$$\begin{aligned}&\frac{1}{2}\left(C_{2n+1}-C_0\right)'(x)D_{2n+1}(x)\\ &\quad= 8(n+1)x(x-\tau)^2\left\{2(2n+1)x^3-3n\tau x^2+(n+1)\left(4\gamma_1-\left(v^2+1\right)\right)x\right.\\ &\qquad\qquad \left.-\tau\left(2(n+1)\gamma_1-\frac{1}{2}\left(1+v^2\right)n\right)\right\}\\ &\quad= 4(n+1)(x-\tau)\left\{4(2n+1)x^5-2(7n+2)\tau x^4+2\left((n+1)\left(4\gamma_1-\left(v^2+1\right)\right)+3n\tau^2\right)x^3\right.\\ &\qquad\qquad -2\tau\left(6(n+1)\gamma_1-\frac{1}{2}(2n+1)\left(1+v^2\right)\right)x^2\\ &\qquad\qquad \left.+2\tau^2\left(2(n+1)\gamma_1-\frac{1}{2}n\left(1+v^2\right)\right)x\right\}.\end{aligned} \tag{5.45}$$

Further, since

$$\begin{aligned}\sum_{\nu=0}^{2n}D_\nu(x) &= \sum_{\nu=0}^{n}D_{2\nu}(x)+\sum_{\nu=0}^{n-1}D_{2\nu+1}(x),\\ \sum_{\nu=0}^{n}D_{2\nu}(x) &= 2(n+1)x\left((n+1)x^2+\left(2\gamma_1-\frac{1}{2}\left(v^2+1\right)-n\tau^2\right)\right),\\ \sum_{\nu=0}^{n-1}D_{2\nu+1}(x) &= 2n(n+1)x(x-\tau)^2,\end{aligned} \tag{5.46}$$

we obtain

$$
\begin{aligned}
D_{2n+1}(x)\sum_{\nu=0}^{2n} D_\nu(x) \\
&= 4(n+1)^2(x-\tau)\{2(2n+1)x^5-2(4n+1)\tau x^4 \\
&\qquad + (4\gamma_1-(v^2+1)+4n\tau^2)x^3-(4\gamma_1-(v^2+1))\tau x^2\}.
\end{aligned}
\tag{5.47}
$$

This leads to (5.40). Similar calculations can be used to prove (5.41). □

6. The integral representations of the second-order self-associated forms

Throughout this section, we will suppose $v \in \mathbb{R} - \{-1, 1\}$. It will be sufficient to consider $0 \le v < 1$ or $v > 1$.

From (3.19), the formal Stieltjes function $S(z_0)$ is given by

$$S(z_0)(z) = \frac{1}{2}\gamma_2^{-1}(z-\tau)^{-1}\{(z^2-1)^{1/2}(z^2-v^2)^{1/2}-2\gamma_2-W(z)\} \tag{6.1}$$

with $W(z) = z^2-(1/2)(v^2+1)$, $z_0 = z_0(\tau, v, \varepsilon)$, and $\gamma_2 = \gamma_2(\tau, v, \varepsilon)$. Putting

$$w(\tau) = w(\tau, v, \varepsilon) = (x-\tau)z_0(\tau, v, \varepsilon), \tag{6.2}$$

we have $S(w(\tau))(z) = (z-\tau)S(z_0)(z)+1$. Therefore, taking (6.1) into account, we get

$$S(w(\tau, v, \varepsilon))(z) = \frac{1}{2}\gamma_2^{-1}Q(z), \tag{6.3}$$

where

$$Q(z) = (z^2-1)^{1/2}(z^2-v^2)^{1/2}-W(z). \tag{6.4}$$

Since $\gamma_2(\tau, v, -\varepsilon) = \gamma_1(\tau, v, \varepsilon)$, we have

$$S(w(\tau, v, -\varepsilon))(z) = \frac{1}{2}\gamma_1^{-1}Q(z). \tag{6.5}$$

Consequently, it is sufficient to study the case $\varepsilon = 1$.

Choosing the branch which is positive when $z^2-1>0$ and $z^2-v^2>0$, we see that Q is regular in the upper half-plane. Moreover, it is easy to prove

$$\sup_{y>0}\int_{-\infty}^{+\infty}|Q(x+iy)|^2dx < +\infty. \tag{6.6}$$

Consequently, the function Q possesses the following representation [2]:

$$Q(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\Im Q(t+i0)}{t-z}dt, \quad \Im z > 0. \tag{6.7}$$

We obtain from (6.4) that

(i) for $0 \leq v < 1$,

$$\Im Q(x+i0) = \begin{cases} 0, & |x| > 1, \\ \operatorname{sgn} x \sqrt{(1-x^2)(x^2-v^2)}, & v < |x| < 1, \\ 0, & |x| < v; \end{cases} \tag{6.8}$$

(ii) for $v > 1$,

$$\Im Q(x+i0) = \begin{cases} 0, & |x| > v, \\ \operatorname{sgn} x \sqrt{(x^2-1)(v^2-x^2)}, & 1 < |x| < v, \\ 0, & |x| < 1. \end{cases} \tag{6.9}$$

In accordance with (6.3), this leads to

$$\langle w(\tau), f \rangle = \frac{1}{2\pi\gamma_2} \int_{-\overline{v}}^{+\overline{v}} \Im Q(x+i0) f(x) dx, \quad f \in \mathcal{P}, \tag{6.10}$$

where

$$\overline{v} := \max(1, v). \tag{6.11}$$

But from (6.2), we have

$$z_0 = \delta_\tau + (x-\tau)^{-1} z(\tau). \tag{6.12}$$

This yields

$$\langle z_0, f \rangle = f(\tau) + \frac{1}{2\pi\gamma_2} \int_{-\overline{v}}^{+\overline{v}} \Im Q(x+i0) \frac{f(x)-f(\tau)}{x-\tau} dx. \tag{6.13}$$

When $\tau \in \mathbb{C} -]-\overline{v}, +\overline{v}[$, we get

$$\langle z_0, f \rangle = \left\{ 1 - \frac{1}{2\pi\gamma_2} \int_{-\overline{v}}^{+\overline{v}} \frac{\Im Q(x+i0)}{x-\tau} dx \right\} f(\tau) + \frac{1}{2\pi\gamma_2} \int_{-\overline{v}}^{+\overline{v}} \frac{\Im Q(x+i0)}{x-\tau} f(x) dx. \tag{6.14}$$

On account of (6.4) and (6.7), we obtain

$$(\tau^2-1)^{1/2}(\tau^2-v^2)^{1/2} - \tau^2 + \frac{1}{2}(v^2+1) = \frac{1}{\pi} \int_{-\overline{v}}^{+\overline{v}} \frac{\Im Q(t+i0)}{t-\tau} dt. \tag{6.15}$$

But $2\gamma_1 = (\tau^2-1)^{1/2}(\tau^2-v^2)^{1/2} - \tau^2 + 1/2(v^2+1)$; accordingly, (6.14) becomes

$$\langle z_0, f \rangle = (1-\gamma_1\gamma_2^{-1}) f(\tau) + \frac{1}{2\pi\gamma_2} \int_{\underline{v}<|x|<\overline{v}} \frac{\operatorname{sgn} x \sqrt{(\overline{v}^2-x^2)(x^2-\underline{v}^2)}}{x-\tau} f(x) dx, \tag{6.16}$$

where $\underline{v} := \min(1, v)$.

When $\tau \in]-\overline{v}, \overline{v}[$, we distinguish two cases.
(a) $\underline{v} \le |\tau| < \overline{v}$. From (6.13), we have

$$\langle z_0, f\rangle = f(\tau) + \frac{1}{2\pi\gamma_2}\int_{\underline{v}<|x|<\overline{v}} \Im Q(x+i0)\frac{f(x)-f(\tau)}{x-\tau}dx \tag{6.17}$$

with

$$\gamma_2(\tau) = \frac{1}{2}(1+v^2) - \tau^2 - \frac{1}{2}Q(\tau+i0). \tag{6.18}$$

It is easy to see that

$$\Re Q(x+i0) = \begin{cases} \sqrt{(x^2-\underline{v}^2)(x^2-\overline{v}^2)} - W(x), & |x| > \overline{v}, \\ -W(x), & \underline{v} \le |x| < \overline{v}, \\ -\sqrt{(\underline{v}^2-x^2)(\overline{v}^2-x^2)} - W(x), & |x| < \underline{v}. \end{cases} \tag{6.19}$$

Consequently,

$$\gamma_2(\tau) = -\frac{1}{2}\left(W(\tau) + i\operatorname{sgn}\tau\sqrt{(\underline{v}^2-\tau^2)(\overline{v}^2-\tau^2)}\right). \tag{6.20}$$

Next, from (6.17), we can have

$$\begin{aligned}\langle z_0, f\rangle = &\left\{1 - \frac{1}{2\pi\gamma_2(\tau)}P\int_{\underline{v}<|x|<\overline{v}} \frac{\Im Q(x+i0)}{x-\tau}dx\right\}f(\tau) \\ &+ \frac{1}{2\pi\gamma_2(\tau)}P\int_{\underline{v}<|x|<\overline{v}} \frac{\Im Q(x+i0)}{x-\tau}f(x)dx,\end{aligned} \tag{6.21}$$

where P means principal value of the integral.
But from (6.7), the following limit relationship holds:

$$\Re Q(x+i0) = \frac{1}{\pi}P\int_{\underline{v}<|t|<\overline{v}} \frac{\Im Q(t+i0)}{t-x}dt, \quad x \in \mathbb{R}. \tag{6.22}$$

With (6.19), this gives

$$\frac{1}{\pi}P\int_{\underline{v}<|t|<\overline{v}} \frac{\Im Q(t+i0)}{t-x}dt = -W(x), \quad \underline{v} < |x| < \overline{v}. \tag{6.23}$$

Consequently, (6.21) becomes

$$\begin{aligned}\langle z_0, f\rangle = &-\frac{1}{2}i\gamma_2^{-1}(\tau)\operatorname{sgn}\tau\sqrt{(\underline{v}^2-\tau^2)(\overline{v}^2-\tau^2)}f(\tau) \\ &+ \frac{1}{2\pi\gamma_2(\tau)}P\int_{\underline{v}<|x|<\overline{v}} \frac{\Im Q(x+i0)}{x-\tau}f(x)dx.\end{aligned} \tag{6.24}$$

(b) $|\tau| < \underline{v}$. From (6.13), we still have (6.17), where here

$$\gamma_2(\tau) = \frac{1}{2}\left(\sqrt{(\underline{v}^2-\tau^2)(\overline{v}^2-\tau^2)} - W(\tau)\right). \tag{6.25}$$

Taking (6.19) and (6.22) into account, we infer that

$$\frac{1}{\pi} P \int_{\underline{v}<|t|<\overline{v}} \frac{\Im Q(t+i0)}{t-\tau} dt = -\left(\sqrt{(\underline{v}^2-\tau^2)(\overline{v}^2-\tau^2)} + W(\tau)\right). \tag{6.26}$$

Thus, we obtain

$$\begin{aligned}\langle z_0, f\rangle &= \gamma_2^{-1}(\tau)\sqrt{(\underline{v}^2-\tau^2)(\overline{v}^2-\tau^2)}\, f(\tau) \\ &\quad + \frac{1}{2\pi\gamma_2(\tau)} \int_{\underline{v}<|x|<\overline{v}} \frac{\Im Q(x+i0)}{x-\tau} f(x)dx.\end{aligned} \tag{6.27}$$

These results are summarized in the following proposition.

Proposition 6.1. *Suppose either* $0 \le v < 1$ *or* $v > 1$. *Let* $\underline{v} := \min(1,v)$ *and* $\overline{v} := \max(1,v)$. *Then the form* z_0 *possesses the following integral representation:*

(1) *for* $\tau \in \mathbb{C} -]-\overline{v}, +\overline{v}[$,

$$\begin{aligned}\langle z_0, f\rangle &= -\gamma_2^{-1}(\tau^2-1)^{1/2}(\tau^2-v^2)^{1/2} f(\tau) \\ &\quad + \frac{1}{2\pi\gamma_2} \int_{\underline{v}<|x|<\overline{v}} \frac{\operatorname{sgn} x\sqrt{(\overline{v}^2-x^2)(x^2-\underline{v}^2)}}{x-\tau} f(x)dx;\end{aligned} \tag{6.28}$$

(2) *for* $\underline{v} < |\tau| < \overline{v}$,

$$\begin{aligned}\langle z_0, f\rangle &= -\frac{1}{2} i\gamma_2^{-1}(\tau)\operatorname{sgn}\tau\sqrt{(\underline{v}^2-\tau^2)(\overline{v}^2-\tau^2)}\, f(\tau) \\ &\quad + \frac{1}{2\pi\gamma_2(\tau)} P\int_{\underline{v}<|x|<\overline{v}} \frac{\operatorname{sgn} x\sqrt{(\overline{v}^2-x^2)(x^2-\underline{v}^2)}}{x-\tau} f(x)dx;\end{aligned} \tag{6.29}$$

(3) *for* $|\tau| \le \underline{v}$,

$$\begin{aligned}\langle z_0, f\rangle &= \gamma_2^{-1}(\tau)\sqrt{(\underline{v}^2-\tau^2)(\overline{v}^2-\tau^2)}\, f(\tau) \\ &\quad + \frac{1}{2\pi\gamma_2(\tau)} \int_{\underline{v}<|x|<\overline{v}} \frac{\operatorname{sgn} x\sqrt{(\overline{v}^2-x^2)(x^2-\underline{v}^2)}}{x-\tau} f(x)dx.\end{aligned} \tag{6.30}$$

Remark 6.2. In the last case $|\tau| \le \underline{v}$, the form z_0 is positive definite since $\gamma_1(\tau) > 0$ and $\gamma_2(\tau) > 0$.

Regarding the moments, from (6.1), we easily obtain

$$\begin{aligned}\left(z_0(\tau,v,+1)\right)_{2n} &= \sum_{\mu=0}^{n} \tau^{2(n-\mu)} d_\mu, \quad n \ge 0, \\ \left(z_0(\tau,v,+1)\right)_{2n+1} &= \tau\left(z_0(\tau,v,+1)\right)_{2n}, \quad n \ge 0,\end{aligned} \tag{6.31}$$

where

$$\begin{gathered}d_0 = 1, \quad d_n = -\frac{1}{2}\gamma_2^{-1}c_{n+1}, \quad n \geq 1, \\ c_n = \frac{1}{4\pi}\sum_{m+k=n}\frac{\Gamma(m-1/2)}{m!}\frac{\Gamma(k-1/2)}{k!}v^{2k}, \quad n \geq 0.\end{gathered} \tag{6.32}$$

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