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A Degenerate Variance Control Problem with Discretionary Stopping

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Abstract: We consider an infinite horizon stochastic control problem with discretionary stopping. The state process is given by a one dimensional stochastic differential equation. The diffusion coefficient is chosen by an adaptive choice of the controller and it is allowed to take the value zero. The controller also chooses the quitting time to stop the system. Here we develop a martingale characterization of the value function and use it and the principle of smooth fit to derive an explicit optimal strategy when the drift coefficient of the state process is of the form $b(x) = -\theta x$ where $\theta > 0$ is a constant.

1. Introduction.

Degenerate variance control problems are those in which the controller has access to the diffusion coefficient in the state dynamics and may even set it to zero. A simple model is the one-dimensional equation

$$X_x^u(t) = x + \int_0^t b(X_x^u(s))ds + \int_0^t u(s)dW(s) \tag{1.1}$$

where x is a real number, $\{W(t) : t \geq 0\}$ is a standard one-dimensional Brownian motion and $u(\cdot)$ is a suitably adapted control process subject to the constraint

$$0 \leq u(t) \leq \sigma_0 \quad \text{for all} \quad t \geq 0. \tag{1.2}$$

Here σ_0 is a given positive constant.

Several control problems based on (1.1)-(1.2) are considered in the literature. Assaf [1] studies minimizing a combination of location and control cost when $\sigma_0 \uparrow \infty$ for a specific control problem generated by a model for dynamic sampling. Papers [10] and [11] generalize Assaf's control structure, but for $\sigma_0 < \infty$. In these papers the cost is a discounted, infinite horizon integral of location cost, which increases as the state approaches the origin, plus a control cost, which increases in the control effort u. The deterministic solutions to $\dot{x} = b(x)$ associated with fully degenerate control ($u \equiv 0$) evolve toward the origin in the direction of higher cost. In [10] and [11], it is shown how to construct cost minimizing controls of bang-bang type that use maximum variance control ($u = \sigma_0$) to move the state to lower cost regions.

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This article presents an example of control of (1.1) combined with discretionary stopping. The object is now to choose a control $u(\cdot)$ and a stopping time τ to maximize the reward functional

$$J(x,u,\tau) = E\int_0^\tau e^{-\alpha t}C(X_x^u(t))dt. \tag{1.3}$$

Here, the discount rate α is a positive constant. The value function is given by

$$V(x) = \sup_{\mathcal{U}} J(x,u,\tau), \tag{1.4}$$

where $\mathcal{U}$ is a collection of all policies $(u(\cdot),\tau)$ to be described precisely below. We have in mind here a situation in which:

(i) the origin is a unique asymptotically stable equilibrium point for $\dot{x} = b(x)$; and

(ii) $C(\cdot)$ is a unimodal function with a unique positive maximum at the origin and $\lim_{x\to-\infty} C(x) = \lim_{x\to\infty} C(x) = -\infty$.

We will place more restrictive hypotheses on $b(\cdot)$ and $C(\cdot)$ for the statements and proofs of the results, but the assumptions (i) and (ii) will serve for motivation. In this case, the solution to $\dot{x} = b(x)$ obtained for the zero variance control $u \equiv 0$ does evolve in a favorable direction, in contrast to the problems in [1], [10] and [11]. It is of interest to ask whether and, if so, where, positive variance control should be employed to boost the expected reward. Consider first the deterministic stopping problem when $u \equiv 0$ is imposed. The discounted reward if $\tau = \infty$ is given by $V_0(x) = E[\int_0^\infty e^{-\alpha t}C(X_x^0(t))\,dt]$. If (i) and (ii) hold and, say, b grows linearly, $V_0(x)$ decreases with increasing $|x|$, and there will be constants $-\infty < a_0 < 0 < b_0 < \infty$ such that $V_0(x) > 0$ if and only if $a_0 < x < b_0$. The optimal choice of τ is then easy; the controller, having the option of stopping, will not accept a negative reward. Hence $\tau = \infty$, if $a_0 < x < b_0$, and $\tau = 0$ otherwise. Consider next adding the possibility of positive variance control. Intuitively, if the state is close to the origin, positive variance control ought never to be applied, as diffusive behavior of the state would lessen the reward. However, let x be a point larger than b_0 but close to it. Then, if $u(\cdot)$ is positive, some sample paths of $X_x^u(\cdot)$ will move more quickly toward the origin than the solution of $\dot{x} = b(x)$ and doing so will enable an overall positive reward; at the same time, the option to stop allows bailing out along sample paths that move the wrong way. Therefore, one should have positive expected reward even for some $x > b_0$. Assuming concavity of C and linearity of b, the main result of this paper verifies this scenario and shows how to construct an optimal feedback control and stopping rule.

For precise results, the following conditions will often be assumed throughout this article.

(1.5) (i) The function b is continuously differentiable on $\mathbf{R}$ and $b(0) = 0$.

(1.6) (ii) $C(\cdot)$ is a twice continuously differentiable, strictly concave function which attains its unique maximum at $x = 0$ and $C(0) = 1$.

The continuous differentiability of $b(\cdot)$ assumed in (1.5) guarantees local existence and uniqueness of solutions to (1.1). The requirement that $C(0) = 1$ is just a normalization convention of no consequence to the results.

Admissible controls are defined precisely as follows: An admissible control system is a quintuple $((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}, W(\cdot), u(\cdot), \tau)$ such that $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\{\mathcal{F}_t\}$ is a right-continuous, complete filtration, $W(\cdot)$ is a one-dimensional Brownian motion adapted to $\{\mathcal{F}_t\}$, such that $W(t+s) - W(t)$ is independent of $\mathcal{F}_t$ for all $t > 0$ and $s > 0$, $u(\cdot)$ is an $\{\mathcal{F}_t\}$-progressively measurable process satisfying (1.2), and τ is an $\{\mathcal{F}_t\}$-stopping time less than or equal to the explosion time of the solution to (1.1). (In the situation of interest in this paper, $xb(x) \geq 0$ for all x and the explosion time is infinite almost surely.) With a slight abuse of notation, henceforth we denote an admissible policy by the pair (u, τ). The class of admissible policies is denoted by $\mathcal{U}$ and this is the class that should be used in (1.4) in the definition of the value function.

Theorem 1.1. *Let the drift coefficient in (1.1) be given by $b(x) = -\theta x$ for all x, where $\theta > 0$ is a positive constant. Assume $C(\cdot)$ satisfies (1.6). Then an explicit representation of the value function $V(\cdot)$ defined in (1.4) is given in (3.11) and the value function is continuously differentiable everywhere. Furthermore, there exist four points $c^* < p^* < 0 < q^* < d^*$ so that the following admissible control strategy (u^*, τ^*) with the corresponding state process $X_x^{u^*}(\cdot)$ is an optimal strategy.*

1. *If $x \leq c^*$ or $x \geq d^*$ then choose $\tau^* = 0$ and stop.*
2. *If $p^* \leq x \leq q^*$, then choose $\tau^* = \infty$, $u^*(t) = 0$ for all t and follow the deterministic motion.*
3. *If $q^* < x < d^*$ then choose $u^*(t) = \sigma_0$ and let $\hat{\tau}$ be the first exit time of the process $X_x^{u^*}(\cdot)$ from the interval (q^*, d^*). Thereafter, follow as in the steps 1 or 2 appropriately.*
   In this case, $\tau^* = \hat{\tau} I_{[X_x^{u^*}(\hat{\tau}) = d^*]} + \infty \cdot I_{[X_x^{u^*}(\hat{\tau}) = q^*]}$.
4. *If $c^* < x < p^*$ then choose $u^*(t) = \sigma_0$ and let $\hat{\tau}$ be the first exit time of the process $X_x^{u^*}(\cdot)$ from the interval (c^*, p^*). Thereafter, follow as in the steps 1 or 2 appropriately.*
   In this case, $\tau^* = \hat{\tau} I_{[X_x^{u^*}(\hat{\tau}) = c^*]} + \infty \cdot I_{[X_x^{u^*}(\hat{\tau}) = p^*]}$.

That allowing positive variance control boosts the expected reward in a way similar to Theorem 1.1 should be a general fact. The linearity of $b(\cdot)$ and the concavity of $C(\cdot)$ are used to derive the particularly simple optimal policy of Theorem 1.1 by smooth fit. The value function $V(\cdot)$ is continuously differentiable and thus "the principle of smooth fit" holds for the first derivative of $V(\cdot)$ and its second derivative has jump discontinuities only at the points c^*, p^*, q^* and d^*. Some of the results preliminary to the proof of Theorem 1.1 are proved under more general assumptions.

Solvable stochastic control problems with discretionary stopping have received attention recently(see [2],[4],[6],[7],[9] and [13]). Variational inequalities related to higher dimensional problems are developed in [9]. A discretionary stopping problem arising in mathematical finance is addressed in [6]. The articles [2] and [4] treat singular stochastic control problems with discretionary stopping, while [13] studies a two player stochastic differential game with degenerate variance control. The existence and characterization of optimal Markov controls for several types of stochastic control problems are developed in [8]. They use a martingale problem approach. To obtain their results, they show that the original stochastic control problem is equivalent to a linear programming problem over a space of measures.

In [7], authors address a finite time horizon problem with combined control and discretionary stopping. Their control process affects only the drift coefficient. Motivated by their martingale characterization of the optimal strategy, we also formulate a martingale characterization for the value function in section 2. We use it in section 3 to construct the optimal state process of Theorem 1.1. Our optimal control is "feed-back" type and hence the optimal state process is a Markov process. As noted in [7], this martingale condition is analogous to the "equalization" condition developed by Dubins and Savage [3] in a discrete time context.

2. A Martingale Formulation

The first result in this section is a martingale characterization of the optimal value function. Then, we derive simple bounds and monotonicity properties of the value function. Since $b(0) = 0$ and $C(x) \le C(0)$, we are also able to show that the origin can be considered as an absorption point without any change in the value function. This enables us to solve the control problem in each of the regions $(-\infty, 0)$ and $(0, +\infty)$ separately and then to paste the two solutions together.
For the results in this section, we do not need the full power of assumptions (1.5) and (1.6). The martingale characterization theorem remains valid under quite general assumptions as listed below. All the other results in this section remain valid if the drift coefficient $b(\cdot)$ is a continuously differentiable function which satisfies $b(0) = 0$ and $C(\cdot)$ is a continuous function which is strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$. We take $C(0) = 1$ for simplicity.
The following theorem requires only that the drift coefficient $b(\cdot)$ in (1.1) be continuous and that the reward function $C(\cdot)$ in (1.3) be continuous and bounded above by a constant.

Theorem 2.1. *Let $Q(\cdot)$ be a non-negative, bounded continuous function defined on $\mathbf{R}$ and let the initial point x be fixed.*

(i) If $Q(X_x^u(t \wedge \tau))e^{-\alpha(t\wedge\tau)} + \int_0^{t\wedge\tau} e^{-\alpha s} C(X_x^u(s))ds$ is a super-martingale for the state process $X_x^u(\cdot)$ corresponding to each admissible control policy (u, τ) in $\mathcal{U}$, then $Q(x) \ge V(x)$.

(ii) If $Q(\cdot)$ satisfies the above condition (i) and if there is a state process $Z_x^{u^*}(\cdot)$ corresponding to an admissible control policy (u^*, τ^*) so that $Q(Z_x^{u^*}(t \wedge \tau^*))e^{-\alpha(t\wedge\tau^*)} + \int_0^{t\wedge\tau^*} e^{-\alpha s} C(Z_x^{u^*}(s))ds$ is a martingale and $Q(Z_x^{u^*}(\tau^*)) = 0$ on the set $[\tau^* < \infty]$, then $Q(x) = V(x)$, $Z_x^{u^*}(\cdot)$ is an optimal state process and (u^*, τ^*) is the corresponding optimal control policy.

Proof. Let $X_x^u(\cdot)$ be a state process corresponding to an admissible control policy (u, τ). Then, using the super-martingale property in condition (i) and the non-negativity of the function $Q(\cdot)$, we obtain

$$Q(x) \ge E\left[\int_0^{t\wedge\tau} e^{-\alpha s} C(X_x^u(s))ds\right] \quad \text{for all } t \ge 0. \tag{2.1}$$

Since $C(\cdot)$ is bounded above, we have

$$\lim_{t\to\infty} E\left[\int_0^{t\wedge\tau} e^{-\alpha s} C(X_x^u(s))ds\right] = E\left[\int_0^{\tau} e^{-\alpha t} C(X_x^u(t))dt\right]. \tag{2.2}$$

Therefore, by (2.1) and (2.2) we obtain

$$Q(x) \geq E\left[\int_0^\tau e^{-\alpha t} C(X_x^u(t))dt\right]$$

and consequently, $Q(x) \geq V(x)$. The proof of part (i) is complete.

Now let (u^*, τ^*) be an admissible control policy with associated state process $Z_x^{u^*}(\cdot)$ which satisfies the assumptions in part (ii). Using the martingale condition, then we have

$$Q(x) = E\left[Q(Z_x^{u^*}(t \wedge \tau^*))e^{-\alpha(t\wedge\tau^*)} + \int_0^{t\wedge\tau^*} e^{-\alpha s} C(Z_x^{u^*}(s))ds\right]. \tag{2.3}$$

Using the fact that $Q(Z_x^{u^*}(\tau^*)) = 0$ on the set $[\tau^* < \infty]$, we obtain

$$E\left[Q(Z_x^{u^*}(t \wedge \tau^*))e^{-\alpha(t\wedge\tau^*)}\right] = E\left[Q(Z_x^{u^*}(t))I_{[t<\tau^*]}\right] e^{-\alpha t}.$$

Since $Q(\cdot)$ is a bounded function, from the above equation it clearly follows that

$$\lim_{t\to\infty} E\left[Q(Z_x^{u^*}(t \wedge \tau^*))e^{-\alpha(t\wedge\tau^*)}\right] = 0.$$

Now letting t tend to infinity in (2.3) and using the above results, we obtain

$$Q(x) = E\int_0^{\tau^*} e^{-\alpha t} C(Z_x^{u^*}(t))dt,$$

and hence (u^*, τ^*) is an optimal control policy. This completes the proof.

Let (u, τ) be an admissible control policy associated with the state process $X_x^u(\cdot)$ which satisfies (1.1). Introduce the stopping time τ_0^x by

$$\begin{aligned} \tau_0^x &= \inf\{t \geq 0 : X_x^u(t) = 0\} \\ &= +\infty \ \text{ if the above set is empty.} \end{aligned} \tag{2.4}$$

Now we introduce the new admissible control process $\tilde{u}(\cdot)$ by

$$\begin{aligned} \tilde{u}(t) &= u(t) \ \text{ for } 0 \leq t \leq \tau_0^x \\ &= 0 \qquad \text{for } \ t > \tau_0^x. \end{aligned} \tag{2.5}$$

Since the drift term $b(\cdot)$ is continuously differentiable, we can consider the associated state process $X_x^{\tilde{u}}(\cdot)$ on the same probability space using the equation (1.1). The condition $b(0) = 0$ implies that $X_x^{\tilde{u}}(t) = X_x^u(t \wedge \tau_0^x)$ for all $t \geq 0$. Hence, we have the following proposition.

Proposition 2.2. *Assume that the drift $b(\cdot)$ is a continuously differentiable function which satisfies $b(0) = 0$ and the reward function $C(\cdot)$ is a continuous function which is strictly increasing in $(-\infty, 0)$, strictly decreasing in $(0, \infty)$ and satisfies $C(0) = 1$. Let the state processes $X_x^u(\cdot)$ and $X_x^{\tilde{u}}(\cdot)$ be defined as above. Then the following results hold.*

(i) $J(x,u,\tau) \le J(x,\tilde{u},\tau)$ *for each stopping time* τ.
Furthermore, if we let $\mathcal{D}$ *be the sub-collection of admissible control policies* $(\tilde{u},\tau)$ *of* $\mathcal{U}$ *so that the corresponding state process* $X_x^{\tilde{u}}(\cdot)$ *is stopped at the origin, then*

$$V(x) = \sup_{\mathcal{D}} J(x,u,\tau) \tag{2.6}$$

(ii) $V(x) \le \frac{1}{\alpha}$ *for all* x *and* $V(0) = \frac{1}{\alpha}$.

Proof. Since $X_x^{\tilde{u}}(t) = X_x^u(t \wedge \tau_0^x)$ for all $t \ge 0$ and the reward function $C(\cdot)$ has a unique maximum at the origin, it follows that $C(X_x^{\tilde{u}}(t)) \ge C(X_x^u(t))$ for all $t \ge 0$. Therefore, $J(x,u,\tau) \le J(x,\tilde{u},\tau)$ for any stopping time τ. As an immediate consequence, $V(x) = \sup\limits_{\mathcal{D}} J(x,u,\tau)$ follows.
To prove part (ii), observe that $C(X_x^u(t)) \le C(0)$ for all t and consequently $J(x,u,\tau) \le \frac{C(0)}{\alpha}$ for each admissible policy (u,τ). Hence $V(x) \le \frac{1}{\alpha}$. If the initial point is at the origin, one can choose $u_0(t) \equiv 0$ and $\tau_\infty = \infty$ to obtain $J(x,u_0,\tau_\infty) = \frac{1}{\alpha}$. Hence $V(0) = \frac{1}{\alpha}$. This completes the proof.

Remark. Notice that the above Proposition 2.2 implies that if the assumptions in part (i) of the Theorem 2.1 holds for the admissible control policies in the sub-collection $\mathcal{D}$, then the conclusion there still remains valid.

The next lemma establishes monotonicity of the value function.

Lemma 2.3. *Under the assumptions of Proposition 2.2, the value function* $V(\cdot)$ *defined in (1.4) is non-negative, monotone increasing on* $(-\infty,0)$ *and monotone decreasing on* $(0,\infty)$.

Proof. If we choose the zero stopping time, $J(x,u,0) = 0$ and hence $V(x) \ge 0$ for all x. Here, we show that $V(\cdot)$ is decreasing on $(0,\infty)$. A similar argument works on $(-\infty,0)$.
Let $x > y > 0$ and let (u,τ) be any admissible control policy. Because of the assumed continuous differentiability of $b(\cdot)$, the solutions to (1.1) are path-wise unique and so $X_y^u(t) \le X_x^u(t)$ for all $t \ge 0$.
Now introduce τ_0^y as in (2.4) and the admissible control process $\tilde{u}(t) = u(t)I_{[0,\tau_0^y]}(t)$ as similar to (2.5). The state process $X_y^{\tilde{u}}(\cdot)$ is given by $X_y^{\tilde{u}}(t) = X_y^u(t \wedge \tau_0^y)$. Then by the proof of Proposition 2.2, it follows that

$$\begin{aligned} J(y,\tilde{u},\tau) &= E\left[\int_0^{\tau\wedge\tau_0^y} e^{-\alpha t} C(X_y^u(t))dt + \int_{\tau\wedge\tau_0^y}^{\tau} e^{-\alpha t} C(0)dt\right] \\ &> J(x,u,\tau). \end{aligned} \tag{2.7}$$

Therefore, $V(y) \ge V(x)$ when $x > y > 0$. This completes the proof.

Next, we show that any smooth solution to the corresponding Hamilton-Jacobi-Bellman(HJB) equation of the discretionary stopping problem is an upper bound for the value function.

Proposition 2.4. *Make the same assumptions as in Proposition 2.2. Let* $Q(\cdot)$ *be a non-negative, bounded and continuously differentiable function which satisfies the following:*

(i) $Q''(\cdot)$ *is continuous everywhere except in a finite set. Furthermore, the one-sided derivatives* $Q''(x-)$ *and* $Q''(x+)$ *exists and are finite for all* x.

(ii) There is a positive constant $M > 0$ *so that* $|Q'(x)| < M$ *for all* x.

(iii) The function $Q(\cdot)$ *satisfies the HJB equation*

$$\max\left\{\sup_{0\le u\le\sigma_0}\frac{u^2}{2}Q''(x)+b(x)Q'(x)-\alpha Q(x)+C(x),-Q(x)\right\}=0$$

for almost every x *in* $\mathbf{R}$.

Then $Q(x)\ge V(x)$ *for all* x.

Proof. We use Proposition 2.2 and consider an initial point $x > 0$. We intend to verify the condition (i) of Theorem 2.1 for all the admissible control policies in $\mathcal{D}$ (see also the remark below the proof of Proposition 2.2). Let $X_x^u(\cdot)$ be the state process which satisfies (1.1) corresponding to an admissible control policy (u,τ) in $\mathcal{D}$. Using a mollification for the function $Q(\cdot)$ to smooth it and using Theorem 7.1 of page 218 and the ex. 7.10 in page 225 of [5] (see also Appendix D, page 301 in [12]), we can extend Itô's lemma to the function $Q(\cdot)$ to obtain $Q(X_x^u(t\wedge\tau))e^{-\alpha(t\wedge\tau)} - \int_0^{t\wedge\tau}(\frac{u(s)^2}{2}Q''(X_x^u(s)) + b(X_x^u(s))Q'(X_x^u(s)) - \alpha Q(X_x^u(s)))e^{-\alpha s}ds$ is a martingale. Therefore, using the assumption (iii), we observe that $Q(X_x^u(t\wedge\tau))e^{-\alpha(t\wedge\tau)}+\int_0^{t\wedge\tau}C(X_x^u(s))e^{-\alpha s}ds$ is a super-martingale. Now the conclusion follows from part (i) of Theorem 2.1.

Remark. Let Q satisfy the conditions of Proposition 2.4. Any function u^* satisfying,

$$\sup_{0\le u\le\sigma_0}\frac{u^2}{2}Q''(x)+b(x)Q'(x)-\alpha Q(x)+C(x)=0,\quad \text{a.e. for } Q(x)>0, \tag{2.8}$$

is natural candidate for an optimal control; it is not necessary to define $u^*(x)$ on the set where $Q(x)=0$, since it is optimal to stop on this set. Solutions u^* to (2.8) are easy to come by; for example, $\sigma_0\mathbf{1}_G(x)$, where $G=\{x:\ Q''(x)>0\}$ will work. But (2.8) does not uniquely specify u^* because it does not prescribe its values at points x such that $Q''(x)=0$ or such that $Q''(x)$ is not defined. Not all choices of u^* will necessarily work. First, it must be chosen so that there is at least a weak solution to (1.1) using u^* as a feedback control. A discussion in [10] shows that this will be the case for a model like (1.1) when u^* is the indicator of an open set, but that other choices of u^* off the set $G=\{x:Q''(x)>0\}$ may not work. Second, one must verify that for the solution X^{u^*} corresponding to the feedback control u^*, the process $Q(X_x^{u^*}(t\wedge\tau))e^{-\alpha(t\wedge\tau)}+\int_0^{t\wedge\tau}C(X_x^{u^*}(s))e^{-\alpha s}ds$ is a martingale, and this requires us to pay some attention to how X^{u*} behaves in the regions where the feedback variance control degenerates.

We do not attempt to frame a general theorem on synthesis of an optimal control under the hypotheses of Proposition 2.4. In the problem that we analyze in the next section, our candidate for an optimal strategy exercises maximum variance control in a disjoint union G of two open intervals only until the corresponding state process X hits a boundary point of G. After this state process X hits a boundary point of G, it will never return to G. Therefore the construction of the candidate process and the proof of optimality present no difficulties.

3. Linear Drift

In this section, we use the assumption (1.6) and the drift term $b(x) = -\theta x$ for all x where $\theta > 0$ is a positive constant. Therefore, for a given control process $u(\cdot)$, the corresponding state process in (1.1) takes the form

$$X_x^u(t) = x - \theta \int_0^t X_x^u(s) ds + \int_0^t u(s) dW(s). \tag{3.1}$$

First we derive the properties of the pay-off function with zero control and infinite stopping time. Notice that when $u(t) = 0$ for all t, the state process is given by $X_x^0(t) = xe^{-\theta t}$. Therefore, the pay-off function from the zero control and the infinite stopping time is given by $J(x, 0, +\infty)$ and for convenience, we label it by $V_0(x)$. Hence,

$$V_0(x) = \int_0^\infty e^{-\alpha t} C(xe^{-\theta t}) dt. \tag{3.2}$$

Notice that the above integral is uniformly convergent on compact sets.

Lemma 3.1. *Let $V_0(\cdot)$ be as in (3.2). Then the following results hold:*

(i) $V_0(\cdot)$ satisfies the differential equation $\theta x V_0'(x) + \alpha V_0(x) = C(x)$ for all x and $V_0(0) = \frac{1}{\alpha}$.
(ii) For $x \neq 0$, $V_0(\cdot)$ is given by $V_0(x) = \frac{1}{\theta x^{\frac{\alpha}{\theta}}} \int_0^x C(r) r^{\frac{\alpha}{\theta} - 1} dr$ and $\lim_{x \to 0} V_0(x) = \frac{1}{\alpha}$.
(iii) $V_0(\cdot)$ is strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, \infty)$. Furthermore, $V_0(\cdot)$ is a strictly concave function which has a unique maximum at $x = 0$.

Proof. The proofs of parts (i) and (ii) are straightforward. The limit $\lim_{x \to 0} V_0(x)$ can be computed using L'Hopital's rule.

To prove part (iii), notice that the following formulas also follow from (3.2):

$$V_0'(x) = \int_0^\infty e^{-(\theta + \alpha)t} C'(xe^{-\theta t}) dt \qquad \text{for all } x \tag{3.3}$$

and

$$V_0''(x) = \int_0^\infty e^{-(2\theta + \alpha)t} C''(xe^{-\theta t}) dt \qquad \text{for all } x. \tag{3.4}$$

Now using the assumption(1.6) for $C(\cdot)$, part (ii) of Proposition 2.2 and the above formulas, the conclusions of part (iii) hold.

Since $V_0(\cdot)$ is a concave function with a unique global maximum at $x = 0$ and $V_0(0) = \frac{1}{\alpha} > 0$, there exist two points a_0 and b_0 so that $a_0 < 0 < b_0$ and $V_0(a_0) = V_0(b_0) = 0$. Furthermore, the set $[V_0 > 0]$ is equal to the open interval (a_0, b_0).

Let us introduce the infinitesimal generator $\mathcal{G}$ related to the Ornstein-Uhlenbeck process corresponding to the constant control $u(t) \equiv \sigma_0$ for all $t \geq 0$ in (3.1) by

$$\mathcal{G} = \frac{\sigma_0^2}{2} \frac{d^2}{dx^2} - \theta x \frac{d}{dx}. \tag{3.5}$$

For a constant $\alpha > 0$, we also write $\mathcal{G} - \alpha$ for

$$\mathcal{G} - \alpha = \frac{\sigma_0^2}{2}\frac{d^2}{dx^2} - \theta x \frac{d}{dx} - \alpha. \tag{3.6}$$

Consider next the family of solutions $Q_d(\cdot)$, for $d \geq b_0$ of

$$\begin{aligned} (\mathcal{G} - \alpha)Q_d(x) + C(x) = 0 \text{ for all } x > 0, \\ Q_d(d) = Q_d'(d) = 0. \end{aligned} \tag{3.7}$$

Our aim is to build the value function on $(0, \infty)$ from $V_0(\cdot)$ and $Q_{d^*}(\cdot)$, where the point $d^* > b_0$ is chosen so that $Q_{d^*}(\cdot)$ meets $V_0(\cdot)$ tangentially.

Lemma 3.2. *Let $V_0(\cdot)$ be as in Lemma 3.1 and the family of functions $Q_d(\cdot)$ be as described above. Then the following hold:*

(i) There is a $\delta_1 > 0$ so that for each d in $(b_0, b_0 + \delta_1)$, $Q_d(\cdot)$ meets $V_0(\cdot)$ at some point in the interval $(0, b_0)$.
(ii) There is a point $l_0 > b_0$ so that for every $d > l_0$, $Q_d(x) > V_0(x)$ for all $x > 0$.

Proof. By part (iii) of Lemma 3.1, we have $V_0'(x) < 0$ for all $x > 0$. By evaluating the differential equation for $V_0(\cdot)$ in the part (i) of Lemma 3.1 at the point b_0 and using $V_0'(b_0) < 0$, we conclude $C(b_0) < 0$. Now consider the function $Q_{b_0}(\cdot)$ which satisfies (3.7) with $d = b_0$. Then $Q_{b_0}(b_0) = Q_{b_0}'(b_0) = V_0(b_0) = 0$ and we evaluate (3.7) for the function $Q_{b_0}(\cdot)$ at the point b_0 and obtain $Q_{b_0}''(b_0) = -\frac{2}{\sigma^2}C(b_0) > 0$. Therefore, the function $Q_{b_0}(\cdot)$ is strictly convex in an interval $(b_0 - \epsilon, b_0 + \epsilon)$ and $V_0(\cdot)$ is strictly concave everywhere. Hence, there is a $\delta_0 > 0$ so that $Q_{b_0}(x) < V_0(x)$ for all x in $(b_0 - \delta_0, b_0)$. The solutions $Q_d(x)$ of (3.7) are jointly continuous in (d, x) and therefore, we can find a $\delta_1 > 0$ so that

$$Q_d(b_0 - \frac{\delta_0}{2}) < V_0(b_0 - \frac{\delta_0}{2})$$

for all d in $[b_0, b_0 + \delta_1)$. By (3.7), $Q_d''(d) = -\frac{2}{\sigma^2}C(d) > 0$, and hence the function $Q_d(\cdot)$ is strictly convex in a neighborhood of the point $x = d$. Consequently, $Q_d'(x) < 0$ in an interval $(d - \epsilon_d, d)$ for some $\epsilon_d > 0$.

For each d in $(b_0, b_0 + \delta_1)$, we intend to show that $Q_d(x) > 0$ for all x in (b_0, d). For this, it suffices to prove $Q_d'(\cdot) < 0$ on the interval (b_0, d). We let

$$\eta = \inf\{x : Q_d'(y) < 0 \text{ on } (x, d)\}.$$

The above set is non-empty, since $Q_d'(\cdot) < 0$ on the interval $(d - \epsilon_d, d)$. Notice that, we attain our conclusion if we can show $\eta \leq b_0$. Suppose that $\eta > b_0$. Then, clearly $Q_d(\eta) > 0$, $Q_d'(\eta) = 0$ and by (3.7), $Q_d''(\eta) = \alpha Q_d(\eta) - \frac{2}{\sigma^2}C(\eta) > 0$. Hence, $Q_d'(x) > 0$ for all x in an interval $(\eta, \eta + \epsilon')$ for some $\epsilon' > 0$. This contradicts with the definition of η and hence we conclude that $\eta \leq b_0$. From this, it follows that $Q_d'(\cdot) < 0$ on (b_0, d). Therefore, $Q_d(x) > 0 > V_0(x)$ on (b_0, d).

We have already shown that $Q_d(b_0 - \frac{\delta_0}{2}) < V_0(b_0 - \frac{\delta_0}{2})$, for each d in $(b_0, b_0 + \delta_1)$. Therefore, $Q_d(\cdot)$ intersects $V_0(\cdot)$ at some point in the interval $(0, b_0)$ for each d in $(b_0, b_0 + \delta_1)$.

Next, we intend to show that for large values of d, $Q_d(\cdot)$ does not intersect $V_0(\cdot)$ at all. First we prove that for each $d > b_0$, $Q_d(\cdot)$ is strictly decreasing on the interval (b_0, d). By evaluating (3.7) at the point d, we know that $Q_d''(d) > 0$ and hence $Q_d(\cdot)$ is strictly decreasing in an interval $(d - \epsilon, d)$ for some $\epsilon > 0$. If $Q_d'(\zeta) = 0$ and

$Q_d(\zeta) > 0$ for some ζ in the interval (b_0, d), then by (3.7), and by the fact that $C(x) < 0$ for all $x > b_0$, we obtain $Q_d''(\zeta) > 0$. Hence, $x = \zeta$ is necessarily a local minimum for $Q_d(\cdot)$. Therefore, $Q_d(\cdot)$ cannot have any local maxima on the interval (b_0, d) and consequently, it is strictly positive and is strictly decreasing on (b_0, d). Next, we show that $\lim_{d\to\infty} Q_d(b_0) = \infty$. By (3.7), we obtain

$$\frac{\sigma^2}{2}Q_d''(x) + C(x) > \theta x Q_d'(x)$$

for all x in (b_0, d). By integrating this, using integration by parts in the right hand side and using the boundary conditions in (3.7) we obtain

$$-\frac{\sigma^2}{2}Q_d'(x) + \int_x^d C(u)du > -\theta x Q_d(x) - \theta \int_x^d Q_d(u)du$$

for all x in (b_0, d). Next, integrating the above inequality again and using the fact that $Q_d(\cdot)$ is decreasing on (b_0, d) we derive
$\frac{\sigma^2}{2}Q_d(b_0) + \int_{b_0}^d \int_x^d C(u)dudx > -\frac{\theta}{2}[(d^2 - b_0^2) + (d - b_0)^2]Q_d(b_0)$.
But $Q_d(b_0) > 0$ and therefore we obtain

$$\left(\frac{\sigma^2}{2} + \theta d^2\right) Q_d(b_0) + \int_{b_0}^d \int_x^d C(u)dudx > 0. \tag{3.8}$$

Since $C(\cdot)$ is a strictly concave, strictly decreasing function and $C(b_0) < 0$, there are two constants k_0 and k_1 so that $k_1 > 0$ and
$C(x) < k_0 - k_1 x$ for all $x > b_0$. Therefore, we obtain the estimate

$$\int_{b_0}^d \int_x^d C(u)dudx < -\frac{k_1}{3}d^3 + \frac{k_0}{2}(d - b_0)^2 + \frac{k_1 b_0}{2}d^2.$$

Consequently, $\lim_{d\to\infty} \frac{1}{d^2}\int_{b_0}^d \int_x^d C(u)dudx = -\infty$. This together with (3.8) implies that $\lim_{d\to\infty} Q_d(b_0) = \infty$. Therefore, we can conclude that there is a point $l_0 > b_0$ so that for every $d > l_0$, $Q_d(b_0) > \frac{1}{\alpha}$. Now let $d > l_0$ and suppose that $Q_d(\cdot)$ intersects $V_0(\cdot)$ at some point in $(0, b_0)$. Then $Q_d(\cdot)$ attains a positive local maximum at some point ζ in $(0, b_0)$ and $Q_d(\cdot)$ is decreasing on (ζ, b_0). Then, by (3.7) we obtain $\frac{\sigma^2}{2}Q_d''(\zeta) + C(\zeta) = \alpha Q_d(\zeta)$. But $\alpha Q_d(\zeta) \geq \alpha Q_d(b_0) > C(0) > C(\zeta)$ and hence $Q_d''(\zeta) > 0$ and $Q_d(\cdot)$ cannot have a local maximum at $x = \zeta$. Consequently, $Q_d(\cdot)$ cannot intersect $V_0(\cdot)$ at any point in $(0, b_0)$, $Q_d(\cdot)$ is strictly decreasing on $(0, \infty)$ and $Q_d(x) > V_0(x)$ for all $x \geq 0$. This implies the proof of part (ii) of the lemma.

Now consider

$$d^* = \sup\{d > b_0 : \exists\ Q_d(\cdot) \text{ which satisfies (3.7) and intersects } V_0(\cdot)\} \tag{3.9}$$

By part (i) of the above lemma, the above set is non-empty and d^* is well defined. By part (ii) of the lemma, d^* is finite and $d^* < l_0$. Next, we consider the function $Q_{d^*}(\cdot)$ and show that its graph intersects the graph of $V_0(\cdot)$ tangentially.

Lemma 3.3. *Let d^* be as in (3.9) and consider the function $Q_{d^*}(\cdot)$ which satisfies (3.7) with $d = d^*$. Then the following results hold:*

(i) There is a point q^* in $(0, b_0)$ so that $Q_{d^*}(\cdot)$ intersects $V_0(\cdot)$ at the point q^* and $Q_{d^*}(\cdot)$ is a strictly decreasing convex function on the interval (q^*, d^*).

(ii) $Q'_{d^*}(q^*) = V'_0(q^*)$.

Proof. If $Q_{d^*}(\cdot)$ does not intersect $V_0(\cdot)$ in the interval $[0, b_0]$, then by the joint continuity of $Q_d(\cdot)$ in the variables (d, x), there is an $\epsilon > 0$ so that $Q_d(\cdot)$ does not intersect $V_0(\cdot)$ on $[0, b_0]$ for each d in $(d^* - \epsilon, d^*)$ and this contradicts with the definition of d^* in (3.9). Hence, $Q_{d^*}(\cdot)$ intersects $V_0(\cdot)$ at least once in the interval $[0, b_0]$. Now let

$$q^* = \sup\{z \text{ in } [0, b_0] : Q_{d^*}(z) = V_0(z)\}.$$

Then, $Q_{d^*}(q^*) = V_0(q^*)$, $Q_{d^*}(x) > V_0(x)$ on (q^*, d^*) and $Q'_{d^*}(q^*) \geq V'_0(q^*)$.
Let us introduce the function $P(x) = Q''_{d^*}(x)$ on $[q^*, d^*]$. Notice that

$$\begin{aligned} \frac{\sigma^2}{2} P(q^*) &= \theta q^* Q'_{d^*}(q^*) + \alpha Q_{d^*}(q^*) - C(q^*) \\ &\geq \theta q^* V'_0(q^*) + \alpha V_0(q^*) - C(q^*) = 0. \end{aligned}$$

By (3.7) and since $d^* > b_0$ we have $P(d^*) > 0$. We intend to show that the function $P(\cdot)$ is increasing on (q^*, d^*). Differentiating (3.7), we derive,

$$\begin{aligned} \frac{\sigma^2}{2} P'(q^*) &= \theta q^* P(q^*) + (\theta + \alpha) Q'_{d^*}(q^*) - C'(q^*) \\ &\geq (\theta + \alpha) V'_0(q^*) - C'(q^*) \\ &= -\theta q^* V''_0(q^*) > 0. \end{aligned} \tag{3.10}$$

Here, we have differentiated the differential equation for $V_0(\cdot)$ in Lemma 3.1 and and used it in the last equality of (3.10). Hence, $P(\cdot)$ is strictly increasing on an interval $(q^*, q^* + \epsilon)$ for some $\epsilon > 0$. Now suppose $P(\cdot)$ has a positive local maximum at some point $\zeta > q^*$ and $P(\cdot)$ is increasing on (q^*, ζ). Then $P(\zeta) > 0$ and $P'(\zeta) = 0$. Furthermore, using parts (ii) and (iii) of Lemma 3.1, we have $V''_0(\zeta) \leq 0$ and

$$Q'_{d^*}(\zeta) > Q'_{d^*}(q^*) \geq V'_0(\beta) > V'_0(\zeta).$$

Therefore,

$$\begin{aligned} \frac{\sigma^2}{2} P'(\zeta) &= \theta \zeta P(\zeta) + (\theta + \alpha) Q'_{d^*}(\zeta) - C'(\zeta) \\ &> \theta \zeta V''_0(\zeta) + (\theta + \alpha) V'_0(\zeta) - C'(\zeta) = 0. \end{aligned}$$

This is a contradiction and hence we can conclude that $P(\cdot)$ is increasing and $P(x) > 0$ on (q^*, d^*). Consequently, $Q_{d^*}(\cdot)$ is a strictly convex function on (q^*, d^*). Since $Q'_{d^*}(d^*) = 0$, it is also strictly decreasing on (q^*, d^*) as well.
Since $V_0(\cdot)$ is a strictly concave function, we can rule out the case $q^* = 0$. Otherwise, there is an $\epsilon > 0$ so that $Q_{d^*}(\cdot) < V_0(\cdot)$ on the interval $(0, \epsilon)$ and now using the joint continuity of $Q_d(x)$ in the variables d and x, we can find $d > d^*$ where $Q_d(\cdot)$ intersects with $V_0(\cdot)$. This is a contradiction and hence $q^* > 0$. Also, since $Q_{d^*}(b_0) > 0$, it is clear that $q^* < b_0$. This completes the proof of part (i).
Now if $Q'_{d^*}(q^*) > V'_0(q^*)$, since $Q_{d^*}(q^*) = V_0(q^*)$, we can find an $\epsilon > 0$ so that $Q_{d^*}(x) < V_0(x)$ for all x in $(\beta^* - \epsilon, q^*)$ and $\beta^* - \epsilon > 0$. Now again using the joint continuity of $Q_d(x)$ in (d, x), we can find $d > d^*$ so that $Q_d(\cdot)$ intersects $V_0(\cdot)$. Hence, we conclude that $Q'_{d^*}(q^*) = V'_0(q^*)$. This completes the proof of the lemma.

By a similar argument, there exist points $c^* < p^* < 0$ and a function $Q_{c^*}(\cdot)$ on $(-\infty, 0)$ satisfying the following: $Q_{c^*}(\cdot)$ is a solution to (3.7) with boundary conditions $Q_{c^*}(c^*) = Q'_{c^*}(c^*) = 0$; $Q_{c^*}(p^*) = V_0(p^*)$, $Q'_{c^*}(p^*) = V'_0(p^*)$, and $Q_{c^*}(\cdot)$

is a strictly increasing convex function on (c^*, p^*).
Theorem 1.1 will follow immediately if we show that the value function for the control problem is given by

$$V^*(x) = \begin{cases} V_0(x) & \text{for} \quad p^* \leq x \leq q^* \\ Q_{c^*}(x) & \text{for} \quad c^* \leq x \leq p^* \\ Q_{d^*}(x) & \text{for} \quad q^* \leq x \leq d^* \\ 0 & \quad \text{otherwise.} \end{cases} \tag{3.11}$$

Proof of Theorem 1.1.
The function $V^*(\cdot)$ is continuously differentiable by Lemma 3.3 and by the discussion above (3.11). Furthermore, $V^{*''}(\cdot)$ is continuous everywhere except at the points c^*, p^*, q^* and d^*. Also, it is easy to check that the one-sided derivatives $V^{*''}(x+)$ and $V^{*''}(x-)$ exists everywhere. Since, $Q_{d^*}(\cdot)$ and $Q_{c^*}(\cdot)$ are convex functions which satisfy the differential equation in (3.7) and since $V_0(\cdot)$ is a concave function which satisfies the differential equation in Lemma 3.1, it is a straight forward computation to check that $V^*(\cdot)$ satisfies all the assumptions in the Proposition 2.4. Therefore, we can conclude that $V^*(x) \geq V(x)$ for all x.

We can apply Itô's lemma to verify that the pay-off function from the admissible control strategy (u^*, τ^*) is indeed $V^*(\cdot)$ and hence $V^*(x) = V(x)$ for all x. This completes the proof.

References

[1] Assaf, D. (1997). Estimating the state of a noisy continuous time Markov chain when dynamic sampling is feasible. *Ann. Appl. Probab.* **3** 822–836.

[2] Davis, M. H. A. and Zervos, M. (1994). A problem of singular stochastic control with discretionary stopping. *Ann. Appl. Probab.* **4** 226–240.

[3] Dubins, L. E. and Savage, L. J. (1965). *How To Gamble If You Must: Inequalities for Stochastic Processes*. McGraw-Hill Book Co., New York.

[4] Karatzas, I., Ocone, D., Wang, H. and Zervos, M. (2000). Finite fuel singular control with discretionary stopping. *Stochastics* **71** 1–50.

[5] Karatzas, I. and Shreve, S. E. (1988). *Brownian motion and stochastic calculus*. Springer-Verlag, New York.

[6] Karatzas, I. and Wang, H. (2000). Utility maximization with discretionary stopping. *SIAM J. Control and Opt.* **39** 306–329.

[7] Karatzas, I. and Zamfirescu, I.-M. (2006). Martingale approach to stochastic control with discretionary stopping. *Appl. Math. Optim.* **53** 163–184.

[8] Kurtz, T. G. and Stockbridge, R. H. (1998). Existence of Markov controls and characterization of optimal Markov controls. *SIAM J. Control and Opt.* **36** 609–653.

[9] Morimoto, H. (2003). Variational inequalities for combined control and stopping. *SIAM J. Control and Opt.* **42** 686–708.

[10] Ocone, D. and Weerasinghe, A. (2000). Degenerate variance control of a one dimensional diffusion process. *SIAM J. Control and Opt.* **39** 1–24.

[11] Ocone, D. and Weerasinghe, A. (2003). Degenerate variance control in the one dimensional stationary case. *Elec. J. of Probab.* **8** 1–27.

[12] Øksendal, B. (2000). *Stochastic differential equations*, fifth edition. Springer-Verlag, New York.

[13] Weerasinghe, A. (2006). A controller and a stopper game with degenerate variance control. *Elec. Comm. of Probab.* **11** 89–99.