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SASAKI MANIFOLDS, KÄHLER CONE MANIFOLDS AND BIHARMONIC SUBMANIFOLDS

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Abstract. For a Legendrian submanifold M of a Sasaki manifold N, we study harmonicity and biharmonicity of the corresponding Lagrangian cone submanifold $C(M)$ of a Kähler manifold $C(N)$. We show that, if $C(M)$ is biharmonic in $C(N)$, then it is harmonic; and M is proper biharmonic in N if and only if $C(M)$ has a nonzero eigen-section of the Jacobi operator with the eigenvalue $m = \dim M$.

1. Introduction

Harmonic maps play a central role in geometry; they are critical points of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$ for smooth maps φ of (M,g) into (N,h). The Euler–Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. In 1983, J. Eells and L. Lemaire [8] extended the notion of harmonic map to biharmonic map, which are, by definition, critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M \left|\tau(\varphi)\right|^2 v_g. \tag{1.1}$$

After G. Y. Jiang [15] studied the first and second variation formulas of E_2, extensive studies in this area have been done (see [2], [4], [17], [18], [20], [25], [26], [11], [12], [30], etc.). Notice that harmonic maps are always biharmonic by definition. We say, for a smooth map $\varphi : (M,g) \to (N,h)$ to be *proper biharmonic* if it is biharmonic, but not harmonic. B. Y. Chen raised ([6]) so called B. Y. Chen's conjecture and later, R. Caddeo, S. Montaldo, P. Piu and C. Oniciuc raised ([4]) the generalized B. Y. Chen's conjecture.

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B. Y. CHEN'S CONJECTURE. *Every biharmonic submanifold of the Euclidean space* $\mathbb{R}^n$ *must be harmonic* (*minimal*).

THE GENERALIZED B. Y. CHEN'S CONJECTURE. *Every biharmonic submanifold of a Riemannian manifold of nonpositive curvature must be harmonic* (*minimal*).

For the generalized Chen's conjecture, Ou and Tang gave ([25]) a counter example in a Riemannian manifold of negative curvature. For the Chen's conjecture, affirmative answers were known for the case of surfaces in the three dimensional Euclidean space ([6]), and the case of hypersurfaces of the four dimensional Euclidean space ([10], [7]). Furthermore, Akutagawa and Maeta gave ([1]) recently a final supporting evidence to the Chen's conjecture.

THEOREM 1.1. *Any complete regular biharmonic submanifold of the Euclidean space* $\mathbb{R}^n$ *is harmonic* (*minimal*).

To the generalized Chen's conjecture, we showed ([24]) the following theorem.

THEOREM 1.2. *Let* (M,g) *be a complete Riemannian manifold, and the curvature of* (N,h), *nonpositive. Then,*

(1) *every biharmonic map* $\varphi\colon (M,g)\to(N,h)$ *with finite energy and finite bienergy must be harmonic.*
(2) *In the case* $\mathrm{Vol}(M,g)=\infty$, *under the same assumption, every biharmonic map* $\varphi\colon (M,g)\to(N,h)$ *with finite bienergy is harmonic.*

We also obtained (cf. [22], [23], [24]):

THEOREM 1.3. *Assume that* (M,g) *is a complete Riemannian manifold,* $\varphi\colon (M,g)\to(N,h)$ *is an isometric immersion, and the sectional curvature of* (N,h) *is nonpositive. If* $\varphi\colon (M,g)\to(N,h)$ *is biharmonic and* $\int_M |\mathbf{H}|^2 v_g<\infty$, *then it is minimal. Here,* $\mathbf{H}$ *is the mean curvature normal vector field of the isometric immersion* φ.

Theorem 1.3 gives an affirmative answer to the generalized B. Y. Chen's conjecture under the L^2-condition and completeness of (M,g).

In this paper, for every Legendrian submanifold $\varphi\colon (M^m,g)\to(N^{2m+1},h)$ of a Sasaki manifold (N^{2m+1},h), and the Lagrangian cone submanifold $\overline{\varphi}\colon (C(M),\overline{g})\to(C(N),\overline{h})$ of a Kähler cone manifold $(C(N),\overline{h})$, we show (Theorems 3.3 and 4.4) that (1) $\overline{\varphi}\colon (C(M),\overline{g})\to(C(N),\overline{h})$ is biharmonic if and only if it is harmonic, which is equivalent to that $\varphi\colon (M,g)\to(N,h)$ is harmonic. (2) $\varphi\colon (M,g)\to(N,h)$ is proper biharmonic if and only if $\tau(\overline{\varphi})$ is a nonzero eigen-section of the Jacobi operator $J_{\overline{\varphi}}$ with the eigenvalue $m=\dim M$. The assertion (2) can be regarded as a biharmonic map version of T. Takahashi's theorem (cf. Theorem 4.5) which claims that each coordinate function of the isometric immersion of (M^m,g) into the unit sphere

$S^n \hookrightarrow \mathbb{R}^{n+1}$ is the eigenfunction of the Laplacian of (M,g) with the eigenvalue $m = \dim M$.

2. Preliminaries

We first prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M,g) \to (N,h)$, of a compact Riemannian manifold (M,g) into another Riemannian manifold (N,h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi) v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M h\big(\tau(\varphi), V\big) v_g = 0, \tag{2.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$ $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M,g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X,Y) &= (\widetilde{\nabla}\, d\varphi)(X,Y) \\ &= (\widetilde{\nabla}_X\, d\varphi)(Y) \\ &= \overline{\nabla}_X\big(d\varphi(Y)\big) - d\varphi(\nabla_X Y), \end{aligned} \tag{2.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇, and ∇^N, are Levi–Civita connections on TM, TN of (M,g), (N,h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), φ is *harmonic* if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M h\big(J(V), V\big) v_g, \tag{2.3}$$

where J is an elliptic differential operator, called the *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{2.4}$$

where $\overline{\Delta} V = \overline{\nabla}^*\overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i))\, d\varphi(e_i)$, and R^N is the curvature tensor of (N,h) given by $R^N(U,V) = \nabla^N_U \nabla^N_V - \nabla^N_V \nabla^N_U - \nabla^N_{[U,V]}$ for $U, V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [8] proposed polyharmonic (k-harmonic) maps and Jiang [15] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2}\int_M \big|\tau(\varphi)\big|^2 v_g, \tag{2.5}$$

where $|V|^2 = h(V,V)$, $V \in \Gamma(\varphi^{-1}TN)$.

The first variation formula of the bienergy functional is given by

$$\frac{d}{dt}\bigg|_{t=0} E_2(\varphi_t) = -\int_M h\big(\tau_2(\varphi), V\big) v_g. \tag{2.6}$$

Here,

$$\tau_2(\varphi) := J\big(\tau(\varphi)\big) = \overline{\Delta}\big(\tau(\varphi)\big) - \mathcal{R}\big(\tau(\varphi)\big), \tag{2.7}$$

which is called the *bitension field* of φ, and J is given in (2.4).

A smooth map φ of (M,g) into (N,h) is said to be *biharmonic* if $\tau_2(\varphi) = 0$. By definition, every harmonic map is biharmonic. We say, for an immersion $\varphi : (M,g) \to (N,h)$ to be *proper biharmonic* if it is biharmonic but not harmonic (minimal).

3. Legendrian submanifolds and Lagrangian submanifolds

In this section, we first show a correspondence between the set of all Legendrian submanifolds of a Sasakian manifold and the one of all Lagrangian submanifolds of a Kähler cone manifold.

An n $(= 2m+1)$ dimensional contact Riemannian manifold (N,h) with a contact form η is said to be a *contact metric manifold* if there exist a smooth $(1,1)$ tensor field J and a smooth vector field ξ on N, called a *basic vector field*, satisfying that

$$J^2 = -\mathrm{Id} + \eta \otimes \xi, \tag{3.1}$$

$$\eta(\xi) = 1, \tag{3.2}$$

$$J\xi = 0, \tag{3.3}$$

$$\eta \circ J = 0, \tag{3.4}$$

$$h(JX, JY) = h(X,Y) - \eta(X)\eta(Y), \tag{3.5}$$

$$\eta(X) = h(X, \xi), \tag{3.6}$$

$$d\eta(X,Y) = h(X, JY), \tag{3.7}$$

for all smooth vector fields X, Y on N. Here, Id is the identity transformation of T_xN $(x \in N)$. A contact metric manifold (N,h,J,ξ,η) is *Sasakian* if $(C(N), \overline{h}, I)$ is a Kähler manifold. Here, a cone manifold $C(N) := N \times \mathbb{R}^+$ where $\mathbb{R}^+ := \{r \in \mathbb{R} | r > 0\}$, $\overline{h}$ is a cone metric on $C(N)$, $\overline{h} := dr^2 + r^2 h$, which is a Hermitian metric with respect to an almost complex structure I on $C(N)$ given by

$$\begin{cases} IY := JY + \eta(Y)\Psi \quad (Y \in \mathfrak{X}(N)), \\ I\Psi := -\xi, \end{cases} \tag{3.8}$$

where $\Psi := r\frac{\partial}{\partial r}$ is called the *Liouville vector field* on $C(N)$. We denote by $\mathfrak{X}(N)$, the set of all smooth vector fields on N. A contact metric manifold (N, h, J, ξ, η) is Sasakian if and only if

$$\big(\nabla^N_X J\big)(Y) = h(X,Y)\xi - \eta(Y)X \quad \big(X, Y \in \mathfrak{X}(N)\big). \tag{3.9}$$

Let us recall the definition below.

DEFINITION 3.1. Let M^m be an m-dimensional manifold, an immersion $\varphi : M^m \to N^{2m+1}$. M^m is called to be a *Legendrian* submanifold of an $(2m+1)$-dimensional Sasakian manifold (N, h, J, ξ, η) (cf. [3], [16], [26]) if $\varphi^*\eta \equiv 0$ which is equivalent to that

$$\varphi_{*x}(X_x) \in \mathrm{Ker}(\eta_{\varphi(x)}) \tag{3.10}$$

for all $X_x \in T_xM$ $(x \in M)$.

A Legendrian submanifold M^m satisfies the following two conditions:

(1) $\varphi_*(T_xM)$ is orthogonal $J(\varphi_*(T_xM))$ with respect to h for all $x \in M$. This is equivalent to that the normal bundle $T^\perp M$ of $\varphi : M \to N$ has the following splitting:

$$T_xM^\perp = \mathbb{R}\xi_{\varphi(x)} \oplus J\varphi_* T_xM \quad (x \in M).$$

(2) The second fundamental form B of $\varphi(M) \subset N$ has its value at $\mathrm{Ker}(\eta)$, that is,

$$B(\varphi_*X, \varphi_*Y) = \nabla^N_X \varphi_*Y - \varphi_*(\nabla_X Y) \in \varphi_*(T_xM)^\perp,$$

where $T_xM^\perp$ is $\varphi_*(T_xM)^\perp$, which is

$$\big\{W_{\varphi(x)} \in T_{\varphi(x)}N | h(W_{\varphi(x)}, \varphi_{*x}X_x) = 0 \ (\forall X_x \in T_xM)\big\}.$$

Here, ∇, ∇^N are Levi–Civita connections of (M, g), (N, h) where g is the induced metric on M by $g := \varphi^*h$.

In the following, we identify $\varphi(M)$ with M, itself. The following theorem is well known, but essentially important for us.

THEOREM 3.2. *Let M^m be an m-dimensional submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$. Then, M is a Legendrian submanifold of a Sasaki manifold N if and only if $C(M) \subset C(N)$ is a Lagrangian submanifold of a Kähler cone manifold $(C(N), \overline{h}, I)$.*

Proof. We have the equivalence that $M \subset N$ is Legendrian if and only if

$$\begin{cases} \xi_x^{\perp} = T_xM \oplus JT_xM, \\ h(T_xM, JT_xM) = \{0\} \end{cases} \tag{3.11}$$

for all $x \in M$. That is, $h(\xi, X) = 0$ and $h(X, JY) = 0$ for all X, $Y \in \mathfrak{X}(M)$. Then, (3.11) is equivalent to that

$$\begin{aligned} \Omega(f_1\Phi + X, f_2\Phi + Y) &= r^2\big\{f_1 h(\xi, Y) - f_2 h(\xi, X) + h(X, JY)\big\} \\ &= 0 \end{aligned} \tag{3.12}$$

for all smooth functions f_1, f_2 on $C(M)$ and X, $Y \in \mathfrak{X}(M)$. Here, Ω is the Kähler form of $C(N)$ which is given by $\Omega = 2r\,dr \wedge \eta + r^2\,d\eta$. Finally, (3.12) is equivalent to that $C(M) \subset C(N)$ is Lagrangian. □

Now our main theorem is as follows.

THEOREM 3.3. *Let $\varphi : (M, g) \to (N, h)$ be a Legendrian submanifold of a Sasakian manifold (N^n, h, J, ξ, η) $(n = 2m + 1)$ and $\overline{\varphi} : (C(M), \overline{g}) \ni (r, x) \mapsto (r, \varphi(x)) \in (C(N), \overline{h}, I)$, a Lagrangian submanifold of a Kähler cone manifold. Here $C(M) := M \times \mathbb{R}^+ \subset C(N) := N \times \mathbb{R}^+$, $\overline{g} = dr^2 + r^2 g$, and $\overline{h} = dr^2 + r^2 h$. Then,*

(1) *it holds that*

$$\tau(\overline{\varphi}) = \frac{1}{r^2}\tau(\varphi). \tag{3.13}$$

Thus, we have the equivalence that $\varphi : (M, g) \to (N, h)$ is harmonic if and only if $\overline{\varphi}(C(M), \overline{g}) \to (C(N), \overline{h})$ is also harmonic.

(2) *Second, it holds that*

$$\tau_2(\overline{\varphi}) = \frac{1}{r^4}\tau_2(\varphi) + \frac{m}{r^2}\tau(\varphi). \tag{3.14}$$

Then, we have the equivalence that $\varphi : (M, g) \to (N, h)$ is proper biharmonic if and only if for $\overline{\varphi} : (C(M), \overline{g}) \to (C(N), \overline{h})$, the tension field $\tau(\overline{\varphi})$ is a nonzero eigen-section of the Jacobi operator $J_{\overline{\varphi}}$ with the eigenvalue $m = \dim M$. And we have the equivalence that $\overline{\varphi} : (C(M), \overline{g}) \to (C(N), \overline{h})$ is biharmonic if and only if it is harmonic, which is equivalent to that $\varphi : (M, g) \to (N, h)$ is harmonic.

(3) *Thirdly, it holds that*

$$\tau_2(\overline{\varphi})^{\perp} = \frac{1}{r^4}\tau_2(\varphi)^{\perp} + \frac{m}{r^2}\tau(\varphi). \tag{3.15}$$

Then, we have the equivalence that $\varphi : (M, g) \to (N, h)$ is minimal if and only if $\overline{\varphi} : (C(M), \overline{g}) \to (C(N), \overline{h})$ is bi-minimal.

(4) *Finally, it holds that*

$$\mathrm{div}_{\overline{g}}\big(I\tau(\overline{\varphi})\big) = \frac{1}{r^2}\,\mathrm{div}_g\big(J\tau(\varphi)\big). \tag{3.16}$$

Then, we have also the equivalence that $\varphi : (M,g) \to (N,h,J,\xi,\eta)$ *is Legendrian minimal if and only if* $\overline{\varphi} : (C(M),\overline{g}) \to (C(N),\overline{h},I)$ *is also Lagrangian minimal.*

To prove Theorem 3.3, we need the following lemma.

LEMMA 3.4. *The Levi–Civita connection* $\nabla^{C(M)}$ *of the cone manifold* $(C(M),\overline{g})$ *of a Riemannian manifold* (M,g)*, where the cone metric* $\overline{g} = dr^2 + r^2 g$*, is given as follows*:

$$\begin{cases} \nabla^{C(M)}_X Y = \nabla_X Y - rg(X,Y)\frac{\partial}{\partial r}, \\ \nabla^{C(M)}_X \frac{\partial}{\partial r} = \frac{1}{r}X, \\ \nabla^{C(M)}_{\frac{\partial}{\partial r}} Y = \frac{1}{r}Y, \\ \nabla^{C(M)}_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0. \end{cases} \tag{3.17}$$

Here, $X,Y \in \mathfrak{X}(M)$*, and* ∇ *is the Levi–Civita connection of* (M,g).

The proof of Lemma 3.4 is a direct computation which is omitted.

To proceed to give a proof of Theorem 3.3, we first take a locally defined orthonormal frame field $\{e_i\}_{i=1}^m$ on (M,g). Define $\overline{e}_i := \frac{1}{r}e_i$ $(i = 1,\dots,m)$, and $\overline{e}_{m+1} := \frac{\partial}{\partial r}$. Then, $\{\overline{e}_i\}_{i=1}^{m+1}$ is a locally defined orthonormal frame field on the cone manifold $(C(M),\overline{g})$.

Let $\varphi : (M^m,g) \to (N^n,h)$ $(n = 2m+1)$ be a Legendrian submanifold of a Sasakian manifold, and $\overline{\varphi} : (C(M),\overline{g}) \to (C(N),\overline{h})$, the corresponding cone submanifold of a Kähler cone $(C(N),\overline{h})$. We should see a relation between the induced bundles $\varphi^{-1}TN$ and $\overline{\varphi}^{-1}TC(N)$. We denote by $\Gamma(E)$, the space of all smooth sections of the vector bundle E. Then, every smooth section W of the induced bundle $\overline{\varphi}^{-1}TC(N)$ can be written as

$$W = V + B\frac{\partial}{\partial r}, \tag{3.18}$$

where V is a smooth section of the induced bundle $\varphi^{-1}TN$ and B is a smooth function on $C(M) = M \times \mathbb{R}^+$. Because, for every point $(x,r) \in C(M) = M \times \mathbb{R}^+$, $\overline{\varphi}(x,r) = (\varphi(x),r)$, and $W_{(x,r)} \in T_{\overline{\varphi}(x,r)}C(N) = T_{(\varphi(x),r)}(N \times \mathbb{R}^+) = T_{\varphi(x)}N \oplus T_r\mathbb{R}^+$, so we can write as $W_{(x,r)} = V_x + B(x,r)\frac{\partial}{\partial r}$, where $V_x \in T_{\varphi(x)}N$ and $B(x,r) \in \mathbb{R}$.

Then, if we denote by $\overline{\nabla}$, and $\overline{\overline{\nabla}}$, the induced connections of the induced bundles $\varphi^{-1}TN$, and $\overline{\varphi}^{-1}TC(N)$ from the connections ∇^N, $\nabla^{C(N)}$ of (N,h) and $(C(N),\overline{h})$, respectively, then we have for every $W\in\Gamma(\overline{\varphi}^{-1}TC(N))$, with $W=V+B\frac{\partial}{\partial r}$ and $V\in\Gamma(\varphi^{-1}TN)$ and $B\in C^\infty(M\times\mathbb{R}^+)$,

$$\begin{cases}\overline{\overline{\nabla}}_X W=\overline{\nabla}_X V+\frac{B}{r}X+(XB)\frac{\partial}{\partial r} \quad (X\in\mathfrak{X}(M)),\\ \overline{\overline{\nabla}}_{\frac{\partial}{\partial r}}W=\frac{\partial B}{\partial r}\frac{\partial}{\partial r}.\end{cases} \tag{3.19}$$

Proof of Theorem 3.3. (1) We have, for $i=1,\dots,m$ $(m=\dim M)$,

$$\begin{aligned}\overline{\varphi}_*\nabla^{C(M)}_{\overline{e}_i}\overline{e}_i&=\overline{\varphi}_*\left(\frac{1}{r^2}\nabla^{C(M)}_{e_i}e_i\right)\\&=\frac{1}{r^2}\overline{\varphi}_*\left(\nabla_{e_i}e_i-rg(e_i,e_i)\frac{\partial}{\partial r}\right)\quad \big(\text{by Lemma 3.4 (3.17)}\big)\\&=\frac{1}{r^2}\left(\nabla_{e_i}e_i-r\frac{\partial}{\partial r}\right)\end{aligned}\tag{3.20}$$

since $\overline{\varphi}$ is the inclusion map of $C(M)$ into $C(N)$. For $i=m+1$, we have

$$\overline{\varphi}_*\big(\nabla^{C(M)}_{\overline{e}_{m+1}}\overline{e}_{m+1}\big)=\overline{\varphi}_*\left(\nabla^{C(M)}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}\right)=0.\tag{3.21}$$

Furthermore, we have, for $i=1,\dots,m$,

$$\begin{aligned}\overline{\nabla}_{\overline{e}_*}\overline{\varphi}_*\overline{e}_i&=\nabla^{C(N)}_{\frac{1}{r}e_i}\frac{1}{r}e_i\\&=\frac{1}{r^2}\left\{\nabla^N_{e_i}e_i-rh(e_i,e_i)\frac{\partial}{\partial r}\right\}\\&=\frac{1}{r^2}\left\{\nabla^N_{e_i}e_i-r\frac{\partial}{\partial r}\right\}\end{aligned}\tag{3.22}$$

since $\overline{\varphi}^*\overline{h}=\overline{g}$ and $\varphi^*h=g$. For $i=m+1$, we have also

$$\overline{\nabla}_{\overline{e}_{m+1}}\overline{\varphi}_*\overline{e}_{m+1}=\nabla^{C(N)}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}=0.\tag{3.23}$$

Thus, we have

$$\begin{aligned}\tau(\overline{\varphi})&=\sum_{i=1}^{m+1}\big\{\overline{\nabla}_{\overline{e}_i}\overline{\varphi}_*\overline{e}_i-\overline{\varphi}_*\big(\nabla^{C(M)}_{\overline{e}_i}\overline{e}_i\big)\big\}\\&=\frac{1}{r^2}\sum_{i=1}^{m}\big\{\nabla^N_{e_i}e_i-\nabla_{e_i}e_i\big\}\quad\big(\text{by (3.20), (3.21), (3.22), (3.23)}\big)\\&=\frac{1}{r^2}\tau(\varphi),\end{aligned}\tag{3.24}$$

which is (3.13).

For (2), we have to see relations between

$$J_\varphi(V) = \overline{\Delta}_\varphi V - \sum_{i=1}^{m} R^N(V, \varphi_* e_i)\varphi_* e_i \quad \big(V \in \Gamma\big(\varphi^{-1}TN\big)\big), \tag{3.25}$$

$$J_{\overline{\varphi}}(W) = \overline{\overline{\Delta}}_{\overline{\varphi}} W - \sum_{i=1}^{m+1} R^{C(N)}(W, \overline{\varphi}_* \overline{e}_i)\overline{\varphi}_* \overline{e}_i \quad \big(W \in \Gamma\big(\overline{\varphi}^{-1}TC(N)\big)\big), \tag{3.26}$$

where

$$\overline{\Delta}_\varphi V := -\sum_{i=1}^{m} \big\{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i V}\big\}, \tag{3.27}$$

$$\overline{\overline{\Delta}}_{\overline{\varphi}} W := -\sum_{i=1}^{m+1} \big\{\overline{\overline{\nabla}}_{\overline{e}_i}(\overline{\overline{\nabla}}_{\overline{e}_i} W) - \overline{\overline{\nabla}}_{\nabla^{C(M)}_{\overline{e}_i}\overline{e}_i} W\big\}. \tag{3.28}$$

Here, $\overline{\nabla}$, and $\overline{\overline{\nabla}}$ are the induced connections of $\varphi^{-1}TN$ and $\overline{\varphi}^{-1}TC(N)$ from the Levi–Civita connections ∇^N and $\nabla^{C(N)}$ of (N, h) and $(C(N), \overline{h})$ with $\overline{h} = dr^2 + r^2 h$, respectively.

The first step. By (3.19), we have

$$\begin{cases} \overline{\overline{\nabla}}_X(\overline{\overline{\nabla}}_Y W) = \overline{\nabla}_X(\overline{\nabla}_Y V) + \frac{B}{r}\nabla^N_X Y + \frac{XB}{r} Y + \frac{YB}{r} X \\ \qquad\qquad + X(YB)\frac{\partial}{\partial r} \quad (X, Y \in \mathfrak{X}(M)), \\ \overline{\overline{\nabla}}_{\frac{\partial}{\partial r}}(\overline{\overline{\nabla}}_{\frac{\partial}{\partial r}} W) = \frac{\partial^2 B}{\partial r^2}\frac{\partial}{\partial r}, \end{cases} \tag{3.29}$$

where we used that $\overline{\overline{\nabla}}_X(\overline{\nabla}_Y V) = \overline{\nabla}_X(\overline{\nabla}_Y V)$, $\overline{\overline{\nabla}}_X Y = \overline{\nabla}_X Y = \nabla^N_X Y$ and $\overline{\overline{\nabla}}_X \frac{\partial}{\partial r} = \frac{1}{r} X$ for every X, $Y \in \mathfrak{X}(M)$. Thus, we obtain, for $W = V + B\frac{\partial}{\partial r} \in \Gamma(\overline{\varphi}^{-1}TC(N))$ with $V \in \Gamma(\varphi^{-1}TN)$ and $B \in C^\infty(M \times \mathbb{R}^+)$,

$$\begin{aligned} \overline{\overline{\Delta}}_{\overline{\varphi}} W = {} & \frac{1}{r^2}\overline{\Delta}_\varphi V - \frac{B}{r^3}\tau(\varphi) - \frac{2}{r^3}\mathrm{grad}_M B \\ & + \left(\frac{1}{r^2}\Delta_M B - \frac{\partial^2 B}{\partial r^2} - \frac{m}{r}\frac{\partial B}{\partial r}\right)\frac{\partial}{\partial r}, \end{aligned} \tag{3.30}$$

where let us recall

$$\overline{\Delta}_\varphi V = -\sum_{i=1}^{m} \big\{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V\big\} \quad \big(V \in \Gamma\big(\varphi^{-1}TN\big)\big),$$

$$\tau(\varphi) = \sum_{i=1}^{m} \big(\nabla^N_{e_i} e_i - \nabla_{e_i} e_i\big), \qquad \mathrm{grad}_M B = \sum_{i=1}^{m} (e_i B) e_i,$$

$$\Delta_M B = -\sum_{i=1}^{m} \big\{e_i(e_i B) - \nabla_{e_i} e_i B\big\} \quad \big(B \in C^\infty\big(M \times \mathbb{R}^+\big)\big).$$

The second step. By a direct computation, we have the curvature tensor field $R^{C(N)}$ of $(C(N),\overline{h})$:

$$\begin{cases} R^{C(N)}(X,Y)Z = R^N(X,Y)Z - h(Y,Z)X + h(X,Z)Y, \\ R^{C(N)}(X,\frac{\partial}{\partial r})\frac{\partial}{\partial r} = 0, \\ R^{C(N)}(\frac{\partial}{\partial r},Y)Z = 0, \end{cases} \tag{3.31}$$

for every X, Y, $Z \in \mathfrak{X}(M)$. Therefore, we obtain

$$\sum_{i=1}^m R^{C(N)}(W,\overline{\varphi}_*\overline{e}_i)\overline{\varphi}_*\overline{e}_i = \frac{1}{r^2}\sum_{i=1}^m R^N(V,\varphi_*e_i)\varphi_*e_i - \frac{m}{r^2}V + \frac{1}{r^2}V^{\mathrm{T}}, \tag{3.32}$$

for $W = V + B\frac{\partial}{\partial r} \in \Gamma(\overline{\varphi}^{-1}TC(N))$, where V^{T} is the tangential part of V.

The third step. Therefore, we have

$$\begin{aligned} J_{\overline{\varphi}}(W) &= \overline{\overline{\Delta}}_{\overline{\varphi}}W - \sum_{i=1}^m R^{C(N)}(W,\overline{\varphi}_*\overline{e}_i)\overline{\varphi}_*\overline{e}_i \\ &= \frac{1}{r^2}\left(\overline{\Delta}_{\varphi}V - \sum_{i=1}^m R^N(V,\varphi_*e_i)\varphi_*e_i\right) + \frac{m}{r^2}V - \frac{1}{r^2}V^{\mathrm{T}} \\ &\quad - \frac{B}{r^3}\tau(\varphi) - \frac{2}{r^3}\mathrm{grad}_M B \\ &\quad + \left(\frac{1}{r^2}\Delta_M B - \frac{\partial^2 B}{\partial r^2} - \frac{m}{r}\frac{\partial B}{\partial r}\right)\frac{\partial}{\partial r}. \end{aligned} \tag{3.33}$$

Here, we have already $\tau(\overline{\varphi}) = \frac{1}{r^2}\tau(\varphi)$ in Theorem 3.3(1) (3.13). For this $W := \tau(\overline{\varphi})$, we have $V = \frac{1}{r^2}\tau(\varphi)$, $B = 0$ and $V^{\mathrm{T}} = 0$, and we have

$$\begin{aligned} J_{\overline{\varphi}}\big(\tau(\overline{\varphi})\big) &= \frac{1}{r^4}\left(\overline{\Delta}_{\varphi}\big(\tau(\varphi)\big) - \sum_{i=1}^m R^N\big(\tau(\varphi),\varphi_*e_i\big)\varphi_*e_i\right) + \frac{m}{r^2}\tau(\varphi) \\ &= \frac{1}{r^4}J_{\varphi}\big(\tau(\varphi)\big) + \frac{m}{r^2}\tau(\varphi). \end{aligned} \tag{3.34}$$

We have (3.14) in (2). By (3.34), we have the equivalence between the biharmonicity of φ and that $\tau(\overline{\varphi})$ is a nonzero eigen-section of the Jacobi operator $J_{\overline{\varphi}}$ with eigenvalue $m = \dim M$. Furthermore, $\tau_2(\overline{\varphi}) = 0$ if and only if $\tau_2(\varphi) + mr^2\tau(\varphi) = 0$ for all $r > 0$, which is equivalent to that $\tau(\varphi) = 0$.

For (3) in Theorem 3.3, we only observe the following orthogonal decompositions:

$$T_xN = T_xM \oplus T_xM^{\perp}, \qquad T_xM^{\perp} = JT_xM \oplus \mathbb{R}\xi_x, \tag{3.35}$$

$$\begin{aligned} T_{(x,r)}C(N) &= T_xN \oplus T_r\mathbb{R}^+ \\ &= T_xM \oplus JT_xM \oplus \mathbb{R}\xi_x \oplus T_r\mathbb{R}^+ \\ &= T_{(x,r)}C(M) \oplus JT_xM \oplus \mathbb{R}\xi_x \\ &= T_{(x,r)}C(M) \oplus T_xM^{\perp}, \end{aligned} \tag{3.36}$$

for every $x \in M \subset N$. So let us decompose $\tau_2(\overline{\varphi}) = \frac{1}{r^4}\tau_2(\varphi)$ following (3.35) and (3.36). Then, we have

$$\tau_2(\overline{\varphi}) = \tau_2(\overline{\varphi})^{\mathrm{T}} + \tau_2(\overline{\varphi})^{\perp}, \tag{3.37}$$

where $\tau_2(\overline{\varphi})^{\mathrm{T}} \in T_{(x,r)}C(M)$ and $\tau_2(\overline{\varphi})^{\perp} \in T_xM^{\perp}$, and also we have

$$\frac{1}{r^4}\tau_2(\varphi) + \frac{m}{r^2}\tau(\varphi) = \frac{1}{r^4}\tau_2(\varphi)^{\mathrm{T}} + \frac{1}{r^4}\tau_2(\varphi)^{\perp} + \frac{m}{r^2}\tau(\varphi), \tag{3.38}$$

where $\tau_2(\varphi)^{\mathrm{T}} \in T_xM$ and $\tau_2(\varphi)^{\perp} \in T_xM^{\perp}$. But, since we have $T_xM \subset T_{(x,r)}C(M)$, we have

$$\begin{cases} \tau_2(\overline{\varphi})^{\mathrm{T}} = \frac{1}{r^4}\tau_2(\varphi)^{\mathrm{T}}, \\ \tau_2(\overline{\varphi})^{\perp} = \frac{1}{r^4}\tau_2(\varphi)^{\perp} + \frac{m}{r^2}\tau(\varphi). \end{cases} \tag{3.39}$$

Then, we have $\tau_2(\varphi)^{\perp} = 0$ if and only if $\tau_2(\varphi)^{\perp} + mr^2\tau(\varphi) = 0$ for all $r > 0$, which is equivalent to that $\tau(\varphi) = 0$.

For (4), we first show that

$$\begin{aligned} I\tau(\overline{\varphi}) &= J\tau(\overline{\varphi}) + \eta\big(\tau(\overline{\varphi})\big)\Psi \\ &= \frac{1}{r^2}J\tau(\varphi) + \frac{1}{r^2}\eta\big(\tau(\varphi)\big)\Psi \\ &= \frac{1}{r^2}J\tau(\varphi). \end{aligned} \tag{3.40}$$

Because for a Legendrian submanifold of a Sasaki manifold, the second fundamental form B takes its value in $\mathrm{Ker}(\eta)$, so $\tau(\varphi) = \mathrm{Trace}(B) \subset \mathrm{Ker}(\eta)$, that is,

$$\eta\big(\tau(\varphi)\big) = 0. \tag{3.41}$$

Then, we have

$$\begin{aligned} \mathrm{div}_{\overline{g}}\big(I\tau(\overline{\varphi})\big) &= \sum_{i=1}^{m+1} \overline{g}\big(\overline{e}_i, \nabla^{C(M)}_{\overline{e}_i}\big(I\tau(\overline{\varphi})\big)\big) \\ &= \frac{1}{r^4}\sum_{i=1}^{m} \overline{g}\big(e_i, \nabla^{C(M)}_{e_i}\big(J\tau(\varphi)\big)\big) \\ &\quad + \frac{1}{r^2}\overline{g}\bigg(\frac{\partial}{\partial r}, \nabla^{C(M)}_{\frac{\partial}{\partial r}}\big(J\tau(\varphi)\big)\bigg). \end{aligned} \tag{3.42}$$

But, the first term of the right-hand side of (3.42) coincides with

$$\begin{aligned} &\frac{1}{r^4}\sum_{i=1}^{m} \overline{g}\bigg(e_i, \nabla_{e_i}\big(J\tau(\varphi)\big) - rg\big(e_i, J\tau(\varphi)\big)\frac{\partial}{\partial r}\bigg) \\ &\quad = \frac{1}{r^2}\sum_{i=1}^{m} g\big(e_i, \nabla_{e_i}\big(J\tau(\varphi)\big)\big) = \frac{1}{r^2}\,\mathrm{div}_g\big(J\tau(\varphi)\big). \end{aligned} \tag{3.43}$$

On the other hand, the second term of the right-hand side of (3.42) coincides with

$$\frac{1}{r^2}\overline{g}\left(\frac{\partial}{\partial r}, \nabla^{C(M)}_{\frac{\partial}{\partial r}}\big(J\tau(\varphi)\big)\right) = \frac{1}{r^3}\overline{g}\left(\frac{\partial}{\partial r}, J\tau(\varphi)\right) = 0 \tag{3.44}$$

because $J\tau(\varphi)$ is tangential to T_xM for the Legendrian immersion $\varphi : (M, g) \to (N, h, J)$. Therefore, we obtain the desired formula:

$$\mathrm{div}_{\overline{g}}\big(I\tau(\overline{\varphi})\big) = \frac{1}{r^2}\,\mathrm{div}_g\big(J\tau(\varphi)\big).$$

We obtain Theorem 3.3. □

REMARK 3.5. The assertion (4) in Theorem 3.3 was given by I. Castro, H. Z. Li and F. Urbano ([5]), and H. Iriyeh ([14]), independently in a different manner from ours.

4. Biharmonic Legendrian submanifolds of Sasakian manifolds

By Theorem 3.3, we turn to review studies of a proper biharmonic Legendrian submanifold of a Sasaki manifold (N^n, h, J, ξ, η) and give Takahashi-type theorem (cf. Theorem 4.4). First, let us recall the equations of biharmonicity of an isometric immersions (cf. [19]).

LEMMA 4.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be an isometric immersion. Then φ is biharmonic if and only if*

$$\begin{cases} \sum_{i=1}^m (\nabla_{e_i} A_{\mathbf{H}})(e_i) + \sum_{i=1}^m A_{\nabla^\perp_{e_i}\mathbf{H}}(e_i) - \sum_{i=1}^m (R^N(\mathbf{H}, e_i)e_i)^{\mathrm{T}} = 0, \\ \Delta^\perp \mathbf{H} + \sum_{i=1}^m B(A_{\mathbf{H}}(e_i), e_i) - \sum_{i=1}^m (R^N(\mathbf{H}, e_i)e_i)^\perp = 0, \end{cases} \tag{4.1}$$

where $\mathbf{H} = \frac{1}{m}\sum_{i=1}^m B(e_i, e_i)$ is the mean curvature vector field along φ, $(\,)^{\mathrm{T}}$, $(\,)^\perp$ are the tangential part and normal part, respectively, B is the second fundamental form, and A is the shape operator for the isometric immersion $\varphi : (M, g) \to (N, h)$.

For an isometric immersion of a Legendrian submanifold into a Sasakian manifold, we have the following theorem.

THEOREM 4.2. *Let $\varphi : (M^m, g) \to (N^n, h, J, \xi, \eta)$ $(n = 2m + 1)$ be an isometric immersion of a Legendrian submanifold of a Sasakian manifold. Then φ is biharmonic if and only if*

$$\begin{aligned} &\sum_{i=1}^m (\nabla_{e_i} A_{\mathbf{H}})(e_i) + \sum_{i=1}^m A_{\nabla^\perp_{e_i}\mathbf{H}}(e_i) \\ &\qquad - \sum_{i,j=1}^m h\big(\big(\nabla^\perp_{e_j} B\big)(e_i, e_i) - \big(\nabla^\perp_{e_i} B\big)(e_j, e_i), \mathbf{H}\big)e_j \\ &\quad = 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned}(4.3)\qquad &\Delta^{\perp}\mathbf{H}+\sum_{i=1}^{m}B\big(A_{\mathbf{H}}(e_i),e_i\big)\\&\quad+\sum_{j=1}^{m}\mathrm{Ric}^N(J\mathbf{H},e_j)Je_j-\sum_{j=1}^{m}\mathrm{Ric}^M(J\mathbf{H},e_j)Je_j\\&\quad-\sum_{i=1}^{m}JA_{B(J\mathbf{H},e_i)}(e_i)+mJA_{\mathbf{H}}(J\mathbf{H})+\mathbf{H}\\&=0.\end{aligned}$$

In the case that (N^{2m+1},h,J,ξ,η) is a Sasaki space form $N^{2m+1}(\varepsilon)$ of constant J-sectional curvature ε whose curvature tensor R^N is given by

$$\begin{aligned}(4.4)\quad R^N(X,Y)Z=&\frac{\varepsilon+3}{4}\big\{h(Y,Z)X-h(Z,X)Y\big\}\\&+\frac{\varepsilon-1}{4}\big\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\&+h(X,Z)\eta(Y)\xi-h(Y,Z)\eta(X)\xi\\&+h(Z,JY)JX-h(Z,JX)JY+2h(X,JY)JZ\big\},\end{aligned}$$

for all $X,Y,Z\in\mathfrak{X}(N)$, we have ([9], [13], [28]):

THEOREM 4.3. *Let $\varphi:(M^m,g)\to N^{2m+1}(\varepsilon)$ be a Legendrian submanifold of a Sasaki space form of constant J-sectional curvature ε. Then φ is biharmonic if and only if*

$$(4.5)\qquad \overline{\Delta}_{\varphi}\mathbf{H}=\frac{\varepsilon(m+3)+3(m-1)}{4}\mathbf{H}$$

which is equivalent to

$$(4.6)\qquad \begin{cases}\sum_{i=1}^{m}(\nabla_{e_i}A_{\mathbf{H}})(e_i)+\sum_{i=1}^{m}A_{\nabla^{\perp}_{e_i}\mathbf{H}}(e_i)=0,\\ \Delta^{\perp}\mathbf{H}+\sum_{i=1}^{m}B(A_{\mathbf{H}}(e_i),e_i)-\frac{\varepsilon(m+3)+3(m-1)}{4}\mathbf{H}=0.\end{cases}$$

Now, let us consider a Legendrian submanifold M^m of the $(2m+1)$-dimensional unit sphere $S^{2m+1}(1)$ with the standard metric ds^2_{std} of constant sectional curvature 1. Then, we have, due to Theorem 3.3, and $J_{\overline{\varphi}}=\overline{\overline{\Delta}}$ which follows from that $R^{C(N)}=0$ because of $(C(N),\overline{h})=(\mathbb{C}^{m+1},ds^2)$:

THEOREM 4.4. *Let $\varphi:(M^m,g)\to(S^{2m+1}(1),ds^2_{\mathrm{std}})$ be a Legendrian submanifold of $(S^{2m+1}(1),ds^2_{\mathrm{std}})$, and $\overline{\varphi}:(C(M),\overline{g})\to(\mathbb{C}^{m+1},ds^2)$, the corresponding Lagrangian cone submanifold of the standard complex space $(\mathbb{C}^{m+1},ds^2)$. Then, it holds that $\varphi:(M^m,g)\to(S^{2m+1}(1),ds^2_{\mathrm{std}})$ is proper biharmonic if and only if $\tau(\overline{\varphi})=\frac{1}{r^2}\tau(\varphi)=\frac{m}{r^2}\mathbf{H}$ is a non-zero eigen-section of the rough Laplacian $\overline{\overline{\Delta}}_{\overline{\varphi}}$ acting on $\Gamma(\overline{\varphi}^{-1}T\mathbb{C}^{m+1})$ with the eigenvalue $m=\dim M$: $\overline{\overline{\Delta}}_{\overline{\varphi}}\tau(\overline{\varphi})=m\tau(\overline{\varphi})$.*

This Theorem 4.4 could be regarded as a biharmonic map version of the following T. Takahashi's theorem ([29]). For Takahashi-type theorem for harmonic maps into Grassmannian manifolds, see pages 42 and 46 in [21].

THEOREM 4.5 (T. Takahashi). *Let* (M^m, g) *be a compact Riemannian manifold, and let* $\varphi : (M^m, g) \to (S^n, ds^2_{\mathrm{std}})$ *be an isometric immersion. We write* $\varphi = (\varphi_1, \ldots, \varphi_{n+1})$ *where* $\varphi_i \in C^\infty(M)$ $(1 \le i \le n+1)$ *via the canonical embedding* $S^n \hookrightarrow \mathbb{R}^{n+1}$. *Then,* $\varphi : (M, g) \to (S^n, ds^2_{\mathrm{std}})$ *is minimal if and only if* $\Delta_g \varphi_i = m\varphi_i$ $(1 \le i \le n+1)$. *Here,* Δ_g *is the positive Laplacian acting on* $C^\infty(M)$.

Certain classification theorems about proper biharmonic Legendrian immersions into the unit sphere $(S^{2m+1}(1), ds^2_{\mathrm{std}})$ were obtained by T. Sasahara ([26], [27], [28]).

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