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REALIZING DIMENSION GROUPS, GOOD MEASURES, AND TOEPLITZ FACTORS

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ABSTRACT. Motivated by connections between dimension groups and good measures or minimal actions on Cantor sets (especially Töplitz), we find realizations of classes of dimension groups as limits of primitive matrices all of which have equal column sums, or equal row sums.

Introduction

There are very strong connections between ordered K_0-groups (*dimension groups*, [E1] and [EHS]) and minimal $\mathbf{Z}$-actions on Cantor sets ([GPS1], [GPS2] and subsequent work). A basic question concerns explicit realizations of the dimension groups, in order to obtain Vershik maps on the corresponding Bratteli diagrams. Another question concerns good measures invariant under minimal actions ([BeH]), and the connection with the realizing Bratteli diagrams. A third problem deals with the presence of factors of the minimal system that are Toeplitz, and how this relates to the dimension group realization.

In this paper, we are mostly dealing with simple dimension groups with unique trace; equivalently, up to strong orbit equivalence class, uniquely ergodic minimal $\mathbf{Z}$-actions on Cantor sets. As with every dimension group, these can be realized as the direct limit, as partially ordered Abelian groups, $\lim A_i : \mathbf{Z}^{n(i)} \to \mathbf{Z}^{n(i+1)}$, where A_i are nonnegative integer matrices, and $\mathbf{Z}^m$ is given the usual coordinatewise ordering [EHS]. This realization is of *size* d if $d = \liminf_{i\to\infty} n(i) \in \mathbf{N} \cup \{\infty\}$. Let $D(G)$ be the minimum of the sizes at which G can be realized.

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Obviously, a lower bound for $D(G)$ is the rank of G as an Abelian group, $\operatorname{rank} G$. Elliott [E2] has given an example of a rank 2 simple dimension group with unique trace for which $D(G) = 3 > \operatorname{rank} G$. In answer to a conjecture of Effros [Ef], Riedel [R1], [R2] showed that if G were free (as an Abelian group) and simple with unique trace, then $D(G) = \operatorname{rank} G$, but that there exist examples of simple dimension groups G with more than one trace that are free, with $D(G) > \operatorname{rank} G$. We show by direct construction and with more requirements on the realization (to be discussed below), that Elliott's example is the worst possible, that is, for any simple dimension group with unique trace, $D(G) \in \{\operatorname{rank} G, 1 + \operatorname{rank} G\}$.

Now drop the assumption of unique trace (unique ergodicity in the dynamical setting). Good measures on Cantor sets were introduced by Akin [Ak1], [Ak2]. In [BeH], this notion was translated, for invariant measures for minimal actions, among other situations, to traces on the corresponding dimension group. Criteria for goodness were obtained, in terms of the affine representation of the dimension group on its normalized trace space (a Choquet simplex). If the trace (measure) has values in the rationals, it need not be good, although it is close to being good. In terms of a realization of G as above, if the column sums of A_i for each $n(i)$ are constant, say equal to r_i, then there is a simple way to construct a trace τ on G by normalizing the rows of the form $(1, 1, \ldots, 1) \in \mathbf{Z}^{1 \times n(i)}$. Such a trace is necessarily faithful (the corresponding measure is faithful, that is, no clopen sets have measure zero), good, and rational-valued. We show that the converse holds; that is, if τ is a faithful, good, rational-valued trace, then there exists a realization of G such that the transition matrices A_i each have equal column sum, and τ is given by the obvious construction.

In the case that the trace is unique (and thus good) and rational-valued, we do better: we find a realization with *equal column sums* of size $(\operatorname{rank} G) + 1$, giving the realizing matrices explicitly in terms of the group extension data. This is in fact how we complete the proof that $D(G) \in \{\operatorname{rank} G, 1 + \operatorname{rank} G\}$ in the unique trace case.

This construction gives us a pool of square matrices that whose column sums are equal. If we transpose them, we obtain matrices whose row sums are equal. Whenever we have a realization of G via matrices (not necessarily square) whose row sums are equal, then there is an ordering on the path space of the corresponding Bratteli diagram so that the minimal system has a factor map onto an odometer, and conversely. So it becomes of interest to characterize the dimension groups which admit such a realization; this is already known [GJ]. However, the contribution of this portion of the paper is to obtain similar bounds (using the transposes) on the matrix sizes in the unique trace situation, with relatively explicit constructions (based on extension data).

We also deal with realizations for which both properties (equal row sum and equal column sum) hold simultaneously. A surprise appears in terms of the supernatural number of the range of the rational trace.

Definitions and outline

All groups are Abelian, *free* means free as an Abelian group, all partially ordered groups are directed (that is, $G = G^+ - G^+$), unperforated (whenever n is a positive integer and $g \in G$, then $ng \in G^+$ implies $g \in G^+$), and torsion-free. Equivalence classes representing elements of the direct limit, $\lim A_n : F_n \to F_{n+1}$ are expressed as $[v, n]$, where v belongs to the nth free Abelian group. The *rank*[1] of a torsion-free Abelian group G is the rational vector space dimension of $G \otimes \mathbf{Q}$, equivalently, the size of a maximal linearly independent subset over the rationals contained in G.

Suppose U is a noncyclic subgroup of the rationals, and let $\tau : G \to U \subset \mathbf{R}$ be an onto group homomorphism from a torsion-free group G to U. We may impose an ordered group structure on G simply by declaring $g \in G^+ \setminus \{0\}$ iff $\tau(g) > 0$. This makes G into a simple dimension group with unique trace, and the trace is rational-valued; all such simple dimension groups are constructed in this manner. That is, G is an extension (in the category of Abelian groups) of a torsion-free Abelian group $\ker \tau$ by the subgroup U of the rationals.

As G is a countable dimension group, by [EHS], it has a representation as *ordered groups*, $G \simeq \lim A_n : \mathbf{Z}^{f(n)} \to \mathbf{Z}^{f(n+1)}$ where $f : \mathbf{N} \to \mathbf{N}$ is a function, we take the usual coordinatewise ordering on each $Z^{f(n)}$ and impose the usual direct limit ordering. The A_n have only nonnegative entries (and, by telescoping, can be made strictly positive when G is simple). However, [EHS] does not give specific representations, that is, the matrices A_n cannot be constructed from the argument, except by extremely complicated machinations. Here we consider the case that G be of rank $k+1$ (so $\ker \tau$ is rank k), and provide explicit realizations for G with the ECS property (*equal column sums*: each of the nonnegative (or strictly positive) matrices A_n has all of its column sums equal).

With an ECS realization, there is a canonical choice of trace, namely (up to scalar multiple), the sequence of normalized multiples of constant rows; in this case, we say the trace has an ECS realization. We show that for general dimension groups with order unit, a trace admits an ECS realization iff it is faithful, rational-valued, and good (in the sense of Akin, after translation to dimension groups as in [BeH]). In this case, there is no control on the matrix sizes, but we do not require simplicity.

[1] Some authors have unfortunately defined the *rank of a dimension group* to be the width of the minimal Bratteli diagram realizing it, that is, what we have defined in the Introduction as $D(G)$. As noted above, this can be strictly larger than the rank of the underlying group.

When we take the transposes of the matrices used for ECS realizations, and thus obtain ERS—equal row sum—realizations, the resulting dimension groups run over all possible simple dimension groups of finite rank with unique trace (not generally rational-valued) which could admit an ERS realization. These are very closely related to Töplitz systems (pairs (X,T) consisting of a Cantor set and a minimal self-homeomorphism, which is an almost everywhere one-to-one extension of an odometer), as explained to me by Chris Skau, whose question about ERS realizations motivated this paper.

An ERS realization of a simple dimension group G with respect to a (noncyclic rank one subgroup) H such that $\tau(H) \neq 0$ and G/H is torsion-free is an ordered group isomorphism $\phi : G \to \lim A_n : \mathbf{Z}^{f(n)} \to \mathbf{Z}^{f(n+1)}$ where $f : \mathbf{N} \to \mathbf{N}$ is some function, A_n are nonnegative integer matrices each having equal row sums, the direct limit ordering is imposed, such that $\phi(H) = \bigcup_n [\mathbf{1}_{f(n)}, n]\mathbf{Z}$, with $\mathbf{1}_{f(n)}$ the column consisting of ones. If $f(n) = s$ for all n, the realization is of size s. A realization is *co-rank one ultrasimplicial* if for all n, the kernel of the map $\mathbf{Z}^{f(n)} \to G$ (given by $v \mapsto [v,n]$) has rank at most one.

If G and H are as in the previous paragraph, and there exists an ERS realization of G with respect to H that is also ECS, then we refer to this as an ECRS realization of G with respect to H.

We establish the following results on ECS, ERS, and ECRS realizations.

Let G be a dimension group.

(i) If G is simple, of rank $k+1$ with unique trace τ, and $\tau(G)$ is a subgroup of the rationals, then there exists an ECS realization of G of size $k+2$ (Theorem 4.1).

(ii) If G is simple and has unique trace, and can be written as a direct limit of finite rank simple dimension groups, then G has a co-rank one ultrasimplicial realization; in particular, if $\operatorname{rank} G = m$, then it admits a size $m+1$ realization (Theorem 5.1).

(iii) Let τ be a trace on G with $\tau(G) \subseteq \mathbf{Q}$. Then there exists an ECS realization of G representing τ if and only if τ is good (in the sense of Akin, as translated to the dimension group setting [BeH]) and faithful (i.e., $\ker \tau \cap G^+ = \{0\}$) (Theorem 6.1(b)).

(iv) Suppose G is simple and has unique trace τ, and H is a noncyclic rank one subgroup of G such that $\tau(H) \neq \{0\}$ and G/H is torsion-free.
   - (a) If $\operatorname{rank} G = k+1$, then there exists a size $k+2$ ERS realization of G with respect to H (Theorem 7.1(a)).
   - (b) There exists an ERS realization of G with respect to H that is co-rank one ultrasimplicial (Theorem 7.1(b)).

(v) Suppose G is as in (iv), and in addition, $\tau(G) \subseteq \mathbf{Q}$.
   - (a) If $\tau(G)$ has no primes of infinite multiplicity (i.e., $\tau(G)$ is not p-divisible for any prime p), then G admits an ECRS realization with

respect to H if and only if $\lambda := |\tau(G)/\tau(H)| \geq \operatorname{rank} G$; when $\lambda < \infty$, there is an ECRS realization *only* of size λ (Theorem 11.9).

(b) If $\tau(G)$ has a prime of infinite multiplicity (i.e., $\tau(G)$ is p-divisible for some prime p), then G admits an ECRS realization with respect to H; this can be constructed to be bounded if $|\tau(G)/\tau(H)| < \infty$ (Theorem 11.13).

Part (iii) above applies to all dimension groups (with order unit), but the other parts require simplicity and unique trace.

Much of the time, we work in the category of Abelian groups with group homomorphism to the reals: a torsion-free Abelian group G together with a group homomorphism $t : G \to \mathbf{R}$ such that $t(G)$ is dense in $\mathbf{R}$; we denote this (G, t). Isomorphism in this category is the obvious one, $(G, t) \simeq (G', t')$ if there exists a group isomorphism $\phi : G \to G'$ such that $t'\phi$ is a nonzero scalar multiple of t. Automatically, this induces an isomorphism $\ker t \simeq \ker t'$.

Suppose that G and G' are noncyclic simple dimension groups with unique trace, τ and τ', respectively. Then $G \simeq G'$ as ordered groups if and only if $(G, \tau) \simeq (G', \tau')$ as Abelian groups with real-valued group homomorphism. One way is trivial. Conversely, suppose $\phi : G \to G'$ is a group isomorphism such that $\tau'\phi = \lambda\tau$ for some nonzero real λ. By replacing ϕ by $-\phi$ if necessary, we may assume $\lambda > 0$. Then ϕ is an isomorphism of ordered groups.

To see this, we note that $g \in G^+ \setminus \{0\}$ iff $\tau(g) > 0$; this is equivalent to $\tau'(\phi(g)) > 0$, which is equivalent to $\phi(g) \in (G')^+ \setminus \{0\}$. As ϕ is a group isomorphism, this says both ϕ and ϕ^{-1} are order preserving, hence ϕ is an order isomorphism.

Hence, to decide if two simple dimension groups with unique trace are order isomorphic, it is sufficient to find a group isomorphism between them that scales the trace(s). This makes life simple, at least when the dimension group has unique trace.

The dimension groups we will be considering for ECS realizations have an additional property, that the range of their trace is (up to nonzero scalar multiple) a subgroup of the rationals. So we consider them as groups with real-valued group homomorphism, (G, t) such that $t(G)$ is rank one and dense in $\mathbf{R}$ (so up to scalar multiple, $t(G) = U \subseteq \mathbf{Q}$).

Although we will often be talking about extensions of Abelian groups, $0 \to C \to G \to U \to 0$, it is too restrictive to deal with the classification as extensions (i.e., within $\operatorname{Ext}^1(C, U)$); instead, we are dealing with the coarser classification, isomorphism for maps $G \to U$, where we are allowed to multiply by ± 1 (and if U is p-divisible, by powers of p). There are still generically uncountably many isomorphism classes of these, since $\operatorname{Aut}(C)$ and $\operatorname{Aut}(U)$ are usually only countable.

As usual, if a group or ordered group is given as a direct limit, $\lim M_n : F_n \to F_{n+1}$ (typically, F_n will be free Abelian groups, and if the ordered direct limit is required, the entries of M_n will be nonnegative), then elements of the direct limit can be written as equivalence classes, $[a,n]$ where $a \in F_n$, and the equivalence relation is generated by $[a,n] = [M_n a, n+1]$.

1. Via subsemigroups

For this section, G need only be a partially ordered group with positive cone G^+. Let P denote the set of nonnegative integers. If $\{a_i\} \subseteq G^+$, we denote by $\sum a_i P$, the set of sums $\{\sum a_i n(i) | n(i) \in P\}$, the semigroup (or subsemigroup) generated by $\{a_i\}$.

Let $\{S_n\}_{n \in \mathbf{N}}$ be a collection of subsemigroups of G^+ with $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$ such that $G^+ = \bigcup S_n$. Suppose S_n is generated by $\{a_i^{(n)}\}_{i=1}^{f(n)}$. Since $S_n \subseteq S_{n+1}$, we can find an $f(n+1) \times f(n)$ matrix A_n (called a *transition* matrix) with entries from P such that for all $i = 1, 2, \ldots, f(n)$,

$$(*) \qquad a_i^{(n)} = \sum_{j=1}^{f(n+1)} (A_n)_{ji} a_j^{(n+1)}.$$

There is usually a great deal of choice available for the matrix entries, since there is no assumption of any sort of unique decomposition. Note the subscript ji, not ij.

Let $F_n = \mathbf{Z}^{f(n)}$, the free Abelian group on $f(n)$ generators (denoted $e_i^{(n)}$ $(i = 1, 2, \ldots, f(n))$, but when superscripted ${}^{(n)}$ is understood, it is deleted), equipped with the usual coordinatewise ordering. Now form the direct limit dimension group from the A_ns, $H = \lim A_n : F_n \to F_{n+1}$. Define $\psi_n : F_n \to G$ via $\psi_n(e_i) = a_i^{(n)}$. This is a well-defined positive group homomorphism from F_n to G. The condition in $(*)$ is precisely what we need in order that $\psi_{n+1} \circ A_n = \psi_n$. Hence, the family $\{\psi_n\}$ induces a positive group homomorphism $\Psi : H \to G$ (explicitly, $\Psi[v,n] = \psi_n(v)$ where $v \in F_n$).

Since $G^+ = \bigcup S_n$, $\Psi(H^+) = G^+$; since G is directed, that is, $G = G^+ - G^+$, it follows that Ψ is onto. If Ψ is one to one, then it is an isomorphism of ordered groups (in particular, G is a dimension group), and we have a realization for it using the matrices A_n. If $\operatorname{rank} H \leq \operatorname{rank} G < \infty$, then Ψ is automatically an isomorphism (since an onto homomorphism from a torsion-free Abelian group of finite rank to a torsion-free group of the same rank is automatically one to one).

The construction of Ψ depends on the choice(s) of the generators for the semigroups S_n, and then on the matrices A_n; different choices for the matrices (even fixing the generators of all the S_n) can result in different Ψ functions, some of which may be one to one while others need not be.

We summarize this in one gigantic statement.

LEMMA 1.1. *Suppose that G is a partially ordered Abelian group with an increasing set of subsemigroups, $S_1 \subseteq S_2 \subset \cdots$ such that $G^+ = \bigcup S_n$, and suppose that A_n is the transition matrix associated to a choice of generators for $S_n \subset S_{n+1}$, with each S_n generated by $\{a_i^{(n)}\}$. Form the dimension group $H = \lim A_n : F_n \to F_{n+1}$.*

(a) *There is a unique positive group homomorphism $\Psi : H \to G$ such that $[e_i^{(n)}, n] \mapsto a_i^{(n)}$; moreover, $\Psi(H^+) = G^+$.*
(b) *If Ψ is one to one, then it is an isomorphism of ordered Abelian groups, and thus G is a dimension group.*
(c) *If G is torsion-free*, $\operatorname{rank} H \leq \operatorname{rank} G$, *and* $\operatorname{rank} H < \infty$, *then Ψ is an ordered group isomorphism.*

2. Realizing G as ECS (free kernel that splits)

Over this and the next few sections, we deal with the simple dimension group G of rank $k+1$ having unique trace τ, and in addition, $\tau(G) = U$ is a rank one (necessarily noncyclic) subgroup of the reals. For expository reasons, we proceed in three steps.

This section deals with a rather special case, that $\ker \tau$ be free of rank k and the extension splits. In the next section, we drop the splitting property (but maintain freeness of the kernel); finally, we deal with the general case, wherein $\ker \tau$ is an arbitrary rank k torsion-free Abelian group, and the extension by U is arbitrary. We could go straight to the general case, but this would have resulted in a very complicated argument. Instead, as we proceed through the cases, we find the extra complications can be dealt with in a relatively smooth manner.

Here we deal with the easiest case, $G = U \oplus \mathbf{Z}^k$ where $G^+ \setminus \{0\} = \{(u, w) | u > 0\}$. Although we know that $G = U \oplus \mathbf{Z}^k$ is a dimension group, and $G^+ \setminus \{0\} = \{(u, w) | u \in \mathbf{Q}^{++}$ and $w \in \mathbf{Z}^k\}$, and thus is a limit of free Abelian groups with their coordinatewise limit by [EHS], the latter does not give an explicit form. Here, we obtain from a natural (but not the most natural) subsemigroup of G^+, an explicit realization with all the matrices being size $k+2$ and column stochastic (all column sums equal for each matrix; this is abbreviated ECS). The following is the result of this section.

PROPOSITION 2.1. *Let $G = U \oplus \mathbf{Z}^k$ where U is a noncyclic subgroup of the rationals, and G is the simple dimension group obtained from the map $G \to U$. Then G can be realized as a direct limit* (*in the category of ordered Abelian groups*) $\lim A_n : \mathbf{Z}^{k+2} \to \mathbf{Z}^{k+2}$ *where A_n are primitive and ECS.*

First, we find a suitable representation of $\mathbf{Z}^k$ as a union of $k+1$ subsemigroups. For $1 \leq i \leq k$, let ε_i denote the standard basis vector of $\mathbf{Z}^k$, and set $\varepsilon_{k+1} = -\sum \varepsilon_i$. Obviously $\sum_{i=1}^{k+1} \varepsilon_i = \mathbf{0}$ and it is easy to verify that $\sum_{i=1}^{k+1} \varepsilon_i P = \mathbf{Z}^k$.

Now let the supernatural number of U be given. We may block (telescope) all the primes and their powers that appear, so that we have a sequence of positive integers $p_1, p_2, \ldots$, with $p_n > (k+1)^2$ for all n and $U \simeq \lim \times p_i : \mathbf{Z} \to \mathbf{Z}$. Let $q_n = \prod_{i=1}^n p_i$. Now define the elements, for $i = 0, 1, 2, \ldots, k+1$,

$$a_i^{(n)} = \begin{cases} (\frac{1}{q_n}, \mathbf{0}) & \text{if } i = 0, \\ (\frac{1}{q_n}, \varepsilon_i) & \text{if } 1 \le i \le k+1. \end{cases}$$

Set $S_n = \sum_{i=0}^{k+1} a_i^{(n)} P$, so that $f(n)$ is constant in n with value $k+2$. Now we can write (in lots of different ways) $a_i^{(n)}$ as a nonnegative linear combination of the $a_i^{(n+1)}$, for example, $a_0^{(n)} = p_{n+1} a_0^{(n+1)}$ and $a_i^{(n)} = \sum_{j \notin \{0,i\}} a_j^{(n+1)} + 2a_i^{(n+1)} + (p_{n+1} - k - 1) a_0^{(n+1)}$ (this exploits the facts that $\sum_{i=1}^{k+1} \varepsilon_i = \mathbf{0}$ and $p_{n+1} > k+1$; in fact, we assumed $p_{n+1} > (k+1)^2$, which we will need later). This yields that $S_n \subseteq S_{n+1}$; the matrices resulting from these representations are not suitable for our purposes, as the resulting map Ψ is not one to one.

It is elementary that $G^+ = \bigcup S_n$; an arbitrary element of G^+ is of the form $x = (b/q_n, v)$ where b is a positive integer and $v \in \mathbf{Z}^k$. Let d be the maximum absolute value of the coordinates of v (as an element of $\mathbf{Z}^k$, i.e., the usual coordinates), and find l so that $p_{n+1} \cdot p_{n+2} \cdot \cdots \cdot p_{n+l} > (k+1)(b+1)d$.

We can find nonnegative integers $r(1), r(2), \ldots, r(k+1)$ with $\sum_{i=1}^{k+1} r(i)\varepsilon_i = v$ such that $\sum r(i) < 2kd$. To see this, let $d_0 = -\inf\{v_i\}$, so that $v - d_0 \varepsilon_{k+1}$ has only nonnegative coefficients, $v_i + d_0$. Thus, $v = d_0 \varepsilon_{k+1} + \sum_{i=1}^k (v_i + d_0)\varepsilon_i$. If $v_i \ge 0$ for all i, there is nothing to do; otherwise, there exists j such that $v_j = -d_0 < 0$. Hence the sum, $d_0 + \sum_1^k (v_i + d_0)$, is bounded above by $d_0 + 0 + (k-1)(d + d_0) \le (2k-1)d$. Then

$$\left(\frac{b}{q_n}, v\right) = \sum_{i=1}^{k+1} r(i) \left(\frac{1}{q_{n+l}}, \varepsilon_i\right) + \left((p_{n+1} \cdot p_{n+2} \cdot \cdots \cdot p_{n+l})b - \sum_{i=1}^{k} r(i)\right)\left(\frac{1}{q_{n+l}}, 0\right).$$

The coefficient of $(1/q_{n+l}, 0)$ is positive, since $(p_{n+1} \cdot p_{n+2} \cdot \cdots \cdot p_{n+l})b > (k+1)(b^2+b)d$, and this exceeds $2kd$. The displayed expression thus expresses x as an element of S_{n+l}.

Now we make a very particular choice of the transition matrices, A_n; not only do they have to satisfy $(*)$, but they have to be rank $k+1$ (or less, but strictly less is not possible, except for finitely many n). Since the matrices are all square of size $k+2$, the rank condition turns out to be not so onerous, especially since imposing the obvious constraint on the trace will force the rank condition to apply.

Temporarily drop the subscript n on some of the variables, so we will obtain a matrix A whose large eigenvalue is p (corresponding to p_n); we insist that

$p > (k+1)^2$. We write,

$$\begin{aligned}
a_0^{(n)} &= (p-k-1)a_0^{(n+1)} + \sum_{i=1}^{k+1} a_i^{(n+1)},\\
a_i^{(n)} &= (p-1)a_0^{(n+1)} + a_i^{(n+1)} \quad \text{for } 1 \le i \le k,\\
a_{k+1}^{(n)} &= \bigl(p-k^2-k-1\bigr)a_0^{(n+1)} + \sum_{i=1}^{k} k a_i^{(n+1)} + (k+1)a_{k+1}^{(n+1)}.
\end{aligned}$$

These relations are trivially verified by using $\sum_{i=1}^{k+1} \varepsilon_i = \mathbf{0}$. The corresponding matrix (we act from the left, so each equation gives rise to a column), is rather simple to describe (but really tedious to TEX). There is a $k \times k$ identity matrix occupying most of the space.

$$A = \begin{bmatrix}
p-k-1 & p-1 & p-1 & p-1 & \dots & p-1 & p-1 & p-k^2-k-1\\
1 & & & & & & & k\\
1 & & & & & & & k\\
\vdots & & & \mathrm{I}_k & & & & \vdots\\
1 & & & & & & & k\\
1 & & & & & & & k\\
1 & 0 & 0 & 0 & \dots & 0 & 0 & k+1
\end{bmatrix}.$$

A consequence of the equations is that the column sums are all p. If we sum all but the first column, the result is $k+1$ times the first column (as follows from $p-k^2-k-1+k(p-1) = (k+1)(p-k-1)$). Hence, $\operatorname{rank} A < k+2$, that is, $\operatorname{rank} A \le k+1$.

Now restore the subscript n; we obtain square matrices A_n of size $k+2$ with nonnegative entries, such that $\operatorname{rank} A_n \le k+1$, each with large eigenvalue p_n (this latter is not needed now). Then the rank of $H = \lim A_n : \mathbf{Z}^{k+2} \to \mathbf{Z}^{k+2}$ is at most $\liminf \operatorname{rank} A_n \le k+1$, and so the positive map $\Psi : H \to G$ obtained from this sequence of relations is an isomorphism of ordered groups. This yields an ECS realization of G.

3. Arbitrary extensions by free Abelian groups

Now we try to find explicit realizations of dimension groups G that are given as arbitrary extensions of $\mathbf{Z}^k$ by U a subgroup of the rationals. Explicitly, we have a short exact sequence of Abelian groups $0 \to \mathbf{Z}^k \to G \to U \to 0$; regarding U as a subgroup of the reals, the quotient map $\tau : G \to U$ yields the ordering: for nonzero g in G, $g \in G^+$ iff $\tau(g) > 0$. This describes all dimension groups with unique trace, whose values lie in the rationals, and whose kernel is free Abelian of finite rank. The previous constructions of course dealt with the case wherein τ is split.

Proposition 3.1. *Let G be a noncyclic simple dimension group with unique trace t such that $t(G) := U$ is a subgroup of the rationals, and such that* $\ker t$ *is free of rank k. Then G admits an ECS realization by primitive matrices of size $k+2$.*

Given the data $(G, \tau, U, \ker \tau)$ we can construct semigroups $S_n \subseteq S_{n+1} \subset \cdots$ of G^+ with the property that $G^+ = \bigcup S_n$. To begin with, write $U = \lim \times p_n : \mathbf{Z} \to \mathbf{Z}$; form $q_n = \prod_{i=1}^n p_i$, and make an initial selection, one for each n, of $g_n \in \tau^{-1}(1/q_n)$. Then $g_n - p_{n+1}g_{n+1} \in \ker \tau$, so we can write $g_n = p_{n+1}g_{n+1} + v^n$ for unique $v^n \in \ker \tau = \mathbf{Z}^k$ (obviously, the sequence (v^n) depends on the selection of the sequence (g_n)). The sequence (v^n, p_{n+1}) determines the isomorphism class of G, but by itself, this is not that useful.

The obvious candidate for the subsemigroup S_n is the semigroup generated by $\{g_n, g_n + \varepsilon_1, \ldots, g_n + \varepsilon_k; g_n + \varepsilon_{k+1}\}$, where ε_i have their usual meaning: standard basis elements if $i < k+1$ and $\varepsilon_{k+1} = -\sum_{j=1}^k \varepsilon_j$; it is convenient to define $\varepsilon_0 = 0$, so we can write $S_n = \sum_{i=0}^{k+1} (g_n + \varepsilon_i)P$. Unfortunately, there is no guarantee that $S_n \subseteq S_{n+1}$ (in other words, that the matrix entries be nonnegative), largely because at this stage, we have no control over v^n.

The idea is to make a better choice of g_n, and then a telescoping (amounting to telescoping the p_n), and if we are careful, we will obtain $S_n \subseteq S_{n+1}$ for all n, and the corresponding transition matrices can be written down; in fact, we will write down the transition matrices, verify the entries are all nonnegative, from which it follows that the S_n are increasing.

Let us see what we need to obtain this; we will write down the relations satisfied by the generators, and hope for the best. Fix n and order $(g_n + \varepsilon_i)$ according to the subscript of the ε_i, with $0 \leq i \leq k+1$. The relations are given by $g_n + \varepsilon_i = \sum_j A_{ji}(g_{n+1} + \varepsilon_j)$, where A_{ji} are integers, hopefully nonnegative, and this forces various equations to hold.

Since $\tau(g_n + \varepsilon_i)$ are all equal to $1/q_n = p_{n+1}/q_{n+1}$ and $\tau(g_{n+1} + \varepsilon_i) = 1/q_{n+1}$, we deduce that for all j, $\sum_i A_{ji} = p_{n+1}$, that is, the row sums of A^T are all p_{n+1}, so that the column sums of A are all p_{n+1}. (This is a useful way of calibrating the matrix—I am always confused as to whether it should be A or A^T, and keeping in mind that the column sums must be equal determines which it is.)

Now fix i; we have the equation, $g_n + \varepsilon_i = p_{n+1}g_{n+1} + \sum_j A_{ji}\varepsilon_j$. Using $g_n = p_{n+1}g_{n+1} + v^n$, we have, for all $i = 0, 1, \ldots, k+1$ (suppressing the subscript n on A_n, as otherwise it gets too crowded),

$$v^n + \varepsilon_i = \sum_{j=0}^{k+1} A_{ji}\varepsilon_j.$$

When $i=0$ (so $\varepsilon_i=0$), we obtain

$$A_{j,0}-A_{k+1,0}=\big(v^n\big)_j \quad \text{if } j>0,$$

$$A_{00}=p_{n+1}-\sum_{i>0}A_{i,0}=p_{n+1}-(k+1)A_{k+1,0}-\sum_{i=1}^{k}\big(v^n\big)_i.$$

Already we see a problem; the coefficients are suppose to be nonnegative, and so we require $p_{n+1}\geq(k+1)A_{k+1,0}+\sum_{i=1}^{k}(v^n)_i$ with $A_{k+1,0}\geq 0$ (we have no control—yet—on the sum of the coefficients of v^n). Anyway, we continue; for $1\leq i\leq k$,

$$\begin{aligned}
A_{i,i}&=A_{k+1,i}+\big(v^n\big)_i+1,\\
A_{j,i}&=A_{k+1,i}+\big(v^n\big)_j \quad \text{if } j\neq i \text{ and } 1\leq j\leq k,\\
A_{0,i}&=p_{n+1}-\sum_{j=1}^{k+1}A_{j,i}=p_{n+1}-(k+1)A_{k+1,i}-\sum_{j=1}^{k}\big(v^n\big)_j-1.
\end{aligned}$$

Finally, with $i=k+1$,

$$A_{j,k+1}=A_{k+1,k+1}+\big(v^n\big)_j-1 \quad \text{if } k+1>j>0,$$

$$A_{0,k+1}=p_{n+1}-\sum_{i>0}A_{i,k+1}-k=p_{n+1}-(k+1)A_{k+1,k+1}-\sum_{i=1}^{k}\big(v^n\big)_i+k.$$

Now set $a_i=A_{k+1,i}$ (obviously this depends on n, but for now we suppress the sub/superscript); then all the entries are linear in the choice of a_i. If the entries do happen to be nonnegative, then the resulting matrix $A_n=(A_{ij})$ (order of the subscripts reversed) will implement the embedding $S_n\subseteq S_{n+1}$. The resulting matrix is similar to the preceding ones, in that the interior $k\times k$ matrix is $v\cdot\mathbf{1}^T+\mathbf{1}\cdot(a_1,\dots,a_k)+\mathrm{I}_k$ where $\mathbf{1}$ is the column of size k consisting of ones, we regard v as a column, and $\cdot$ represents the usual product of matrices. Notice that $v\cdot\mathbf{1}^T$ is $k\times k$ but $v^T\mathbf{1}$ is just the sum of the coefficients of v, $\sum_{i=1}^{k}(v^n)_i$. We sometimes suppress the sub/superscripts n or $n+1$ in v^n and p_{n+1}, and the implicit superscripts in $a_i^{(n)}$.

$$(1)\quad A_n=\begin{pmatrix}
p_{n+1}-(v^n)^T\mathbf{1}-(k+1)a_0 & (p_{n+1}-(v^n)^T\mathbf{1}-1)\mathbf{1}^T-(k+1)(a_1,a_2,\dots,a_k) & *\\
v^n+a_0\mathbf{1} & v^n\mathbf{1}^T+\mathbf{1}(a_1,\dots,a_k)+\mathrm{I}_k & v^n+(a_{k+1}-1)\mathbf{1}\\
a_0 & a_1,a_2,\dots,a_k & a_{k+1}
\end{pmatrix},$$

where the $(0,k+1)$ entry (the upper right; left blank, because of overflow) is $p_{n+1}-v^T\mathbf{1}-(k+1)a_{k+1}+k$. The column sums are all p_{n+1}, as follows from the choice of generators of the subsemigroups. Without yet worrying about positivity or rank, we can calculate the eigenvalues and their geometric multiplicities, by explicitly computing the left eigenvectors.

First, $\mathbf{1}_{k+2}^T$ is the left eigenvector for p. Next, define ${}^{\perp}v := \{w \in \mathbf{Z}^{1\times k} | wv = 0\}$ (we use $\mathbf{Z}^m$ to mean $\mathbf{Z}^{m\times 1}$, i.e., the default is columns). For each $u \in {}^{\perp}v$, the row of size $k+2$, $(0, u, -u^T\mathbf{1})$, is a left eigenvector for the eigenvalue 1. If $v \neq \mathbf{0}$ (as we are assuming implicitly anyway), then ${}^{\perp}v$ is rank $k-1$, and thus even the geometric multiplicity of 1 as an eigenvalue is at least $k-1$.

This leaves two eigenvalues. We may find $u_0 \in \mathbf{Q}^{1\times k}$ such that $u_0 v = 1 - p + v^T\mathbf{1}_k$; then $(1, u_0, k+1-u_0\mathbf{1}_k)$ is another left eigenvector for the eigenvalue 1 (we may multiply by an integer and so obtain an integer eigenvector if desired), and since its first coordinate is not zero, it is not in the $\mathbf{R}$-span of the previous eigenvectors for 1; hence the multiplicity of 1 is at least k.

There is one remaining eigenvalue, in addition to $p, 1^k$, and it is easily determined from the trace; the trace of the matrix is $p + k + \sum_{i=1}^{k+1} a_i - (k+1)a_0$, hence the last remaining eigenvalue is $\sum_{i=1}^{k+1} a_i - (k+1)a_0$. Since we want the rank of the matrix to be $k+1$, we are free to choose any selection of integers a_i such that $\sum_{i=1}^{k+1} a_i = (k+1)a_0$ (i.e., a_0 is the average of all the others). When this is imposed, we see quickly that the corresponding relation holds for the columns, that is, the sum of all but the first column is $k+1$ times the first. In other words, if we set $z = (k+1, -1, -1, \dots, -1)^T \in \mathbf{Z}^{k+2}$, then $Az = \mathbf{0}$. Moreover, z is independent of the choice of n (i.e., $A_n z = \mathbf{0}$ for all n).

A particular consequence is that $W := {}^{\perp}z = \{w \in \mathbf{Z}^{1\times(k+2)} | wz = 0\}$ is a common A_n-invariant subgroup (on the left, of course, meaning $WA_n \subseteq W$ for all n); moreover, the eigenvalues of A_n restricted to this subgroup are exactly $p, 1^k$ (the zero eigenvector has conveniently been eliminated, since z spans, as a real vector space, the right zero-eigenspace of all the A_n).

Now we modify the sequence (g_n) and corresponding (v^n) to permit a selection of integers $a_{i,n}$ (and with $\sum_{i=1}^{k+1} a_i = (k+1)a_0$) so that the matrix A_n has only nonnegative entries.

Let G be given by the sequence (p_{n+1}, v^n). Let $E \in \mathrm{GL}(k, \mathbf{Z})$ and $W \in \mathbf{Z}^k$; then the group extension given by the sequence $(p_{n+1}, Ev^n + (p_{n+1}-1)W)$ is equivalent. To see this, form the square matrices of size $k+1$, $C_n = \left(\begin{smallmatrix} p_{n+1} & \mathbf{0} \\ v^n & \mathrm{I}_k \end{smallmatrix}\right)$ and $F = \left(\begin{smallmatrix} 1 & \mathbf{0} \\ W & E \end{smallmatrix}\right)$. Then $F^{-1} = \left(\begin{smallmatrix} 1 & \mathbf{0} \\ -E^{-1}W & E^{-1} \end{smallmatrix}\right)$, and

$$D_n := FC_nF^{-1} = \begin{pmatrix} p_{n+1} & \mathbf{0} \\ Ev^n + (p_{n+1}-1)W & \mathrm{I} \end{pmatrix}.$$

Now $G \simeq \lim C_n : \mathbf{Z}^{k+1} \to \mathbf{Z}^{k+1}$ as Abelian groups, and the map on each copy of $\mathbf{Z}^{k+1}$ given by F induces a group isomorphism from G to $G' := \lim D_n : \mathbf{Z}^{k+1} \to \mathbf{Z}^{k+1}$. The corresponding data for the sequence of D_n's is $(p_{n+1}, Ev^n + (p_{n+1}-1)W)$ and these maps preserve the map to U. Of course the drawback with this equivalence relation is that it applies to all the v^n's at once.

Lemma 3.2. *Suppose $V \in (\mathbf{R}^+)^k$. There exist $E \in \mathrm{GL}(k,\mathbf{Z})$ and $W \in \mathbf{Z}^k$ such that the coefficients of $EV - W$ are all nonnegative, and sum to less than one.*

Proof. Define $W^0 \in \mathbf{Z}^k$ via $(W^0)_i = \lfloor V_i \rfloor$ (the floor function). Then $V_0 := V - W^0$ has all its coefficients nonnegative and strictly less than one. If either all or all but one of the entries of V_0 is zero, we are done.

Otherwise, let $s = \max\{V_i\}$ and $t = \max\{V_i \setminus \{s\}\}$ (the notation is not very clear, but if there is a tie for maximum, then $t = s$). We apply the division algorithm to s and t; there exists an integer $m > 1$ such that $s = mt + s'$ where $0 \le s' < t$; this is implemented by an elementary transformation, hence by an element of $\mathrm{GL}(k,\mathbf{Z})$, and the new vector (replacing s in one of its positions by the smaller s') either has strictly smaller maximal entry, or the multiplicity of its maximal entry has been reduced. If in the resulting vector, there is still more than one entry, we can continue the process.

The process either terminates in a single nonzero entry (which occurs precisely when all the nonzero entries of V are rational multiples of each other), or we can make the maximal entry as small as we like, say less than $1/k$. Either way, we have constructed E as a product of elementary transformations (hence in $\mathrm{GL}(k,\mathbf{Z})$) such that $E(V - W^0)$ has only nonnegative entries and whose entries sum to less than 1. Now set $W = EW^0 \in \mathbf{Z}^k$. □

Lemma 3.3. *Suppose that the extension G of $\mathbf{Z}^k$ by U is implementable by (p_{n+1}, v^n) such that v^n/p_{n+1} converges (in $\mathbf{R}^k$) and $p_{n+1} \to \infty$. Then the corresponding dimension group G with unique trace being the map to U is realizable as a limit $A_n : \mathbf{Z}^{k+2} \to \mathbf{Z}^{k+2}$ with A_n of the form (1) primitive.*

Proof. Set $V = \lim v^n/p_{n+1}$. By the preceding, there exists $E \in \mathrm{GL}(k,\mathbf{Z})$ and $W \in \mathbf{Z}^k$ such that $EV - W$ has only nonnegative entries adding to $\lambda < 1$. Now the extension corresponding to (p_{n+1}, v^n) is equivalent to $(p_{n+1}, (v^n)' := Ev^n - (p_{n+1} - 1)W)$, so it suffices to show that $(p_{n+1}, (v^n)')$ can be realized by a sequence of primitive matrices of the form (1).

We observe that $(v^n)'/p_{n+1} = Ev^n/p_{n+1} - W(p_{n+1} - 1)/p_{n+1}$, and this sequence converges to $EV - W$. Thus, given $\varepsilon < \min\{(1-\lambda)/3(k+1), \lambda/3k\}$, for all sufficiently large n, we have $-\varepsilon\mathbf{1} \le (v^n)'/p_{n+1}$ and the sum of the entries is less than $\lambda + \varepsilon$. Thus,

$$-\varepsilon p_{n+1} \le \left(v^n\right)_i' \quad \text{and} \quad \sum \left(v^n\right)_i' < (\lambda + \varepsilon) p_{n+1}.$$

Set $a_0 = a_1 = \cdots = a_{k+1}$ to be 1 if $\min(v^n)_i' \ge 0$ and equal to $1 - \min(v^n)_i'$ otherwise.

If $\min(v^n)_i' \ge 0$, then we note that from $\sum (v^n)_i' \le (\lambda + \varepsilon)p_{n+1}$, we obtain an upper bound on the sum, $p_{n+1}\mu$, where $\mu = 1 - (1-\lambda)(1 - 1/3k)$ (what is important is that the coefficient is bounded above away from one, uniformly in sufficiently large n). Then $p_{n+1} - \sum (v^n)_i' - (k+1) - 1 \ge p_{n+1}(1-\mu) - (k+2)$.

Since $p_{n+1} \to \infty$, for all further sufficiently large n, this expression is positive; thus the matrix A_n in (1) has only nonnegative entries.

If $\min(v^n)'_i < 0$, then for any j,

$$\begin{aligned} p_{n+1} - \sum (v^n)'_i - (k+1)a_j - 1 &\geq p_{n+1} - p_{n+1}(\lambda+\varepsilon) - (k+1) \\ &\quad - (k+1)p_{n+1}\varepsilon - 1 \\ &\geq p_{n+1}\bigl(1-\lambda-(k+2)\varepsilon\bigr) - (k+1). \end{aligned}$$

Now $1-\lambda-(k+1)\varepsilon > 1-\lambda-(1-\lambda)/3 = 2(1-\lambda/3) > 0$. Hence by further increasing n, we have that for all sufficiently large n, the matrix entries of A_n are nonnegative. We can always delete a finite number of the matrices at the outset. Because $(k+1)a_0 = \sum_{i=1}^{k+1} a_i$, the rank of each A_n is $k+1$, so that Lemma 1.1 applies. □

LEMMA 3.4. *Let G be a group extension of $\mathbf{Z}^k$ by noncyclic $U \subseteq \mathbf{Q}$ with data (p_{n+1}, v^n). Then there is an equivalent representation, $(q_{n+1}, (v^n)')$, such that q_{n+1} is increasing, $q_{n+1} \to \infty$, and $(v^n)'/q_{n+1}$ converges.*

Proof. First, we may make an initial telescoping, and thus may assume that p_{n+1} are increasing to infinity at the outset. Now we perform the following substitution transform, to ensure that the resulting v_n entries are all between 0 and $p_{n+1}-1$. Suppose we have done this up to $n = m-1$; that is, $g_i = p_{i+1}g_{i+1} + v^i$ for $1 \leq i \leq m-1$. Now set $g'_{m+1} = g_{m+1} + u^m$, the u^m to be determined. Then we have $g_m = p_{m+1}g'_{m+1} + v^m - p_{m+1}u^m$ (and the subsequent relations, for larger m, are also affected, but we come to them by the induction argument). We can obviously choose $u^m \in \mathbf{Z}^k$ so that all the entries of $v^m - p_{m+1}u^m$ lie in the set $\{0,1,2,\ldots,p_{m+1}-1\}$. This completes the induction, and allows us to assume that the newly relabelled v^n satisfy $\mathbf{0} \leq v^n \leq (p_{n+1}-1)\mathbf{1}$.

In particular, with the current notation, $\{v^n/p_{n+1}\}$ is a bounded sequence in $[0,1]^k$. Hence there exists a subsequence indexed by $n(i) \in \mathbf{N}$ such that $\{v^{n(i)}/p_{n(i)+1}\}$ converges, say to $V \in [0,1]^k$. The integers $n(1) < n(2) < n(3) < \cdots \to \infty$ suggest a telescoping; set $M_j = \left(\begin{smallmatrix} p_{j+1} & \mathbf{0} \\ v^j & \mathrm{I}_k \end{smallmatrix}\right)$, discard the M_j for $j < n(1)$, and define

$$M^{n(i)} = M_{n(i+1)-1} \cdot \cdots \cdot M_{n(i)+1} \cdot M_{n(i)};$$

the upper left entry is $q_i = \prod_{j=1}^{n(i+1)-n(i)} p_{n(i)+j}$, and of course the lower right $k \times k$ is the identity matrix. The column of size k to the left of the identity is obtained by an easy induction argument. This yields (again by induction):

$$\frac{v^{(i)}}{q_i} = \frac{v^{n(i)}}{p_{n(i)+1}} + \frac{v^{n(i)+1}}{p_{n(i)+1}p_{n(i)+2}} + \frac{v^{n(i)+2}}{p_{n(i)+1}p_{n(i)+2}p_{n(i)+3}} + \cdots,$$

$$\left\| \frac{v^{(i)}}{q_i} - \frac{v^{n(i)}}{p_{n(i)+1}} \right\|_\infty \leq \sup \left\| \frac{v^j}{p_{j+1}} \right\| \left(\frac{1}{p_{n(i)+2}} + \frac{1}{p_{n(i)+2}p_{n(i)+3}} + \cdots \right)$$

$$\leq \delta\left(\frac{1}{p_{n(i)+2}}\left(1+\frac{1}{p_{n(i)+2}}+\left(\frac{1}{p_{n(i)+2}}\right)^2 + \left(\frac{1}{p_{n(i)+2}}\right)^3+\cdots\right)\right) = \frac{\delta}{p_{n(i)+2}-1}.$$

The δ was obtained from $\{v^n/p_{n+1}\}$ being a bounded sequence. We have used $p_{n+2} \geq p_{n+1}$ to convert the estimate into a geometric series. Finally, since $p_{n+1} \to \infty$, the sequence $\{\frac{v^{(i)}}{q_i}\}$ converges. □

The previous two results yield Proposition 3.1.

4. Arbitrary extensions

In this section, we deal with arbitrary extensions of U by arbitrary finite rank torsion-free Abelian groups, instead of $\mathbf{Z}^k$; that is, let C be a rank k torsion-free Abelian group, U a noncyclic subgroup of the rationals, and consider extensions $0 \to C \to G \to U \to 0$. We realize the dimension group G with strict ordering induced by $G \to U \subset \mathbf{R}$ as a limit of ECS matrices of size $k+2$. The arguments are similar to those of the previous section, but involve a couple of extra features.

Let $C = \ker t$; this has rank k, so we can write C as a limit (as Abelian groups) $B_n : \mathbf{Z}^k \to \mathbf{Z}^k$ for some choice of B_n with $\det B_n \neq 0$. We can incorporate the identity as many times as we wish, say $B_1, \mathrm{I}_k, \dots, \mathrm{I}_k, B_2, \mathrm{I}_k, \dots, \mathrm{I}_k$, $B_3, \dots$, and this gives the same Abelian group (the idea is that we will be telescoping the $k+1$ size matrices, and we want to ensure that the absolute column sums of the B_n are $\mathbf{o}(p_{n+1})$). Re-indexing, we can obtain G (the extension) as the Abelian group direct limit arising from the square matrices of size $k+1$, $\left(\begin{smallmatrix} p_{n+1} & \mathbf{0} \\ v^n & B_n \end{smallmatrix}\right)$, and we can assume that $\|B_n\|_{\infty,\infty} = \mathbf{o}(\sqrt{p_{n+1}})$.

When we realize the corresponding semigroup coming from the relations, we obtain rather similar matrices to those previously encountered. Let ε_i^n be the standard basis elements for $\mathbf{Z}^k$ at the nth level, and define $\varepsilon_0^n = \mathbf{0}$ and $\varepsilon_{k+1}^n = -\sum_{i=1}^k \varepsilon_i^n$.

We can express G (as an Abelian group with real-valued homomorphism) as the limit, $G = \lim M_n := \left(\begin{smallmatrix} p_{n+1} & \mathbf{0} \\ v^n & B_n \end{smallmatrix}\right) : \mathbf{Z}^{k+1} \to \mathbf{Z}^{k+1}$, where the common left eigenvector $x = (1, 0, 0, \dots, 0)$ induces the map to U. Call the nth copy of $\mathbf{Z}^{k+1}$, F_n, so the elements of G are the equivalence classes $[a, n] = [M_n a, n+1]$ where $a \in F_n$, and the map to U is given by $t : [a, n] \mapsto xa/p_1 \cdots p_n$. In particular, the corresponding positive cone (of the dimension group, once we put the strict ordering arising from t) will be $\bigcup t^{-1}(1/p_1\cdots p_n)P$.

Set $g_n = [x^T := (1,0,\ldots,0)^T, n]$; so $t(g_n) = 1/p_1\cdots p_n$. Then $g_n - p_{n+1}g_{n+1} = [M_n x^T - x^T, n+1]$, and this is simply the column whose top entry is 0 and the rest of which is v^n, which we write as $\sum_{i=1}^k v_i^n \varepsilon_i^{n+1}$.

Take as our generators for the positive cone at the nth level, $g_n + \varepsilon_i^n$ (now $i = 0, 1, \ldots, k+1$ to incorporate the two extra elements required); we rewrite this as $g_n + \sum_{j=1}^k (B_n)_{ji}\varepsilon_j^{n+1}$ (arising from the effect of B_n; note that the coefficients are transposed). From the equation $g_n = p_{n+1}g_{n+1} + \sum_{i=1}^k v_i^n \varepsilon_i^{n+1}$, we want to find a $(k+2)\times(k+2)$ matrix $A^n := (A^n_{ij})$ ($0 \le i,j \le k+1$; this is not supposed to represent the nth power of some matrix A, but is merely superscripting the index—the previous notation was A_n; we frequently drop the super/subscript n for simplicity) such that for all i,

$$g_n + \sum_{j=1}^{k} (B_n)_{ji}\varepsilon_j^{n+1} = \sum_{j=0}^{k+1} \bigl(A^n\bigr)_{ji}\bigl(g_{n+1} + \varepsilon_j^{n+1}\bigr).$$

We are free to choose the entries of A^n (subject of course to positivity constraints) so long as this set of equations holds. First, from $t(g_n + \varepsilon_j^n) = 1/p_1\cdots p_n$, the column sums of A^n must all be p_{n+1}. Hence, we require $\sum_{j=0}^{k+1} A^n_{ji} = p_{n+1}$ for all i. Using the relation $g_n - p_{n+1}g_{n+1} = \sum_{i=1}^k v_i^n \varepsilon_i^{n+1}$, the previous equations become

$$\sum_{j=0}^{k+1} A^n_{ji}\varepsilon_j^{n+1} = \sum_{l=1}^{k} v_l^n \varepsilon_l^{n+1} + \begin{cases} 0 & \text{if } i = 0,\\ \sum_1^k (B_n)_{ji}\varepsilon_j^{n+1} & \text{if } 1 \le i \le k,\\ -\sum_{l=1}^k \sum_{j=1}^k (B_n)_{jl}\varepsilon_j^{n+1} & \text{if } i = k+1. \end{cases}$$

Restrict to the case $j > 0$ (each entry in the top row is determined by the remaining ones in its column, as the column sums are all p_{n+1}); taking coordinates, we obtain the following equations, successively obtained by setting $i = 0$, $1 \le i \le k$, and $i = k+1$,

$$\begin{aligned} A_{j0} &= v_j^n + A_{k+1,0},\\ A_{ji} &= v_j^n + (B_n)_{ji} + A_{k+1,j} \quad \text{for } 1 \le i \le k,\\ A_{j,k+1} &= v_j^n - \sum_{l=1}^{k} (B_n)_{jl} + A_{k+1,k+1}. \end{aligned}$$

Set $a_i = A_{k+1,i}$ (of course, we should really write this as $a_{(i,n)} = A^n_{k+1,i}$ to indicate dependence on n), so we obtain these as free parameters, which determine all the rest of the entries (and $A_{0,i} = p_{n+1} - \sum_{j=1}^{k+1} A_{ji}$). The matrix

A^n has the following form (recalling that $\mathbf{1}$ is the column of 1's of size k):

$$(**)\qquad A^n = \begin{pmatrix} p_{n+1} - v^T\mathbf{1} - (k+1)a_0 & (p_{n+1} - v^T\mathbf{1})\mathbf{1}^T - \mathbf{1}^T B_n - (k+1)(a_1, a_2, \ldots, a_k) & * \\ v^n + a_0\mathbf{1} & B_n + v^n\mathbf{1}^T + \mathbf{1}(a_1, \ldots, a_k) & v^n - B_n\mathbf{1} + a_{k+1}\mathbf{1} \\ a_0 & a_1, a_2, \ldots, a_k & a_{k+1} \end{pmatrix},$$

where the $(0, k+1)$ entry (the upper right; left blank, because of horizontal overflow) is $p_{n+1} - v^T\mathbf{1} - (k+1)a_0 + \mathbf{1}^T B_n\mathbf{1}$. The matrix $B_n + v^n\mathbf{1}^T + \mathbf{1}(a_1, \ldots, a_k)$ appearing in the middle is a $k \times k$ block. If we sum all of the columns except the leftmost, we obtain $(k+1)v^n + (\sum_{i=1}^{k+1} a_i)\mathbf{1}$; thus if we impose the condition $\sum_{i=1}^{k+1} a_i = (k+1)a_0$, the rank of the matrix A^n is at most $k+1$.

The only restriction to deal with is positivity. We find a telescoping of M_n, so that in the resulting telescoping and transformation, the column sums of the corresponding B_n's plus the sum of the entries of v plus $(k+1)a_i$ is less than the (new) p_{n+1} obtained from the telescoping.

We begin, as in the preceding case, with the original relations, $g_n = p_{n+1}g_{n+1} - v^n$. Replace g_{n+1} by $g'_{n+1} = g_{n+1} + u^n$ (where u_n is to be determined), so that the new relation is $g_n = p_{n+1}g'_{n+1} + v^n - p_{n+1}u^n$. So we may choose u^n so that $v'_n = v^n - p_{n+1}u^n$ has all its entries in $\{0, 1, 2, \ldots, p_{n+1} - 1\}$. Now the relation for g'_{n+1} in terms of g_{n+2} can be adjusted, and we continue by induction. Relabelling everything in sight (including the matrices M_n), we are now in the situation that $v^n \geq 0$ and $\|v^n\| < p_{n+1}$.

Since $\{v^n/p_{n+1}\}$ is a bounded set of $\mathbf{R}^k$, it contains a convergent subsequence, say $v^{n(i)}/p_{n(i)+1} \to V \in [0,1]^k$. This yields an obvious telescoping; set $M^{(i)} = M_{n(i+1)} \cdot M_{n(i+1)-1} \cdots M_{n(i)+1}$ and $q_{i+1} = \prod_{j=n(i)+1}^{n(i+1)} p_j$, and $B^{(i)} = B_{n(i+1)} \cdot B_{n(i+1)-1} \cdots B_{n(i)+1}$; then $M^{(i)} = \left(\begin{smallmatrix} q_{i+1} & \mathbf{0} \\ v^{(i)} & B^i \end{smallmatrix}\right)$.

The column $v^{(i)}$ has a relatively simple expression,

$$v^{(i)} = q_{i+1}\left(\frac{v^{n(i+1)}}{p_{n(i+1)+1}} + \frac{B_{n(i+1)-1}v^{n(i+1)} - 1}{p_{n(i+1)+1}p_{n(i+1)}} + \frac{B_{n(i+1)-1}B_{n(i+1)-2}v^{n(i+1)-1} - 1}{p_{n(i+1)+1}p_{n(i+1)}p_{n(i+1)-1}} + \cdots\right).$$

Hence,

$$\left\|\frac{v^{(i)}}{q_{i+1}} - \frac{v^{n(i+1)}}{p_{n(i+1)+1}}\right\| \leq \sum \left\|\frac{v^{n(i+1)-j}}{p_{n(i+1)-j+1}}\right\| \cdot \left\|\frac{B_{n(i+1)-j+1}}{p_{n(i+1)-j+2}}\right\| \cdot \cdots \cdot \left\|\frac{B_{n(i+1)-1}}{p_{n(i+1)}}\right\|.$$

(The norm on the matrices is the maximum absolute column sum, which is either $\infty - \infty$ or $1 - 1$.) Since we have made the norms of B_n be $\mathbf{o}(p_{n+1})$, this goes to zero. Hence, $v^{(i)}/q_{i+1} \to V$. As before, we find $W \in \mathbf{Z}^k$ and

$E \in \mathrm{GL}(k, \mathbf{Z})$ such that the absolute sum of the entries of $EV - W$ is less than one. Now conjugate the matrices simultaneously with $D = \left(\begin{smallmatrix} 1 & \mathbf{0} \\ W & E \end{smallmatrix}\right)$ as before, and the new matrices are of the form $\left(\begin{smallmatrix} q_{i+1} & \mathbf{0} \\ v^{(i)} & B^i \end{smallmatrix}\right)$ where we can make the substitution for $a_i^{(n)}$ as we did in the previous case with a slight modification; as before, we make $a_{(i,n)} = a^{(n)}$ equal to each other for $1 \leq i \leq k$, and we require that

$$\frac{\min\{p_{n+1} - (v^n)^T \mathbf{1}, p_{n+1} - (v^n)^T \mathbf{1} + \mathbf{1}^T B_n \mathbf{1}\}}{k+1} \\ \geq a^{(n)} \geq \max\left\{0, \mathbf{1}^T B_n \mathbf{1} - v_i^n, -v_i^n, -(B_n)_{ij} - v^n \mathbf{1}^T\right\}_{1 \leq i,j \leq k}$$

in order that the resulting matrices be nonnegative, and of rank $k+1$. But $\|B_n\|_{\infty,\infty} = \mathbf{o}(\sqrt{p_{n+1}})$ (which obviously persists after the telescoping), so the entries of B_n/p_{n+1} go to zero; dividing the expressions by p_{n+1}, the B_n entries contribute negligibly to the obstruction. Now Lemma 1.1 applies, and we have a realization of G by ECS matrices of size $k+2$. This yields the following theorem.

THEOREM 4.1. *Let $t : G \to U \subseteq \mathbf{Q}$ be a simple dimension group with unique trace t, such that t is rational-valued. If* $\operatorname{rank} G = k+1$, *then G admits an ECS realization by matrices of size $k+2$, and they are of the form* $(**)$.

5. Nearly ultrasimplicial dimension groups

Effros called a dimension group *ultrasimplicial* if it has a realization as ordered direct limit (with nonnegative matrices) $G \simeq \lim A_n : \mathbf{Z}^{f(n)} \to \mathbf{Z}^{f(n+1)}$ where $\ker A_n = \{0\}$ for all n (so the obvious map $\mathbf{Z}^{f(n)} \to G$ is one to one). Elliott [E1], [E2] showed that the simple dimension group with unique trace $\mathbf{Z}[1/2] \oplus \mathbf{Z}$ (the trace is the projection onto $\mathbf{Z}[1/2]$; this is the split case, covered by Proposition 2.1 with $p_n = 2$ for all n and $k = 1$) is not ultrasimplicial, whereas any totally ordered group is ultrasimplicial, and Riedel [R1] showed that if G is free of finite rank and with unique trace, then it is ultrasimplicial. It follows easily from Riedel's result that if G is a simple dimension group with unique trace τ and $\operatorname{rank} \tau(G) > 1$, then G is ultrasimplicial ([H2]). This is practically the complementary class to the dimension groups considered here (which are characterized by $\operatorname{rank} \tau(G) = 1$ and $\operatorname{rank} G < \infty$) among simple dimension groups with unique trace.

Motivated by the ECS results, we say a dimension group is *co-rank one ultrasimplicial* if there exists a realization as partially ordered groups $G \simeq \lim A_n : \mathbf{Z}^{f(n)} \to \mathbf{Z}^{f(n+1)}$ (as usual, A_n have only nonnegative entries, and the free groups are equipped with the coordinatewise ordering) such that the kernel of any telescoping (with $m > n$) $A_m A_{m-1} \cdots A_{n+1} A_n$ has rank at most one (alternatively, the map $\mathbf{Z}^{f(n)} \to G$ given by $x \mapsto [x, n]$ has kernel of rank at most one). Then among other things, combining the ultrasimplicial results ([H2], Corollary 4) with Theorem 4.1, we obtain that any finite rank simple dimension group with unique trace is co-rank one ultrasimplicial. A simple direct limit argument extends this to the following theorem.

THEOREM 5.1. *Every simple dimension group with unique trace which is a limit of finite rank simple dimension groups with unique trace is co-rank one ultrasimplicial.*

A better result would be that infinite rank simple dimension groups with unique trace admit corank one ultrasimplicial realizations. This would be true if every infinite rank simple dimension group with unique trace were a direct limit of finite rank simple dimension groups with unique trace. This occurs if the range of the trace has rank exceeding one, as is easy to check; so Theorem 5.1 applies in this instance.

On the other hand, there exists a simple dimension group with unique trace which is not a direct limit of simple dimension groups of finite rank (with no constraint on their trace spaces). A *discrete trace* on a partially ordered Abelian group H is a trace τ such that $\tau(H)$ is cyclic. Simple dimension groups other than $\mathbf{Z}$ do not admit discrete traces, so Example 5.2 yields a simple dimension group with unique trace which cannot be a direct limit of finite rank simple dimension groups.

EXAMPLE 5.2. A countable simple dimension group with unique trace G such that if $\phi : H \to G$ is a nonzero positive homomorphism of partially ordered Abelian groups with $\operatorname{rank} H < \infty$, then H must admit a discrete trace.

REMARK. In particular, if H is a simple dimension group of finite rank and not cyclic, then there are no nonzero positive homomorphisms $\phi : H \to G$. In particular, G cannot be a direct limit of finite rank simple dimension groups.

Let $G = \mathbf{Z}[x] = \bigoplus_{i=0}^{\infty} x^i\mathbf{Z}$ be the polynomial ring over the integers; it is free as an Abelian group. Define the ring homomorphism $\tau : \mathbf{Z}[x] \to \mathbf{R}$ determined by $\tau(x) = 1/2$. Then $\tau(G) = \mathbf{Z}[1/2]$ is dense in the reals, so that with the positive cone $G^+ := \tau^{-1}(\mathbf{R}^{++}) \cup \{0\}$, G becomes a simple dimension group with unique trace [EHS].

Let $\phi : H \to G$ be a nonzero positive homomorphism, where H is of finite rank. Since $\phi \neq 0$, $\phi(H^+) \neq 0$, so there exists $h \in H^+$ such that $\phi(h)$ is an order unit (the positive cone of G consists of 0 and order units). Since H is of finite rank, so is $\phi(H)$; a finite rank subgroup of a free group (G) is itself free, therefore finitely generated. Hence, there exists n such that $\phi(H) \subset \bigoplus_{i=0}^{n} x^i\mathbf{Z}$.

Thus $\tau \circ \phi(H) \subset 2^{-n}\mathbf{Z}$, and moreover, $\tau \circ \phi(h) > 0$, as $\phi(h)$ is an order unit. Hence, $\tau \circ \phi$ is nonzero and a trace on H with discrete value group, that is, $\tau \circ \phi$ is a discrete trace of H.

An earlier version of this article claimed (essentially) that every simple dimension group with unique trace was a direct limit of simple dimension groups of finite rank, each with unique trace. The referee queried this, and this example popped out. So it is still open whether every simple dimension group with unique trace, whose value group is rank one, is corank one ultrasimplicial.

Riedel [R2] also showed that some free rank three simple dimension groups with two pure traces are not ultrasimplicial. It is possible that every simple dimension group is co-rank one ultrasimplicial, although this seems unlikely.

6. Good and not-so-good traces

The previous results showed that if G is a simple dimension group with unique trace, the trace is rational-valued, and G is of rank $k+1$, then it admits an ECS representation of size $k+2$. For this section, we drop the requirements that G be simple and have unique trace. We show that if (G,τ) is a dimension group (having order units) and τ is a faithful rational-valued trace, then G admits an ECS realization with τ obtained from the sequence of rows consisting of multiples of $\mathbf{1}^T$ if and only if τ is good (as defined in [BeH] and below; when the trace is unique, it is automatically good). However, even in the finite rank case, the argument does not yield *bounded* ECS realizations.

Suppose $G = \lim M_i : \mathbf{Z}^{n(i)} \to \mathbf{Z}^{n(i+1)}$ is an ECS representation of the dimension group G, with the ith matrix having column sum c_i. Then ECS merely says that $\mathbf{1}^T_{n(i+1)} M_i = c_i \mathbf{1}^T_{n(i)}$. This allows us to define a trace τ on G, via $\tau([w,j]) = \mathbf{1}^T_{n(j)} w / \prod_{i=1}^{j-1} c_i$. We call this trace the *trace associated to the representation of G via M_i*. Obviously τ is *faithful* (i.e., $\ker \tau \cap G^+ = \{0\}$).

Different ECS realizations of the same group G can yield inequivalent traces, moreover, some of the traces so obtained can be pure, while others need not, and their value groups may differ. For example, consider the situation with $M_j = \begin{pmatrix} 1 & 2^j \\ 2^j & 1 \end{pmatrix}$, a well-known construction with two pure traces; the trace obtained from this ECS representation is not pure.

For a particular ECS realization, the value group of the trace is $\tau(G) = \bigcup \frac{1}{\prod_{i=1}^{j} c_i} \mathbf{Z}$. Thus, $\tau(G) \subseteq \mathbf{Q}$. In particular, if each c_j is a power of the same integer k, then $\tau(G) = \mathbf{Z}[1/k]$.

The set of order units of G will be denoted G^{++}. Following [BeH], a trace $\tau : G \to \mathbf{R}$ is *good* if for all $b \in G^+$, $\tau([0,b]) = \tau(G) \cap [0, \tau(b)]$; it is *order unit good* when this property holds for all b in G^{++}. Notation that is not explained here will probably be found in [BeH].

THEOREM 6.1. *Suppose that (G,u) is a dimension group with order unit and let τ be a normalized trace thereon.*

(a) *Suppose there is an ECS realization of G implementing τ. Then τ is a faithful good trace with $\tau(G) \subseteq \mathbf{Q}$.*

(b) *Let (G,u,t) be a countable dimension group with order unit and trace such that $t(G)$ is a subgroup of the rationals. If t is faithful and good (as a trace), then there exists a realization of G as a direct limit of simplicial groups whose realizing matrices have the equal column sum property and for which t is the corresponding trace.*

It is possible that the hypothesis in (b) that t be good can be weakened to refinability ([BeH]) of t.

Toward (a), we have already observed that τ is a faithful rational-valued trace. To show that τ is good, we have an elementary lemma.

LEMMA 6.2. *Let n be a positive integer, and $\mathbf{Z}^n$ the simplicial group of rank n. Define a trace t on $\mathbf{Z}^n$ by $t(v) = \mathbf{1}_n^T v$ (so the vector v is sent to the sum of its coordinates). Then t is good.*

Proof. Select nonnegative vectors $a = (a_i)^T$, $b = (b_i)^T$ in $\mathbf{Z}^n$ such that $0 < \sum a_i < \sum b_i$. We fix b and systematically alter a. Let $S_-(a) = \{i | a_i > b_i\}$, $S_+(a) = \{j | a_j < b_j\}$, and $S_0(a) = \{i | a_i = b_i\}$. Obviously $S_+(a)$ is not empty. If $S_-(a)$ is empty, we are finished; otherwise, we proceed by induction on $\sum_{S_-(a)}(a_i - b_i)$. Select $j \in S_+(a)$, and $k \in S_-(a)$, and define a' by subtracting 1 from a_j and adding 1 to a_k, and leaving the rest of the entries unchanged. Then $\mathbf{1}_n^T a' = \mathbf{1}_n^T a$, $S_+(a) \subset S_+(a') \cup S_0(a')$, and $S_-(a') \subseteq S_-(a)$, and moreover, $\sum_{S_-(a')}(a'_i - b_i) \leq \sum_{S_-(a)}(a_i - b_i) - 1$. The transformation $a \mapsto a'$ is repeated until the S_--set is empty, and we are done. $\square$

Proof of Theorem 6.1(a). Now τ is the inverse limit of traces obtained in Lemma 6.2, so is the limit of good traces, hence is good.

Proof of Theorem 6.1(b). Start with an arbitrary $\mathbf{Z}^n$ with basis $\{e_j\}$ and map $e_j \mapsto g_j \neq 0$ in G^+. Let $t(G) = \bigcup_k \prod_{i \leq k} m(i)^{-1}\mathbf{Z}$ (i.e., the $m(i)$'s are the successive factors realizing the supernatural number of $t(G)$); set $\mathcal{M}_k = \prod_{i \leq k} m(i)$. There exists k such that each $t(g_j) = a_j/\mathcal{M}_k$ for some positive integer a_j.

By goodness, there exists $h \in G^+$ such that $t(h) = 1/\mathcal{M}_k$, so again by goodness, there exists h_{jl} in G^+ such that $g_j = \sum_{l=1}^{a_j} h_{jl}$. This allows us to create a simplicial map $\mathbf{Z}^n \to \mathbf{Z}^{\sum a_j}$ by sending $e_j \mapsto \sum_{l=1}^{a_j} E_{jl}$, and we also have the obvious map from $\mathbf{Z}^{\sum a_j}$ to G via $E_{jl} \mapsto h_{jl}$; then the maps to G are compatible.

The upshot of this preliminary construction is that all the basis elements of the new simplicial group are sent to the same value under t. Now we apply the usual construction (as in [EHS]), that is, adjoin the next pre-selected generator of the positive cone, make it the image of a map, and fix up the kernel, so we arrive at the following (all maps are positive):

$$\begin{tikzcd}
\mathbf{Z}^{n(1)} \arrow[r] \arrow[d] & \mathbf{Z}^{n'} \arrow[d] \\
G \arrow[r, "="] & G
\end{tikzcd}$$

such that the kernel of the left vertical map is contained in the kernel of horizontal map, and $\mathbf{Z}^{n(1)}$ is the $\mathbf{Z}^{\sum a_j}$ of two paragraphs above. We extend the horizontal map to a better simplicial group.

The standard basis elements E_i of the left simplicial group map to elements h_i with the property that $t(h_i) = 1/\mathcal{M}_k$ for some k. Let F_j be the standard basis elements of the right, say with images g_j (we have re-indexed the bases). Then $h_i = \sum b(i,j) g_j$ for some integers $b(i,j)$, so that $1/\mathcal{M}_k = \sum b(i,j) t(g_j)$. Then applying the method of the preliminary construction, we obtain a map $\mathbf{Z}^{n'} \to \mathbf{Z}^{n''}$ (together with a map to G) such that the images of the new basis elements all have value at t equalling $1/\mathcal{M}_{k'}$ for some $k' \geq k$ (we can make sure that $k' > k$ for infinitely many iterations of this process).

So we are in the following situation:

$$\begin{tikzcd} \mathbf{Z}^{n(1)} \arrow[r] \arrow[d] & \mathbf{Z}^{n''} \arrow[d] \\ G \arrow[r, "="] & G \end{tikzcd}$$

with the generators of the left group mapping to $1/\mathcal{M}_k$ under t and the generators of the right group mapping to $1/\mathcal{M}_{k'}$ under t; and of course, the kernel of the vertical map from the preceding $\mathbf{Z}^n$ is contained in the kernel of $\mathbf{Z}^n \to \mathbf{Z}^{n(i)}$, etc. The image of E_i in the right group is the ith column of the transition matrix; if the image of E_i is $\sum c_j F_j$, applying t, we obtain $1/\mathcal{M}_k = \sum c_j/\mathcal{M}_{k'}$. Hence $\sum c_j$, the column sum, is independent of the choice of column. So the transition matrix has equal column sums.

Now we repeat this process with the new $\mathbf{Z}^{n''}$ (adjoin the next element of the positive cone etc.). Since this sequence of transition matrices just obtained intertwines the sequence built up via the [EHS] method, both give the same dimension group as limit. □

There is a simple rank two dimension group with two pure traces, such that the value groups are both $\mathbf{Z}[1/2]$ and their kernels are discrete [BeH, Example 6]. In fact, in that example, there are no additive functions (let alone traces, pure or impure) $t : G \to \mathbf{Q}$ such that the kernel is not cyclic; in particular, none of the countably many traces with rational value groups is good (by [BeH, Corollary 1.8], the kernel of a good trace τ has dense range in $\tau^{\vdash}$).

To prove this, we note that G is strongly indecomposable and an extension of a cyclic group by $\mathbf{Z}[1/2]$; now if $\ker t$ were not cyclic, there would be (up to isomorphism) a noncyclic subgroup of $\mathbf{Q}$ sitting inside G. Applying one of traces to this subgroup, we see that it must be disjoint from the kernel, so that its image in $\mathbf{Z}[1/2]$ is an isomorphic copy. But this forces the supernatural number of the subgroup to be 2^∞, hence the subgroup is 2-divisible, hence the restriction of the trace is of finite index, and therefore we have a splitting from a finite index subgroup of G, which is impossible, as G is strongly indecomposable.

Hence, the kernel of any trace of G with rational values is either zero or cyclic. Since G is simple, this means that no trace with rational values can be good, and thus G cannot be represented by an ECS limit.

It is interesting to ask when other positive maps $\mathbf{Z}^n \to \mathbf{Z}$ are good or (better, for our purposes, order unit good, since a limit of order unit good traces is still order unit good, and if the limit group happens to be simple, the limit trace is then good). In fact, no others are good, but some others are order unit good.

LEMMA 6.3. *Let* $w = (c(i)) \in \mathbf{Z}^{1\times n}$ *be a nonnegative row, for which* $\gcd\{c(i)\} = 1$. *Then the trace* $\mathbf{Z}^n \to \mathbf{Z}$ *given by* $v \mapsto wv$ *is good iff all the nonzero* $c(i)$ *are* 1.

Proof. We may discard the zero entries, and so reduce to the case wherein all the $c(i) > 0$. If they are not all equal, by permuting the entries, we may assume $c(1) < c(2)$. Set $b = (0, 1, 0, \dots, 0)^T$ and $a = (1, 0, \dots, 0)^T$, so that $wa = c(1) < c(2) = wb$. However, b is an atom, so the value of any nonnegative less than b at the trace is zero. $\square$

There is a characterization of order unit good traces for $\mathbf{Z}^n$ with the usual ordering [H6, Appendix 1], but it is far more complicated.

7. Introductory section on ERS

As usual, $\mathbf{1}_s$ denotes the column of size s all of whose entries are 1. When s is understood, it may be deleted.

Let G be a dimension group (with order unit) that is not simplicial, and H be a rank one subgroup such that $G^{++} \cap H \neq \{0\}$. Suppose we have an order isomorphism of G with a limit of maps,

$$G \simeq \lim A_n : F_n \to F_{n+1},$$

where $F_n = \mathbf{Z}^{f(n)}$ is the usual simplicially ordered free Abelian group of columns of size $f(n)$, and A_n are $f(n+1) \times f(n)$ matrices with nonnegative integer entries, and suppose in addition, we have the following properties:

(a) for all n, there exists a (positive) integer p_{n+1} such that $A_n \mathbf{1}_{f(n)} = p_{n+1}\mathbf{1}_{f(n+1)}$;

(b) the isomorphism from G to the direct limit sends the subgroup H to $\bigcup_n [\mathbf{1}_{f(n)}, n]\mathbf{Z}$.

We make a couple of observations. Condition (a) says that each A_n has all of its row sums equal (to p_{n+1}); we say the matrix A_n *satisfies ERS* when this occurs. Condition (a) also implies $[\mathbf{1}_{f(n)}, n]\mathbf{Z} \subseteq [\mathbf{1}_{f(n+1)}]\mathbf{Z}$, so the union of rank one groups is an ascending union of rank one groups (and thus is a group, and of rank one). We also note that $\mathbf{1}_{f(n)}$ is an order unit in F_n and its image under A_n is an order unit in F_{n+1} (by (a)). Hence $[\mathbf{1}_{f(n)}, n]$ is an order unit in the direct limit. Moreover, if G_0 denotes the direct limit, and H_0 denotes $\bigcup_n [\mathbf{1}_{f(n)}, n]\mathbf{Z}$, then G_0/H_0 is torsion-free (just observe that if $kg_0 \in H_0$, then g_0 must be represented by an element of the form $t[\mathbf{1}_{f(n)}, n]$). We call the sequence (or G_0) *an ERS realization of G with respect to H* when (a) and (b) hold. This of course forces G/H to be torsion-free and $H \cap G^{++} \neq \{0\}$. Moreover, $p_{n+1} > 1$ for infinitely many n, or else the limit is simplicial, which we forbid; hence H is not cyclic.

Sometimes, if H is understood, or we are talking about whether there exists an H for which an ERS realization exists with respect to H, we say an *ERS realization for G exists.* If the matrix sizes, $\{f(n)\}$ are bounded, then there is a telescoping so that the sizes are all equal, say to s, and then the matrices have $\mathbf{1}_s$ as a common right eigenvector. In that case, we say that G has a *bounded* (or *size s*) ERS realization (with respect to H).

For example, if as an Abelian group, $G \simeq U \oplus \mathbf{Z}^k$ where $U \subseteq \mathbf{Q}$, then there is only one choice for H, namely U, and an ERS realization also requires that none of the traces kill U. If instead, the underlying group of G is $\mathbf{Z}[1/3] \oplus \mathbf{Z}[1/2]$ and the only trace is given by summing (i.e., $(a,b) \mapsto a+b$, so G is a simple dimension group with unique trace, and the trace has kernel $\{(m,-m)\}_{m\in\mathbf{Z}} \simeq \mathbf{Z}$), then there are exactly two choices for H, $(\mathbf{Z}[1/2], 0)$ and $(0, \mathbf{Z}[1/3])$. On the other hand, if G has the same underlying group, but has as pure traces the two coordinatewise projections, then G is a simple dimension group with two pure traces, but there are no candidates for H (so no ERS realizations exist for G).

If G is simple with unique trace τ, the conditions on H are equivalent to $\tau(H) \neq 0$ (equivalently, since H is rank one, $\ker \tau \cap H = \{0\}$) and G/H is torsion-free. The last property is a pink herring[2] because for every rank one subgroup H_0 of a torsion free group J, there is a unique rank one subgroup H of J such that $H_0 \subseteq H$ and J/H is torsion-free.

Our results on ERS realizations show that for simple dimension groups with unique trace, the obvious necessary conditions are sufficient, and we obtain a bound on the size in terms of the rank. All our dimension groups are countable. It has been known for some time that the necessary conditions on H and G are sufficient, in the case of arbitrary simple dimension groups (not necessarily with unique trace), as a result of Host on rational eigenvalues of minimal dynamical systems, to obtain an ERS realization of G with respect to H. In the unique trace case, we find upper bounds for the size (in terms of the rank of the underlying group), and co-rank ultrasimpliciality.

[2] Not as misdirecting as a red herring.

THEOREM 7.1. *Let G be a simple dimension group with unique trace τ, together with a noncyclic rank one subgroup H such that $\tau(H) \neq 0$ and G/H is torsion-free.*

(a) *If* $\operatorname{rank} G = k+1$, *then there exists an ERS realization of G with respect to H of size $k+2$.*
(b) *There exists a co-rank one ultrasimplicial ERS realization of G with respect to H.*

Part (a) (proved in the next section as part of Theorem 8.5) includes an explicit bound in terms of the rank (which is sharp: some of these dimension groups cannot be realized—even without the ERS property—at the same size as their rank). Part (b) (established in Section 9) is a routine consequence of (a), and of course permits infinite rank (which means that the $f(n)$ have to be unbounded).

We have a huge class of ERS representations available: begin with an ECS realization of a dimension group by square matrices, for example as obtained in Theorem 4.1, and take the sequence of transposes. The resulting dimension groups are not that closely related to the original ones from which they emanated. For example, although the dimension group defined by the transposes obtained from the previous construction will have unique trace, generically, *this trace is not rational-valued.* (This will become clear later.)[3]

We have to enter the looking-glass world of torsion-free Abelian groups, and as a result, intuition goes out the window. For example, the group $G = \mathbf{Z}[1/2] \oplus \mathbf{Z}[1/3]$ is a simple-minded direct sum of two rank one groups; however, the addition map $\mathbf{Z}[1/2] \oplus \mathbf{Z}[1/3] \to \mathbf{Z}[1/6]$ ($(a,b) \mapsto a+b$) is onto and has kernel isomorphic to $\mathbf{Z}$ (explicitly, $(1,-1)\mathbf{Z}$); hence we have a nonsplit extension of G, $\mathbf{Z} \to G \to \mathbf{Z}[1/6]$, by rank one groups, completely different from the direct summands. More generally, if $\{m(i)\}_{i=1}^{k}$ are pairwise relatively prime integers each exceeding one with $m = \prod m(i)$, then $G = \bigoplus \mathbf{Z}[1/m(i)]$ is an extension of $\mathbf{Z}^{k-1}$ by $\mathbf{Z}[1/m]$.

8. Transposes

Suppose J is an Abelian group, and is given as an Abelian group extension $0 \to L \to J \to M \to 0$, with $\tau : J \to M$ denoting the quotient map. We say the

[3] It is not true in general that if G is a limit of square strictly positive matrices (so is a simple dimension group) and G has unique trace, then the limit dimension group of their transposes need have unique trace (although it is simple). This is left as an exercise to the reader, but with a hint: first find an example with upper triangular 2×2 matrices where the number of traces—corresponding to certain eigenvectors—can easily be made to change by transposition, then perform a perturbation so the matrices are strictly positive.

extension is *nearly split*[4] if there exists a subgroup J_0 of J such that $L \subseteq J_0$, $J_0 = L \oplus H_0$ for some subgroup H_0 of J and $|J/J_0| < \infty$. Equivalently, there exists a subgroup H_0 of J such that $H_0 \cap L = \{0\}$ and $\tau(H_0)$ is of finite index in M.

In the following, the norms on rows are the maximum of the absolute values, and the norms on matrices are the maximum absolute column sums.

LEMMA 8.1. *Let $t : G \to V$ be an onto group homomorphism from a torsion-free group G of rank s to a dense subgroup V of the reals. Let H be a noncyclic rank one subgroup of G such that $\ker t \cap H = \{0\}$ and G/H is torsion-free. Then there exists a realization of G as an Abelian group, as the direct limit of matrices of the form*

$$\lim M_n := \begin{pmatrix} p_{n+1} & u^n \\ \mathbf{0} & B_n \end{pmatrix} : \mathbf{Z}^s \to \mathbf{Z}^s$$

with $p_{n+1} > 1$, $B_n \in \mathbf{Z}^{(s-1)\times(s-1)}$, $\det B_n \neq 0$, and $u^n \in \mathbf{Z}^{1\times(s-1)}$ such that

(i) $H \simeq \lim \times p_{n+1} : \mathbf{Z} \to \mathbf{Z}$.
(ii) *G/H is given as $\lim B_n : \mathbf{Z}^{s-1} \to \mathbf{Z}^{s-1}$, each B_n of nonzero determinant, and the trace t is given up to rational multiple by a sequence of rows of the form $r^i = (1/p_2 \cdots p_i, \rho_i) \in \mathbf{R}^{s\times 1}$ satisfying $r^{i+1} M_i = r^i$, with $t[a,i] = r^i a$.*
(iii) *The isomorphism of G with the direct limit identifies H with $\bigcup_{k\in\mathbf{N}} [(1,0,0,\ldots,0)^T, k]\mathbf{Z}$.*
(iv) $\|B_n\| \leq p_{n+1}^{1/8s}/(s!)^{2/s}$ *and* $\|u^n\| \leq p_{n+1}^{1/4}$.

Moreover, if G/H is free, then $\ker t$ is free; if additionally, $t(G)$ is rank one, then the image of $\ker t$ in G/H is of finite index, the extension $\ker t \to G \to t(G)$ is nearly split, and we can take $B_n = \mathrm{I}_{s-1}$.

REMARK. When we change the matrices B_n to the identity, the corresponding u^n will also change.

Proof of Lemma 8.1. We can write V first as countably generated, say by $\{l_n\} \subset \mathbf{R}$, and $t(H) = \bigcup (1/q_{n+1})\mathbf{Z}$ where $q_n > 1$ divides q_{n+1} and form the subgroups $V_n = (1/q_{n+1})\mathbf{Z} + \sum_{i=1}^n l_i \mathbf{Z}$, so that $V_n \subseteq V_{n+1}$. Next, consider $\ker t$; we can write this as an increasing union of free Abelian groups, $J_n \subset J_{n+1}$, all having the same rank as $\operatorname{rank} \ker t = s - \operatorname{rank} V$ (this is true

[4] Nearly split is almost the same as *quasi-split* used in Abelian group theory (that there exist a map $\sigma : M \to J$ such that $\tau\sigma$ is n times the identity for some nonnegative integer n). However, quasi-split is also used in other contexts, and I thought it would be confusing here. Different is the notion of *almost split*, used in representation theory of finite-dimensional algebras.

In [R], nearly split is defined for extensions of nonabelian groups; it agrees with the definition here when restricted to torsion-free Abelian groups. The equivalence classes of nearly split extensions are closed under Baer sums and differences, hence form a subgroup of Ext, although a very small one. We never use the additive structure of the group of extensions.

of any finite rank torsion-free Abelian group). Select $h'_n \in H$ and $g_n \in G$ such that $t(h'_n) = 1/q_{n+1}$ and $t(g_n) = l_n$, and form the group G_n generated by $\{J_n, h'_n, g_1, g_2, \ldots, g_n\}$; this is finitely generated, hence being a subgroup of a torsion-free group, is free; moreover, its rank must $\operatorname{rank} J_n + \operatorname{rank} V = \operatorname{rank} \ker t + \operatorname{rank} V = s$.

Then $G_n \subseteq G_{n+1}$, and since $\ker t = \bigcup J_n \subset \bigcup G_n$, and $\bigcup G_n \to V$ is onto, it follows that $G = \bigcup G_n$. Now define $H_n = H \cap G_n$; this is cyclic and its image under t contains (possibly strictly) $(1/q_{n+1})\mathbf{Z}$. We may choose its generator, h_n, so that $t(h_n) > 0$ (which of course uniquely determines it). Since G/H is torsion-free, so is $G_n/H_n = G_n/(h_n\mathbf{Z})$. Hence for each n, there is an ordered $\mathbf{Z}$-basis for G_n whose first entry is h_n.

The matrix implementing $G_n \subseteq G_{n+1}$ with respect to the two bases is precisely of the form displayed (but without the estimates in (iv) being satisfied), where p_{n+1} is uniquely determined by $h_n = h_{n+1}p_{n+1}$. Condition (i) is straightforward to verify. We have seen that $G = \bigcup G_n$, so we obtain a sequence of matrices whose limit Abelian group is G. The matrices B_n are the maps $G_n/h_n\mathbf{Z} \to G_{n+1}/h_{n+1}\mathbf{Z}$, and the limit of these is G/H. From the rank conditions, $\operatorname{rank} B_n = s-1$ for almost all n, so $\det B_n \neq 0$ for almost all n (and so by deleting an initial segment of the direct limit, we can ensure that $\det B_n \neq 0$ for all n). The second part of (ii) just follows from the definitions. Condition (iii) comes from the construction.

Now we want to adjust the sequence in order to arrange that (iv) holds.

Having the original construction of B_n as the quotient maps on $G_n/h_n\mathbf{Z} \simeq \mathbf{Z}^s/h_n\mathbf{Z}$, let $f : \mathbf{N} \to \mathbf{N}$ be any strictly increasing function. Define $G^n = G_n + h_{f(n)}\mathbf{Z}$. Then $G^n \subseteq G^{n+1}$, and $t(h_{f(n)})/t(h_{f(n+1)}) = p_{f(n+1)+1}p_{f(n+1)} \cdot \cdots \cdot p_{f(n)+2}$. In particular, we can take the basis for G_n given by $(h_n, y_{n,1}, \ldots, y_{n,k})$, and observe that $(h_{f(n)}, y_{n,1}, \ldots, y_{n,k})$ is a $\mathbf{Z}$-basis for G^n. The map $M^n : G^n \to G^{n+1}$ with respect to this basis then has its first column simply $(p_{f(n+1)+1}p_{f(n+1)} \cdot \cdots \cdot p_{f(n)+2}, 0, 0, \ldots, 0)^T$. Moreover, the induced map $G^n/h_{f(n)}\mathbf{Z} \to G^{n+1}/h_{f(n+1)}\mathbf{Z}$ is naturally the same as the induced map $G^n/h_n\mathbf{Z} \to G^{n+1}/h_{n+1}\mathbf{Z}$, that is B_n. Hence, the form of the transition matrices M^n is

$$\begin{pmatrix} p_{f(n+1)+1}p_{f(n+1)} \cdot \cdots \cdot p_{f(n)+2} & u^n \\ \mathbf{0} & B_n \end{pmatrix}$$

for some (different, but relabelled) $u^n \in \mathbf{Z}^{1\times s}$. Thus the new p_{n+1} is the product $p_{f(n+1)+1}p_{f(n+1)} \cdot \cdots \cdot p_{f(n)+2}$, which we can make as large as we like (by choosing f to grow fast), while fixing B_n. Relabel the upper left corner p_{n+1}. Thus, we can ensure that $\|B_n\|^{2s} \leq \sqrt{p_{n+1}}/(s!)^2$ (or smaller if we like) and p_n increasing.

Having this, we can now ensure that $\|u^n\| < p_{n+1}^{1/4}$. Set $U_n = \left(\begin{smallmatrix} 1 & y_n \\ \mathbf{0} & \mathrm{I}_{s-1} \end{smallmatrix}\right)$ where $y_n \in \mathbf{Z}^{1\times(s-1)}$ is to be determined. Each U_n is in $\mathrm{GL}(s, \mathbf{Z})$ and $U_n^{-1} = \left(\begin{smallmatrix} 1 & -y_n \\ \mathbf{0} & \mathrm{I}_{s-1} \end{smallmatrix}\right)$. Then $\lim M^n : \mathbf{Z}^s \to \mathbf{Z}^s$ is isomorphic to $\lim U_{n+1}M^nU_n^{-1} : \mathbf{Z}^s \to$

$\mathbf{Z}^s$ (via $[a,m] \mapsto [U_n a, m]$). We calculate

$$U_{n+1} M^n U_n^{-1} = \begin{pmatrix} p_{n+1} & u^n - p_{n+1} y_n + y_{n+1} B_n \\ \mathbf{0} & B_n \end{pmatrix}.$$

Set $y_1 = \mathbf{0}$. Obviously, $0 \neq |\det B_n| \leq \|B_n\|^{s-1} \cdot (s-1)! < p_{n+1}^{1/4}$. Now B_n^{-1} exists as a matrix with rational entries, and $\det B_n \cdot (B_n)^{-1}$ is simply the adjoint matrix of B, so has integer entries. Let $d_n = |\det B_n|$. Then we have $\mathbf{Z}^{1\times(s-1)} d_n B_n^{-1} \subseteq \mathbf{Z}^{1\times(s-1)}$. Applying B_n, we have $d_n \mathbf{Z}^{1\times(s-1)} \subseteq \mathbf{Z}^{1\times(s-1)} B_n$.

This means that for any vector $z \in \mathbf{Z}^{1\times(s-1)}$, we can find $y \in \mathbf{Z}^{1\times(s-1)}$ such that $\|z - yB_n\| < d_n$ ($\leq d_n/2$ can be arranged, but is unnecessary here). Given $y_1, \dots, y_n$, we can thus find y_{n+1} inductively so that $\|(u^n - p_{n+1} y_n) - y_{n+1} B_n\| < d_n$. After relabelling $U_{n+1} M^n U_n^{-1}$ to M_n, the resulting upper right corner entry (again called u^n) thus satisfies $\|u^n\| < d_n < p_{n+1}^{1/4}$.

Each (newly relabelled) h_n appears as $[(1,0,\dots,0)^T, n]$ from the $\mathbf{Z}$-basis construction, and since $H = \bigcup h_n \mathbf{Z}$, the identification with H follows again.

Now we deal with the *Moreover* statement. The map $\ker t \to G/H$ is one to one; so if G/H is free, then $\ker t$, being a subgroup, is free as well. Since G has finite rank, G/H is free of rank $s-1$. If additionally, $t(G)$ has rank one, then $\ker t$ has rank $s-1$, the same as that of G/H, and since both are free, the image $\ker t$ is of finite index in G/H.

Since $G/H \simeq \lim B_n : \mathbf{Z}^{s-1} \to \mathbf{Z}^{s-1}$ ($\mathbf{Z}^{s-1}$ is an abbreviation for $\mathbf{Z}^s/h_n\mathbf{Z}$), and G/H is free of maximal rank, it must happen that $|\det B_n| = 1$ for all but finitely many n (from finite generation of the direct limit). If G/H is free, then $\ker t \oplus H$ is of finite index in G: to see this, note that $G \to G/H$ splits, so there exists a subgroup J of G such that $H \oplus J = G$ and J maps isomorphically to G/H. There is no guarantee that $\ker t \subseteq J$; however, the exact sequence $H \to H \oplus \ker t \to L$ (where L is the image of $\ker t$ in G/H) yields $G/(H \oplus \ker t)$ is finite, since it embeds in $(G/H)/L$, which is finite.

Still in the case that G/H is free, we may discard an initial segment of nonelements of $\mathrm{GL}(s-1, \mathbf{Z})$, so assume each B_n is in $\mathrm{GL}(s-1, \mathbf{Z})$. Then we can systematically pre- and post-multiply the M_n by matrices of the form $\mathrm{diag}(1, E_n)$ where $E_n \in \mathrm{GL}(s-1, \mathbf{Z})$ to arrange that the lower right blocks are all the identity. □

For the general case, the matrices B_n can be put in Hermite normal form (the normal forms arising from the action of $\mathrm{GL}(s-1, \mathbf{Z})$ on $\mathbf{Z}^{(s-1)\times(s-1)}$ from the left). It is not clear whether this would be useful.

An immediate observation is that $e := (1,0,\dots,0)^T$ is a common right eigenvector for all the matrices M_n appearing there, with eigenvalue p_{n+1}, and if we identify G with the direct limit, then $H = \bigcup [e,k]\mathbf{Z}$. We can also recalculate t in terms of the direct limit.

Lemma 8.2. *Let $B_i : \mathbf{Z}^d \to \mathbf{Z}^d$ be a sequence of matrices, and let J be the direct limit as an Abelian group. Suppose that for all i, the left kernel of B_i,* *that is,* $\{w \in \mathbf{Z}^{1\times l} | wB_i = \mathbf{0}\}$, *is the same,* $\mathbf{Z}z$, *for some* $z \in \mathbf{Z}^{1\times l}$. *We may assume that* z *is unimodular. Set* $W = z^{\perp} = \{v \in \mathbf{Z}^l | zv = 0\}$; *then* $B_i W \subseteq W$ *and form the direct limit,* $J_0 := \lim C_i : W \to W$, *where* $C_i = B_i | W$. *Then the natural map* $J_0 \to J$ *given by* $[v,s]_W \mapsto [v,s]$, *is an isomorphism (of Abelian groups).*

Proof. Since $z(B_i W) = 0$, not only is $B_i W \subseteq W$, but in fact $B_i(\mathbf{Z}^l) \subset W$. If $B_{n+t} \cdot B_{n+t-1} \cdots B_{n+1} v = \mathbf{0}$ for v as an element of W, then it is obviously true as an element of $\mathbf{Z}^l$, and it follows that the map $J_0 \to J$ is well defined and one to one. Next, if $y \in \mathbf{Z}^l$, then $B_s y \in W$, so that $[y,s] = [B_s y, s+1]$ which is in the image of the map $J_0 \to J$. Hence, the map is onto. □

In the ECS cases discussed earlier, the sequence of vectors (v^n) is compatible with the addition operation on the Ext group, that is, with the Baer sum $((p_{n+1}, v^n + (v^n)')$ represents the Baer sum of the extensions arising from (p_{n+1}, v^n) and $(p_{n+1}, (v^n)'))$; however, many different sequences can represent the same equivalence class, and it is very difficult to decide when they do. The same applies here, although if G/H is free, then as Abelian groups (but not as extensions), $G \simeq \mathbf{Z}^{s-1} \times H$.

Here ρ denotes the spectral radius. The following is well known in a more general setting, dealing with projective convergence and weak ergodicity. But we do not need this generality in our situation.

LEMMA 8.3. *Let* $G = \lim C_i : \mathbf{Z}^s \to \mathbf{Z}^s$ *be a sequence of primitive matrices for which there exists positive real numbers* $f(m,n)$ *with* $m > n$ *such that* $\lim_{m\to\infty} f(m,k+1)/f(m,k) \to 1/\rho(C_k)$ *for all* k, *and for all* n

$$\lim_{m\to\infty \& m>n} \frac{C_m C_{m-1} \cdots C_n}{f(m,n)} = V_n$$

exists and is strictly positive. Then the candidate map $V : G \to \mathbf{R}^s$ *via* $V[a,k] = V_k / \prod_{i=1}^{k-1} \rho(C_i)$ *is well-defined, and every pure trace of* G *factors through it. In particular,* G *has unique trace iff* $\operatorname{rank} V_n = 1$ *for almost all* n.

REMARK. The simplest situation in which the hypotheses hold occur when $\rho(C_n C_{n-1} \cdots C_m) = \prod_m^n \rho(C_i)$ for all $m > n$, that is, when the spectral radius is multiplicative on the matrices. For example, this occurs when the C_i have a common right Perron eigenvector, or a common left Perron eigenvector.

Proof of Lemma 8.3. $[a,k] = [C_k a, k+1]$, and the latter is sent to $V_{k+1} C_k a / \prod_{i=1}^k \rho(C_i)$. Now $C_m \cdots C_{k+1} C_k / f(m,k) = (C_m \cdots C_{k+1} / f(m, k+1)) C_k (f(m,k+1)/f(m,k))$. The left-hand side converges to V_k; the right-hand side converges to $V_{k+1} C_k / \rho(C_k)$. Hence $V_{k+1} C_k a = \rho(C_k) V_k a$, so V is well-defined.

Next, suppose that $[a,k]$ is an order unit in G; then there exists $m > k$ such that $C_{m-1} C_{m-2} \cdots C_k a$ is strictly positive; as V_m has only nonzero entries, this means $V_m C_{m-1} C_{m-2} \cdots C_k a$ is nonnegative and not all entries are zero, and thus $V[a,k]$ is nonnegative and nonzero (as an element of $\mathbf{R}^s$), and thus V is a positive group homomorphism. Each row of V_n is either zero, or induces a trace on G (via $\mathbf{R}^s \to \mathbf{R}$). Discard any zero rows from V_n (for all sufficiently large n) obtaining a newly-labelled V_n which is now a map from G to $\mathbf{R}^{s'}$ with $s \leq s'$ such that every row of V_n is not zero. Then the map V sends order units if G to order units of $\mathbf{R}^{s'}$.

Conversely, if $[a,k]$ is an arbitrary element of G such that $V[a,k] > 0$, then there exists $m > k$ such that $\|(C_{m-1}C_{m-2}\cdots C_k)/f(m,k)a - V_k a\|$ is smaller than the infimum of the entries of $V_k a$, and thus $(C_{m-1}C_{m-2}\cdots C_k)/f(m,k)a$ is strictly positive, hence $C_{m-1}C_{m-2}\cdots C_k a$ is strictly positive, and thus $[a,k]$ is an order unit of G. Now consider all the traces on G obtained by composing V with any positive vector space map $\mathbf{R}^s \to \mathbf{R}$. What we just obtained is that these are enough to determine the order units of G, and this implies that these traces include all the extreme points in the trace space of G, hence the factorization for pure traces.

If $\operatorname{rank} V_n = 1$, then the trace space is 0-dimensional (after normalization, a single point); conversely, if G has unique trace, then all the composed traces are equal up to normalization, and it follows immediately that $\operatorname{rank} V_n = 1$ for almost all n. $\square$

Suppose A_i are primitive matrices of the same size with common right Perron eigenvector. Then the spectral radius is multiplicative on products of the A_i, and moreover, $A_i/\rho(A_i)$ are uniformly bounded, as are their products. Hence there exists a subsequence, $1 = n(1) < n(2) < n(3)\cdots$, such that for the sequence $(C_i := A_{n(i+1)-1}\cdots A_{n(1)+1}\cdot A_{n(1)})$, we have for all k,

$$\frac{C_m \cdot C_{m-1}\cdot \cdots \cdot C_k}{\prod_{i=k}^m \rho(C_i)}$$

converges to a matrix, necessarily strictly positive (since the row sums are all one, all the rows have nonzero entries; it now follows since C_i are strictly positive, that the limit matrices are strictly positive). Hence by suitably telescoping, we use Lemma 8.3 to derive the pure traces from rows of the limit matrices, and if G has unique trace, the limit matrices eventually have rank one, so we can pick any fixed nonzero row.

Denote by $\mathcal{B}(p,B,v,a)$ for p and a positive integers, $v \in \mathbf{Z}^k$ and $B \in \mathbf{Z}^{k\times k}$, the square matrix of size $k+2$,

$$\mathcal{B}(p,B,v,a) = \begin{pmatrix} * & * & * \\ v + a\mathbf{1} & B + v\mathbf{1}^T + a\mathbf{1}\mathbf{1}^T & v + (a-1)\mathbf{1} \\ a & a\mathbf{1}^T & a \end{pmatrix},$$

where $\mathbf{1}$ is the column of size k consisting of ones, and the column sums are all p (hence the entries marked with an asterisk are uniquely determined); in particular, $\mathbf{1}^T\mathcal{B} = p\mathbf{1}^T$. We will eventually transpose these matrices. It is easy to check that $\mathcal{B}(p,B,v,a)\mathcal{B}(p',B',v',a')=\mathcal{B}(pp',BB',p'v+Bv',p'a)$. Setting $A_n=\mathcal{B}(p_{n+1},B_n,v^n,A_n)$, then inductively

$$A_nA_{n+1}\cdots A_{n+j}=\mathcal{B}\left(\prod_{i=0}^{j}p_{n+i+1},B_n\cdots B_{n+j},v^{(n,j)},a_n\prod_{i=1}^{j}p_{n+i+1}\right),$$

where

$$v^{(n,j)}=p_{n+1}\cdots p_{n+j+1}\left(\frac{v^n}{p_{n+1}}+\frac{B_nv^{n+1}}{p_{n+1}p_{n+2}}+\frac{B_nB_{n+1}v^{n+2}}{p_{n+1}p_{n+2}p_{n+3}}+\cdots\right.$$
$$\left.+\frac{B_n\cdots B_{n+j-1}v^{n+j}}{p_{n+1}\cdots p_{n+j+1}}\right).$$

We assume as we may that $\|B_n\|=\mathbf{O}(p_{n+1}^{1/2})$, and that $\{v^n/p_{n+1}\}$ is bounded. Then $\lim_{j\to\infty}v^{(n,j)}/p_{n+1}\cdots p_{n+j+1}$ exists (provided $p_n\to\infty$); call it $V^{\infty,n}$; this forces $A_{n+j}^TA_{n+j-1}^T\cdots A_n^T/p_{n+1}\cdots p_{n+j+1}$ to converge to the rank one matrix

$$\mathbf{1}_{k+2}\left(1-\left(V^{\infty,n}\right)^T\mathbf{1}-\frac{(k+1)a^n}{p_{n+1}},V^{\infty,n}+\frac{a^n}{p_{n+1}}\mathbf{1}^T,\frac{a^n}{p_{n+1}}\right).$$

Call the row appearing in this factorization, $W^{\infty,n}$; if the A_n are primitive, then it is strictly positive. The family $\{W^{\infty,n}\}$ satisfies $W^{\infty,n}A_n^T=W^{\infty,n-1}$, hence induces a trace on the dimension group $G=\lim A_n^T:\mathbf{Z}^{k+2}\to\mathbf{Z}^{k+2}$ via $\tau[x,m]=W^{\infty,m}x$. As G has unique trace, this is it, up to scalar multiple.

Now we consider G as an Abelian group with trace; then we obtain a group isomorphism from the restriction to $(z^T)^\perp$ (where $z=(-(k+1),1,1,\dots,1)^T$); using as ordered $\mathbf{Z}$-basis for the latter, the columns $((1,\dots,1)^T,(0,1,0,\dots,0,-1)^T,\dots,(0,0,\dots,0,1,-1)^T)$ (it is easy to check that this *is* a $\mathbf{Z}$-basis), the group isomorphism from Lemma 8.2 is with the group given as $J:=\lim M_n=\left(\begin{smallmatrix}p_{n+1}&(v^n)^T\\0&B_n^T\end{smallmatrix}\right):\mathbf{Z}^{k+1}\to\mathbf{Z}^{k+1}$. Moreover, the effect of $W^{\infty,1}$ on the basis yields the group homomorphism obtained from the rows $R_n:=(1,V^{\infty,1})$; that is, the corresponding homomorphism from J to $\mathbf{R}$ is given by $t[x,k]=R_kx$. Since each M_n is one to one, a group homomorphism from J is uniquely determined by its affect on the first level, that is, on elements of the form $[x,1]$.

In the following, the norms on rows are the maximum of the absolute values, and the norms on matrices are the maximum absolute column sums.

Lemma 8.4. *Let $p_{n+1}\uparrow\infty$, let B_i be $k\times k$ integer matrices such that $\det B_i\neq 0$ and $\|B_i\|=\mathbf{o}(p_i^{1/2})$, and let $z_i\in\mathbf{Z}^{1\times k}$ with $\|z_i\|<p_{i+1}$; let $r^1=(1,\rho^1)$ where $\rho^1\in\mathbf{R}^{1\times k}$. Then there exist a sequence $\{w^i\}$, with $w^i\in\mathbf{Z}^k$ and $\|w^i\|<\|z_i\|+(p_{i+1}+\|B_i\|)/2$ for all $i>1$ together with group isomor-*

phisms $F_i : \mathbf{Z}^{k+1} \to \mathbf{Z}^{k+1}$ *such that the following diagram*

$$
\begin{tikzcd}
 & \mathbf{Z}^{k+1} \arrow[r, "\left(\begin{smallmatrix} p_2 & z_1 \\ 0 & B_1 \end{smallmatrix}\right)"] \arrow[dl, "r^1"'] \arrow[dd, "F_1"'] & \mathbf{Z}^{k+1} \arrow[r, "\left(\begin{smallmatrix} p_3 & z_2 \\ 0 & B_2 \end{smallmatrix}\right)"] \arrow[dd, "F_2"'] & \mathbf{Z}^{k+1} \cdots \arrow[dd, "F_3"'] & \mathbf{Z}^{k+1} \arrow[r, "M_n := \left(\begin{smallmatrix} p_{n+1} & z_n \\ 0 & B_n \end{smallmatrix}\right)"] \arrow[dd, "F_n"'] & \mathbf{Z}^{k+1} \arrow[r, "\cdots"] \arrow[dd, "F_{n+1}"'] & {} \\
\mathbf{R} & & & & & & \\
 & \mathbf{Z}^{k+1} \arrow[r, "\left(\begin{smallmatrix} p_2 & w_1 \\ 0 & B_1 \end{smallmatrix}\right)"] \arrow[ul, "r^1"] & \mathbf{Z}^{k+1} \arrow[r, "\left(\begin{smallmatrix} p_3 & w_2 \\ 0 & B_2 \end{smallmatrix}\right)"] & \mathbf{Z}^{k+1} \cdots & \mathbf{Z}^{k+1} \arrow[r, "\left(\begin{smallmatrix} p_{n+1} & w_n \\ 0 & B_n \end{smallmatrix}\right)"] & \mathbf{Z}^{k+1} \arrow[r, "\cdots"] & {}
\end{tikzcd}
$$

commutes, and such that

$$\rho^1 = \frac{w^1}{p_2} + \frac{w^2 B_1}{p_3 p_2} + \frac{w^3 B_2 B_1}{p_4 p_3 p_2} + \frac{w^4 B_3 B_2 B_1}{p_5 p_4 p_3 p_2} + \cdots . \tag{2}$$

Proof. We will define $F_i = \left(\begin{smallmatrix} 1 & y_i \\ 0 & \mathrm{I}_k \end{smallmatrix}\right)$ (where $y_i \in \mathbf{Z}^{1\times k}$), and then define $w_i \in \mathbf{Z}^{1\times k}$ so that all the properties hold. First, set $y_1 = \mathbf{0}$. Now define

$$
\begin{aligned}
z^\infty &= \frac{z_1}{p_2} + \frac{z_2 B_1}{p_3 p_2} + \frac{z_3 B_2 B_1}{p_4 p_3 p_2} + \cdots \\
&= \frac{z_1}{p_2} + \sum_{i=2}^{\infty} \frac{z_i B_{i-1} B_{i-2} \cdots B_1}{p_{i+1} p_i \cdots p_2}.
\end{aligned}
$$

That the sum exists is a consequence of $\|z_i\|/p_{i+1}$ being bounded, $\|B_i\| = \mathbf{o}(\sqrt{p_{i+1}})$, and summability of $1/\sqrt{p_i p_{i-1} \cdots p_2}$.

Now define

$$y_{n+1} = \left[p_{n+1} p_n \cdot \cdots \cdot p_2 \left(\rho^1 - z^\infty\right) (B_n \cdot B_{n-1} \cdot \cdots \cdot B_1)^{-1}\right];$$

of course, the inverses of B_i exist as matrices with rational entries. Here the integer function $[\,\cdot\,]$ means to take the nearest integer in each entry. Let Y_{n+1} denote the thing on the right before we take the integer function; it is an element of $\mathbf{R}^{1\times k}$. Then obviously we have $y_{n+1} \in \mathbf{Z}^{1\times k}$ and $\|y_{n+1} - Y_{n+1}\| \leq 1/2$.

Finally, set $w^n = z_n + y_{n+1} B_n - p_{n+1} y_n$. It is easy to check that the squares in the diagram all commute. We show that ρ^1 is the infinite sum in (2).

Let S_n be the sum of the first n terms on the right-hand side of (2). When we substitute $w^i = z_i + y_{i+1} B_n - p_{n+1} y_i$, we find that the series partially telescopes:

$$S_n = \left(\frac{z_1}{p_2} + \sum_{i=2}^{n} \frac{z_i B_{i-1} B_{i-2} \cdots B_1}{p_{i+1} p_i \cdots p_2}\right) + \frac{y_{n+1} B_n B_{n-1} \cdots B_1}{p_{n+1} \cdots p_1},$$

as follows immediately by induction. Now $Y_{n+1}B_n \cdot \cdots \cdot B_1(p_{n+1}\cdots p_2)^{-1} = \rho^1 - z^\infty$, hence $\|y_{n+1}B_n \cdot \cdots \cdot B_1(p_{n+1}\cdots p_2)^{-1} - (\rho^1 - z^\infty)\| < 1/\sqrt{p_{n+1}\cdot \cdots \cdot p_1}$. Thus, $\lim S_n$ exists and

$$\lim S_n = z^\infty + \left(\rho^1 - z^\infty\right) = \rho^1.$$

Next, we estimate $\|w^i/p_{i+1}\|$. We have

$$\begin{aligned}\left\|\frac{w^n - z_n}{p_{n+1}}\right\| &= \left\|\frac{y_{n+1}B_n - p_{n+1}y_n}{p_{n+1}}\right\| \\ &\leq \left\|\frac{Y_{n+1}B_n - p_{n+1}Y_n}{p_{n+1}}\right\| + \left\|\frac{(Y_{n+1} - y_{n+1})B_n + p_{n+1}(Y_n - y_n)}{p_{n+1}}\right\| \\ &\leq 0 + \left\|\frac{B_n}{2p_{n+1}}\right\| + \frac{1}{2}.\end{aligned}$$

Thus, $\|w^n/p_{n+1}\| \leq \|z_n/p_{n+1}\| + (1/2)(\|B_n\|/p_{n+1} + 1)$. □

We are permitted to telescope the bottom row, and then apply the same transformation to the resulting upper right corner entries as we did in the ECS case (for the lower left corners), conjugating by a block upper triangular element of $\mathrm{GL}(k+2, \mathbf{Z})$, to ensure we could choose the a^n so that the resulting matrices are positive. This yields the following.

THEOREM 8.5. *Let G be a simple dimension group of rank $k+1$ with unique trace τ, and let H be a noncyclic rank one subgroup of G such that G/H is torsion-free and $\tau(H) \neq 0$. Then there exists an ERS realization of G of size $k+2$ such that the image of H in the direct limit is $\bigcup_{j\in\mathbf{Z}}[(1,1,\ldots,1)^T, j]\mathbf{Z}$. If G/H is free, then the extension $0 \to \ker\tau \to G \to \tau(G) \to 0$ is nearly split.*

This does not require the trace to be rational-valued; since there is no restriction on $\tau(G)$ except $\tau(H) \neq 0$, the value group, $\tau(G)$ can be an arbitrary subgroup of $\mathbf{R}$ containing $\tau(H)$ and of rank at most $k+1$ (when equality occurs, G is totally ordered).

9. Infinite rank ERS

The following is a routine argument involving direct limits, but it allows us to prove Theorem 7.1(b) via Theorem 7.1(a), as well as results on simultaneous ERS and ECS realizations (ECRS).

LEMMA 9.1. *Let G_n be a family of dimension groups and let $\phi^n : G_n \to G_{n+1}$ be ordered group homomorphisms that send order units to order units. Let G be the ordered group $\lim \phi^n : G_n \to G_{n+1}$.*

(a) *Suppose H_n are noncyclic rank one subgroups of G_n such that $H_n \cap G^{++} \neq 0$ for all n, $\phi^n(H_n) \subseteq H_{n+1}$, and each G_n admits an ERS realization with respect to H_n. Define $H = \lim \phi_n|H_n$. Then G admits an ERS realization with respect to H obtained from telescoping.*

(b) *Suppose that* (G,u) *is given as the direct limit of* $\psi^j : G^j \to G^{j+1}$ *where each* G^j *is a simple dimension group with unique trace, and each admits an ECS realization. Then* G *admits an ECS realization with respect to its unique trace.*

(c) *If each of the* G_n *admit an ERS realization with respect to* H_n *that is simultaneously ECS, then* G *admits an ERS realization with respect to* H *that is also ECS.*

Proof. (a) For each n, let $G_n \simeq \lim_i \phi_i^n : F_i^n \to F_{i+1}^n$ (where $F_i = \mathbf{Z}^{f(i,n)}$ with the simplicial ordering) be an ERS realization of G_n with respect to H_n. Let $\{e_{ji}^n\}$ be the standard basis of F_i^n. We may of course replace $\simeq$ by equality. Since ϕ^n is positive, given i, there exists $m \equiv m(i,n)$ such that for all $j \le f(i,n)$, $\phi^n[e_{ji}^n] = [v^{j,i,m}, m]$ where $v^{j,i,m}$ has all of its entries nonnegative.

Since $\phi^n(H_n) \subseteq H_{n+1}$, we have $\phi^n[\mathbf{1}_{f(i,n)}, i] = p[\mathbf{1}_{f(l,n+1)}, l]$ for some integers $l > n$ and $p \ge 1$. Thus $\sum_{j=1}^{f(i,n)} [e_{ji}^n, n] = p[\mathbf{1}_{f(l,n+1)}, l]$. Hence there exists $m' \equiv m'(i,n)$ such that for all $j \le f(i,n)$, $\phi^n[e_{ji}^n] = [w^{j,n+1}, m']$ where $w^{j,n+1} \ge 0$ and $\sum_{j=1}^{f(m',n+1)} w^{j,n+1}$ is a multiple of $\mathbf{1}_{f(m',n+1)}$. This means we can define a positive matrix $C_i^n : F_i^n \to F_{m'(i,n)}^{n+1}$ which sends $\mathbf{1}_{f(i,n)}$ to a multiple of $\mathbf{1}_{f(m',n+1)}$; in particular, C_i^n has equal row sums.

Beginning with $i = 1$, we obtain a telescoping of the sequence for G_2 by composing and then relabelling $F_{m'(1,1)}^2$ as F_1^2 (and telescoping and relabelling the mappings), $F_{\max\{m'(1,1),m'(2,1)\}}^2$ as F_2^2, etc., so that now the matrices C_i^1 go straight down, that is map $F_i^1 \to F_i^2$ (in the new notation). We may iterate this construction inductively. It is now straightforward that $G \simeq \lim C^{n,n} \circ \phi_n^n : F_n^n \to F_{n+1}^{n+1}$, the order-isomorphism sending H to the obvious limit of H_n, that is we have an ERS realization of G with respect to H.

(b) That G has unique trace is trivial. As in the preceding argument, we may telescope the various rows, and assume that each ϕ^j is implemented by *nonnegative* matrices $A_i^j : F_i^j \to F_i^{j+1}$, and we may assume that no A_i^j has any zero rows (in fact, since each G^j is simple, it is easy to arrange that the matrices be strictly positive).

The element u comes from some G^j, so we may normalize the unique trace τ_j of G^j at its pre-image. Then $\tau_{j+1} \circ \phi^j = \tau^j$ (from uniqueness); hence, $\tau^j[f,n] = \tau^{j+1}[A_n^j f, n]$ (where $f \in F_n^j$). Since we have assumed each realization is ECS, this says $\tau^j[e_i, n] = \tau^j[e_{i'}, n]$ for all standard $\mathbf{Z}$-basis elements $e_i, e_{i'}$ of F_n^j, for all j, and in particular, $\tau^j[f,n] = \lambda_{j,n} \sum f_i$, where $f = \sum f_i e_i$ for some positive rational number $\lambda_{j,n}$.

Thus for each basis element e_i, we have (where $\mathbf{1}^T$ represents the row of the appropriate size consisting of ones)

$$\lambda_{j+1,n} \mathbf{1}^T A_n^j e_i = \tau^{j+1}\left[A^j e_i, n\right] = \tau_j[e_i, n] = \lambda_{j,n}.$$

Since the last term is independent of the choice of i, we have that all the $\mathbf{1}^T A_n^j e_i$ are the same, as i varies. This means exactly that the column sums of A_n^j are all equal. Now the diagonal argument (as in (a)) can be applied.

(c) In the simultaneous case, we first ensure that the process in (a) is carried out, then apply the method of (b). $\square$

LEMMA 9.2. *Let G be a simple dimension group with unique trace τ, and let H be a noncyclic rank one subgroup such that $\tau(H) \neq 0$ and G/H is torsion-free. Then we can write $G = \bigcup G_n$ where $G_n \subset G_{n+1}$ are simple dimension groups with unique trace* (*in the relative ordering*), *each of finite rank, and each containing H.*

Proof. Consider the subgroup of the reals, $\tau(G)$; since this is countable, and $\tau(H)$ is contained in it, we can find a countable set of elements $\{r_n\}$ such that with $J_n := \tau(H) + \sum_{i=1}^n r_i \mathbf{Z} \subset \mathbf{R}$, we have $\tau(H) \subseteq J_n \subseteq J_{n+1}$ and $\bigcup J_n = \tau(G)$. Select $g_i \in G$ such that $\tau(g_i) = r_i$.

Now $\ker \tau$ is a countable torsion-free Abelian group (and nothing else: every countable torsion-free Abelian group can appear as a $\ker \tau$); we may thus write it as an increasing union of free Abelian groups of finite rank (this is completely elementary: list the elements, then take increasing finite subsets), say $\ker \tau = \bigcup T_n$, each T_n of finite rank.

Finally, set $G_n = T_n + H + \sum_{1 \leq i \leq n} g_i \mathbf{Z}$. Then $H \subseteq G_n \subset G_{n+1} \subset \cdots$. Since $\tau(H) \subseteq \tau(G_n)$, the range of $\tau | G_n$ is dense, and it is immediate that with the relative ordering inherited from G, G_n is a simple dimension group with unique trace τ. Next let $G_0 = \bigcup G_n \subseteq G$; we note that $\ker \tau = \bigcup A_n \subset G_0$, and $\tau(G_0) = \tau(G)$ by construction. Hence $G_0 = G$. Since each of T_n, H, and $\sum_{1 \leq i \leq n} g_i \mathbf{Z}$ is of finite rank, so is G_n. $\square$

COROLLARY 9.3 (Theorem 7.1(b)). *Let G be a* (*countable*) *simple dimension group with unique trace τ, and let H be a noncyclic rank one subgroup of G such that $\tau(H) \neq 0$ and G/H is torsion-free. Then there is a co-rank one ultramatricial ERS realization of G with respect to H.*

Proof. By the preceding, we can write $G = \bigcup G_n$ with $H \subset G_n$, where each G_n is a finite rank simple dimension group with unique trace given by the restriction of τ. Since G/H is torsion-free and $G_n/H \subseteq G/H$, we have G_n/H is torsion-free. Hence, each G_n admits an ERS realization with respect to H. The inclusion maps $G_n \to G_{n+1}$ send H onto H, and implement a realization of G as a direct limit of the G_ns, hence Lemma 9.1 applies. $\square$

There are still questions about realizations that both ERS and ECS (simultaneously; that is, the matrices have their row sums equal, and their column sums equal). These will be addressed in the next two sections.

10. ECRS and nearly split extensions

A realization is ECRS if it is simultaneously ECS and ERS. The trace induced by normalized multiples of the rows $\mathbf{1}_{f(n)}$ is automatically rational-valued, and will be denoted τ. If G admits an ECRS realization wherein, viewed as an ERS realization, it is with respect to H, then we shall write, *an ECRS realization with respect to H.* It is routine to see that if G admits an ECRS realization with respect to H that is of size s (so all the matrices have both $\mathbf{1}_s$ and $\mathbf{1}_s^T$ as their right and left Perron eigenvectors, respectively), then $|\tau(G)/\tau(H)|$ divides s: the image of the trace on G is $\bigcup(1/p_{j+1}\cdots p_2)\mathbf{Z}$, while on the image of the subgroup $\bigcup[\mathbf{1}_s,k]\mathbf{Z}$, it is $\bigcup(s/p_{j+1}\cdots p_2)\mathbf{Z}$, and the one by the other is a quotient of $\mathbf{Z}/s\mathbf{Z}$, hence has order dividing s.

This puts a fairly stringent condition on the matrix sizes required for bounded ECRS realizations. Of course, unbounded ECRS realizations can be obtained as direct limits (obtained from unions) of bounded ones (exactly as in the case of ERS realizations).

LEMMA 10.1. *Suppose $t: G\to U\subseteq \mathbf{Q}$ is obtained as the direct limit $G\simeq \lim C_n : \mathbf{Z}^s \to \mathbf{Z}^s$ where the map t is obtained from a common left eigenvector w of all the C_n (with corresponding eigenvalue c_{n+1}), via $t[a,n]=wa/c_1\cdots c_n$. Suppose in addition, the C_n have a common right eigenvector v and $wv\neq 0$. Then the extension $0\to \ker t\to G\to U\to 0$ is nearly split.*

Proof. We note that the eigenvalue of v for C_n must be the same as that of w, c_{n+1}, since $vw\neq 0$. Set $H=\bigcup[v,n]\mathbf{Z}\subset G$. Then $\bigcup(c_2\cdots c_n)^{-1}vw\mathbf{Z}\subseteq H$, and this is obviously of finite index in $U=t(G)=\bigcup(c_1\cdots c_n)^{-1}\mathbf{Z}$. Thus $\ker t\oplus H$ is of finite index in G, so the extension is nearly split. □

When $\ker t$ is free, the index of the image of $\ker t$ in G/H is finite. In that case, $\ker t\oplus H$ has finite index in G, so that the extension $\ker t\to G\to U$ is nearly split. Thus, we have the following.

LEMMA 10.2. *For any simple dimension group of finite rank with unique trace, which is rational-valued, $t: G\to U\subseteq \mathbf{Q}$ such that $\ker t$ is free, and admits a bounded ERS realization, the extension $\ker t\to G\to U$ is nearly split.*

Although freeness of J/H implies $J\to J/H$ splits—yielding a group isomorphism $J\simeq \mathbf{Z}^{s-1}\times H$—this does not imply that $J\to U$ splits. In the stationary and ECRS example, $G=\lim\left(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right):\mathbf{Z}^2\to\mathbf{Z}^2$, we have $\tau(G):=U=\mathbf{Z}[1/3]$ and $\tau(H)=2\mathbf{Z}[1/3]$ (where $H=\bigcup_n[\mathbf{1}_2,n]\mathbf{Z}$), so the extension $0\to\mathbf{Z}\to G\to U\to 0$ does not split, although $G\simeq \mathbf{Z}\oplus\mathbf{Z}[1/3]$ (as Abelian groups) and $t|H\neq 0$. As we will see in the next section, this is fairly typical.

Dropping the strong assumption that $\ker t$ or J/H be free, a sufficient condition for the extension $J\to U$ to be nearly split, that is, $\ker t\oplus H$ be of finite index in J, is that the image of $\ker t$ be of finite index in J/H.

This places restrictions on realizations of dimension groups by commuting primitive matrices. For example, if G is a simple dimension group with unique trace, and it is realized by commuting nonnegative matrices, we can telescope and kill off zero rows, and arrange that the matrices additionally be primitive. Hence, they will have common left and common right Perron eigenvectors and the unique trace is determined from the left Perron eigenvector. Thus, the corresponding extension $0 \to \ker \tau \to G \to \tau(G) \to 0$ given by the image of the trace must be nearly split.

If for example, $\ker \tau = \mathbf{Z}^k$ and $\tau(G) = U$ has an interesting supernatural number (e.g., every prime has multiplicity at most one), then the set of isomorphism classes of nearly split extensions within the class of extensions of $\mathbf{Z}^k$ by U is negligible. So most extensions cannot be given by commuting families of matrices. Certainly a realization of bounded matrix size that is both ERS and ECS qualifies for Lemma 10.1, as does a stationary dimension group.

In particular, Example 10.4 below is a simple dimension group with unique and rational-valued trace, of rank two, that cannot be realized by any sequence of (square) primitive matrices which have common right and common left eigenvectors; in particular, it cannot be realized by a bounded sequence of simultaneously ERS and ECS primitive matrices. It *can* be realized by a sequence of increasing size strictly positive rectangular matrices, $M_n : \mathbf{Z}^{f(n)} \to \mathbf{Z}^{f(n+1)}$, where $f(n) \to \infty$, and each M_n is both ERS and ECS, as we will see in the next section. It can be shown that for any such realization, $f(n+1)/f(n)$ must be divisible by 3 for infinitely many n.

There is a trivial case in which the extension must be nearly split.

LEMMA 10.3. *Suppose $t : G \to U$ is a finite rank torsion-free group such that $U = \mathbf{Z}[1/p]$ for some prime p. If G contains a noncyclic rank one subgroup that is disjoint from* $\ker t$, *then the extension* $0 \to \ker t \to G \to U \to 0$ *is nearly split.*

REMARK. Stationary examples show that such extensions need not be split.

Proof of Lemma 10.3. Obviously, t induces an embedding $H \to t(G)$. The extension is nearly split because every noncyclic subgroup of $\mathbf{Z}[1/p]$ is of finite index! $\square$

Let p be a prime; all noncyclic subgroups of $U = \mathbf{Z}[1/p]$ are therefore of finite index. Hence, if in the situation of Lemma 10.3, $t(G)$ is isomorphic to $\mathbf{Z}[1/p]$ (and H is not cyclic, which is part of the hypotheses), then the corresponding extension is nearly split (Lemma 10.3). Another situation arises when the realization is by commuting matrices, or more generally, when the implementing matrices have common left eigenvector and common right eigenvector. In the situation arising from positive matrices, these must be the Perron eigenvectors, hence correspond to the same eigenvalue (for each n).

EXAMPLE 10.4. An example of a simple dimension group with unique, rational-valued trace, which is ERS-realizable, but for which the corresponding extension, $0 \to \ker t \to G \to t(G) = U \to 0$, is not nearly split.

We construct a simple example for which the subgroup $H \simeq \mathbf{Z}[1/3]$ and $t(G) \simeq \mathbf{Z}[1/6]$. In this case, the extension $\mathbf{Z} \to G \to \mathbf{Z}[1/6]$ is not nearly split, but the corresponding dimension group admits an ERS realization (of size three). It also admits an ECS realization, but cannot have a simultaneously ECS and ERS realization of bounded size (since that would imply common left and common right eigenvectors, which entails nearly splitting by Lemma 10.1).

Construct an extension $0 \to \mathbf{Z} \to G_0 \to \mathbf{Z}[1/2] \to 0$ for which there are no 2-divisible elements in G_0, equivalently, the extension is not nearly split. Let $t_0 : G \to \mathbf{Z}[1/2]$ denote the map. We may regard G_0 as a subgroup of its divisible hull, which is of course $\mathbf{Q}^2$; t_0 extends uniquely to a group homomorphism $T : \mathbf{Q}^2 \to \mathbf{Q}$. Pick an element of G_0, $u \in t_0^{-1}(1)$, and form $G = G_0 + u\mathbf{Z}[1/3]$ (inside $\mathbf{Q}^2$). Then T restricts to a map, called $t : G \to \mathbf{Q}$, with values in $\mathbf{Z}[1/2] + \mathbf{Z}[1/3] = \mathbf{Z}[1/6]$.

Now we show that $\ker t = \ker t_0 = \mathbf{Z}$, so that $\mathbf{Z} \to G \to \mathbf{Z}[1/6]$ is the corresponding extension. Elements of G are of the form $g = g_0 - um/3^k$ for $g_0 \in G_0$, m an integer, and k a nonnegative integer. If $t(g) = 0$, then $t(g_0) = m/3^k$; there exist integers l and nonnegative j such that $t(g_0) = l/2^j$. Hence $3^k l = 2^j m$. This forces 3^k to divide m, so $g \in G_0$. Hence $\ker t_0 = \ker t$.

Next, we show that G contains no 2-divisible elements. Select $g = g_0 + um/3^k$ in G with $g_0 \in G_0$ as in the previous paragraph. If g were 2-divisible, for all positive integers l, we could solve the equations

$$g_0 + u\frac{m}{3^k} = 2^l\left(g_l + u\frac{m_l}{3^{k(l)}}\right),$$

where $g_l \in G_0$; we may assume that m and m_l are relatively prime to 3. Suppose for now that $k, k(l) > 0$. Fix l and multiply by 3^k. This yields $3^k g_0 + mu = 3^k 2^l g_l + 2^l 3^{k-k(l)} m_l u$. Thus $2^l m_l 3^{k-k(l)} u \in G_0$, so its value at t_0, $2^l m_l 3^{k-k(l)} \in \mathbf{Z}[1/2]$. Thus if $k(l) > k$, we must have 3 dividing m_l, a contradiction. Thus $k \geq k(l)$.

This yields $3^k(g_0 - 2^l g_l) = u(m - 2^l 3^{k-k(l)})$. Evaluating at t_0, we obtain $3^k t_0(g_0 - 2^l g_l) = m - 2^l 3^{k-k(l)}$. If $k > k(l)$, then 3 divides m (as the values of t_0) lie in $\mathbf{Z}[1/2]$, again a contradiction. Hence $k = k(l)$, so that $m - 2^l \equiv 0 \mod 3^k$. Since m and k are fixed, but the $\bmod 3$ equivalence classes of 2^l alternate between $1, 2$, this is impossible.

Let us dispose of the remaining possibilities; first, if $k = 0$, we have the equations $g + mu = 2^l g_l + 2^l m_l 3^{-k(l)} u$, so $2^l m_l 3^{-k(l)} u \in G_0$; evaluating at t_0, we obtain $2^l m_l 3^{-k(l)} \in \mathbf{Z}[1/2]$; since 3 does not divide m_l, we must have $k(l) = 0$ for all l. But then the element $g + mu = 2^l(g_l + m_l u)$ is 2-divisible within G_0, a contradiction.

Next, if $k(l)=0$ for one value of $l>0$, then $um3^{-k}\in G_0$, which forces $k=0$ (evaluate at t_0 again), and we are in the preceding case.

Thus G contains no 2-divisible subgroup. Since any subgroup of finite index in $\mathbf{Z}[1/6]$ must be 2-divisible, the extension cannot be nearly split. On the other hand, the subgroup $H=u\mathbf{Z}[1/3]$ of G is 3-divisible, so there is a group realization of the form described in Corollary 9.3, that is, common right eigenvector, and a corresponding ERS realization for the dimension group. But there cannot be a dimension group realization that is both ERS and ECS simultaneously when the matrix size is bounded.

In terms of the B_n, a necessary condition for $G\to\tau(G)$ to split is that if $d_n=|\det B_n|$, then H is of finite index in $\bigcup(1/\prod_{i=1}^n d_i)\mathbf{Z}+t(H)$ (e.g., if p_{n+1} are powers of the same prime p, this would force almost all the d_n to be powers of p). But this is not sufficient.

11. ECRS

Suppose that (G,H) is a simple dimension group with noncyclic rank one subgroup such that G/H is torsion-free, and in addition that G admits a unique trace τ. Moreover, assume that $\tau(G):=U$ is a subgroup of the rationals, and $\tau(H)\neq 0$. These conditions (except the uniqueness of the trace) are necessary for an ECRS realization of G with respect to H.

The converse is not quite true. We will show that if U is p-divisible for some prime p (i.e., at least one prime has infinite multiplicity in the supernatural number of U), then the converse is true. However, in case U is not p-divisible for any prime p, then an ECRS realization exists with respect to H exists if and only if $\operatorname{rank} G\leq|\tau(G)/\tau(H)|$. In this formulation, we allow ∞ as a value, and this corresponds to unbounded realizations. In the cases that $\operatorname{rank} G<\infty$, we have some control on the size of the realization.

In particular, if $\tau(G)$ has no primes with infinite multiplicity, and $\operatorname{rank} G>1$ (the case of $\operatorname{rank} G=1$ is trivial), then the split case, $G=U\oplus\ker\tau$ with the strict ordering from the projection onto U, does not admit an ECRS realization. If in addition, $\ker\tau$ is free of finite rank, by earlier results, then G admits both an ECS realization and an ERS realization with respect to H, of the same size, but no ECRS realizations at all.

We begin with the case that $|\tau(G)/\tau(H)|<\infty$. This of course implies that $G\to U$ is nearly split. For now, we also assume G/H is free and finite rank. Then we can write $G=H\oplus\mathbf{Z}^k$ (with $\tau(H)=n\tau(G)$ for some integer n), but we must recall that $\ker\tau$ is not the copy of $\mathbf{Z}^k$ that appears as a direct summand.

For a row or column v consisting of integers, the *content* of v, denoted $c(v)$ is the greatest common divisor of the nonzero entries of v.

LEMMA 11.1. *Let $\lambda,p_{n+1}>1$ be positive integers such that for all n, $p_{n+1}\equiv 1 \mod \lambda$, and let $\rho\in\mathbf{Z}^k$ be a vector such that $(c(\rho),\lambda)=1$; set $q_n=p_2\cdot\cdots\cdot p_n$.*

For each n, define $M_n = \begin{pmatrix} p_{n+1} & 0 \\ 0 & \mathrm{I}_k \end{pmatrix}$, and $r^n = (\lambda/p_2 \cdot \cdots \cdot p_n, \rho)$ for $n > 1$, and $r_1 = (\lambda, \rho) \in \mathbf{Z}^{1\times(s+1)}$; with $G = \lim M_n$ (as Abelian groups) define $t : G \to \mathbf{Q}$ by $t[w,n] = r^n w$. Then there exist $v_n = \rho(p_{n+1}-1)/\lambda, y_n = \rho(q_n - 1)/\lambda \in \mathbf{Z}^{1\times s}$ such that for all n, the following diagram commutes,

$$\begin{tikzcd}
 & \mathbf{Z}^{k+1} \arrow[r, "M_1"] \arrow[dl, "r^1"'] \arrow[dd, "\begin{pmatrix} 1 & y_1 \\ 0 & \mathrm{I}_k \end{pmatrix}"'] & \mathbf{Z}^{k+1} \arrow[r, "M_2"] \arrow[dd, "\begin{pmatrix} 1 & y_2 \\ 0 & \mathrm{I}_k \end{pmatrix}"'] & \mathbf{Z}^{k+1} \arrow[r, phantom, "\cdots"] \arrow[dd, "\begin{pmatrix} 1 & y_3 \\ 0 & \mathrm{I}_k \end{pmatrix}"'] & \mathbf{Z}^{k+1} \arrow[r, "M_n"] \arrow[dd, "\begin{pmatrix} 1 & y_n \\ 0 & \mathrm{I}_k \end{pmatrix}"'] & \mathbf{Z}^{k+1} \arrow[r, "\cdots"] \arrow[dd, "\begin{pmatrix} 1 & y_{n+1} \\ 0 & \mathrm{I}_k \end{pmatrix}"'] & {} \\
\mathbf{Q} & & & & & & \\
 & \mathbf{Z}^{k+1} \arrow[r, "\begin{pmatrix} p_2 & v_1 \\ 0 & \mathrm{I}_k \end{pmatrix}"] \arrow[ul, "r^1"] & \mathbf{Z}^{k+1} \arrow[r, "\begin{pmatrix} p_3 & v_2 \\ 0 & \mathrm{I}_k \end{pmatrix}"] & \mathbf{Z}^{k+1} \arrow[r, phantom, "\cdots"] & \mathbf{Z}^{k+1} \arrow[r, "\begin{pmatrix} p_{n+1} & v_n \\ 0 & B_n \end{pmatrix}"] & \mathbf{Z}^{k+1} \arrow[r, "\cdots"] & {}
\end{tikzcd}$$

and in addition, $r^i M_i = r^{i-1}$ and r^1 is a common left eigenvector of all the matrices $\begin{pmatrix} p_{n+1} & v_n \\ 0 & B_n \end{pmatrix}$, with corresponding eigenvalue p_{n+1}.

If we set $H = \bigcup [\mathbf{1}_{k+1}, n]\mathbf{Z}$, then $t(H) = \lambda t(G)$.

Proof. Set $y_1 = \mathbf{0}$. To have r^1 as common left eigenvector, we must have $(\lambda/q_n)y_n + \rho = \rho/q_n$, that is, $y_n = \rho(q_n - 1)/\lambda$; as $p_i \equiv 1 \mod \lambda$, λ divides $q_n - 1$, hence y_n has only integer entries.

For the square to commute (now that we have define all the y's), it is equivalent to $y_{n+1} = p_{n+1}y_n + v_n$, that is, we set $v_n = y_{n+1} - p_{n+1}y_n = ((q_{n+1} - 1)/\lambda - p_{n+1}(q_n - 1)/\lambda)\rho$, and this simplifies to $v_n = \rho(p_{n+1} - 1)/\lambda$.

At the nth level, the trace is given by $(\lambda/q_n, \rho)$, so its image is $q_n^{-1}(\lambda\mathbf{Z} + q_n c(\rho)\mathbf{Z})$. Since $\gcd(q_n, \lambda) = \gcd(c(\rho), \lambda) = 1$, we have $\gcd(\lambda, q_n) = 1$. Hence the range of the trace on the nth level is $q_n^{-1}\mathbf{Z}$, so that $t(G) = \bigcup q_n^{-1}\mathbf{Z}$. On the other hand, $t[(1,0,\dots,0)^T, n] = \lambda/q_n$. Hence, the range of t on $H = \bigcup [(1,0,\dots,0)^T, n]\mathbf{Z}$ is $\bigcup \lambda q_n^{-1} = \lambda t(G)$. □

Under the assumptions of Lemma 11.1, set $v_0 = (1,1,0,\dots,0) \in \mathbf{Z}^k$. There exists $E_0 \in \mathrm{GL}(k,\mathbf{Z})$ such that $\rho E_0^{-1} = c(\rho)(1,0,\dots,0)$. Then

$$v_0(1 - p_{n+1}) + \frac{p_{n+1} - 1}{\lambda}\rho E_0^{-1} = \frac{p_{n+1} - 1}{\lambda}\big(c(\rho) + \lambda, \lambda, 0, \dots, 0\big).$$

Now $\gcd\{\lambda, c(\rho) + \lambda\} = \gcd\{\lambda, c(\rho)\} = 1$. There thus exists $E_1 \in \mathrm{GL}(k,\mathbf{Z})$ such that $(c(\rho) + \lambda, \lambda, 0, \dots, 0)E_1^{-1} = (1,1,\dots,1)$. Setting $v = v_0 E_0^{-1} E_1$ and $E = E_1 E_0$ then for all n, $v(1 - p_{n+1}) + \rho(c(\rho) + \lambda, \lambda, 0, \dots, 0)E = \frac{p_{n+1}-1}{\lambda}(1,1,\dots,1)$.

Let $u = (1, 1, \ldots, 1) \in \mathbf{Z}^k$. Now for any choice of integer p (such that λ divides $p - 1$), we have

$$\begin{pmatrix} 1 & v \\ 0 & E \end{pmatrix} \begin{pmatrix} p & \frac{p-1}{\lambda}\rho \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & E \end{pmatrix}^{-1} = \begin{pmatrix} p & -pvE^{-1} + (\frac{p-1}{\lambda}\rho + v)E^{-1} \\ 0 & E \end{pmatrix}$$
$$= \begin{pmatrix} p & \frac{p-1}{\lambda}u \\ 0 & I \end{pmatrix}.$$

Hence, after conjugating every $\left(\begin{smallmatrix} p_{n+1} & v_n \\ 0 & \mathrm{I} \end{smallmatrix}\right)$ by the same matrix, we reduce to the case that the transition matrices are $\left(\begin{smallmatrix} p_{n+1} & \frac{p_{n+1}-1}{\lambda}u \\ 0 & \mathrm{I} \end{smallmatrix}\right)$, having $(\lambda, \rho E^{-1})$ as common eigenvector, and since it is an eigenvector of the matrices, it follows that $\rho E^{-1} = (1, 1, \ldots, 1) \in \mathbf{Z}^k$, and the trace on the group with homomorphism is given by the suitably normalized eigenvector, $(\lambda, 1, 1, \ldots, 1)/q_n$ at the nth level.

At this stage, we note that if $\lambda = k+1$, there is a simple finishing argument. Add the first row of each matrix, that is, $(p_{n+1}, \frac{p_{n+1}-1}{\lambda}u)$ to all the other rows, and then subtract all the columns from the first. This amounts to conjugating every one of the matrices with same elementary matrix. The entries are suddenly strictly positive, and since the inner product of the left and right unimodular Perron eigenvectors is $\lambda = k+1$, and they consist strictly positive of strictly positive integers, they must all be exactly one. (We will review this argument.)

We record the following elementary criterion.

Lemma 11.2. *Let A be a primitive integer matrix of size s, whose Perron eigenvalue is an integer, and let V and W be the corresponding left and right Perron eigenvectors consisting of integers, such that $c(V) = c(W) = 1$. If $VW = s$, then all row and column sums are equal.*

Proof. The Perron eigenvectors consist of strictly positive real numbers, and since they are all integers, each is at least one; as they are of size s, the only way VW is as small as s is if every entry of each is one. Hence the column and row sums are all equal. $\square$

In the case that $\lambda > k$, our strategy is to embroider a block of $\lambda - k - 1$ zero rows and corresponding nonzero columns (or zero columns and nonzero rows) around each of our current matrices in such a way that the resulting matrices still have common left and common right eigenvectors corresponding to p_{n+1}, and such that their unimodularized inner product (the VW of Lemma 11.2) is still λ. Then we conjugate all the matrices (with the same matrix), so that as in the $\lambda = k+1$ case outlined above, the resulting matrices are primitive.

The embroidered pieces actually vary in n (in order to guarantee that the eigenvectors do *not* vary in n), and must be carefully chosen.

If $\lambda < k+1$, we run into a technical difficulty when we try this, and indeed, an easy result shows that it is impossible to proceed.

Suppose $\lambda \geq k+1$; this bifurcates into $\lambda - (k-1) \leq k$ and $\lambda - (k-1) \geq k$ (for which the treatments are different but similar).

We first justify the process of embroidering; this is elementary, and completely derivative of symbolic dynamical techniques.

LEMMA 11.3. *Let* $M_n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$ *be block partitions of* $s \times s$ *integer matrices* (*with* A_n *square of size* a *and* D_n *square of size* $s-a$) *of full rank*, s. *Form the* $S \times S$ *matrices*

$$M'_n = \begin{pmatrix} A_n & B_n & 0 \\ C_n & D_n & 0 \\ 0 & X_n & 0 \end{pmatrix}, \qquad M''_n = \begin{pmatrix} A_n & B_n & Z_n \\ C_n & D_n & Y_n \\ 0 & 0 & 0 \end{pmatrix},$$

where X_n *are* $(S-s) \times s$, Y_n *are* $s \times (S-s)$, *and* Z_n *are* $a \times (S-s)$ *integer matrices. Then there are natural isomorphisms* $G' = \lim M'_n \to G = \lim M_n$ *and* $G \to G'' = \lim M''_n$, *induced by* $\mathbf{Z}^S \to \mathbf{Z}^s$ (*projection onto first* s *coordinates*) *and the natural inclusion of* $\mathbf{Z}^s$ *in* $\mathbf{Z}^S$.

Moreover, if $v = (\alpha, \beta)$ *is a left eigenvector for* M_n (*with corresponding block decomposition*), *then* $v' = (\alpha, \beta, \mathbf{0})$ *is a left eigenvector for* M'_n.

Proof. Let V be the subgroup of $\mathbf{Z}^S$ with zeros in the top s entries, and let $\phi : \mathbf{Z}^S \to \mathbf{Z}^s$ be the projection onto the top s coordinates, so that $VU = \ker \phi$. Then $\phi M'_n = M_n \phi$, so ϕ induces a group homomorphism between the limit groups, which is clearly onto. Since $\operatorname{rank} M_n = s$ and this is full, it easily follows that $\operatorname{rank} M'_n = s$, hence $\operatorname{rank} G' \leq s$. As $G' \to G$ is onto, and the rank of the latter (s) is at least as large as that of the former, the map must be one to one.

Define $\psi : \mathbf{Z}^s \to \mathbf{Z}^S$ to be the inclusion (viewing $\mathbf{Z}^s$ as the subgroup whose bottom $S-s$ entries are zero). Then it is trivial that $M''_n \psi = \psi M_n$, so ψ induces a map $G'' \to G$, which is obviously one to one. Since $M''_n(\mathbf{Z}^S) \subset \phi(\mathbf{Z}^s)$, the map is onto (in the direct limit).

The eigenvector property is trivial. $\square$

First, consider the case $\lambda - k - 1 \leq k$ (and $\lambda \geq k+1$). Relabel our current matrices

$$M_n = \begin{pmatrix} p_{n+1} & \frac{p_{n+1}-1}{\lambda} u \\ 0 & \mathrm{I} \end{pmatrix};$$

this has left eigenvector (λ, u) and right eigenvector $(1, 0, \ldots, 0)^T$ for p_{n+1} (recall $u = (1, 1, \ldots, 1)$). Here, $a = 1$ and $s = k+1$. We set $X_n = (\mathrm{I}_{\lambda-k-1}\mathbf{0})$ (the big zero is the block of size $(\lambda - k - 1) \times (k - \lambda)$, so X_n is $(\lambda - k - 1) \times$

$(k-(\lambda-k-1))$, so we have

$$M'_n = \begin{pmatrix} p_{n+1} & \frac{p_{n+1}-1}{\lambda}u & 0 & 0 & \dots & 0 \\ 0 & & 0 & 0 & \dots & 0 \\ 0 & & 0 & 0 & \dots & 0 \\ \vdots & \mathrm{I}_k & \vdots & & & \vdots \\ 0 & & 0 & 0 & \dots & 0 \\ \mathbf{0} & \mathrm{I}_{\lambda-k-1}\mathbf{0} & & & \mathbf{0} & \end{pmatrix}.$$

Now we perform the elementary column operations which simply add the first $\lambda-k-1$ columns of the second block to their counterparts in the third (so the columns get shifted to the right by k). The inverse operation is to subtract the corresponding rows of the third block from their counterparts in the second. The two operations together amount to simultaneous conjugation by the same element of $\mathrm{GL}(\lambda, \mathbf{Z})$, and lead to the following matrices,

$$\begin{pmatrix} p_{n+1} & \frac{p_{n+1}-1}{\lambda}(1,1,\dots,1) & \frac{p_{n+1}-1}{\lambda}(1,1,\dots,1) \\ \mathbf{0}_{1\times(\lambda-k-1)} & \mathbf{0}_{(\lambda-k-1)\times k} & \mathbf{0}_{(\lambda-k-1)\times(\lambda-k-1)} \\ \mathbf{0}_{k-(\lambda-k-1)} & \mathbf{0}_{(k-(\lambda-k-1))\times(\lambda-k-1)}\mathrm{I}_{k-(\lambda-k-1)} & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_{\lambda-k-1}\mathbf{0} & \mathrm{I}_{\lambda-k-1} \end{pmatrix}.$$

Now we add the first row to each of the others, and correspondingly subtract all the columns from the first; again, these are implemented simultaneously in n by a single product of elementary matrices, and results in all the entries being nonnegative, and moreover, all the matrices are primitive (since the first column and the first row are strictly positive), and with the same zero pattern (so products will still be primitive). Call these matrices A_n.

The content one left and right eigenvectors of M'_n for p_{n+1} are $V' = (\lambda, u, \mathbf{0})$ and $W' = (1, 0, \dots, 0)^T$, hence their inner product $V'W' = \lambda$. This is preserved by simultaneous conjugation; as each A_n is primitive of size λ, it follows from Lemma 11.2 above that the left and right Perron eigenvectors of A_n consist entirely of ones, hence the column and row sums are equal. The simultaneous conjugations obviously induce isomorphism of the groups with homomorphism induced by the common left eigenvector, so we have a realization of G as $\lim A_n$, which is ECRS.

In case $\lambda = k+1$, we skip the embroidery (X_n), and just proceed via conjugations with elements of $\mathrm{GL}(k+1, \mathbf{Z})$. If $\lambda - k - 1 = k$, then there are no extra zero blocks, and the same process works. In this case, the realization is ultrasimplicial.

The process for $\lambda \geq k+1$ and $\lambda - k - 1 \geq k$ (i.e., $\lambda \geq 2k+1$) is almost the same. We embroider the matrix with $\lambda - k - 1$ columns of zeros at the right (as we did before) and the same number of rows at the bottom, and with X_n being $\binom{\mathrm{I}_k}{\mathbf{0}}$. Then we add the corresponding columns to the third block and subtract the rows from the second analogously with what we did before, and we can again just perform the last operation, adding the first row to all the others and subtracting the columns from the first.

So far, we have the following.

PROPOSITION 11.4. *Let G be a simple dimension group of finite rank containing a rank one noncyclic subgroup H such that G/H is free and $H \cap G^{++} \neq \emptyset$, and suppose G has a unique trace τ, and $\tau(G)$ is a rank one subgroup of $\mathbf{Q}$ whose supernatural number contains no primes of infinite multiplicity. Then G admits an ECRS realization of size $\lambda := |\tau(G)/\tau(H)|$ with respect to H if $\lambda \geq \operatorname{rank} G$.*

Proof. If $\operatorname{rank} G = 1$, then there is almost nothing to do. Otherwise, $\lambda > 1$. For subgroups $V \subset U$ of $\mathbf{Q}$, $|U/V| < \infty$ implies there exists m such that $V = mU$, and if U has no primes of infinite multiplicity, then $|U/mU| = m$. Set $U = \tau(G)$, and discard from the supernatural number all the primes (including multiples) that divide λ; the resulting subgroup U_0 is isomorphic to U, and correspondingly, U_0/mU_0 is isomorphic to U/mU. Consider the set of primes (together with their multiplicities) dividing U_0; since they are all relatively prime to λ, we may telescope them to obtain sequence of positive integers representing U_0, $\{p_{n+1}\}_{n=1}^{\infty}$, such that $p_{n+1} \equiv 1 \mod \lambda$.

Since G/H is free, the extension $\ker \tau \to G \to U$ is nearly split. Hence, we can write $G = H \oplus \mathbf{Z}^k$, and the trace, given by the row r_1 at the first level, is of the form described in the top row of the statement of Lemma 11.1. The bottom row of the statement yields a representation of G as a direct limit of Abelian groups, with group homomorphism induced by the common left eigenvector, $[w, n] \mapsto r^1 w/q_n$.

The comment subsequent to Lemma 11.1 allows us to assume that the realizing matrices are all in the form $\left(\begin{smallmatrix} p_{n+1} & \frac{p_{n+1}-1}{\lambda}u \\ 0 & I \end{smallmatrix}\right)$, having common left eigenvector $(\lambda, 1, 1, \ldots, 1)$. Now the embroidery process, together with Lemma 11.3, and subsequent simultaneous conjugation, gives an isomorphism of groups with group homomorphism to the direct limit of primitive matrices with equal row and column sums, as described above. $\square$

Now we show that if U has no primes of infinite multiplicity, then $|t(G)/t(H)| \geq \operatorname{rank} G$ is a necessary condition for G to have a bounded ECRS realization.

LEMMA 11.5. *Let U be a noncyclic subgroup of rank one with no primes of infinite multiplicity. If l is an integer exceeding 1, then $U/lU \simeq \mathbf{Z}/l\mathbf{Z}$.*

Proof. First, if $j > 1$, then $U \neq jU$, otherwise $\times j$ is a group automorphism of U, hence $\times 1/j$ is also an automorphism, and it easily follows that if p is a prime dividing j, it must have infinite multiplicity in U. Thus if j properly divides l, then $jU \neq lU$. As every subgroup of finite index in U is of the form nU for some integer n, there is an obvious bijection between the intermediate subgroups $lU \subset U_0 \subset U$ and those of $l\mathbf{Z} \subset \mathbf{Z}_0 \subset \mathbf{Z}$, thus the map $\mathbf{Z} \to U \to U/lU$ has kernel $l\mathbf{Z}$, and is obviously onto. $\square$

Suppose G, with unique trace, has a realization as $\lim A_n : \mathbf{Z}^s \to \mathbf{Z}^s$ which is ECRS, where H is identified with $\bigcup [\mathbf{1}_s, n]\mathbf{Z}$. Then the trace is given by the normalized constant row, and we see immediately that $\tau(H) = s\tau(G)$. Hence if $\tau(G)$ has no primes with infinite multiplicity, we have $|\tau(G)/\tau(H)| = s \geq \operatorname{rank} G$. However, $\tau(G)/\tau(H)$ is an invariant of (G, H), as G has unique trace.

COROLLARY 11.6. *Suppose G is a finite rank simple dimension group with unique trace τ, such that $\tau(G)$ is a rank one noncyclic subgroup of $\mathbf{R}$ with no prime divisors of infinite multiplicity. If G admits an ECRS representation with respect to H, then $\tau(G)/\tau(H)$ is finite and must be at least as large as* $\operatorname{rank} G$.

Proof. Finiteness comes from the extension $G \to \tau(G)$ being nearly split (Lemma 10.1). The rest is from the comment just above. $\square$

THEOREM 11.7. *Suppose that G is a finite rank simple dimension group with unique trace τ, having rational values, and H is a rank one noncyclic subgroup such that G/H is free and $\tau(G)$ is not p-divisible for any prime p. Then G admits an ECRS realization (with respect to H) if and only if* $|\tau(G)/\tau(H)| \geq \operatorname{rank} G$.

For example, if $G = U \oplus \mathbf{Z}^k$ where U is an infinite multiplicity-free noncyclic subgroup of $\mathbf{Q}$, and we impose the strict ordering induced by the projection onto U, then the extension is split, and obviously $|\tau(G)/\tau(H)| = 1$; so G admits an ECRS realization (there is only one choice for H, namely U) if and only if $k = 0$, and the latter is uninteresting. If instead, we impose as trace $\tau(u, v) = lu + v_1$ ($lu +$ the first entry of v), then $\tau(G) = U$, but $\tau(U) = lU$, so that $|\tau(G)/\tau(H)| = l$, then G admits an ECRS realization if and only if $l \geq k + 1 = \operatorname{rank} G$.

Now we assume G simple dimension group with unique trace τ, $\tau(G)$ is rank one [and being dense, is noncyclic] H is a noncyclic rank one subgroup of G such that G/H is torsion-free, and $\tau(H) \neq 0$. We permit $\operatorname{rank} G$ and $\tau(G)/\tau(H)$ to be infinite.

THEOREM 11.8. *Suppose that G is a simple dimension group with unique trace τ, the value group of τ is $\tau(G) = U \subseteq \mathbf{Q}$, and U has no primes of infinite multiplicity. Assume that H is a rank one noncyclic subgroup of G such that $\tau(H) \neq 0$ and G/H is torsion-free. Then G admits an ECRS realization with respect to H if either of the conditions below hold.*

(a) $|\tau(G)/\tau(H)| = \infty$, *regardless of* $\operatorname{rank} G$ (*which can be infinite*); *in this case the realization must be unbounded.*
(b) $\infty > |\tau(G)/\tau(H)| \geq \operatorname{rank} G$, *and in this case, the realization is bounded.*

Proof. First, we note that if $U_0 \subset U$ are noncyclic rank one subgroups of $\mathbf{Q}$, then there exists an infinite increasing chain of subgroups, $U_0 \subset U_1 \subset U_2 \subset \cdots \subset U$ such that $U = \bigcup U_i$ and $|U/U_i| < \infty$. Applying this with $U_0 = \tau(H)$ and $U = \tau(G)$, set $G_i^0 = \tau^{-1}(U_i)$. Moreover, U_i/U_0, being finite, is cyclic. Hence, there exists $g_i \in G_i$ such that $\tau(G_i) = \tau(H) + \tau(g_i)\mathbf{Z}$.

Since $\ker\tau$ is torsion free, we may find an increasing union of finitely generated groups $F_1 \subseteq F_2 \subseteq F_3 \subseteq$ such that $\ker\tau = \bigcup F_i$; by interposing as many equalities as we like, and telescoping the G_i, we may assume $j + \operatorname{rank} F_j < |\tau(G_j)/\tau(H)|$.

Set $G_j = F_j + H + \sum_{l \leq j} g_l\mathbf{Z}$; then $G_j \subseteq G_{j+1}$ and $G = \bigcup G_j$. Moreover, $\operatorname{rank} G_j \leq \operatorname{rank} F_j + 1 + j \leq |\tau(G_j)/\tau(H)|$. In addition, G_j/H is finitely generated, and a subgroup of G/H, hence is torsion-free, hence is free. Since $\tau(H)$ is dense in $\mathbf{R}$, G_j with the relative ordering is a simple dimension group with unique trace, the restriction of τ. Thus $\ker\tau \cap G_j \to G_j \to \tau(G_j)$ is nearly split, and the condition $|\tau(G_j)/\tau(H)| \geq \operatorname{rank} G_j$ ensures that G_j has a bounded ECRS realization with respect to H.

Since G is obviously the direct limit of G_j, by Lemma 9.1(c), G has an ECRS realization with respect to H. In case (a), it must be unbounded (since bounded ERS realizations yield $|\tau(G)/\tau(H)| < \infty$). In case (b), the realization is obtained from telescoping a uniformly bounded family of realizations (using the method of Lemma 9.1(c)), so is bounded (or see the observation in the next paragraph). $\square$

Now we have an elementary observation about unbounded ECRS realizations, when $\tau(G)$ has no infinite prime divisors. Suppose $G = \lim A_n : \mathbf{Z}^{f(n)} \to \mathbf{Z}^{f(n+1)}$ is an ECRS realization, with $\sup f(n) = \infty$. The sequence of row vectors $(\mathbf{1}^T_{f(n)}/p_2 \cdot \cdots \cdot p_n)$, where $\mathbf{1}^T_{f(n)} A_n = \mathbf{1}^T_{f(n+1)} p_{n+1}$ (defining p_{n+1}, the constant column sum of A_n), induces a trace $\tau[y, n] = \mathbf{1}_{f(n)} y/p_2 \cdot \cdots \cdot p_n$. If G is simple with unique trace, then τ is the unique trace. Then $\tau(G)$ is $\bigcup_n 1/p_2 \cdot \cdots \cdot p_n$. With H identified with $\bigcup_n [\mathbf{1}_{f(n)}, n]\mathbf{Z}$, we see that $|\tau(G)/\tau(H)| \geq f(n)$ for all n (this follows from no p-divisible subgroups for all primes p).

Combining everything in sight, we have the following complete characterization of ECRS realizations when $\tau(G)$ has no p-divisible elements for any prime p.

THEOREM 11.9. *Let G be simple dimension group with unique trace, τ. Suppose that $\tau(G)$ is a subgroup of $\mathbf{Q}$ whose supernatural number has no primes of infinite multiplicity. Let H be a rank one noncyclic subgroup of G such that $\tau(H) \neq 0$ and G/H is torsion-free. Then G admits an ECRS realization with respect to H if and only if* $\operatorname{rank} G \leq |\tau(G)/\tau(H)|$; *this includes the case that one or both of* $\operatorname{rank} G$ *and* $|\tau(G)/\tau(H)|$ *are infinite. Finally, every ECRS realization is of size* $|\tau(G)/\tau(H)|$ (*i.e., unbounded if and only if* $\tau(G)/\tau(H)$ *is infinite*).

When G is p-divisible for some prime, the situation is different; no restriction on $\tau(G)/\tau(H)$ is required.

Now we assume that $t(G)$ is divisible by p^∞ and to begin with, we also assume G/H is free and $\lambda := |\tau(G)/t(H)| < \infty$. If p is any prime infinitely dividing $\tau(G)$, then it also divides $\tau(H)$; hence $\gcd(\lambda, p) = 1$ for any prime p dividing H (which is isomorphic to $t(H)$). If $\lambda = 1$, we are in the split case, for which there is an interesting argument, obtaining a realization by commuting matrices.

Set $G = U \oplus \mathbf{Z}^k$ with the projection onto U as the unique trace—this is the split case—we show that G admits a bounded ECRS realization (with respect to $H = U$, the only possible choice for H) under the assumption that U is p-divisible for some prime p.

Find a power, $q = p^a > k-1$. Then the matrix $M := \left(\begin{smallmatrix} q & 0 \\ 0 & -I_k \end{smallmatrix}\right)$ (note the appearance of the *negative* of the identity matrix) satisfies all the conditions of [BoH, Theorem 3.3]. Hence, there exists a primitive matrix M' that is algebraically shift equivalent to M. By [M, Theorem 5], there exists a primitive matrix A having equal row and column sums (so that $\mathbf{1}^T$ and $\mathbf{1}$ are respectively left and right Perron eigenvectors of A for the eigenvalue q) shift equivalent to M. In particular, A is algebraically shift equivalent to M.

If the supernatural number has only finitely many other primes of multiplicity at least one, then $U = \mathbf{Z}[1/p]$ and then G admits a stationary realization with $A_n = A$ (the argument to show this will be included in what follows). Otherwise, we may telescope the other primes (including their multiplicities), so that $U_0 := \lim \times p_i : \mathbf{Z} \to \mathbf{Z}$ (p_i are products of the other primes) is relatively prime to p and $U = \mathbf{Z}[1/p] \otimes U_0$. Since $\gcd(p_i, p) = 1$, so $\gcd(p_i, p^a) = 1$, hence by a further telescoping, we may also assume that $p_i \equiv 1 \mod q = p^a$.

Now we use the following lemma to contort A.

LEMMA 11.10. *Let $m > 1$ be an integer, and suppose l is a positive integer with $l \equiv \pm 1 \mod m$. Then there exists $f \in \mathbf{Z}[x]^+$ (polynomials with nonnegative integer coefficients) such that $f(m) = l$ and $f(-1) \in \{\pm 1\}$.*

Proof. We find $f_0 \in \mathbf{Z}[x]^+$ such that $f_0(m) = l$; then we modify it inductively until $|f(-1)| = 1$. Expand $l = \sum_{i=0}^t a_i m^i$ with $0 \le a_i < m$ as an m-adic expansion. Then set $f_0 = \sum a_i x^i$. Obviously $f_0(m) = l$

If $f_0(-1) > 1$, then $\sum a_i(-1)^i > 1$. If a_0 is the only even-indexed coefficient that is strictly greater than zero, then $f(m) \le a_0 < m < l$, a contradiction. Hence, there must exist $i = 2j$ such that $a_i > 0$. Replace a_i by $a_i - 1$ and a_{i-1} by $a_{i-1} + m$, to create f_1. Then $f_1(m) - f_0(m) = -m^i + m^i = 0$, so $f_1(m) = l$, and $f_1(-1) = f_0(-1) - m - 1$.

For any polynomial $g \in \mathbf{Z}[x]$, $g(m) \equiv g(-1) \mod m+1$. Hence $l = f_0(m) - f_0(-1)$ is a multiple of $m+1$; writing $l = km + 1$ (if $l \equiv 1 \mod m$), we have $km + 1 - f_0(-1) = s(m+1)$, so $f_0(-1) = 1 + km - s(m+1)$ and so $km \ge s(m+1)$. Also, $f_1(-1) = km - (s+1)(m+1) + 1$. If this is negative, then

$(s+1)m > km+1 \geq s(m+1)+1$, so $m > s$. Also, $s+1 > k \geq s+s/m$. This is impossible. Hence $f_1(-1) \geq 1$. If it equals 1, we are done. If not, the process can be repeated, each time reducing the value at -1 by $m+1$, and it must eventually hit 1. A similar process works if $l \equiv -1 \mod m$, except that the value at -1 eventually hits -1.

If $f_1(-1) < -1$, the process is similar, but easier (we do not have to worry about large a_0). There must exist $i = 2j+1$ such that $a_i > 0$; replace a_i by $a_i - 1$ and a_{i-1} by $a_{i-1} + m$. The resulting f_1 satisfies $f_1(m) = l$ and $f_1(-1) = f_0(-1) + m + 1$. A similar argument to that of the preceding allows us to conclude that $f_1(-1) < 0$ (if $l \equiv -1 \mod m$) or $f_1(-1) \leq 1$, whence either it is ± 1, or strictly less than -1, and the process can be iterated. $\square$

For each $p_i \equiv 1 \mod q$, there exists $f_i \in \mathbf{Z}[x]^+$ such that $f_i(q) = p_i$ and $f_i(-1) = 1$. Set $A_n = Af_n(A)$; as f_n has only nonnegative coefficients, so does A_n; since each A_n is a polynomial in A, its large eigenvalue is $qf_n(q) = qp_n$, they commute with each other, and have the same Perron eigenvectors, $\mathbf{1}^T$ and $\mathbf{1}$.

Suppose the matrix size of A is y (all we know is that $y \geq k+1$; otherwise, we have very little control over it).

Now form $M_n = Mf_n(M) = \left(\begin{smallmatrix} qp_n & 0 \\ 0 & I_k \end{smallmatrix}\right)$. Suppose the algebraic shift equivalence between M and A is given by X and Y; that is, $XM = AX$, $MY = YA$, and $XY = A^t$, $YX = M^t$ (t is called the *lag*). Then for every nonzero power of A, we have $XM^r = A^rX$ and $M^rY = YA^r$; hence for every polynomial $g \in \mathbf{Z}[x]$ such that $g(0) = 0$, we have $Xg(M) = g(A)X$ and $g(M)Y = Yg(A)$. Hence the map $X : \mathbf{Z}^{k+1} \to \mathbf{Z}^y$ induces a group homomorphism $G = \lim M_n \to G' = \lim A_n$ by $[z, m] \mapsto [Xz, m]$, and similarly, Y induces a group homomorphism $G' \to G$ via $[w, m] \mapsto [Yw, m]$. The products of the two group homomorphisms are given by $\widehat{A}$ and $\widehat{M}$, respectively, both of which are immediately seen to be group automorphisms of G' and G respectively. Hence, the maps induced by X and Y are isomorphisms.

Moreover, they take the eigenspaces of nonzero eigenvalues for A to those of M (and vice versa), and in particular, they must send the common eigenvectors for q, and thus send $(1, 0, \dots)$ to $(1, 1, \dots, 1)$ and the same with the transposes. They thus induce an isomorphism of the groups with group homomorphism. Moreover, it is easy (trivial) to see that G' has a unique trace (when given the direct limit ordering), so that the group isomorphism is an order isomorphism from G (with ordering induced by the common left eigenvector of M_n for qp_n) to G' (with direct limit ordering). It is trivial that G is simply the split extension.

The upshot is a special case.

LEMMA 11.11. *Suppose $G = U \oplus \mathbf{Z}^k$ where U is a rank one subgroup of $\mathbf{Q}$ that is p-divisible for some prime p, and the unique trace on G is the* *projection onto U (the split case). Then with $H = U$, there exists a bounded ECRS realization of G with respect to H by commuting matrices.*

The matrices constructed in the other realizations need not commute. A similar argument can be made to work in some nonsplit cases with a prime having infinite multiplicity. We can of course extend this via the direct limit argument of Lemma 9.1(c).

So we may assume that $\lambda > 1$.

If all primes infinitely divide H, then H (and thus $\tau(G)$) are rational vector spaces, $\lambda = 1$, the extension splits (indeed, there is only one extension). If $H \simeq \mathbf{Z}[1/n]$ for some integer n, then the system is stationary, and the result follows from [M]. If there are only finitely many primes with finite multiplicity, we reduce to the last case immediately.

Otherwise, there exist infinitely many primes each with finite and nonzero multiplicity, in addition to at least one prime p with infinite multiplicity. Throwing away all the primes that divide λ amounts to throwing away a finite set of primes each with only finite multiplicity, hence does not change anything.

There exists a power of p, $q = p^a$ such that $q \equiv 1 \mod \lambda$. We may also arrange, by taking a multiple of a if necessary, that $q\lambda > k^2 + k$. We may telescope the other primes with their powers, so obtain $\tau(G)$ as $\mathbf{Z}[p^{-1}] \otimes \lim p_i : \mathbf{Z} \to \mathbf{Z}$ where $\gcd(p_i, \lambda) = \gcd(q, p_i) = 1$. Now we can implement the same isomorphism as in Lemma 11.1, with qp_{i+1} replacing p_{i+1} (and $(qp_{i+1} - 1)/\lambda$ replacing $(p_{i+1} - 1)/\lambda$), where we set $y_1 = \mathbf{0}$, $v_i = \rho \cdot (qp_{i+1} - 1)/\lambda$, and $y_n = \rho \cdot (qq_n - 1)/\lambda$. This yields a group isomorphism to the limit group obtained as $\lim \left(\begin{smallmatrix} qp_{i+1} & w_i \\ \mathbf{0} & \mathrm{I}_k \end{smallmatrix}\right)$, and the group homomorphism has been converted to left multiplication by the common eigenvector, (λ, ρ).

As in the previous case, we can simultaneously conjugate all the current matrices by $\left(\begin{smallmatrix} 1 & v \\ \mathbf{0} & E \end{smallmatrix}\right)$ where $v \in \mathbf{Z}^{1\times k}$ and $E \in \mathrm{GL}(k, \mathbf{Z})$. This replaces the upper right entry by $\lambda^{-1}(qp_{i+1} - 1)(\rho - \lambda v)$. We could have previously conjugated the matrices with $\left(\begin{smallmatrix} 1 & \mathbf{0} \\ \mathbf{0} & J \end{smallmatrix}\right)$ where $J \in \mathrm{GL}(k, \mathbf{Z})$, and so have assumed that $\rho = c(\rho)(1, 0, \dots, 0)$ (this applies to the left eigenvector as well). Now set $v = (0, 1, 0, \dots, 0)$, so $\rho - \lambda v = (c(\rho), -\lambda)$; hence $c(\rho - \lambda v) = 1$. Thus, there exists $E \in \mathrm{GL}(k, \mathbf{Z})$ such that $(\rho - \lambda v)E^{-1} = (1, 1, 1, \dots, 1) = \mathbf{1}_k^T$, which as before, we call u.

Hence, we are in the situation wherein the matrices are of the form $\left(\begin{smallmatrix} qp_{i+1} & (qp_{i+1}-1)u/\lambda \\ \mathbf{0} & \mathrm{I}_k \end{smallmatrix}\right)$, their common eigenvector is (λ, u) (for the eigenvalue qp_{i+1}), and the group homomorphism is obtained by left multiplication by suitable multiples of the eigenvector.

Now we embroider $\lambda q - k - 1$ rows and columns around the matrix; the only nonzero entries of the newly embroidered part occur in the top row, where we put $p_{i+1}\mathbf{1}^T$. This creates new matrices

$$B_n = \begin{pmatrix} qp_{n+1} & \frac{qp_{n+1}-1}{\lambda}(1,1,\dots,1)^T & p_{n+1}(1,1,\dots,1) \\ \mathbf{0} & \mathrm{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

which are of size $q\lambda$. Miraculously, B_n have a common left eigenvector, $(\lambda q, qu, \lambda\mathbf{1}^T)$, where the third block is of size $\lambda q - k - 1$ (it is not in general true that embroidery where the right-hand side is not zero, in this case, Z_n in Lemma 11.3, will preserve the common left eigenvector property).

In order to perform the desired column and row operations, we need an estimate. From $q(k^2 - \lambda^2 - k + \lambda) - k^2/p_{i+1} + k + 1 \geq k$ (easy), we see that

$$\frac{qp_{i+1} - 1}{\lambda} \geq k + \frac{q}{\lambda};$$

hence there exists a multiple of k, $t = sk$, such that $q/\lambda \leq t/k \leq (qp_{i+1} - 1)/\lambda$. Now to each B_n, subtract s times each of the first k columns from their counterpart in the second block. The inequalities we just used are equivalent to the resulting top row consists of positive entries, and the sum of all but the first is less that qp_{i+1}. The inverse operation is to subtract the bottom rows from their counterparts, but this has no effect. This amounts to a conjugacy (which of course yields an isomorphism with group homomorphism), and now we simply add the top row to each of the others, and all the columns but the first from the first column. As before, the result is a primitive matrix, of size λq; the inner product (as is easy to see) is the same as the size, so the matrix has equal row and column sums.

This yields the following rather surprising result.

Proposition 11.12. *Let G be a simple dimension group of finite rank with unique trace τ, such that $\tau(G)$ is p-divisible for at least one prime p. Suppose H is a noncyclic rank one subgroup such that $\tau(H) \neq 0$ and G/H is free. Then there exists an ECRS realization of G with respect to H.*

And the direct limit argument, using Lemma 9.1 and Theorem 11.8, yields the definitive result.

Theorem 11.13. *Let G be a simple dimension group with unique trace τ, such that $\tau(G)$ is p-divisible for at least one prime p. Suppose H is a rank one noncyclic subgroup of G such that $\tau(H) \neq 0$ and G/H is torsion-free. Then G admits an ECRS realization with respect to H.*

So we have a dichotomy: if $\tau(G)$ is not p-divisible for every prime p, then the condition $|\tau(G)/\tau(H)| \geq \operatorname{rank} G$ is necessary and sufficient (allowing ∞ as possible values); but if $\tau(G)$ is p-divisible for some prime p, there is no such constraint.

12. Comments

As discussed earlier, the primordial example of Elliott [E1], [E2] was the rank two split extension dimension group $\mathbf{Z}[1/2]\oplus\mathbf{Z}$ (with the strict ordering), which cannot be realized as a limit of simplicial groups of rank two (i.e., any direct limit realization requires almost all the free Abelian groups to be of rank at least three). He also showed that this dimension group can be realized as a limit of rank three simplicial groups, and is stationary (via a size three primitive matrix algebraically shift equivalent to $\mathrm{diag}(2,1)$). This is in fact what led me to think about using semigroups to obtain realization of the transfer matrices.

This paper was motivated by a question of Christian Skau: given the split extension $G = U \oplus \mathbf{Z}^k$, with $U \subseteq \mathbf{Q}$ and the projection onto U yielding the ordering (so as to be a dimension group with unique trace), does it admit a nice ERS representation? (As we have seen, there is only one possible choice for the rank one noncyclic subgroup H such that G/H is torsion free, namely U itself, so H is unambiguous.) This appears as a special case, and the implementing matrices are the transposes of the matrices of the form A appearing in section two (with parameters $p = p_{n+1}$, once we ensure that $p_{n+1} > (k+1)^2$). I would like to thank Christian for his repeated insistence on solving this problem.

Skau's question was motivated by questions concerning Töplitz $\mathbf{Z}$-actions on Cantor sets (systems which admit factor maps onto odometers). A particular consequence of the results here is that among uniquely ergodic minimal actions of $\mathbf{Z}$ on Cantor sets, those that are strongly orbit equivalent to a Töplitz, and those that are orbit equivalent, are characterized.

It has been known for over a decade that dimension groups which are rational vector spaces admit ERS realizations with respect to any dimension one subspace containing an order unit (this appears in [GJ]). The recipe is to begin with any realization of the dimension group, find an increasing sequence $h_n\mathbf{Z} \subset h_{n+1}\mathbf{Z}$ whose union is H where h_n is an order unit, telescope the realization, so that a cofinal collection of the h_n appear, each at the nth level, say by a strictly positive vector v_n, apply the obvious diagonal matrix $\Delta_n \in \mathrm{GL}(f(n),\mathbf{Q})$ so that $\Delta_n v_n$ is a multiple of $\mathbf{1}$, replace the nth matrix A_n by $\Delta_{n+1}A_n\Delta_n^{-1}$, then multiply each by a positive integer to ensure that the entries are all nonnegative integers. Since the dimension group G satisfies $G\otimes\mathbf{Q}\simeq G$, it follows immediately that the new improved direct limit yields G, and the elements of H are implemented by constant vectors in the limit.

It has also been known for around a decade that if (G,u) is a simple dimension group and H is a rank one noncyclic subgroup containing the order unit u, then there exists an ERS realization so that the order unit is $[\mathbf{1}^T, 1]$ in the direct limit. This does not of course give any indication of the size(s) of the transition matrices that can be used.

There is a substantial literature on realizing shift equivalence classes of integer matrices with nonnegative entries ([BoH], [H1], [H3], [H4] and the references therein), corresponding to stationary direct limits (i.e., G is a limit with the same matrix repeated, as an Abelian group with real-valued homomorphism emanating from the largest eigenvalue and corresponding left eigenvector). Here we have a generally easier problem, since we are permitted to telescope matrices, something not allowed in the matrix realization problem. On the other hand, there are situations in dimension group realization questions (such as $\tau(G)$ being a subgroup of the rationals with no primes of infinite multiplicity) which do not arise in the matrix realization case.

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