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THE ENGEL ELEMENTS IN GENERALIZED FC-GROUPS

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Abstract. We generalize to FC^*, the class of generalized FC-groups introduced in [*Serdica Math. J.* **28** (2002) 241–254], a result of Baer on Engel elements. More precisely, we prove that the sets of left Engel elements and bounded left Engel elements of an FC^*-group G coincide with the Fitting subgroup; whereas the sets of right Engel elements and bounded right Engel elements of G are subgroups and the former coincides with the hypercentre. We also give an example of an FC^*-group for which the set of right Engel elements contains properly the set of bounded right Engel elements.

1. Introduction

Let n be a positive integer and x, y be elements of a group G. The commutator $[x,_n y]$ is defined inductively by the rules

$$[x,_1 y] = x^{-1}x^y \quad \text{and, for } n \geq 2, \quad [x,_n y] = \big[[x,_{n-1} y], y\big].$$

An element $a \in G$ is called a *left Engel element* if for any $g \in G$ there exists $n = n(a,g) \geq 1$ such that $[g,_n a] = 1$. If n can be chosen independently of g, then a is called a *left n-Engel element*. Moreover, a is a *bounded left Engel element* if it is left n-Engel for some $n \geq 1$. Similarly, an element $a \in G$ is called a *right Engel element* if the variable g appears on the right, that is, for any $g \in G$ there exists $n = n(a,g) \geq 1$ such that $[a,_n g] = 1$; in addition, if $n = n(a)$, then a is a *right n-Engel element* or simply a *bounded right Engel element*. By a well-known result of Heineken [11, Theorem 7.11], the inverse of any right Engel element is a left Engel element and the inverse of any right n-Engel element is a left $(n+1)$-Engel element.

Following [11], we denote by $L(G)$ and $\overline{L}(G)$ the sets of left Engel elements and bounded left Engel elements of G, respectively; and by $R(G)$ and $\overline{R}(G)$

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the sets of right Engel elements and bounded right Engel elements of G, respectively. Thus,

$$R(G)^{-1} \subseteq L(G) \quad \text{and} \quad \overline{R}(G)^{-1} \subseteq \overline{L}(G). \tag{1}$$

It is also clear that these four subsets are invariant under automorphisms of G, but it is still unknown whether they are subgroups. This is a very long-standing problem, even if Bludov announced recently that there exists a group G for which $L(G)$ is not a subgroup [3].

We mention that $L(G)$ contains the Hirsch–Plotkin radical $HP(G)$ of G and $\overline{L}(G)$ contains the Baer radical $B(G)$ of G; whereas, $R(G)$ contains the hypercentre $\overline{Z}(G)$ of G and $\overline{R}(G)$ contains $Z_\omega(G)$, the ω-hypercentre of G [11, Lemma 7.12]. Recall that $HP(G)$ is the unique maximal normal locally nilpotent subgroup containing all normal locally nilpotent subgroups of G [11, Part 1, p. 58]; and $B(G)$ is the subgroup generated by all elements $x \in G$ such that $\langle x \rangle$ is subnormal in G. Notice also that, by a famous example of Golod [6], $L(G)$ can be larger than $HP(G)$. However, if G is a soluble group, then $L(G) = HP(G)$ and $\overline{L}(G) = B(G)$ [11, Theorem 7.35]. This latter result is due to Gruenberg, who also proved that in this case $R(G)$ and $\overline{R}(G)$ are always subgroups and that there exists a soluble group G such that $Z_\omega(G) \subset \overline{R}(G), \overline{Z}(G) \subset R(G)$ and $\overline{R}(G) \subset R(G)$ [7]. On the other hand, a remarkable theorem of Baer shows that groups satisfying the maximal condition have a fine Engel structure:

Theorem 1.1 (see Theorem 7.21 of [11]). *Let G be a group which satisfies the maximal condition. Then $L(G)$ and $\overline{L}(G)$ coincide with the Fitting subgroup of G, and $R(G)$ and $\overline{R}(G)$ coincide with the hypercentre of G, which equals $Z_k(G)$ for some finite k.*

There are a series of wide generalizations of Theorem 1.1 (see [11, 7.2 and 7.3] and [1], [2] for an account). For instance, in [10], Plotkin proved that $L(G) = HP(G)$ and $R(G)$ is a subgroup whenever G is a group with an ascending series whose factors satisfy max locally (i.e., every finitely generated subgroup has the maximal condition).

The aim of this note is to extend Theorem 1.1 to the class of FC^n-groups, which has been introduced in [5] as follows. Let FC^0 be the class of finite groups, and suppose by induction hypothesis that for some positive integer n a group class FC^{n-1} has been defined. A group G is called an *FC^n-group* if for any element $x \in G$ the factor group $G/C_G(x^G)$ belongs to the class FC^{n-1}, where x^G is the normal closure of $\langle x \rangle$ in G. It is easy to see that the set $FC^n(G) = \{x \in G | G/C_G(x^G) \text{ is an } FC^{n-1}\text{-group}\}$ is a subgroup of G, the so-called *FC^n-centre* of G. Hence, G is an FC^n-group if and only if $G = FC^n(G)$. Of course, FC^1 is the class of FC-groups, namely groups with finite conjugacy classes. More generally, a group is an *FC^*-group* if it is an FC^n-group for some $n \geq 0$.

The investigation of properties, that are common to finite groups and nilpotent groups, has been satisfactory for FC^*-groups [5], [12], [13], [9]. It turns out that every finite-by-nilpotent group is an FC^*-group and, conversely, every FC^*-group is locally (finite-by-nilpotent) [5, Proposition 3.6]. A group G is said to be *extended residually finite*, or briefly an *ERF-group*, if every subgroup is closed in the profinite topology, that is, every subgroup of G is an intersection of subgroups of finite index. A complete classification of ERF-groups in the class of FC^*-groups is given in [12].

In Section 2, we prove that if G is an FC^n-group, then $L(G)$ and $\overline{L}(G)$ coincide with the Fitting subgroup of G; whereas $R(G)$ and $\overline{R}(G)$ are subgroups of G and, in particular, $R(G)$ coincides with the hypercentre of G, which equals $Z_{\omega+(n-1)}(G)$. It remains an open question whether $\overline{R}(G)$ coincides with the ω-hypercentre, when G is an FC^n-group. Nevertheless, we show that $R(G)=\overline{R}(G)=Z_\omega(G)$ under the additional assumption that G is a periodic ERF-group. We also give an example of a non-periodic FC^2-group G such that G is ERF and $\overline{R}(G)\subset R(G)$.

2. The results

Given an arbitrary group X, we denote by $F(X)$ the Fitting subgroup of X.

Lemma 2.1. *Let G be an FC^n-group. Then the normal closure of any left Engel element of G is nilpotent and, consequently,*

$$L(G)=\overline{L}(G)=F(G).$$

Proof. Let $a\in L(G)$. By [5, Lemma 3.7] the quotient group $a^G/Z_n(a^G)$ is finite. Applying Theorem 1.1, we have

$$L\big(a^G/Z_n\big(a^G\big)\big)=F\big(a^G/Z_n\big(a^G\big)\big)=F\big(a^G\big)/Z_n\big(a^G\big),$$

where $F(a^G)$ is nilpotent because so is $F(a^G/Z_n(a^G))$. From $aZ_n(a^G)\in L(a^G/Z_n(a^G))$, we get $a\in F(a^G)$. But $F(a^G)$ is characteristic in a^G and hence normal in G. Thus $a^G=F(a^G)$ and a^G is nilpotent. In particular $a\in F(G)$, that is $L(G)\subseteq F(G)$. It follows that $L(G)=\overline{L}(G)=F(G)$, because $F(G)\subseteq B(G)\subseteq \overline{L}(G)\subseteq L(G)$ [11, Lemma 7.12]. □

We recall that any FC^*-group G satisfies max locally [5, Proposition 3.6] and therefore, in according to Plotkin [10], the set of right Engel elements of G is always a subgroup.

Lemma 2.2. *Let G be an FC^n-group and $a\in\gamma_n(G)\cap R(G)$. Then $a\in Z_k(G)$ for some $k=k(a)$.*

Proof. Let N be the normal closure of a in G. Since $\gamma_n(G)$ is contained in the FC-centre of G [5, Theorem 3.2], we have that $G/C_G(N)$ is finite. Then $G=HC_G(N)$, where H is a finitely generated subgroup of G.

Now HN is finitely generated and so it satisfies the maximal condition, by [5, Proposition 3.6]. Hence, Theorem 1.1 shows that $R(HN) = Z_k(HN)$ for some k. But $N \le R(G)$, so that $N \le R(HN) = Z_k(HN)$. For any $1 \le i \le k$, let $g_i = x_i h_i \in G$ with $x_i \in C_G(N)$ and $h_i \in H$. Thus $[a, g_1, \dots, g_k] = [a, h_1, \dots, h_k] = 1$ and $a \in Z_k(G)$, as desired. □

Let G be a group. Following [7], we denote by $\rho(G)$ the set of all elements $a \in G$ such that $\langle x\rangle$ is ascendant in $\langle x, a^G\rangle$, for any $x \in G$; and by $\overline{\rho}(G)$ the set of all elements $a \in G$ such that $\langle x\rangle$ is subnormal in $\langle x, a^G\rangle$ of defect at most $k = k(a)$, for any $x \in G$. By [11, Lemma 7.31], the sets $\rho(G)$ and $\overline{\rho}(G)$ are characteristic subgroups of G satisfying the following inclusions:

$$\overline{Z}(G) \subseteq \rho(G) \subseteq R(G) \quad \text{and} \quad Z_\omega(G) \subseteq \overline{\rho}(G) \subseteq \overline{R}(G). \tag{2}$$

The subgroups $\rho(G)$ and $R(G)$ can be different (see, for instance, [11, Part 2, p. 59]) and it is possible that $\overline{Z}(G) = 1$ and $\rho(G) = R(G) \ne 1$ [7]. In contrast with this, for FC^n-groups, we have the following lemma.

Lemma 2.3. *Let G be an FC^n-group. Then*
(i) $R(G) = \rho(G) = \overline{Z}(G) = Z_{\omega+(n-1)}(G)$;
(ii) $\overline{R}(G) = \overline{\rho}(G)$.
In particular, if G is an FC-group, then

$$R(G) = \overline{R}(G) = Z_\omega(G).$$

Proof. (i) Clearly $Z_{\omega+(n-1)}(G) \subseteq \overline{Z}(G) \subseteq R(G)$, by (2). Let $a \in R(G)$. As $R(G)$ is normal in G, for any $x_1, \dots, x_{n-1} \in G$, we have $[a, x_1, \dots, x_{n-1}] \in \gamma_n(G) \cap R(G)$ which is contained in $Z_\omega(G)$, by Lemma 2.2. Hence $a \in Z_{\omega+(n-1)}(G)$ and $R(G) \subseteq Z_{\omega+(n-1)}(G)$.

(ii) Let $a \in \overline{R}(G)$. By Lemma 2.1, jointly with (1), we have that a^G is nilpotent. It follows that $a \in \overline{\rho}(G)$, by [8, Theorem 1.6], and $\overline{R}(G) \subseteq \overline{\rho}(G)$. Thus $\overline{R}(G) = \overline{\rho}(G)$, by (2). □

A group G is called an *Engel group* if $R(G) = G$ or, equivalently, $L(G) = G$. Of course locally nilpotent groups are Engel, but Golod's example [6] shows that Engel groups need not be locally nilpotent. As a consequence of Lemma 2.3(i), every Engel FC^n-group is hypercentral and its upper central series has length at most $\omega + (n-1)$ (compare with [5, Theorem 3.9(b)]). Moreover, this bound cannot be replaced by ω when $n > 1$, see Example 2.7.

By combining Lemma 2.1 and Lemma 2.3, our main result follows.

Theorem 2.4. *Let G be an FC^n-group. Then $L(G)$ and $\overline{L}(G)$ coincide with the Fitting subgroup of G; whereas $R(G) = \rho(G)$ coincides with the hypercentre of G, which equals $Z_{\omega+(n-1)}(G)$, and $\overline{R}(G) = \overline{\rho}(G)$.*

The respective position of these subgroups is indicated in the following diagram (see also the diagram in [11, Part 2, p. 63]).

$$\begin{tikzcd}
F(G)=\overline{L}(G)=L(G) \arrow[d, dash] & \\
Z_{\omega+(n-1)}(G)=\overline{Z}(G)=\rho(G)=R(G) \arrow[dd, dash] \arrow[dr, dash] & \\
& \overline{\rho}(G)=\overline{R}(G) \arrow[dl, dash] \\
Z_\omega(G) &
\end{tikzcd}$$

Notice that if G is a finitely generated FC^*-group, then G is finite-by-nilpotent [5, Proposition 3.6] and so, by the next result, $R(G)$ and $\overline{R}(G)$ coincide with the ω-hypercentre of G.

Proposition 2.5. *Let G be a finite-by-nilpotent group. Then*

$$R(G)=\overline{R}(G)=Z_\omega(G).$$

Proof. By (2), we have $Z_\omega(G)\subseteq\overline{R}(G)\subseteq R(G)$. Since G is finite-by-nilpotent, there exists $i\geq 0$ such that $G/Z_i(G)$ is finite [11, Theorem 4.25]. Then, by Theorem 1.1, we have

$$R(G)Z_i(G)/Z_i(G)\subseteq R\big(G/Z_i(G)\big)=Z_j\big(G/Z_i(G)\big)=Z_{i+j}(G)/Z_i(G)$$

for some $j\geq 0$. It follows that $R(G)\subseteq Z_{i+j}(G)$, so that $R(G)\subseteq Z_\omega(G)$. $\square$

In the sequel, we restrict our attention to FC^*-groups belonging to the class of ERF-groups. Let G be any FC^n-group and denote by $T(G)$ its torsion subgroup [5, Corollary 3.3]. By [12, Theorem 3.6], the group G is ERF if and only if the following conditions hold:

(i) Sylow subgroups of G are Abelian-by-finite with finite exponent;
(ii) Sylow subgroups of $\gamma_{n+1}(G)$ are finite;
(iii) $G/T(G)$ is torsion-free nilpotent of finite rank and no quotient of its subnormal subgroups is of p^∞-type for any prime p.

Proposition 2.6. *Let G be an FC^*-group which is ERF. Then every periodic right Engel element of G belongs to $Z_k(G)$ for some $k=k(a)$. Hence, if G is periodic, then*

$$R(G)=\overline{R}(G)=Z_\omega(G).$$

Proof. First notice that, if N is a finite subgroup of G of order m contained in $Z_i(G)$ for some $i\geq 1$, then $N\leq Z_m(G)$. This is true for any arbitrary group and its proof is a straightforward induction on m.

Let a be any nontrivial right Engel element of G. We may assume that a is a p-element, where p is prime. With $x_1,\dots,x_n\in G$, by Lemma 2.2, we have $[a,x_1,\dots,x_n]\in Z_i(G)$ for some i. Suppose $[a,x_1,\dots,x_n]\neq 1$ and denote by N

the normal closure of $[a, x_1, \ldots, x_n]$ in G. Then $N \leq P \cap Z_i(G)$, where $i \geq 1$ and P is a Sylow p-subgroup of $\gamma_{n+1}(G)$. Now P is finite, say of order m. Then the previous remark implies that $N \leq Z_m(G)$ and so $[a, x_1, \ldots, x_n] \in Z_m(G)$. But Sylow p-subgroups of G are isomorphic [12, Theorem 3.9] and so m is independent of $x_1, \ldots, x_n$. Hence, $a \in Z_k(G)$ where $k = m + n$. $\square$

Next we show that, in our context, the set of bounded right Engel elements can be properly contained in the set of right Engel elements.

EXAMPLE 2.7. There exists a non-periodic met-Abelian FC^2-group G such that

$$\overline{R}(G) = Z_\omega(G) \quad \text{and} \quad R(G) = Z_{\omega+1}(G) = G.$$

Further, $Z_\omega(G)$ is periodic and G is an ERF-group.

Proof. Let $p_1 < p_2 < \cdots$ be a sequence of odd primes and $1 < n_1 < n_2 < \cdots$ be a sequence of integers. For any $i \geq 1$, put

$$P_i = \langle a_i, b_i \rangle,$$

where a_i has order $p_i^{n_i}$, b_i has order $p_i^{n_i - 1}$ and $a_i^{b_i} = a_i^{1+p_i}$. Then $[a_i, b_i] = a_i^{p_i}$ and therefore, for any $m \geq 1$, we have $[a_i, {}_m b_i] = a_i^{p_i^m}$. In particular $[a_i, {}_{n_i} b_i] = 1$ and, consequently, the commutator $[a_i, {}_{n_i - 1} b_i]$ is a nontrivial element of $Z(P_i)$. This leads to $[a_i, b_i] \in Z_{n_i - 1}(P_i)$, so that $P_i = \langle a_i, b_i \rangle$ is nilpotent of class exactly n_i.

Let $b_i^j a_i^k$ be an arbitrary element of P_i, with $0 \leq j < p_i^{n_i - 1}$ and $0 \leq k < p_i^{n_i}$. By [4, Lemma 4], the map α_i defined by

$$\left(b_i^j a_i^k\right)^{\alpha_i} = \left(b_i a_i^p\right)^j a_i^k$$

is an automorphism of P_i. Clearly, $a_i^{\alpha_i} = a_i$ and $b_i^{\alpha_i} = b_i a_i^{p_i}$.

Now form the semidirect product

$$G = \langle x \rangle \ltimes P,$$

where $\langle x \rangle$ is infinite cyclic, $P = Dr_{i \geq 1} P_i$ and

$$a_i^x = a_i, \qquad b_i^x = b_i a_i^{p_i}.$$

If $A = Dr_{i \geq 1} \langle a_i \rangle$, then G/A is Abelian and $A \leq C_G(x)$. It follows that $C_G(x^G) = C_G(x)$ and $G/C_G(x^G)$ is Abelian, that is $x \in FC^2(G)$. On the other hand y^G is finite for any $y \in P$. Then so is $G/C_G(y^G)$, which embeds in $Aut(y^G)$. Hence $P \leq FC(G) \leq FC^2(G)$ and $G = FC^2(G)$, namely G is an FC^2-group. Of course G is ERF by construction. Notice also that $R(G) = Z_{\omega+1}(G)$, by Lemma 2.3. But $a_i \in Z_{n_i}(G)$, so that $A \leq Z_\omega(G)$ and $Z_{\omega+1}(G) = G$.

Finally let $g = x^r y$ be an arbitrary element of $\overline{R}(G)$, where $r \in \mathbb{Z}$ and $y \in P$. Since y is a periodic (right Engel) element, then $y \in Z_\omega(G) \subseteq \overline{R}(G)$ by Proposition 2.6. It follows that x^r is an m-right Engel element, for some $m \geq 1$. From $[b_i, x^r] = a_i^{r p_i}$, we get $1 = [x^r, {}_m b_i]^{-1} = (a_i^r)^{p_i^m}$, for any i. This forces $r = 0$ and therefore $g = y \in P \cap Z_\omega(G)$. We conclude that $\overline{R}(G) = Z_\omega(G) \leq P$. □

It is well known that, if G is an FC-group, then $G/Z(G)$ is periodic. This fails for FC^*-groups, even if they are ERF: there exists a non-periodic FC^2-group with trivial centre which is ERF [12, Example 4.4]. One more example, but with nontrivial centre, is then the group given in Example 2.7.

References

[1] A. Abdollahi, *Engel elements in groups*, Groups St Andrews 2009 in Bath, vol. 1, London Math. Soc. Lecture Note Ser., vol. 387, Cambridge University Press, Cambridge, 2011, pp. 94–117. MR 2858851

[2] A. Abdollahi, R. Brandl and A. Tortora, *Groups generated by a finite Engel set*, J. Algebra **347** (2011), 53–59. MR 2846396

[3] V. Bludov, *An example of not Engel group generated by Engel elements*, abstracts of “A conference in honor of Adalbert Bovdi's 70th birthday”, November 18–23, 2005; available at `www.math.klte.hu/bovdiconf/abstracts.pdf`.

[4] R. M. Davitt, *The automorphism group of a finite metacyclic p-group*, Proc. Amer. Math. Soc. **25** (1970), 876–879. MR 0285594

[5] F. de Giovanni, A. Russo and G. Vincenzi, *Groups with restricted conjugacy classes*, Serdica Math. J. **28** (2002), 241–254. MR 1952010

[6] E. S. Golod, *Some problems of Burnside type*, Proc. Internat. Congr. Math. (Moscow, 1966), Izdat. “Mir”, Moscow, 1968, pp. 284–289. Amer. Math. Soc. Transl. (2) **84** (1969), 83–88. MR 0238880

[7] K. W. Gruenberg, *The Engel elements of a soluble group*, Illinois J. Math. **3** (1959), 151–168. MR 0104730

[8] K. W. Gruenberg, *The upper central series in soluble groups*, Illinois J. Math. **5** (1961), 436–466. MR 0136657

[9] G. Kaplan and G. Vincenzi, *On the Wielandt subgroup of generalized FC-groups*, Internat. J. Algebra Comput. **24** (2014), 1031–1042. MR 3286149

[10] B. I. Plotkin, *Radicals and nil-elements in groups*, Izv. Vyssh. Uchebn. Zaved. Mat. **1** (1958), 130–135. MR 0125880

[11] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups. Part 1 and Part 2*, Springer, Berlin, 1972. MR 0332989

[12] D. J. S. Robinson, A. Russo and G. Vincenzi, *On the theory of generalized FC-groups*, J. Algebra **326** (2011), 218–226. MR 2746062

[13] E. Romano and G. Vincenzi, *Pronormality in generalized FC-groups*, Bull. Aust. Math. Soc. **83** (2011), 220–230. MR 2784779

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