Illinois Journal of Mathematics
Volume 56, Number 4, Winter 2012, Pages 1149–1168
S 0019-2082

GROUP ACTIONS ON LABELED GRAPHS AND THEIR C^*-ALGEBRAS

TERESA BATES, DAVID PASK AND PAULETTE WILLIS

ABSTRACT. We introduce the notion of the action of a group on a labeled graph and the quotient object, also a labeled graph. We define a skew product labeled graph and use it to prove a version of the Gross–Tucker theorem for labeled graphs. We then apply these results to the C^*-algebra associated to a labeled graph and provide some applications in non-Abelian duality.

1. Introduction

A labeled graph $(E, \mathcal{L})$ is a directed graph $E = (E^0, E^1, r, s)$ together with a function $\mathcal{L} : E^1 \to \mathcal{A}$ where $\mathcal{A}$ is called the alphabet. Labeled graphs are a model for studying symbolic dynamical systems; the labeled path space is a shift space whose properties may be inferred from the labeled graph presentation (cf. [12]). Labeled graph algebras were introduced in [2], [3], their theory has been developed in [1], [7], [8] and has found applications in mirror quantum spheres in [16].

The main purpose of this paper is to introduce the notion of a group action on a labeled graph and study the crossed products formed by the induced action on the associated C^*-algebra. Before we do this, we update the definition of the C^*-algebra associated to a labeled graph. In order to circumvent a technical error in the literature, we add a new condition to ensure that the resulting C^*-algebra satisfies a version of the gauge-invariant uniqueness theorem. Since a directed graph is a labeled graph where $\mathcal{L}$ is injective, we will

Received November 14, 2011; received in final form May 16, 2013.

The second author was supported by the Australian Research Council. The third author was supported by the NSF Mathematical Sciences Postdoctoral Fellowship DMS-1004675, the University of Iowa Graduate College Fellowship as part of the Sloan Foundation Graduate Scholarship Program, and the University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242.

2010 *Mathematics Subject Classification.* Primary 46L05. Secondary 37B10.

be generalizing a suite of results for directed graphs and their C^*-algebras (see [5], [11], [9]). This is not as straightforward as it may seem since two distinct edges may carry the same label, so new techniques will be needed to prove our results.

An action of a group G on a labeled graph $(E, \mathcal{L})$ is an action of G on E together with a compatible action of G on $\mathcal{A}$ so that we may sensibly define the quotient object $(E/G, \mathcal{L}/G)$ as a labeled graph. In [6], Gross and Tucker introduce the notion of a skew product graph $E \times_c G$ formed from a map $c\colon E^1 \to G$ and show that G acts freely on $E \times_c G$ with quotient E. The Gross–Tucker theorem [6, Theorem 2.1.2] takes a free action of G on E and recovers (up to equivariant isomorphism) the original graph and action from the quotient graph E/G. One might speculate that a similar result holds for free actions on labeled graphs. In Section 4, we describe a skew product construction for labeled graphs and prove a version of the Gross–Tucker theorem for free actions on labeled graphs (Theorem 5.10). Since a group action on a labeled graph is a pair of compatible actions, a new approach is needed: In Definition 4.1, we define a skew product labeled graph $(E \times_c G, \mathcal{L}_d)$ to be a skew-product graph $E \times_c G$ together with a labeling $\mathcal{L}_d : (E \times_c G)^1 \to \mathcal{A} \times G$ which is defined using a new function $d : E^1 \to G$. The purpose of the new function d is to accommodate the possibility that two edges carry the same label. In Remark 5.11, we discuss the importance of d.

We then turn our attention to applications of our results on labeled graph actions to the C^*-algebras, $C^*(E, \mathcal{L})$ we have associated to labeled graphs.

A function $c\colon E^1 \to G$ on a directed graph gives rise to a coaction δ of G on $C^*(E)$ such that $C^*(E) \times_\delta G \cong C^*(E \times_c G)$ (cf. [9]). In Proposition 6.2, we show that a skew product labeled graph $(E \times_c G, \mathcal{L}_d)$ gives rise to a coaction δ of G on $C^*(E, \mathcal{L})$ provided that $c : E^1 \to G$ is consistent with the labeling map $\mathcal{L} : E^1 \to \mathcal{A}$. Then in Theorem 6.7 we show that $C^*(E, \mathcal{L}) \times_\delta G \cong C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ where $\mathbf{1} : E^1 \to G$ is given by $\mathbf{1}(e) = 1_G$ for all $e \in E^1$. Since this isomorphism is equivariant for the dual action of G on $C^*(E, \mathcal{L}) \times_\delta G$ and the action of G on $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ induced by left translation of G on $(E \times_c G, \mathcal{L}_{\mathbf{1}})$, Takai duality then gives us

$$C^*(E \times_c G, \mathcal{L}_{\mathbf{1}}) \times_{\tau, r} G \cong C^*(E, \mathcal{L}) \otimes \mathcal{K}\big(\ell^2(G)\big)$$

in Corollary 6.8. Indeed if d is consistent with the labeling map $\mathcal{L} : E^1 \to \mathcal{A}$, then $C^*(E \times_c G, \mathcal{L}_d)$ is equivariantly isomorphic to $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ (see Proposition 6.3).

For a directed graph E, a function $c\colon E^1 \to \mathbb{Z}$ given by $c(e) = 1$ for all $e \in E^1$ gives rise to a skew product graph $E \times_c G$ whose C^*-algebra which is strongly Morita equivalent to the fixed point algebra $C^*(E)^\gamma$ for the gauge action. In the case of labelled graphs, if $c, d : E^1 \to \mathbb{Z}$ are given by $c(e) = 1$, $d(e) = 0$ for all $e \in E^1$, then $C^*(E \times_c G, \mathcal{L}_d)$ is strongly Morita equivalent to $C^*(E, \mathcal{L})^\gamma$ (see Theorem 6.10).

An action α of G on a directed graph E induces an action of G on $C^*(E)$, moreover if the action is free, then using the Gross–Tucker theorem we have

$$C^*(E) \times_{\alpha,r} G \cong C^*(E/G) \otimes \mathcal{K}\big(\ell^2(G)\big) \tag{1.1}$$

by [11, Corollary 3.10]. In Theorem 3.2, we show that an action of G on $(E,\mathcal{L})$ induces an action of G on $C^*(E,\mathcal{L})$. If we wish to use the Gross–Tucker theorem for labeled graphs to prove the labeled graph analog (1.1), we need to know when the maps $c,d : (E/G)^1 \to G$ provided by Theorem 5.10 are consistent with the quotient labeling $\mathcal{L}/G$. The answer to this question is provided by Theorem 7.3: It happens precisely when the action α has a fundamental domain. Hence, if the free action of G on $(E,\mathcal{L})$ has a fundamental domain, then in Corollary 7.4 we show that

$$C^*(E,\mathcal{L}) \times_{\alpha,r} G \cong C^*(E/G,\mathcal{L}/G) \otimes \mathcal{K}\big(\ell^2(G)\big).$$

2. Labeled graphs and their C^*-algebras

We begin with a collection of definitions, which are taken from [2]. A *directed graph* $E = (E^0, E^1, r, s)$ consists of a vertex set E^0, an edge set E^1, and range and source maps $r, s : E^1 \to E^0$. We shall assume throughout this paper that E is row-finite and essential, that is

$$r^{-1}(v) \neq \emptyset \quad \text{and} \quad 1 \leq \#s^{-1}(v) < \infty$$

for all $v \in E^0$. We let E^n denote the set of paths of length n and set $E^+ = \bigcup_{n\geq 1} E^n$.

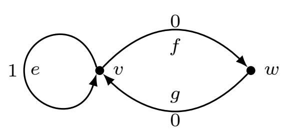
Definition 2.1. A *labeled graph* $(E,\mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E together with a labeling map $\mathcal{L} : E^1 \to \mathcal{A}$.

We may assume that $\mathcal{L} : E^1 \to \mathcal{A}$ is surjective. Let $\mathcal{A}^*$ be the collection of all *words* in the symbols of $\mathcal{A}$. For $n \geq 1$, the map $\mathcal{L}$ extends naturally to a map $\mathcal{L} : E^n \to \mathcal{A}^*$: for $\lambda = \lambda_1 \cdots \lambda_n \in E^n$ we set $\mathcal{L}(\lambda) = \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n)$ and we say that λ is a *representative* of the labeled path $\mathcal{L}(\lambda)$. Let $\mathcal{L}(E^n)$ denote the collection of all labeled paths in $(E,\mathcal{L})$ of length n. Then $\mathcal{L}^+(E) = \bigcup_{n\geq 1} \mathcal{L}(E^n)$ denotes the collection of all labeled paths in $(E,\mathcal{L})$, that is all words in the alphabet $\mathcal{A}$ which may be represented by paths in E.

Examples 2.2.

(a) Every directed graph E gives rise to a labeled graph $(E,\mathcal{L}_\tau)$ over the alphabet E^1 where $\mathcal{L}_\tau : E^1 \to E^1$ is the identity map.
(b) The directed graph E whose edges e, f, g have been labeled using the alphabet $\{0,1\}$ as shown below is an example of a labeled graph

$(E,\mathcal{L}) :=$



Let $(E, \mathcal{L})$ be a labeled graph. Then for $\beta \in \mathcal{L}^+(E)$ we set

$$r(\beta) = \{r(\lambda) : \mathcal{L}(\lambda) = \beta\}, \qquad s(\beta) = \{s(\lambda) : \mathcal{L}(\lambda) = \beta\}.$$

For $A \subseteq E^0$ and $\beta \in \mathcal{L}^+(E)$, the *relative range of β with respect to A* is

$$r(A, \beta) = \{r(\lambda) : \lambda \in E^+, \mathcal{L}(\lambda) = \beta, s(\lambda) \in A\}.$$

The labeled graph $(E, \mathcal{L})$ is *left-resolving*, if for all $v \in E^0$ the map $\mathcal{L}$ restricted to $r^{-1}(v)$ is injective. The labeled graph $(E, \mathcal{L})$ is *weakly left-resolving* if for all $A, B \subseteq E^0$ and $\beta \in \mathcal{L}^+(E)$ we have

$$r(A \cap B, \beta) = r(A, \beta) \cap r(B, \beta).$$

If $(E, \mathcal{L})$ is left-resolving, then it is weakly left-resolving. Examples 2.2(a) and (b) are examples of left-resolving labeled graphs.

A collection $\mathcal{B} \subseteq 2^{E^0}$ of subsets of E^0 is *closed under relative ranges for* $(E, \mathcal{L})$ if for all $A \in \mathcal{B}$ and $\beta \in \mathcal{L}^+(E)$ we have $r(A, \beta) \in \mathcal{B}$. If $\mathcal{B}$ is closed under relative ranges for $(E, \mathcal{L})$, contains $r(\beta)$ for all $\beta \in \mathcal{L}^+(E)$ and is also closed under finite intersections and unions, then $\mathcal{B}$ is *accommodating* for $(E, \mathcal{L})$ and the triple $(E, \mathcal{L}, \mathcal{B})$ is called a *labeled space*. Let $\mathcal{E}^{0,-}$ be the smallest accommodating collection of subsets of E^0 for $(E, \mathcal{L})$.

Definition 2.3. For $A \subseteq E^0$ and $n \geq 1$, let $\mathcal{L}_A^n := \{\beta \in \mathcal{L}(E^n) : A \cap s(\beta) \neq \emptyset\}$ denote those labeled paths of length n whose source intersects A nontrivially.

Though E is row finite it is possible for $\mathcal{L}_A^1$ to be infinite; for example if $\mathcal{L}$ is trivial, then $\mathcal{L}_{E^0}^1 = E^1$, which is infinite if E^1 is infinite. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is *set-finite* if $\mathcal{L}_A^1$ is finite for all $A \in \mathcal{B}$. The following definition is given in [2].

Definition 2.4. A representation of a weakly left-resolving, set-finite labeled space $(E, \mathcal{L}, \mathcal{B})$ consists of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that

(i) If $A, B \in \mathcal{B}$, then $p_A p_B = p_{A \cap B}$ and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$, where $p_\emptyset = 0$.
(ii) If $a \in \mathcal{A}$ and $A \in \mathcal{B}$, then $p_A s_a = s_a p_{r(A,a)}$.
(iii) If $a, b \in \mathcal{A}$, then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$.
(iv) For $A \in \mathcal{B}$, we have

$$p_A = \sum_{a \in \mathcal{L}_A^1} s_a p_{r(A,a)} s_a^*.$$

$C^*(E, \mathcal{L}, \mathcal{B})$ is the universal C^*-algebra generated by a representation of $(E, \mathcal{L}, \mathcal{B})$. Let $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E, \mathcal{L}, \mathcal{B})$ be the gauge action determined by

$$\gamma_z p_A = p_A, \qquad \gamma_z s_a = z s_a \quad \text{for } A \in \mathcal{B}, a \in \mathcal{A}.$$

REMARK 2.5. The gauge invariant uniqueness theorem for $C^*(E,\mathcal{L},\mathcal{B})$ as stated in [2, Theorem 5.3] is incorrect. The authors are grateful to Gow for pointing out the error. The problem arises in [2, Lemma 5.2(ii)] as it not possible to prove that the projection r is nonzero under the hypotheses used in [2]. We are also grateful to Jeong and Kim for pointing out an mistake in the formula [3, Remark 3.5] and in [7, Example 2.4] which is a direct result of the error discovered by Gow.

The problem in [2, Lemma 5.2(ii)] arises because, under the hypotheses on a labeled space used in [2], it is possible to have $A \supsetneq B \in \mathcal{B}$ with $p_A = p_B$ in $C^*(E,\mathcal{L},\mathcal{B})$. To rectify this problem, we must assume that $\mathcal{B}$ is closed under relative complements; that is if $A, B \in \mathcal{B}$ are such that $A \supsetneq B$, then $A \setminus B \in \mathcal{B}$. If $\mathcal{B}$ is closed under relative complements, then we also recover the formula in [3, Remark 3.5].

Before stating the Gauge Invariant Uniqueness theorem, we give a corrected version of [2, Lemma 5.2] using the new hypothesis.

LEMMA 2.6. *Let $(E,\mathcal{L},\mathcal{B})$ be a weakly left-resolving, set-finite labeled space where $\mathcal{B}$ is closed under relative complements and $\{s_a, p_A\}$ be a representation $(E,\mathcal{L},\mathcal{B})$. Let $Y = \{s_{\alpha_i} p_{A_i} s^*_{\beta_i} : i = 1, \ldots, N\}$ be a set of partial isometries in $C^*(E,\mathcal{L},\mathcal{B})$ which is closed under multiplication and taking adjoints. If q is a minimal projection in $C^*(Y)$, then either*

(i) $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i}$ *for some* $1 \le i \le N$

(ii) $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q'$ *where* $q' = \sum_{l=1}^m s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ *and* $1 \le i \le N$*; moreover there is a nonzero* $r = s_{\alpha_i\beta} p_{r(A_i,\beta)} s^*_{\alpha_i\beta} \in C^*(E,\mathcal{L},\mathcal{B})$ *such that* $q'r = 0$ *and* $q \ge r$.

Proof. By [2, Lemma 4.4], any projection in $C^*(Y)$ may be written as

$$\sum_{j=1}^n s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} - \sum_{l=1}^m s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}},$$

where the projections in each sum are mutually orthogonal and for each l there is a unique j such that $s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} \ge s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$.

If $q = \sum_{j=1}^n s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} - \sum_{l=1}^m s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ is a minimal projection in $C^*(Y)$, then we must have $n = 1$. If $m = 0$, then $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i}$ for some $1 \le i \le N$. If $m \ne 0$, then

$$q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - \sum_{\ell=1}^m s_{\alpha_{k(\ell)}} p_{A_{k(\ell)}} s^*_{\alpha_{k(\ell)}},$$

where $A_i, A_{k(\ell)} \in \mathcal{B}$ for $1 \le \ell \le m$. If we apply Definition 2.4(iv), we may write

$$q = \sum_{j=1}^n s_{\alpha_i\beta_j} p_{r(A_i,\beta_j)} s^*_{\alpha_i\beta_j} - \sum_{h=1}^t \sum_{\ell=1}^m s_{\alpha_{k(\ell)}\kappa_h} p_{r(A_{k(\ell)},\kappa_h)} s^*_{\alpha_{k(\ell)}\kappa_h},$$

where all $\alpha_i\beta_j$ and $\alpha_{k(\ell)}\kappa_h$ have the same length. Since q is a nonzero projection there is $1 \le j \le n$ and $H_j \subseteq \{1,\dots,t\} \times \{1,\dots,m\}$ such that $\alpha_i\beta_j = \alpha_{k(\ell)}\kappa_h$ for all $(h,\ell) \in H_j$ and

$$Y_j := \bigcup_{(h,\ell)\in H_j} r(A_{k(\ell)},\kappa_h) \subsetneq r(A_i,\beta_j).$$

Since $\mathcal{B}$ is closed under finite unions we have $Y_j \in \mathcal{B}$. Then for this j define $X_j = r(A_i,\beta_j) \setminus Y_j \neq \emptyset$, then $X_j \in \mathcal{B}$ since $\mathcal{B}$ is closed under relative complements. Hence, the projection $r = s_{\alpha_i\beta_j} p_{X_j} s^*_{\alpha_i\beta_j}$ is nonzero and $q \ge r$ since $X_j \subset r(A_i,\beta_j)$. If we set $q' = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q$, then since $X_j \cap Y_j = \emptyset$ we have $q'r = 0$ as required. □

THEOREM 2.7 (Gauge invariant uniqueness theorem). *Let $(E,\mathcal{L},\mathcal{B})$ be a weakly left-resolving, set-finite labeled space where $\mathcal{B}$ is closed under relative complements and $\{S_a, P_A\}$ be a representation $(E,\mathcal{L},\mathcal{B})$ on Hilbert space. Take $\pi_{S,P}$ to be the representation of $C^*(E,\mathcal{L},\mathcal{B})$ satisfying $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_A) = P_A$. Suppose that $P_A \neq 0$ for all $\emptyset \neq A \in \mathcal{B}$ and that there is a strongly continuous action γ' of $\mathbb{T}$ on $C^*(\{S_a, P_A\})$ such that for all $z \in \mathbb{T}$, $\gamma'_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$. Then $\pi_{S,P}$ is faithful.*

Proof. The proof is the same as given in [2, Theorem 5.3], using Lemma 2.6 instead of [2, Lemma 5.2]. □

DEFINITION 2.8. Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labeled graph, then we define $\mathcal{E}(r,\mathcal{L})$ to be the smallest accommodating collection of subsets of E^0 which is closed under relative complements.

REMARK 2.9. Every $A \in \mathcal{E}^{0,-}$ can be written as $A = \bigcup_{j=1}^n A_j$ where $A_j = \bigcap_{i=1}^{m(j)} r(\beta_i^j)$ and $\beta_i^j \in \mathcal{L}^+(E)$ for all i,j. Hence, by applications of de Morgan's laws we may show that every $A \in \mathcal{E}(r,\mathcal{L})$ can be written in the form $A = \bigcup_{j=1}^n A_j$ where $A_j = \bigcap_{i=1}^{m(j)} r(\alpha_i^j) \setminus r(\beta_i^j)$ where $r(\alpha_i^j) \supsetneq r(\beta_i^j)$ and $\alpha_i^j, \beta_i^j \in \mathcal{L}^+(E)$ for all i,j

This remark motivates the following definition.

DEFINITION 2.10. Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labeled graph. A *Cuntz–Krieger $(E,\mathcal{L})$-family* consists of commuting projections $\{p_{r(\beta)} : \beta \in \mathcal{L}^+(E)\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ with the properties that:

(CK1a) For all $\beta,\omega \in \mathcal{L}^+(E)$, $p_{r(\beta)} p_{r(\omega)} = 0$ if and only if $r(\beta) \cap r(\omega) = \emptyset$.
(CK1b) For all $\beta,\omega,\kappa \in \mathcal{L}^+(E)$, if $r(\beta) \cap r(\omega) = r(\kappa)$, then $p_{r(\beta)} p_{r(\omega)} = p_{r(\kappa)}$, if $r(\beta) \cup r(\omega) = r(\kappa)$, then $p_{r(\beta)} + p_{r(\omega)} - p_{r(\beta)} p_{r(\omega)} = p_{r(\kappa)}$ and if $r(\beta) \supsetneq r(\omega)$, then $p_{r(\beta)} - p_{r(\omega)} \neq 0$.
(CK2) If $a \in \mathcal{A}$ and $\beta \in \mathcal{L}^+(E)$, then $p_{r(\beta)} s_a = s_a p_{r(\beta a)}$.
(CK3) If $a,b \in \mathcal{A}$, then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$.

(CK4) For $\beta \in \mathcal{L}^+(E)$, if $\mathcal{L}^1_{r(\beta)}$ is finite and nonempty, then we have

$$p_{r(\beta)} = \sum_{a \in \mathcal{L}^1_{r(\beta)}} s_a p_{r(\beta a)} s_a^*. \tag{2.1}$$

Let $C^*(E,\mathcal{L})$ be the universal C^*-algebra generated by a Cuntz–Krieger $(E,\mathcal{L})$-family.

Let $\gamma' : \mathbb{T} \to \operatorname{Aut} C^*(E,\mathcal{L})$ be the gauge action determined by

$$\gamma'_z p_{r(\beta)} = p_{r(\beta)}, \qquad \gamma'_z s_a = z s_a \quad \text{for } \beta \in \mathcal{L}^+(E), a \in \mathcal{A}.$$

THEOREM 2.11. *Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labeled graph. Then $C^*(E,\mathcal{L})$ is isomorphic to $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$; moreover*

$$C^*(E,\mathcal{L}) = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha,\beta \in \mathcal{L}^+(E), A \in \mathcal{E}(r,\mathcal{L})\}.$$

Proof. Let $\{s_a, p_{r(\beta)}\}$ be a universal Cuntz–Krieger $(E,\mathcal{L})$-family and $\{t_a, q_A\}$ be a universal representation of the labeled space $(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$. For $a \in \mathcal{A}$, set $T_a = s_a$.

By (CK1a), we may define $Q_\emptyset = 0$. For $\alpha,\beta \in \mathcal{L}^+(E)$, we may define $Q_{r(\alpha)\cap r(\beta)} = Q_{r(\alpha)} Q_{r(\beta)}$ and $Q_{r(\alpha \cup r(\beta)} = Q_{r(\alpha)} + Q_{r(\beta)} - Q_{r(\alpha)\cap r(\beta)}$ in $C^*(E,\mathcal{L})$. If $r(\alpha) \supsetneq r(\beta)$, then we may define $Q_{r(\alpha)\setminus r(\beta)} = Q_{r(\alpha)} - Q_{r(\beta)} \neq 0$ in $C^*(E,\mathcal{L})$. By Remark 2.9 and using the inclusion/exclusion law we may define Q_A in $C^*(E,\mathcal{L})$ for all $A \in \mathcal{E}(r,\mathcal{L})$.

It is a routine calculation to show that $\{T_a, Q_A\}$ is a representation of the labeled space $(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ in $C^*(E,\mathcal{L})$. By the universal property of $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ there exists a homomorphism $\Phi: C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L})) \to C^*(E,\mathcal{L})$ such that $\Phi(t_a) = T_a$ and $\Phi(q_A) = Q_A$. It is straightforward to see that $\gamma'_z \circ \Phi = \Phi \circ \gamma_z$ for $z \in \mathbb{T}$. The first statement then follows by Theorem 2.7, and the final statement follows by applying Φ to an arbitrary element of $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ (see [2, Lemma 4.4]). □

3. Automorphisms of labeled graphs and their C^*-algebras

We begin by defining what a labeled graph morphism is and use the definition to define a labeled graph automorphism. Then in Theorem 3.2 we show that a labeled graph automorphism of $(E,\mathcal{L})$ induces an automorphism of $C^*(E,\mathcal{L})$.

DEFINITION 3.1. Let $(E,\mathcal{L})$ and $(F,\mathcal{M})$ be labeled graphs over alphabets $\mathcal{A}_E$ and $\mathcal{A}_F$ respectively. A *labeled graph morphism* is a triple $\phi := (\phi^0, \phi^1, \phi^{\mathcal{A}_E}) : (E,\mathcal{L}) \to (F,\mathcal{M})$ such that

(a) for all $e \in E^1$, we have $\phi^0(r(e)) = r(\phi^1(e))$ and $\phi^0(s(e)) = s(\phi^1(e))$;
(b) $\phi^{\mathcal{A}_E} : \mathcal{A}_E \to \mathcal{A}_F$ is a map such that $\mathcal{M} \circ \phi^1 = \phi^{\mathcal{A}_E} \circ \mathcal{L}$.

If the maps $\phi^0, \phi^1, \phi^{\mathcal{A}_E}$ are bijective, then the triple $\phi := (\phi^0, \phi^1, \phi^{\mathcal{A}_E})$ is called a *labeled graph isomorphism*. In the case that $F = E$, $\mathcal{A}_E = \mathcal{A}_F$ and $\mathcal{L} = \mathcal{M}$, we call $(\phi^0, \phi^1, \phi^{\mathcal{A}})$ a *labeled graph automorphism.*

For a labeled graph morphism $\phi = (\phi^0, \phi^1, \phi^{\mathcal{A}_E})$, we shall omit the superscripts on ϕ when the context in which it is being used is clear.

The set $\operatorname{Aut}(E,\mathcal{L}) := \{\phi : \phi \text{ is a labeled graph automorphism of } (E,\mathcal{L})\}$ forms a group under composition. The following result follows easily from the universal definition of $C^*(E,\mathcal{L})$.

THEOREM 3.2. *Let ϕ be an automorphism of a weakly left-resolving, set-finite labeled graph $(E,\mathcal{L})$ and $\{s_a, p_{r(\beta)}\}$ be a universal Cuntz–Krieger $(E,\mathcal{L})$-family. The maps $s_a \mapsto s_{\phi(a)}$ and $p_{r(\beta)} \mapsto p_{\phi(r(\beta))}$ induce an automorphism of $C^*(E,\mathcal{L})$.*

4. Skew product labeled graphs and group actions

In this section, we shall define a skew product labeled graph and define what it means for a group to act on a labeled graph.

DEFINITION 4.1. Let $(E,\mathcal{L})$ be a labeled graph and let $c, d : E^1 \to G$ be functions. The *skew product labeled graph* $(E \times_c G, \mathcal{L}_d)$ over alphabet $\mathcal{A} \times G$ consists of the skew product graph $(E^0 \times G, E^1 \times G, r_c, s_c)$ where

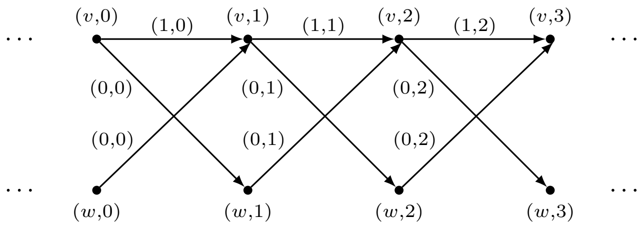
$$r_c(e,g) = \big(r(e), gc(e)\big), \qquad s_c(e,g) = \big(s(e), g\big)$$

together with the labeling $\mathcal{L}_d : (E \times_c G)^1 \to \mathcal{A} \times G$ given by $\mathcal{L}_d(e,g) := (\mathcal{L}(e), gd(e))$.

Since the labels received by $(v,g) \in (E \times_c G)^0$ are in one-to-one correspondence with the labels received by $v \in E^0$ it follows that if $(E,\mathcal{L})$ is left-resolving, then so is $(E \times_c G, \mathcal{L}_d)$.

EXAMPLE 4.2. For the labeled graph $(E,\mathcal{L})$ of Examples 2.2(b) let $c, d : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ and $d(e) = 0$ for all $e \in E^1$. Then

$(E \times_c \mathbb{Z}, \mathcal{L}_d) :=$



REMARK 4.3. We shall use the following simpler description of the path space of $E \times_c G$. For $v \in E^0, e \in E^1, g \in G$ set $v_g = (v,g)$, $e_g = (e,g)$. Then for $\mu \in E^n$ where $n \geq 2$ and $g \in G$ set

$$\mu_g = (\mu_1, g)\big(\mu_2, gc(\mu_1)\big) \cdots \big(\mu_n, gc(\mu')\big) \in (E \times G)^n.$$

For $\mu \in E^*$ the map $(\mu, g) \mapsto \mu_g$ identifies $E^* \times G$ with $(E \times_c G)^*$. Then for $(\mu, g) \in E^* \times G$ we have

$$s(\mu,g) = \big(s(\mu), g\big) \quad \text{and} \quad r(\mu,g) = \big(r(\mu), gc(\mu)\big). \tag{4.1}$$

Let $(E,\mathcal{L})$ be a labeled graph over the alphabet $\mathcal{A}$. A *labeled graph action of* G *on* $(E,\mathcal{L})$ is a triple $((E,\mathcal{L}),G,\phi)$ where $\phi: G\to \operatorname{Aut}(E,\mathcal{L})$ is a group homomorphism. In particular, for all $e\in E^1$ and $g\in G$ we have

$$\mathcal{L}\big(\phi_g(e)\big)=\phi_g\big(\mathcal{L}(e)\big). \tag{4.2}$$

If we ignore the label maps, a labeled graph action $((E,\mathcal{L}),G,\phi)$ restricts to a graph action of G on E; we denote this restricted action by (E,G,ϕ). The labeled graph action $((E,\mathcal{L}),G,\alpha)$ is *free* if $\phi_g(v)=v$ for some $v\in E^0$, then $g=1_G$ and if $\phi_g(a)=a$ some $a\in\mathcal{A}$, then $g=1_G$.

The following lemma shows that skew product labeled graphs provide a rich source of examples of free labeled graph actions. As the proof is routine, we omit it.

Lemma 4.4. *Let $(E,\mathcal{L})$ be a labeled graph, $c,d: E^1\to G$ be functions and $(E\times_c G,\mathcal{L}_d)$ be the associated skew product labeled graph. Then*

- (i) *For $(x,h)\in(E\times_c G)^i$, $(a,h)\in\mathcal{A}\times G$, $g\in G$ and $i=0,1$ let $\tau_g^i(x,h)=(x,gh)$ and $\tau_g^{\mathcal{A}}(a,h)=(a,gh)$. Then $\tau_g=(\tau_g^0,\tau_g^1,\tau_g^{\mathcal{A}})$ is a labeled graph automorphism.*
- (ii) *The map $\tau=(\tau^0,\tau^1,\tau^{\mathcal{A}}): G\to\operatorname{Aut}(E\times_c G,\mathcal{L}_d)$ defined by $g\mapsto\tau_g$ is a homomorphism.*
- (iii) *The triple $((E\times_c G,\mathcal{L}_d),G,\tau)$ is a free labeled graph action.*

Definition 4.5. The map $\tau=(\tau^0,\tau^1,\tau^{\mathcal{A}}): G\to\operatorname{Aut}(E\times_c G,\mathcal{L}_d)$ as given in Lemma 4.4(ii) is called the *left labeled graph translation map*, and the action $((E\times_c G,\mathcal{L}_d),G,\tau)$ the *left labeled graph translation action.*

Two labeled graph actions $((E,\mathcal{L}),G,\phi)$ and $((F,\mathcal{M}),G,\psi)$ are *isomorphic* if there is a labeled graph isomorphism $\varphi:(E,\mathcal{L})\to(F,\mathcal{M})$ which is *equivariant* in the sense that $\varphi\circ\phi_g=\psi_g\circ\varphi$ for all $g\in G$.

Theorem 4.6. *Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labeled graph, and $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action. Let $\{s_a,p_{r(\beta)}\}$ be a universal Cuntz–Krieger $(E,\mathcal{L})$-family. Then for $h\in G$ the maps*

$$\alpha_h s_a=s_{\alpha_h a}\quad \textit{and}\quad \alpha_h p_{r(\beta)}=p_{\alpha_h r(\beta)}$$

determine an action of G on $C^*(E,\mathcal{L})$. If $((E,\mathcal{L}),G,\phi)$ and $((F,\mathcal{M}),G,\psi)$ are isomorphic then $C^*(E,\mathcal{L})\times_\phi G\cong C^*(F,\mathcal{M})\times_\psi G$.

Proof. Follows by a straightforward application of Theorem 3.2 and the universal property of crossed products. □

5. Gross–Tucker theorem

In this section, we prove a version of the Gross–Tucker theorem for labeled graphs. For directed graphs, the Gross–Tucker theorem says, roughly speaking, that up to equivariant isomorphism, every free action α of a group G on a directed graph E is a left translation automorphism τ on a skew product graph $(E/G) \times_c G$ built from the quotient graph E/G. Our aim is to prove a similar result for labeled graphs. The new ingredient is the map $d : E^1 \to G$ found in the definition of a skew product labeled graph for labeled graphs. Before giving our main result, Theorem 5.10, we introduce some notation.

DEFINITIONS 5.1. Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action. For $i = 0,1$ and $x \in E^i$ let $Gx := \{\alpha_g^i(x) : g \in G\}$ and $(E/G)^i = \{Gx : x \in E^i\}$. For $a \in \mathcal{A}$ let $Ga = \{\alpha_g^{\mathcal{A}}(a) : g \in G\}$ and $\mathcal{A}/G = \{Ga : a \in \mathcal{A}\}$.

The proof of the following lemma is straightforward, so we omit it.

LEMMA 5.2. *Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action. The maps $r,s : (E/G)^1 \to (E/G)^0$ given by*

$$r(Ge) = Gr(e) \quad \text{and} \quad s(Ge) = Gs(e) \quad \text{for } Ge \in (E/G)^1 \tag{5.1}$$

and the map $\mathcal{L}/G : (E/G)^1 \to \mathcal{A}/G$ given by $(\mathcal{L}/G)(Ge) = G\mathcal{L}(e)$ are well defined. Consequently, $(E/G,\mathcal{L}/G)$ is a labeled graph over the alphabet $\mathcal{A}/G$.

The map $q = (q^0,q^1,q^{\mathcal{A}}) : (E,\mathcal{L}) \to (E/G,\mathcal{L}/G)$ given by $q^i(x) = Gx$ for $i = 0,1, x \in E^i$ and $q^{\mathcal{A}}(a) = Ga$ for $a \in \mathcal{A}$ is a surjective labeled graph morphism.

DEFINITION 5.3. Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action. The *quotient labeled graph* $(E/G,\mathcal{L}/G)$ is the labeled graph described in Lemma 5.2, the map $q : (E,\mathcal{L}) \to (E/G,\mathcal{L}/G)$ is the *quotient labeled map*.

The following Proposition is an analog of [6, Theorem 2.2.1] whose proof is routine, and so we omit it.

PROPOSITION 5.4. *Let $(E,\mathcal{L})$ be a labeled graph, $c,d : E^1 \to G$ be functions and $(E \times_c G,\mathcal{L}_d)$ be the associated skew product labeled graph. Let $((E \times_c G,\mathcal{L}_d),G,\tau)$ be the left labeled graph translation action. Then*

$$\big((E \times_c G)/G,\mathcal{L}_d/G\big) \cong (E,\mathcal{L}).$$

EXAMPLE 5.5. Recall the labeled graphs $(E,\mathcal{L})$ and $(E \times_c \mathbb{Z},\mathcal{L}_d)$ from Example 4.2. For the left labeled graph translation action $((E \times_c \mathbb{Z},\mathcal{L}_d),\mathbb{Z},\tau)$, we have $((E \times_c \mathbb{Z})/\mathbb{Z},\mathcal{L}_d/\mathbb{Z}) \cong (E,\mathcal{L})$ by Proposition 5.4.

The Gross–Tucker theorem is a converse to Proposition 5.4. It states that if we have a free action of a group on a labeled graph, then we can recover the original graph from the quotient via a skew product. Recall the following definition for directed graphs.

DEFINITION 5.6. Let F,E be directed graphs. A surjective graph morphism $p : F \to E$ has the *unique path lifting property* if given $u \in F^0$ and $e \in E^1$ with $s(e) = p^0(u)$ there is a unique edge $f \in F^1$ with $s(f) = u$ and $p^1(f) = e$.

REMARK 5.7. Let (E,G,α) be a free graph action. Then the quotient map $q : E \to E/G$ has the unique path lifting property (see [11, Section 5] or [6, p. 67], for instance).

DEFINITIONS 5.8. Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action and $q=(q^0,q^1,q^{\mathcal{A}}):(E,\mathcal{L})\to(E/G,\mathcal{L}/G)$ be the quotient labeled map. A *section* for q^i is a map $\eta^i:(E/G)^i\to E^i$ for $i=0,1$ such that $q^i\circ\eta^i=\mathrm{id}_{(E/G)^i}$. A *section* for $q^{\mathcal{A}}$ is $\eta^{\mathcal{A}}:\mathcal{A}/G\to\mathcal{A}$ such that $q^{\mathcal{A}}\circ\eta^{\mathcal{A}}=\mathrm{id}_{\mathcal{A}/G}$.

LEMMA 5.9. *Let (E,G,α) be a graph action and $q=(q^0,q^1):E\to E/G$ be the quotient map. Given a section η^0 for q^0 there is a unique section η^1 for q^1 such that*

$$s\big(\eta^1(Ge)\big)=\eta^0\big(s(Ge)\big)\quad \textit{for all } e\in E^1. \tag{5.2}$$

Proof. By Remark 5.7 the quotient map $q:E\to E/G$ has the unique path lifting property. Hence if we fix $Gv\in(E/G)^0$, then for each $Ge\in(E/G)^1$ with $s(Ge)=Gv$ there is a unique $f\in E^1$ with $q^1(f)=Ge=Gf$ and $s(f)=\eta^0(Gv)$. Put $\eta^1(Ge)=f$, then $\eta^1:(E/G)^1\to E^1$ is well defined and the source map on $(E/G)^1$ is well defined. Since $q^1(\eta^1(Ge))=q^1(f)=Ge$ it follows that η^1 is a section satisfying (5.2). Uniqueness of η^1 follows from the unique path lifting property of q. □

The following is a version of the Gross–Tucker theorem (cf. [6, Theorem 2.2.2]) for labeled graphs.

THEOREM 5.10. *Let $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action. Let $\eta^0,\eta^{\mathcal{A}}$ be sections for $q^0,q^{\mathcal{A}}$ respectively. There are functions $c,d:(E/G)^1\to G$ such that $((E,\mathcal{L}),G,\alpha)$ is isomorphic to $((E/G\times_c G,(\mathcal{L}/G)_d),G,\tau)$.*

Proof. Fix a section $\eta^0:(E/G)^0\to E^0$ for q^0. By Lemma 5.9, there is a section η^1 for q^1satisfying (5.2). For $Ge\in(E/G)^1$ set $f=\eta^1(Ge)$, then

$$q^0\big(r\big(\eta^1(Ge)\big)\big)=q^0\big(r(f)\big)=Gr(f)=r(Gf)=r(Ge)=q^0\big(\eta^0\big(r(Ge)\big)\big).$$

As (E,G,α) is free, there is a unique $h\in G$ such that $\alpha^0_h\eta^0(r(Ge))=r(\eta^1(Ge))$ and we may set $c(Ge)=h$. Define $\phi:E/G\times_c G\to E$ by

$$\phi^0_c(Gv,g)=\alpha^0_g\eta^0(Gv)\quad\text{and}\quad\phi^1_c(Ge,g)=\alpha^1_g\eta^1(Ge)$$

for $(Gv,g)\in(E/G\times_c G)^0$ and $(Ge,g)\in(E/G\times_c G)^1$. One checks that $\phi_c:(E/G\times_c G)\to E$ is an isomorphism of directed graphs.

We claim that ϕ_c is equivariant. Notice that for all $(Gv,h)\in(E/G\times_c G)^0$ and $g\in G$ we have

$$\phi^0_c\big(\tau^0_g(Gv,h)\big)=\phi^0_c(Gv,gh)=\alpha^0_{gh}\eta^0(Gv)=\alpha^0_g\alpha^0_h\eta^0(Gv)=\alpha^0_g\phi^0_c(Gv,h)$$

and so $\phi^0_c\circ\tau^0_g=\alpha^0_g\circ\phi^0_c$ for all $g\in G$. The argument for ϕ^1_c is similar and our claim follows.

We now construct an equivariant bijection $\phi_d^{\mathcal{A}/G\times G}:\mathcal{A}/G\times G\to\mathcal{A}$ which satisfies condition (b) of Definition 3.1. Fix a section $\eta^{\mathcal{A}}:\mathcal{A}/G\to\mathcal{A}$ for $q^{\mathcal{A}}$.

We now define a map $d\colon (E/G)^1 \to G$. Fix $Ge \in (E/G)^1$ and set $f = \eta^1(Ge)$ so that $q^1(f) = Ge$. Since

$$q^{\mathcal{A}}\eta^{\mathcal{A}}\big(\mathcal{L}/G(Ge)\big) = q^{\mathcal{A}}\eta^{\mathcal{A}}\big(G\mathcal{L}(f)\big) = q^{\mathcal{A}}\mathcal{L}\eta^1(Ge)$$

and the graph action $((E,\mathcal{L}),G,\alpha)$ is free, there is a unique $k \in G$ such that $\alpha_k^{\mathcal{A}}\eta^{\mathcal{A}}((\mathcal{L}/G)(Ge)) = \mathcal{L}(\eta^1(Ge))$ and we may define $d(Ge) = k$. The function $d : (E/G)^1 \to G$ described in this way is such that $d(Ge)$ is the unique element of G with the property that

$$\alpha^{\mathcal{A}}_{d(Ge)}\eta^{\mathcal{A}}\big((\mathcal{L}/G)(Ge)\big) = \mathcal{L}\big(\eta^1(Ge)\big). \tag{5.3}$$

For each $(Ga,g) \in \mathcal{A}/G \times G$ we define $\phi_d^{\mathcal{A}/G\times G} : \mathcal{A}/G \times G \to \mathcal{A}$ by $\phi_d^{\mathcal{A}/G\times G}(Ga, g) = \alpha_g^{\mathcal{A}}\eta^{\mathcal{A}}(Ga)$. We claim that $\phi_d^{\mathcal{A}/G\times G}$ satisfies $\phi_d^{\mathcal{A}/G\times G} \circ (\mathcal{L}/G)_d = \mathcal{L} \circ \phi_c^1$: By (5.3) for all $(Ge,h) \in (E/G \times_c G)^1$ we have

$$\begin{aligned}\phi_d^{\mathcal{A}/G\times G} \circ (\mathcal{L}/G)_d(Ge,h) &= \alpha_h^{\mathcal{A}}\alpha^{\mathcal{A}}_{d(Ge)}\eta_{\mathcal{A}}\big(\mathcal{L}/G(Ge)\big)\\ &= \mathcal{L}\big(\alpha_h^1\eta^1(Ge)\big) = \mathcal{L} \circ \phi_c^1(Ge,h)\end{aligned}$$

as required.

It is straightforward to see that $\phi_d^{\mathcal{A}/G\times G}$ is bijective. To see that $\phi_d^{\mathcal{A}/G\times G}$ is equivariant notice that we have

$$\begin{aligned}\phi_d^{\mathcal{A}/G\times G}\big(\tau_g^{\mathcal{A}/G\times G}(Ge,h)\big) &= \phi_d^{\mathcal{A}/G\times G}(Ge,gh) = \alpha_g^{\mathcal{A}}\alpha_h^{\mathcal{A}}\eta^{\mathcal{A}}(Ge)\\ &= \alpha_g^{\mathcal{A}}\phi_d^{\mathcal{A}/G\times G}(Ge,h)\end{aligned}$$

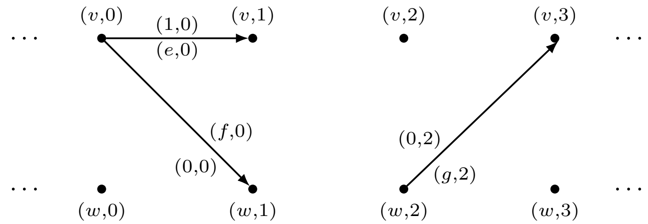
for all $(Ge,h) \in (E/G \times G)^1$ and $g \in G$. Thus $\phi_{c,d} = (\phi_c^0, \phi_c^1, \phi_d^{\mathcal{A}/G\times G})$ is the required labeled graph isomorphism. $\square$

REMARK 5.11. The possibility that two edges in the quotient graph have the same label means that we must choose a separate section $\eta^{\mathcal{A}}$ for $q^{\mathcal{A}}$. In turn means that the function d given in the definition of a skew product labeled graph plays a crucial role in the reconstruction of the labeled graph action in Theorem 5.10.

EXAMPLE 5.12. Recall from Example 5.5 the labeled graph $(E \times_c \mathbb{Z}, \mathcal{L}_d)$ has a free action of $\mathbb{Z}$ such that the quotient labeled graph is $(E,\mathcal{L})$. We use this example to illustrate the point made in Remark 5.11:

Suppose we choose a section $\eta^0 : E^0 \to (E \times_c \mathbb{Z})^0$ such that $\eta^0(v) = (v,0)$ and $\eta^0(w) = (w,2)$, then the section $\eta^1 : E^1 \to (E \times_c \mathbb{Z})^1$ as defined in Lemma 5.9 is given by $\eta^1(e) = (e,0)$, $\eta^1(f) = (f,0)$, and $\eta^1(g) = (g,2)$ whose image in

$(E \times_c \mathbb{Z}, \mathcal{L}_d)$ is as shown below.



Note that $c(e) = 1$, $c(f) = -1$, and $c(g) = 3$.

Observe that $f, g \in E^1$ are such that $\mathcal{L}(f) = \mathcal{L}(g) = 0$ however,

$$\mathcal{L}\big(\eta^1(f)\big) = \mathcal{L}(f,0) = (0,0) \neq (0,2) = \mathcal{L}(g,2) = \mathcal{L}\big(\eta^1(g)\big).$$

The function d accounts for this difference. By Equation (5.3), we have $d(g) = 2$, since $\alpha_2^{\mathcal{A}}(0,0) = (0,2)$, whereas $d(f) = 0$. Observe that $d(g) \neq d(f)$ even though $\mathcal{L}(g) = \mathcal{L}(f)$.

6. Coactions on labeled graph algebras

In [9] it is shown that a function $c : E^1 \to G$ induces a coaction δ of G on the graph algebra $C^*(E)$ such that $C^*(E) \times_\delta G \cong C^*(E \times_c G)$. One should expect, therefore, that the functions $c, d : E^1 \to G$ would induce a coaction δ of G on $C^*(E, \mathcal{L})$ such that $C^*(E, \mathcal{L}) \times_\delta G \cong C^*(E \times_c G, \mathcal{L}_d)$. However in order to obtain such a result we must assume that both functions c, d are label consistent (see Definition 6.1 below). For further information about coactions of discrete groups see [15], amongst others.

Definition 6.1. Let $(E, \mathcal{L})$ be a labeled graph over alphabet $\mathcal{A}$. A function $c : E^1 \to G$ is *label consistent* if there is a function $C : \mathcal{A} \to G$ such that $c = C \circ \mathcal{L}$.

For any labeled graph $(E, \mathcal{L})$ the function $\mathbf{1} : E^1 \to G$ given by $\mathbf{1}(e) = 1_G$ for all $e \in E^1$ is label consistent. First, we show that if c is label consistent then there is a coaction of G on $C(E, \mathcal{L})$.

Proposition 6.2. *Let $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labeled graph, G be a discrete group, and $c : E^1 \to G$ be a label consistent function. Then there is a maximal, normal coaction $\delta : C^*(E, \mathcal{L}) \to C^*(E, \mathcal{L}) \otimes C^*(G)$ such that*

$$\delta(s_a) = s_a \otimes u_{C(a)} \quad \text{and} \quad \delta(p_{r(\beta)}) = p_{r(\beta)} \otimes u_{1_G}, \tag{6.1}$$

where $\{s_a, p_{r(\beta)}\}$ is a universal Cuntz–Krieger $(E, \mathcal{L})$-family and $\{u_g : g \in G\}$ are the canonical generators of $C^*(G)$.

Proof. The first part of the result follows by the same argument given in [9, Lemma 3.2]. That the coaction δ is normal and maximal follows by essentially the same arguments as the ones given in [5, Lemma 3.3] and [13, Theorem 7.1(v)]. □

The next result shows that if d is label consistent then we may as well assume that $d = \mathbf{1}$.

PROPOSITION 6.3. *Let $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labeled graph and $c : E^1 \to G$ a function. If $d_1, d_2 : E^1 \to G$ are label consistent functions, then $((E \times_c G, \mathcal{L}_{d_1}), G, \tau) \cong ((E \times_c G, \mathcal{L}_{d_2}), G, \tau)$ where τ is the left translation action. Hence, if $d : E^1 \to G$ is a label consistent function then there is an isomorphism from $C^*(E \times_c G, \mathcal{L}_d)$ to $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ which is equivariant for the G-action induced by τ.*

Proof. For the first statement, let $\phi^i : (E \times_c G)^i \to (E \times_c G)^i$ be the identity map for $i = 0, 1$ and define $\phi^{\mathcal{A} \times G} : \mathcal{A} \times G \to \mathcal{A} \times G$ by

$$\phi^{\mathcal{A} \times G}(a, g) = \big(a, g D_1^{-1}(a) D_2(a)\big).$$

For $(e, g) \in (E \times_c G)^1$, after a short calculation we have

$$\phi^{\mathcal{A} \times G} \mathcal{L}_{d_1}(e, g) = \big(\mathcal{L}(e), d_2(e)\big) = \mathcal{L}_{d_2}(e, g).$$

It is then straightforward to check that $\phi = (\phi^0, \phi^1, \phi^{\mathcal{A} \times G})$ is a labeled graph isomorphism. Since for all $h \in G$ we have

$$\tau_h\big(\phi^{\mathcal{A} \times G}(a, g)\big) = \big(a, hg D_1^{-1}(a) D_2(a)\big) = \phi^{\mathcal{A} \times G}\big(\tau_h(a, g)\big)$$

it follows that $((E \times_c G, \mathcal{L}_{d_1}), G, \tau) \cong ((E \times_c G, \mathcal{L}_{d_2}), G, \tau)$.

The final statement follows from Theorem 4.6. □

REMARK 6.4. Thanks to Proposition 6.3 we may, without loss of generality, assume that $d = \mathbf{1}$ when we are working with label consistent d-functions. On the other hand it is not hard to see that a different choice of label consistent functions c will yield non-isomorphic skew-product graphs.

Next, we shall show that if $d = \mathbf{1}$ then there is a natural identification $\mathcal{L}_{\mathbf{1}}^+(E \times_c G)$, the labeled path space of $(E \times_c G, \mathcal{L}_{\mathbf{1}})$ with $\mathcal{L}^+(E) \times G$.

LEMMA 6.5. *Let $(E, \mathcal{L})$ be a labeled graph and $c : E^1 \to G$ label consistent. For $\mu \in E^+$ and $g \in G$ the map*

$$\mathcal{L}_{\mathbf{1}}(\mu, g) \mapsto \big(\mathcal{L}(\mu), g\big)$$

establishes a bijection from $\mathcal{L}_{\mathbf{1}}^+(E \times_c G)$ to $\mathcal{L}^+(E) \times G$.

Proof. From Remark 4.3 it follows that for $n \geq 1$ every path in $(E \times_c G)^n$ has the form $(\mu, g) = (\mu_1, g)(\mu_2, gc(\mu_1)) \cdots (\mu_n, gc(\mu'))$, for some $\mu \in E^n$ and $g \in G$. Then by definition we have

$$\mathcal{L}_{\mathbf{1}}(\mu, g) = \big(\mathcal{L}(\mu_1), g\big)\big(\mathcal{L}(\mu_2), gc(\mu_1)\big) \cdots \big(\mathcal{L}(\mu_n), gc(\mu')\big). \tag{6.2}$$

If we define the right-hand side of (6.2) to be $(\mathcal{L}(\mu), g)$ the result follows. $\square$

The following lemma indicates the behavior of the range map under the identification of $\mathcal{L}^+_{\mathbf{1}}(E \times_c G)$ with $\mathcal{L}^+(E) \times G$.

Lemma 6.6. *Let $(E, \mathcal{L})$ be a labeled graph and $c: E^1 \to G$ be a label consistent function. Let $a \in \mathcal{A}$, $\beta \in \mathcal{L}^+(E)$, and $g \in G$. Then under the identification of $\mathcal{L}^+(E) \times G$ with $\mathcal{L}^+_{\mathbf{1}}(E \times_c G)$ we have $r(\beta, g) = (r(\beta), gC(\beta)) \in \mathcal{E}(r, \mathcal{L}) \times G$.*

Proof. Observe that for $(\beta, g) \in \mathcal{L}^+(E) \times G$, we have

$$\begin{aligned} r(\beta, g) &= \{r(\mu, g) : (\mu, g) \in E^* \times G, \mathcal{L}(\mu) = \beta\} \quad \text{by (6.2)} \\ &= \{(r(\mu), gC(\beta)) : \mathcal{L}(\mu) = \beta\} \quad \text{by (4.1)} \end{aligned} \tag{6.3}$$

since the function $c: E^1 \to G$ is label consistent. Hence, we may identify $r(\beta, g)$ with $(r(\beta), gC(\beta)) \in \mathcal{E}(r, \mathcal{L}) \times G$. $\square$

With the above identifications in mind, we turn our attention to the main result of this section. By Theorem 4.6 the left labeled graph translation action $((E \times_c G, \mathcal{L}_{\mathbf{1}}), G, \tau)$ defined in Definition 4.5 induces an action $\tau : G \to \operatorname{Aut} C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$. When we identify $\mathcal{L}^+_{\mathbf{1}}(E \times_c G)$ with $\mathcal{L}^+(E) \times G$ this action may be described on the generators of $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ as follows: For $h, g \in G$, $a \in \mathcal{A}$, and $\beta \in \mathcal{L}^+(E)$ we have

$$\tau_h(s_{(a,g)}) = s_{(a,hg)} \quad \text{and} \quad \tau_h(p_{(r(\beta),g)}) = p_{(r(\beta),hg)}. \tag{6.4}$$

The method of proof for the next result closely follows that of [9, Theorem 2.4], however we give some of the details as they rely heavily on the identification we made in Lemma 6.6.

Theorem 6.7. *Let $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labeled graph. Suppose that G is a discrete group, $c: E^1 \to G$ is a label consistent function, and δ is the coaction from Proposition 6.2. Let $j_{C^*(E,\mathcal{L})}, j_G$ denote the canonical covariant homomorphisms of $C^*(E, \mathcal{L})$ and $C^*(G)$ into $M(C^*(E, \mathcal{L}) \times_\delta G)$ and $\{s_{(a,g)}, p_{(r(\beta),g)}\}$ be the canonical generating set of $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$. Then the map $\phi : C^*(E \times_c G, \mathcal{L}_{\mathbf{1}}) \to C^*(E, \mathcal{L}) \times_\delta G$ given by*

$$\begin{aligned} \phi(s_{(a,g)}) &= j_{C^*(E,\mathcal{L})}(s_a) j_G(\chi_{C(a)^{-1}}), \\ \phi(p_{(r(\beta),g)}) &= j_{C^*(E,\mathcal{L})}(p_{r(\beta)}) j_G(\chi_{g^{-1}}) \end{aligned}$$

is an isomorphism.

Sketch of proof. For each $g \in G$, let $C^*(E, \mathcal{L})_g = \{b \in C^*(E, \mathcal{L}) : \delta(b) = b \otimes u_g\}$ denote the corresponding spectral subspace; we write b_g to denote a generic element of $C^*(E, \mathcal{L})_g$. Then $C^*(E, \mathcal{L}) \times_\delta G$ is densely spanned by the set $\{(b_g, h) : b_g \in C^*(E, \mathcal{L})_g \text{ and } g, h \in G\}$, and the algebraic operations are given on this set by

$$\begin{aligned} (b_g, x)(b_h, y) &= (b_g b_h, y) \quad \text{if } y = h^{-1}x \text{ (and 0 if not)}, \quad \text{and} \\ (b_g, x)^* &= (b_g^*, gx). \end{aligned}$$

If $(j_{C^*(E,\mathcal{L})}, j_G)$ denotes the canonical covariant homomorphism of $C^*(E,\mathcal{L})$ into the multiplier algebra of $C^*(E,\mathcal{L}) \times_\delta G$, then (b_g, x) is by definition $(j_{C^*(E,\mathcal{L})}(b_g) j_G(\chi_{\{x\}}))$.

Using Lemma 6.6, we may show that for $(a,g) \in \mathcal{A} \times G$, $\beta \in \mathcal{L}^+(E)$ and $g \in G$

$$t_{(a,g)} = \big(s_a, C(a)^{-1} g^{-1}\big) \quad \text{and} \quad q_{(r(\beta),g)} = \big(p_{r(\beta)}, g^{-1}\big)$$

is a Cuntz–Krieger $(E \times_c G, \mathcal{L}_{\mathbf{1}})$-family in $C^*(E,\mathcal{L}) \times_\delta G$.

By universality of $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ there is a homomorphism $\pi_{t,q}$ from $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ to $C^*(E,\mathcal{L}) \times_\delta G$ such that $\pi_{t,q}(s_{(a,g)}) = t_{(a,g)}$ and $\pi_{t,q}(p_{(r(\beta),g)}) = q_{(r(\beta),g)}$ which we may show is injective using the argument from [9, Theorem 2.4] and Theorem 2.7.

Next, we show that $\pi_{t,q}$ is surjective. Observe that $C^*(E,\mathcal{L}) \times_\delta G$ is generated by (s_a, g) and $(p_{r(\beta)}, h)$. Since $\pi_{t,q}(s_{(a,g^{-1}C(a)^{-1})}) = t_{(a,g^{-1}C(a)^{-1})} = (s_a, C(a)^{-1}C(a)g)$, and $\pi_{t,q}(p_{(r(\beta),h^{-1})}) = (p_{r(\beta)}, h)$ we see that $\pi_{t,q}$ is surjective. Hence, $\pi_{t,q}$ is the desired isomorphism.

We need to check that $\pi_{t,q}$ is equivariant for the G actions, that is $\pi_{t,q} \circ \tau_g = \widehat{\delta}_g \circ \pi_{t,q}$ for all $g \in G$. It is enough to check on generators: Notice that for all $s_{(a,h)} \in C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$

$$\begin{aligned}\pi_{t,q} \circ \tau_g(s_{(a,h)}) &= \pi_{t,q}(s_{(a,gh)}) = \big(s_a, C(a)^{-1} h^{-1} g^{-1}\big) \\ &= \widehat{\delta}_g\big(s_a, C(a)^{-1} h^{-1}\big) = \widehat{\delta}_g \circ \pi_{t,q}(s_{(a,h)})\end{aligned}$$

and similarly $\pi_{t,q} \circ \tau_g(p_{(r(\beta),h)}) = \widehat{\delta}_g \circ \pi_{t,q}(p_{(r(\beta),h)})$ for $p_{(r(\beta),h)} \in C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$.

We claim that $\pi_{t,q}$ is equivariant for the $\mathbb{T}$ actions, that is $\pi_{t,q} \circ \gamma_z = (\gamma_z \times G) \circ \pi_{t,q}$ for all $z \in \mathbb{T}$. It is enough to check this on generators: Notice that for all $s_{(a,h)} \in C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ and $z \in \mathbb{T}$ we have

$$\begin{aligned}\pi_{t,q} \circ \gamma_z(s_{(a,h)}) &= \pi_{t,q}(z s_{(a,h)}) = \big(z s_a, C(a)^{-1} h^{-1}\big) = (\gamma_z \times G)\big(s_a, C(a)^{-1} h^{-1}\big) \\ &= (\gamma_z \times_\delta G) \circ \pi_{t,q}(s_{(a,h)}).\end{aligned}$$

Similarly, $\pi_{t,q} \circ \gamma_z(p_{(r(\beta),h)}) = (\gamma_z \times G) \circ \pi_{t,q}(p_{(r(\beta),h)})$ for all $p_{(r(\beta),h)} \in C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$. $\square$

Corollary 6.8. *Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labeled graph. Suppose that G is a discrete group, $c \colon E^1 \to G$ be a label consistent function, and τ the induced action of G on $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$. Then*

$$C^*(E \times_c G, \mathcal{L}_{\mathbf{1}}) \times_{\tau,r} G \cong C^*(E,\mathcal{L}) \otimes \mathcal{K}\big(\ell^2(G)\big).$$

Proof. Since the isomorphism of $C^*(E \times_c G, \mathcal{L}_{\mathbf{1}})$ with $C^*(E,\mathcal{L}) \times_\delta G$ is equivariant for the G-actions $\tau, \widehat{\delta}$, respectively, it follows that

$$C^*(E \times_c G, \mathcal{L}_{\mathbf{1}}) \times_{\tau,r} G \cong C^*(E,\mathcal{L}) \times_\delta G \times_{\widehat{\delta},r} G.$$

Following the argument in [9, Corollary 2.5], Katayama's duality theorem [10] gives us that $C^*(E,\mathcal{L}) \times_\delta G \times_{\widehat{\delta},r} G$ is isomorphic to $C^*(E,\mathcal{L}) \otimes \mathcal{K}(\ell^2(G))$, as required. □

In order to provide a version of Corollary 6.8 for group actions, we must first characterise when the functions c, d in the Gross–Tucker Theorem 5.10 are label consistent maps. We will do this in the next section.

Recall from [15, p. 209] that a coaction δ of a discrete group G on a C^*-algebra A is *saturated* if for each $s \in G$ we have $\overline{A_s A_s^*} = A^\delta$ where A_s is the spectral subspace $A_s = \{b \in A : \delta(b) = b \otimes u_s\}$ and A^δ is the fixed point algebra for δ

$$A^\delta := \big\{b \in A : \delta(a) = a \otimes u_{1_G}\big\}.$$

Lemma 6.9. *Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labeled graph and $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. Then the coaction δ of $\mathbb{Z}$ on $C^*(E,\mathcal{L})$ induced by c is saturated.*

Proof. The coaction δ of $\mathbb{Z}$ on $C^*(E,\mathcal{L})$ defined in Proposition 6.2 is such that the fixed point algebra $C^*(E,\mathcal{L})^\delta$ is precisely the fixed point algebra $C^*(E,\mathcal{L})^\gamma$ for the canonical gauge action of $\mathbb{T}$ on $C^*(E,\mathcal{L})$ by the Fourier transform (cf. [4, Corollary 4.9]). By an argument similar to that in [14, Section 2], we have

$$C^*(E,\mathcal{L})^\gamma = \overline{\mathrm{span}}\big\{s_\alpha p_A s_\beta^* : \alpha,\beta \in \mathcal{L}^n(E), A \in \mathcal{E}(r,\mathcal{L})\big\}.$$

Since E has no sinks it follows by a similar argument to that in [14, Lemma 4.1.1] that $C^*(E,\mathcal{L})$ is saturated. □

Theorem 6.10. *Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labeled graph. Then $C^*(E,\mathcal{L})^\gamma$ is strongly Morita equivalent to $C^*(E \times_c \mathbb{Z}, \mathcal{L}_{\mathbf{1}})$ where $c : E^1 \to \mathbb{Z}$ is given by $c(e) = 1$ for all $e \in E^1$.*

Proof. Since c is label consistent it follows by Theorem 6.7 that

$$C^*(E \times_c \mathbb{Z}, \mathcal{L}_{\mathbf{1}}) \cong C^*(E,\mathcal{L}) \times_\delta \mathbb{Z}.$$

By Lemma 6.9, the coaction is δ is saturated and since $C^*(E,\mathcal{L})^\delta \cong C^*(E,\mathcal{L})^\gamma$ the result follows. □

7. Free group actions on labeled graphs

In this section, we examine conditions on the free labeled graph action $((E,\mathcal{L}), G, \alpha)$ which ensure that the functions c, d from Theorem 5.10 are label consistent.

Recall that a fundamental domain for a graph action (E, G, α) is a subset T of E^0 such that for every $v \in E^0$ there exists $g \in G$ and a unique $w \in T$ such that $v = \alpha_g^0 w$. Every free graph action has a fundamental domain.

DEFINITION 7.1. Let $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action. A *fundamental domain for* $((E,\mathcal{L}),G,\alpha)$ is a fundamental domain $T \subseteq E^0$ for the restricted graph action such that for every $e,f \in E^1$ we have

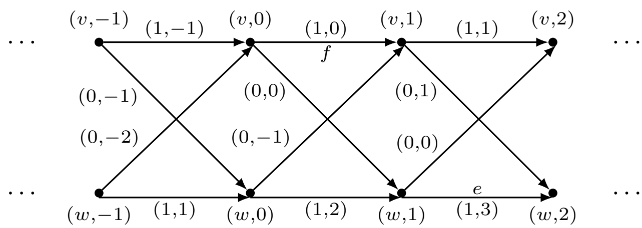
(a) if $r(e), r(f) \in T$ and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$ and
(b) if $s(e), s(f) \in T$ and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$.

In Examples 7.2(i) below, we see that not every free action of a group on a labeled graph has a fundamental domain.

EXAMPLES 7.2.

(i) Consider the following labeled graph

$(E,\mathcal{L}) :=$



The group $\mathbb{Z}$ acts freely on $(E,\mathcal{L})$ by addition in the second coordinate of the vertices, edges and labels as indicated in the picture above; call this action α. Let $T = \{(v,0),(w,1)\}$, then T is a fundamental domain for the restricted graph action $(E,\mathbb{Z},\alpha)$. However when considering the labeled graph action $((E,\mathcal{L}),\mathbb{Z},\alpha)$ the set T does not satisfy Definition 7.1(b).

Consider the edges e,f as shown above with $\mathcal{L}(e) = (1,3)$ and $\mathcal{L}(f) = (1,0)$ respectively. We have $s(e) = (w,1) \in T$ and $s(f) = (v,0) \in T$ and $\mathbb{Z}\mathcal{L}(e) = \mathbb{Z}\mathcal{L}(f) = \{(1,n) : n \in \mathbb{Z}\}$, however $\mathcal{L}(e) = (1,3) \neq (1,0) = \mathcal{L}(f)$. Indeed any fundamental domain for the restricted action $(E,\mathbb{Z},\alpha)$ will also fail Definition 7.1(b).

(ii) Let $c,d : E^1 \to G$ be label consistent functions and $((E \times_c G, \mathcal{L}_d), G, \tau)$ be the associated left labeled graph translation action. Then one checks that $T = \{(v, 1_G) : v \in E^0\}$ is a fundamental domain for $((E \times_c G, \mathcal{L}_d), G, \tau)$.

The following result shows that when we add the fundamental domain hypothesis to the free labeled graph action, the functions $c,d : (E/G)^1 \to G$ in the labeled graph version of the Gross–Tucker theorem (Theorem 5.10) may be chosen to be label consistent.

THEOREM 7.3. *Let $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action with a fundamental domain. Then there are label consistent functions $c,d : (E/G)^1 \to G$ such that $((E,\mathcal{L}),G,\alpha) \cong ((E/G) \times_c G, (\mathcal{L}/G)_d), G, \tau)$.*

Proof. Let T be a fundamental domain for $((E,\mathcal{L}),G,\alpha)$. For every $Gv \in (E/G)^0$ there exists a unique $w \in T$ such that $Gw = Gv$. Hence, if we define

$\eta^0(Gv) = w$, then $\eta^0 : (E/G)^0 \to T$ is a section for q^0. Then we may define η^1, c, d, and $\eta^{\mathcal{A}}$ as in Theorem 5.10.

It suffices to show that c and d are label consistent. To see that d is label consistent suppose $Ge, Gf \in (E/G)^1$ are such that $(\mathcal{L}/G)(Ge) = (\mathcal{L}/G)(Gf) = Ga \in \mathcal{A}/G$. Let $b = \eta^{\mathcal{A}}(Ga) \in \mathcal{A}$, $d(Ge) = k \in G$, and $d(Gf) = l \in G$. Then by the definition of d we have

$$\mathcal{L}\big(\eta^1(Ge)\big) = \alpha_k^{\mathcal{A}} \eta^{\mathcal{A}}(\mathcal{L}/G)(Ge) = \alpha_k^{\mathcal{A}} b, \tag{7.1}$$

$$\mathcal{L}\big(\eta^1(Gf)\big) = \alpha_l^{\mathcal{A}} \eta^{\mathcal{A}}(\mathcal{L}/G)(Gf) = \alpha_l^{\mathcal{A}} b. \tag{7.2}$$

This implies that $G\mathcal{L}(\eta^1(Ge)) = Ga = G\mathcal{L}(\eta^1(Gf))$ and so $\mathcal{L}(\eta^1(Ge)) = \mathcal{L}(\eta^1(Gf))$ since $s(\eta^1(Ge)), s(\eta^1(Gf)) \in T$. From Equations (7.1) and (7.2) we have $\alpha_k^{\mathcal{A}} b = \alpha_l^{\mathcal{A}} b$ and so $k = l$ since the G action on $\mathcal{A}$ is free. Therefore, d is label consistent.

To see that c is label consistent suppose that $Ge, Gf \in (E/G)^1$ are such that $(\mathcal{L}/G)(Ge) = (\mathcal{L}/G)(Gf) = Ga \in \mathcal{A}/G$, say. Let $b = \eta^{\mathcal{A}}(Ga) \in \mathcal{A}$, $c_\eta(Ge) = k \in G$, and $c(Gf) = l \in G$. Then by the definition of c we have

$$r\big(\eta^1(Ge)\big) = \alpha_k^0 \eta^0\big(r(Ge)\big), \tag{7.3}$$

$$r\big(\eta^1(Gf)\big) = \alpha_l^0 \eta^0\big(r(Gf)\big). \tag{7.4}$$

Then if we let $e = \alpha_{-k}^1(\eta^1(Ge))$ and $f = \alpha_{-l}^1(\eta^1(Gf))$ we have $e, f \in E^1$ with $r(e) = \eta^0(r(Ge)), r(f) = \eta^0(r(Gf)) \in T$ and $G\mathcal{L}(e) = G\mathcal{L}(f)$. Since T is a fundamental domain, we have $\mathcal{L}(e) = \mathcal{L}(f)$ and hence $\alpha_{-k}^{\mathcal{A}}(\mathcal{L}(\eta^1(Ge))) = \mathcal{L}(e) = \mathcal{L}(f) = \alpha_{-l}^{\mathcal{A}}(\mathcal{L}(\eta^1(Gf)))$. Since $\mathcal{L}(\eta^1(Ge)) = \mathcal{L}(\eta^1(Gf))$ we can conclude that $k = l$ as in the previous paragraph. Therefore, c is label consistent and our result is established. $\square$

Corollary 7.4. *Let $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labeled graph. Suppose that $((E, \mathcal{L}), G, \alpha)$ is a free labeled graph action which admits a fundamental domain. Then*

$$C^*(E, \mathcal{L}) \times_{\alpha, r} G \cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}\big(\ell^2(G)\big).$$

Proof. By Theorem 7.3, there are label consistent functions $c, d : E^1/G \to G$ such that

$$\big((E, \mathcal{L}), G, \alpha\big) \cong \big(\big(E/G \times_c G, (\mathcal{L}/G)_d\big), G, \tau\big),$$

so we have

$$C^*(E, \mathcal{L}) \times_{\alpha, r} G \cong C^*\big(E/G \times_c G, (\mathcal{L}/G)_d\big) \times_{\tau, r} G.$$

By Proposition 6.3 and Corollary 6.8, we have

$$\begin{aligned} C^*\big(E/G \times_c G, (\mathcal{L}/G)_d\big) \times_{\tau, r} G &\cong C^*\big(E/G \times_c G, (\mathcal{L}/G)_{\mathbf{1}}\big) \times_{\tau, r} G \\ &\cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}\big(\ell^2(G)\big) \end{aligned}$$

which gives the desired result. $\square$

References

[1] T. Bates, T. Carlsen and D. Pask, *C^*-algebras of labelled graphs III—K-theory*, available at arXiv:1203.3072v1 [math.OA].

[2] T. Bates and D. Pask, *C^*-algebras of labeled graphs*, J. Operator Theory **57** (2007), 207–226. MR 2304922

[3] T. Bates and D. Pask, *C^*-algebras of labeled graphs II—Simplicity results*, Math. Scand. **104** (2009), 249–274. MR 2542653

[4] T. Crisp, *Corners of graph algebras*, J. Operator Theory **60** (2008), 253–271. MR 2464212

[5] K. Deicke, D. Pask and I. Raeburn, *Coverings of directed graphs and crossed products of C^*-algebras by coactions of homogenous spaces*, Internat. J. Math. **14** (2003), 773–789. MR 2000743

[6] J. Gross and T. Tucker, *Topological graph theory*, Series in Discrete Mathematics and Optimization, Wiley, New York, 1987. MR 0898434

[7] J. Jeong and S. Kim, *On simple labelled graph C^*-algebras*, J. Math. Anal. Appl. **386** (2012), 631–640. MR 2834773

[8] J. Jeong, S. Kim and G. Park, *The structure of gauge invariant ideals of labelled graph C^*-algebras*, J. Funct. Anal. **262** (2012), 1759–1780. MR 2873859

[9] S. Kaliszewski, J. Quigg and I. Raeburn, *Skew products and crossed products by coactions*, J. Operator Theory **46** (2001), 411–433. MR 1870415

[10] Y. Katayama, *Takesaki's duality for a non-degenerate co-action*, Math. Scand. **55** (1985), 141–151. MR 0769030

[11] A. Kumjian and D. Pask, *C^*-algebras of directed graphs and group actions*, Ergodic Theory Dynam. Systems **19** (1999), 1503–1519. MR 1738948

[12] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995. MR 1369092

[13] D. Pask, J. Quigg and I. Raeburn, *Coverings of k-graphs*, J. Algebra **289** (2005), 161–191. MR 2139097

[14] D. Pask and I. Raeburn, *On the K-theory of Cuntz–Krieger algebras*, Publ. RIMS Kyoto **32** (1996), 415–443. MR 1409796

[15] J. Quigg, *Discrete C^*-coactions and C^*-algebraic bundles*, J. Austral. Math. Soc. Ser. A **60** (1996), 204–221. MR 1375586

[16] D. Robertson and W. Szymański, *C^*-algebras associated to C^*-correspondences and applications to mirror quantum spheres*, Illinois J. Math. **55** (2011), 845–870. MR 3069287

Teresa Bates, School of Mathematics and Statistics, The University of NSW, UNSW Sydney 2052, Australia

E-mail address: teresa@unsw.edu.au

David Pask, School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia

E-mail address: dpask@uow.edu.au

Paulette Willis, Department of Mathematics 651 PGH, University of Houston, Houston, TX 77204-3008, USA

E-mail address: pnwillis@math.uh.edu