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DESCRIPTIVE THEORY OF NEAREST POINTS IN BANACH SPACES

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To Don Burkholder, leader and guide

ABSTRACT. Let X be a separable Banach space, Y a closed, non-reflexive, linear subspace, and P the set of points admitting a nearest approximation in Y. Then P is an analytic set, and has three obvious algebraic properties. By adjusting the norm of X, any analytic set of this kind can be realized as the set of elements *proximal* to Y.

Let X be a Banach space with norm $|\cdot|$ and Y a closed linear subspace. An element of X is called *proximal* if it admits a closest point in Y. The set of proximal elements is called P (or $P(|\cdot|)$ to emphasize the dependence on the norm). Clearly, $P=X$ if Y is reflexive. P has three further algebraic properties: (a) $P\supset Y$, (b) $P+Y=Y$, (c) $tP=P$ if $t\neq 0$. A set with these properties is called *stable*. We assume throughout that Y isn't reflexive, as otherwise $P=X$. When X is separable, then P is the projection into X of a certain closed subset of $X\times Y$, and is therefore analytic [10].

When X/Y has dimension 1, then $P=X$ or $P=Y$. This special case is a disguised form of the classical problem of *norm-attaining* linear functionals. When X/Y has dimension at least 2, then the set P can fail to be a Borel set. Subspaces Y such that all elements of X are proximal are called *proximinal*. It seems to be unknown whether there is always a proximinal subspace of codimension 2.

THEOREM. *Suppose X is separable, Y is not reflexive, and $\mathbf{A}$ is stable and analytic. The extremes $\mathbf{A}=X$ and $\mathbf{A}=Y$ are allowed. Then there is an equivalent norm $\|\cdot\|$ on X such that $P(\|\cdot\|)=\mathbf{A}$.*

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In the following paragraphs, we collect lemmas from functional analysis and topology, proceeding to the special case $\mathbf{A}=Y$. The general case depends on analysis in the space $l^1(N)$. Fatou's lemma is invoked frequently, and a converse is needed at the conclusion.

1. LUR norm

A norm $|\cdot|$ is LUR (locally uniformly rotund) at an element w if the conditions $\lim|z_n| = |w|$, $\lim|z_n + w| = 2|w|$ imply that $\lim z_n = w$. By a theorem of Kadec (1958) every separable space Z, in particular $Z = X/Y$, admits an LUR norm ([6], pp. 42–49.)

Since Z is a quotient space, we denote by by π the map of X onto Z and an LUR norm on Z by $\|\cdot\|_*$. The norm promised in our theorem will yield this norm as the quotient norm on Z. To begin, we define a norm with that property: $|x|' = c|x| \vee \|\pi x\|_*$ with a small constant $c>0$.

2. Bartle–Graves selector

The map π of X onto X/Y admits a continuous right inverse, that is, a map θ of X/Y into X such that $\pi\theta z = z$ on X/Y ([3], [9], pp. 1–6). Thus, θ is a continuous selector for the multivalued map π^{-1}. It's easily seen that θ can be taken as an odd mapping; with a bit more work we can assume that θ is bounded in the sense that $|\theta z| \le c\|z\|_*$ with a constant c (whose value isn't important). We remark that in certain cases, θ cannot be made uniformly continuous [1], [2], [11], but this doesn't cause any difficulties. A study of Lipschitz-continuous selectors, and more about uniformly continuous selectors, is presented in [8].

3. A special basic sequence

In this paragraph, we use a theorem of Pelczynski: Y contains a bounded basic sequence $[e_n]$, such that $f^*(e_n) = 1, n \ge 1$, with some element f^* of X^* [13]. Thus, the basic sequence is *nonshrinking*; conversely, from a nonshrinking basic sequence we can obtain the sequence $[e_n]$ by standard methods.

4. Discontinuous functions on a metric space

Let U be an open set in a metric space M, and ϕ its characteristic function. Then $\phi = \sum w_n$, where $w_1, w_2, w_3, \ldots$ are continuous and $\sum |w_n| \le 1$. To see this, we take a continuous function v such that $v = 0$ off U and $0 < v < 1$ on U. Then $w_1 = v, w_2 = v^{1/2} - v, w_3 = v^{1/4} - v^{1/2}, \ldots$ are the functions we sought. The same can be accomplished for the difference $U \setminus V$ of open sets U and V, except that the inequality on the sum of absolute values becomes $\sum |w_n| \le 2$. This follows from the identity $U \setminus V = U \setminus U \cap V$.

We apply this in the metric space $Z = X/Y$ with open sets U and V symmetric about 0. Thus, all functions w_n can be made even. Let (f_j^*) be a sequence in Z^* whose common null-space is (0)—that is, the sequence is total over Z. Let $\nu(z)$ be the first j such that $f_j^*(z) \neq 0$ (if $z \neq 0$). Then the set $(\nu = 1)$ is open, while the sets $(\nu = 2)$, $(\nu = 3), \ldots$ are differences of open sets and are plainly symmetric. We define

$$h(z) = 2^{-j} \arctan f_j^*(z) \quad \text{if } \nu(z) = j, \qquad h(0) = 0.$$

Then we have $h = \sum w_n$, w_n odd and continuous, $\sum |w_n| \leq 4$. It is clear that the function h cannot be continuous when $\dim Z > 1$. Borsuk's "Antipodal Theorem" (1937) ([7], pp. 347–350) is a profound generalization of this.

5. A special case

This treats the case $\mathbf{A} = Y$. We recall that $Z = X/Y$, and then define three odd maps of Z into $\overline{\text{sp}}[e_n]$

$$r(z) = \sum w_n(z) e_n,$$
$$g_k(z) = r(z) - h(z) e_k, \quad k \geq 1,$$
$$g_k'(z) = k(k+1)^{-1}\theta(z) + g_k(z).$$

Let B'' be the ball of radius 1/2 around 0, defined by the norm $|\cdot|'$ in the paragraph on LUR norms, and $\|\cdot\|$ the norm whose closed unit ball is the closed convex hull of the set

$$S = B'' \cup \{g_k'(z) : k \geq 1, \|z\|_* = 1\}.$$

Then the quotient norm of $\|\cdot\|$ is the LUR norm $\|\cdot\|_*$ on Z. We claim that when $\|x\| = 1$ then $\|\pi x\|_* < 1$, that is, $P = Y$. In the contrary case $\|x_0\| = 1, \|\pi x_0\|_* = 1$, it is clear that the set B'' plays only a negligible role and can be omitted from the set S. Thus, x_0 is a (norm) limit of sums $\sum t_j g_k'(z_j)$ where $t_j \geq 0, \sum t_j = 1$ and k is a variable depending on j. Applying the quotient mapping π leaves only terms $k(k+1)^{-1} z_j$. Since $\|x_0\|_* = 1$, the variable k must tend to ∞ "almost everywhere", that is, we can replace $k(k+1)^{-1}$ by 1 in what follows. (Besides using "almost everywhere" in this colloquial way, we omit a special notation for the limiting process.) Thus, $\sum t_j z_j$ must approach πx_0. From the LUR property of the norm in Z, we conclude that $\sum t_j \|z_j - \pi x_0\|_* \to 0$ and from the continuity of θ at πx_0 we conclude that $\sum t_j \theta(z_j)$ must converge in norm. (Thus, continuity at πx_0 is sufficient in our theorem.)

We remark that a weaker property of the norm—abbreviated ALUR—is sufficient in the previous step. A comparison of ALUR and LUR may be found in [6], pp. 72, 135–138.

The mapping r is continuous into a (very) weak topology on $\overline{\text{sp}}[e_n]$, namely convergence of the biorthogonal functionals. We call this τ-convergence, and observe that $\lim e_k = 0$ in this sense. The $\tau - \lim$ of the convex sums

$\sum t_j g_k(z_j)$ will therefore be $r(\pi x_0)$ since $\sum t_j \|z_j - \pi x_0\|_* \to 0$. But this cannot be a limit in the norm of X since the functions $r(z) - h(z)e_k$ are in the null-space of f^*, while $f^*(r(z)) = h(z)$, and $h(z) = 0$ only when $z = 0$. This contradiction completes the proof in the special case $\mathbf{A} = Y$.

The argument just completed is valid when X/Y is separable (or, more generally, when X/Y admits an LUR norm and a total sequence of linear functionals) and Y isn't reflexive, but the conclusion fails for certain spaces X, as we now explain. Suppose that a Banach space W has the property that for each norm $|\cdot|$ in W there are elements u and v such that $|au+bv| = |a|+|b|$ for all real a, b; and J is James' space: J^{**}/J has dimension 1. Then every norm $\|\cdot\|$ on $X = W \oplus J$ will present elements proximal to J but not in J.

To verify this, we take for $|\cdot|$ the distance to J, an equivalent norm on W, and denote by u and v the elements defined above, relative to the norm $|\cdot|$. Then there are bounded sequences p_n and q_n in J such that $\lim \|p_n - u\| = |u|$ and $\lim \|q_n - v\| = |v|$. Now there are constants a and b, not both 0, such that the sequence $au_n + bv_n$ has a weakly convergent subsequence, with a limit L in J. Then $|L - au - bv| \leq |a| + |b|$, so $au + bv$ is proximal but not in J.

The space m_0 described in [12], [6], pp. 76–79, has the property imposed on W (and much more). It is possible that more transparent examples could be found, following [5], pp. 516, 521–522, or [4]. Unlike the first two examples, the third makes no use of uncountable sets.

6. Conclusion

Let Σ be the closed set in X defined by the equations $\|\pi x\|_* = 1$ and $x = \theta\pi x$. We construct the norm $\|\cdot\|$ so that $P \cap \Sigma = \mathbf{A} \cap \Sigma$. This equality quickly yields our main theorem, as we now demonstrate. Suppose, for example, that $x \in P$ but $x \notin Y$. Then $tx + y \in \Sigma$, with certain $t > 0$ and y in Y. By the stability of P, $tx + y \in P$, hence $tx + y \in \mathbf{A}$. By the stability of $\mathbf{A}$, $x \in \mathbf{A}$. The reverse implication follows similarly by stability. Defining $\mathbf{A} \cap \Sigma = \mathbf{B}$, we observe that $\mathbf{B}$ is analytic and symmetric; we can suppose that $\mathbf{B} \neq \emptyset$, since the contrary case was treated above. Let BN be the standard product space N^N (Baire null-space, homeomorphic to the set of irrationals) and BN' the set of pairs $\sigma' = (\varepsilon, \sigma)$ where $\varepsilon = -1, 1$ and σ belongs to BN. Then $\mathbf{B}$, being analytic and symmetric, is a continuous image $\psi(BN')$, with an odd mapping ψ, i.e. $\psi(-1, \sigma) = -\psi(1, \sigma)$. We define three maps of BN' into X as follows

$$T_1(\sigma') = \sum 2^{-k} e(n_k) \quad \text{when } \sigma = (n_1, n_2, \ldots, n_k, \ldots);$$
$$T_2(\sigma') = r \circ \pi \circ \psi(\sigma');$$
$$T_3(\sigma') = \psi(\sigma') + T_2(\sigma') - h \circ \pi \circ \psi(\sigma') \cdot T_1(\sigma').$$

Thus, T_1 is continuous and even with respect to the sign ε.

We then define $S' = T_3(BN')$, and now $\|\cdot\|$ is the norm on X whose closed unit ball is the closed convex hull of $S \cup S'$; its quotient norm is again $\|\cdot\|_*$. The elements of $\mathbf{B}$ have distance 1 from Y and are proximal, by the formula for T_3; this is the easy half of the equality $\mathbf{B} = P \cap \Sigma$. Suppose now that $x_0 \in P \cap \Sigma$, that is, $\|x_0\|_* = 1$ and $\|y - x_0\| = 1$ for some y in Y. Then, as before, y is a limit of sums $\sum t_j s''_j, 0 \le t_j, \sum t_j = 1$ and each $s''_j \in S \cup S'$. Again, we can omit B'' from the estimation. It will be convenient to write each s''_j as $a_j + b_j - c_j$ following the order in which T_3 and g'_k were defined. The same analysis as before yields $\lim \sum t_j a_j = x_0$. Also $\sum b_j$ has a $\tau - \lim r \circ \pi x_0$; applying f^* to this we get $h \circ \pi(x_0) \neq 0$. For definiteness, we assume this is positive.

The sums $\sum t_j c_j$ demand closer analysis. The part of the sum extended over S has $\tau - \lim 0$. The remainder, that is, the sum extended over S', has to be divided in two pieces. To explain this, we write $\nu(\pi(x_0)) = \nu_0$. (i) In this piece, $\nu(\pi \circ \psi(\sigma')) < \nu_0$. Now $\pi \circ \psi(\sigma') \to \pi(x_0)$ almost everywhere. The definition of ν shows that the value of h tends almost everywhere to 0 in the sum over (i). (ii) Here, ν takes the value ν_0 so h tends to $h \circ \pi(x_0)$ almost everywhere. We add that cases (i) and (ii) account for all but a negligible part of the sum of $t_j c_j$ over S'. We can pass to a subsequence so that all three pieces have τ-limits. Now we can conclude that case (ii) covers almost all of the sum, for the remaining cases would otherwise produce a positive jump in the value of f^*; by Fatou's lemma such a jump would not be balanced by a contrary jump arising from case (ii). In the last assertion, we refer to the formula for T_1.

Now we have to look at sums $\sum t_j T_1(\sigma'_j)$ τ-convergent to a limit L, such that $f^*(L) = 1$. We treat this limit in two ways.

First, we treat them as non-negative elements of $l^1(N)$ converging everywhere on N, such that the sum (or integral) of the limit sequence is the limit of the sum. By an argument of Kadec–Klee type (explained below), the limit must be a limit in the norm of l^1. This can be stated in terms of the remainders R_p: the remainder of a series $a_1 + a_2 + a_3 + \cdots$ is $a_p + a_{p+1} + \cdots$. On a sequence which converges in norm, the remainders R_p must converge to 0 uniformly as $p \to \infty$.

The second approach is to treat the sums $\sum t_j(\sigma')$ as integrals over BN' of T_1 with respect to a sequence of (nearly) probability distributions, say (λ_m). We also know that $\psi\sigma' - x_0$ tends to 0 in the sequence of measure spaces defined by the probabilities. We observe that BN' is a set of type G_δ in a compact metric space Γ—for example, the space obtained by adjoining ∞ to each of the factors N. Hence, the sequence λ_m has a subsequence converging weak* to a probability measure λ on Γ. Our aim is to prove that the limit measure is concentrated on BN' (and a bit more than this). We do so by proving that the sequence is *tight*: for each $r \ge 1$ there is a compact set Γ_r *contained in* BN' such that $\lambda_m(\Gamma_r) \ge 1 - r^{-1}$ for all m. (This notion occurs in the theory of stochastic processes.) We can express this by means of the digits n_k, treating them as continuous functions on BN'. (The signs $-1, 1$ do not affect the compactness.) We claim that for each $k \geq 1$ and $r \geq 1$ there is a number $c = c_{k,r}$ such that $\lambda_m(n_k \leq c) \geq 1 - r^{-1}$ for all r. From this, the tightness follows. When $n_k > p$, then R_p gains at least 2^{-k}. If our claim were false for some k and r, the remainders of the sums $\sum t_j T_1(\sigma')$ would not converge uniformly to 0.

From the tightness of the sequence, we find a measure concentrated on BN' such that $\psi(\sigma') = x_0$ a.e. Thus, x_0 belongs to $\mathbf{B}$ and the proof is complete.

We referred to an argument related to the Kadec–Klee property of norms on Banach spaces: on the unit sphere of $l^1(N)$, weak* convergence implies convergence in norm.

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