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MULTIPLICATIVE FUNCTIONAL FOR REFLECTED BROWNIAN MOTION VIA DETERMINISTIC ODE

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Dedicated to D. Burkholder

ABSTRACT. We prove that a sequence of semi-discrete approximations converges to a multiplicative functional for reflected Brownian motion, which intuitively represents the Lyapunov exponent for the corresponding stochastic flow. The method of proof is based on a study of the deterministic version of the problem and the excursion theory.

1. Introduction

This article is the first part of a project devoted to path properties of a stochastic flow of reflected Brownian motions. We will first outline the general direction of the project and then we will comment on the results contained in the current article.

Consider a bounded C^2 domain $D \subset \mathbf{R}^n$, $n \geq 2$, and for any $x \in \overline{D}$, let X_t^x be reflected Brownian motion in D, starting from $X_0^x = x$. Construct all processes X^x so that they are driven by the same n-dimensional Brownian motion. It has been proved in [BCJ] that in some planar domains, for any $x \neq y$, the limit $\lim_{t\to\infty} \log|X_t^x - X_t^y|/t = \Lambda(D)$ exists a.s. Moreover, an explicit formula has been given for the limit $\Lambda(D)$, in terms of geometric quantities associated with D. Our ultimate goal is to prove an analogous result for domains in $\mathbf{R}^n$ for $n \geq 3$.

The higher dimensional case is more difficult to study for several reasons. First, we believe that the multidimensional quantity analogous to $\Lambda(D)$ in the two dimensional case cannot be expressed directly in terms of geometric

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properties of D. Instead, it has to be expressed using the stationary distribution for the normalized version of the multiplicative functional studied in the present paper. Second, non-commutativity of projections is a more challenging technical problem in dimensions $n \geq 3$.

The result of [BCJ] mentioned above contains an implicit assertion about another limit, namely, in the space variable for a fixed time. In other words, one can informally infer the existence and value of the limit $\lim_{\varepsilon \downarrow 0}(X_t^{x+\varepsilon \mathbf{v}} - X_t^x)/\varepsilon = \widetilde{\mathcal{A}}_t \mathbf{v}$, for $\mathbf{v} \in \mathbf{R}^n$. The limit operator $\widetilde{\mathcal{A}}_t$, regarded as a function of time, is a linear multiplicative functional of reflected Brownian motion. Its form is considerably more complex and interesting in dimensions $n \geq 3$ than in two dimensions.

Our overall plan is to begin with the differentiability in the space variable of the limit operator $\widetilde{\mathcal{A}}_t$, which has been proved in the companion paper [Bu2]. Then we will prove the existence and uniqueness of the stationary distribution for the normalized version of $\widetilde{\mathcal{A}}_t$. And then we will prove the formula for the rate of convergence of $|X_t^x - X_t^y|$ to 0, as $t \to \infty$.

The immediate goal of the present paper is much more modest than the overall plan outlined above. We will deal with some foundational issues related to the application of our main method, excursion theory, to the convergence of semi-discrete approximations to the multiplicative functional described above. We will briefly review some of the existing literature on the subject, so that we can place out own results in an appropriate context.

The multiplicative functional $\widetilde{\mathcal{A}}_t$ appeared in a number of publications discussing reflected Brownian motion, starting with [A], [IW1], and later in [IW2], [H]. None of these publications contains the analysis of the deterministic version of the multiplicative functional. This is what we are going to do in Section 2. In a sense, we are trying to see whether the approach of [LS] could be applied in our case; that approach was to develop a deterministic theory that could be applied to stochastic processes path by path. Unfortunately, our result on deterministic ODE's does not apply to reflected Brownian motion, roughly speaking, for the same reason why the Riemann–Stieltjes integral does not work for integrals with respect to Brownian motion.

Nevertheless, our deterministic results are not totally disjoint from the second, probabilistic section. In fact, our basic approach developed in Lemma 2.9 is just what we need in Section 3. Also, Lemma 2.2 proved to be very useful as one of the key ingredients of the proof of the main theorem in [Bu2].

The main theorem of Section 3 proves existence of the multiplicative functional using semi-discrete approximations. The result does not seem to be known in this form, although it is obviously close to some theorems in [A], [IW1], [H]. However, the main point is not to give a new proof to a slightly different version of a known result but to develop estimates using excursion techniques that are analogous to those in [BCJ], and that can be applied to study $X_t^x - X_t^y$.

We continue with some general review of literature. The differentiability of X_t^x in the initial data was proved in [DZ] for reflected diffusions. The main difference between our project and that in [DZ] is that that paper was concerned with diffusions in $(0, \infty)^n$, and our main goal is to study the effect of the curvature of ∂D. Deterministic transformations based on reflection were considered, for example, in [LS], [DI], [DR]. Synchronous couplings of reflected Brownian motions in convex domains were studied in [CLJ1], [CLJ2], where it was proved that under mild assumptions, $X_t^x - X_t^y$ is not 0 at any finite time. Our estimates in Section 3 are so robust that they indicate that Theorem 3.2 holds for the trace of a degenerate diffusion on ∂D, defined as in [CS], [MO], with the density of jumps having different scaling properties than that for reflected Brownian motion. In other words, the main theorem of Section 3 is likely to hold in the case when the trace of the reflected diffusion is any "stable-like" process on ∂D. We do not present this generalization because, as far as we can tell, the multiplicative functional $\widetilde{\mathcal{A}}_t$ does not represent the limit $\lim_{\varepsilon \downarrow 0}(X_t^{x+\varepsilon\mathbf{v}} - X_t^x)/\varepsilon$ for flows of degenerate reflected diffusions.

2. Deterministic differential equation

2.1. Geometric preliminaries. Throughout this section, M will be a C^2, properly embedded, orientable hypersurface (i.e., submanifold of codimension 1) in $\mathbf{R}^n$, endowed with a C^1 unit normal vector field $\mathbf{n}$. The properness condition means that the inclusion map $M \hookrightarrow \mathbf{R}^n$ is a proper map (the inverse image of every compact set is compact), which is equivalent to M being a closed subset of $\mathbf{R}^n$. For any $R > 0$, let M_R denote the intersection of M with the closed ball of radius R around the origin in $\mathbf{R}^n$, and note that M_R is a compact subset of M.

We consider M as a Riemannian manifold with the induced metric. We use the notation $\langle \cdot, \cdot \rangle$ for both the Euclidean inner product on $\mathbf{R}^n$ and its restriction to $\mathcal{T}_x M$ for any $x \in M$, and $|\cdot|$ for the associated norm.

For any $x \in M$, let $\pi_x : \mathbf{R}^n \to \mathcal{T}_x M$ denote the orthogonal projection onto the tangent space $\mathcal{T}_x M$, so

$$\pi_x \mathbf{z} = \mathbf{z} - \langle \mathbf{z}, \mathbf{n}(x) \rangle \mathbf{n}(x), \tag{2.1}$$

and let $\mathcal{S}(x) : \mathcal{T}_x M \to \mathcal{T}_x M$ denote the *shape operator* (also known as the *Weingarten map*), which is the symmetric linear endomorphism of $\mathcal{T}_x M$ associated with the second fundamental form. It is characterized by

$$\mathcal{S}(x)\mathbf{v} = -\partial_{\mathbf{v}} \mathbf{n}(x), \quad \mathbf{v} \in \mathcal{T}_x M, \tag{2.2}$$

where $\partial_{\mathbf{v}}$ denotes the ordinary Euclidean directional derivative in the direction of $\mathbf{v}$. The eigenvalues of $\mathcal{S}(x)$ are the principal curvatures of M at x, and its determinant is the Gaussian curvature. We extend $\mathcal{S}(x)$ to an endomorphism of $\mathbf{R}^n$ by defining $\mathcal{S}(x)\mathbf{n}(x)=0$. It is easy to check that $\mathcal{S}(x)$ and π_x commute, by evaluating separately on $\mathbf{n}(x)$ and on $\mathbf{v}\in\mathcal{T}_xM$.

If $\gamma:[0,T]\to M$ is a curve in M, a *vector field along* γ is a map $\mathbf{v}:[0,T]\to M$ such that $\mathbf{v}(t)\in\mathcal{T}_{\gamma(t)}M$ for each t. This is equivalent to the equation $\langle\mathbf{v}(t),\mathbf{n}(\gamma(t))\rangle=0$ for all t, or more succinctly $\langle\mathbf{v},\mathbf{n}\circ\gamma\rangle\equiv 0$. If $\mathbf{v}$ is a C^1 vector field along a C^1 curve γ, the *covariant derivative of* $\mathbf{v}$ *along* γ is the vector field $\mathcal{D}_t\mathbf{v}(t)$ along γ given by the orthogonal projection onto TM of the ordinary derivative of $\mathbf{v}(t)$:

$$\mathcal{D}_t\mathbf{v}(t):=\pi_{\gamma(t)}\mathbf{v}'(t)=\mathbf{v}'(t)-\langle\mathbf{v}'(t),\mathbf{n}(\gamma(t))\rangle\mathbf{n}(\gamma(t)).$$

Because $\langle\mathbf{v}(t),\mathbf{n}(\gamma(t))\rangle\equiv 0$, the product rule yields $\langle\mathbf{v}'(t),\mathbf{n}(\gamma(t))\rangle=-\langle\mathbf{v}(t),(\mathbf{n}\circ\gamma)'(t)\rangle$, and therefore the covariant derivative can also be written

$$\begin{aligned}\mathcal{D}_t\mathbf{v}(t)&=\mathbf{v}'(t)+\langle\mathbf{v}(t),(\mathbf{n}\circ\gamma)'(t)\rangle\mathbf{n}(\gamma(t))\\&=\mathbf{v}'(t)-\langle\mathbf{v}(t),\mathcal{S}(\gamma(t))\gamma'(t)\rangle\mathbf{n}(\gamma(t)).\end{aligned}$$

The following lemma expresses some elementary observations that we will use below. Most of these follow easily from the fact that C^1 maps satisfy uniform local Lipschitz estimates, so we leave the proof to the reader. For any linear map $\mathcal{A}:\mathbf{R}^n\to\mathbf{R}^n$, we let $\|\mathcal{A}\|$ denote the operator norm.

LEMMA 2.1. *For any $R>0$ and $T>0$, there exists a constant K depending only on M, R, and T such that the following estimates hold for all $x,y\in M_R$, $0\le l,r\le T$, $t\ge 0$ and $\mathbf{z}\in\mathbf{R}^n$*:

$$\|\pi_x-\pi_y\|\le K|x-y|, \tag{2.3}$$

$$\|\mathcal{S}(x)\|\le K, \tag{2.4}$$

$$\|\mathcal{S}(x)-\mathcal{S}(y)\|\le K|x-y|, \tag{2.5}$$

$$\left\|e^{t\mathcal{S}(x)}\right\|\le e^{Kt}, \tag{2.6}$$

$$\left\|e^{-t\mathcal{S}(x)}\right\|\le e^{Kt}, \tag{2.7}$$

$$\left\|e^{l\mathcal{S}(x)}-\mathrm{Id}\right\|\le Kl, \tag{2.8}$$

$$\left\|e^{l\mathcal{S}(x)}-e^{l\mathcal{S}(y)}\right\|\le Kl|x-y|, \tag{2.9}$$

$$\left\|e^{l\mathcal{S}(x)}-e^{r\mathcal{S}(x)}\right\|\le K|l-r|, \tag{2.10}$$

$$|\mathbf{n}(x)-\mathbf{n}(y)|\le K|x-y|. \tag{2.11}$$

Another useful estimate is the following.

LEMMA 2.2. *For any $R>0$, there exists a constant C depending only on M and R such that for all $w,x,y,z\in M_R$, the following operator-norm estimate holds*:

$$\|\pi_z\circ(\pi_y-\pi_x)\circ\pi_w\|\le C(|w-y||y-z|+|w-x||x-z|).$$

Proof. Using the fact that $\mathbf{n}$ is a unit vector field and expanding $|\mathbf{n}(x)-\mathbf{n}(y)|^2$ in terms of inner products, we obtain

$$\langle \mathbf{n}(x),\mathbf{n}(y)\rangle = \frac{1}{2}\big(|\mathbf{n}(x)|^2+|\mathbf{n}(y)|^2-|\mathbf{n}(x)-\mathbf{n}(y)|^2\big) = 1-\frac{1}{2}|\mathbf{n}(x)-\mathbf{n}(y)|^2.$$

Suppose $w,x,y,z\in M_R$ and $\mathbf{v}\in\mathbf{R}^n$. If $\pi_w\mathbf{v}=0$, then the estimate holds trivially, so we may as well assume that $\mathbf{v}\in\mathcal{T}_wM$. Expanding the projections as in (2.1) and using the fact that $\pi_w\mathbf{v}=\mathbf{v}$, we obtain

$$\begin{aligned}
&\pi_z(\pi_y-\pi_x)\pi_w\mathbf{v}\\
&\quad=\pi_z\big(\mathbf{v}-\langle\mathbf{v},\mathbf{n}(y)\rangle\mathbf{n}(y)\big)-\pi_z\big(\mathbf{v}-\langle\mathbf{v},\mathbf{n}(x)\rangle\mathbf{n}(x)\big)\\
&\quad=\big(\mathbf{v}-\langle\mathbf{v},\mathbf{n}(y)\rangle\mathbf{n}(y)\big)\\
&\qquad-\big(\langle\mathbf{v},\mathbf{n}(z)\rangle\mathbf{n}(z)-\langle\mathbf{v},\mathbf{n}(y)\rangle\langle\mathbf{n}(y),\mathbf{n}(z)\rangle\mathbf{n}(z)\big)\\
&\qquad-\big(\mathbf{v}-\langle\mathbf{v},\mathbf{n}(x)\rangle\mathbf{n}(x)\big)\\
&\qquad+\big(\langle\mathbf{v},\mathbf{n}(z)\rangle\mathbf{n}(z)-\langle\mathbf{v},\mathbf{n}(x)\rangle\langle\mathbf{n}(x),\mathbf{n}(z)\rangle\mathbf{n}(z)\big)\\
&\quad=-\langle\mathbf{v},\mathbf{n}(y)\rangle\mathbf{n}(y)+\langle\mathbf{v},\mathbf{n}(x)\rangle\mathbf{n}(x)\\
&\qquad+\langle\mathbf{v},\mathbf{n}(y)\rangle\left(1-\frac{1}{2}|\mathbf{n}(y)-\mathbf{n}(z)|^2\right)\mathbf{n}(z)\\
&\qquad-\langle\mathbf{v},\mathbf{n}(x)\rangle\left(1-\frac{1}{2}|\mathbf{n}(x)-\mathbf{n}(z)|^2\right)\mathbf{n}(z)\\
&\quad=-\langle\mathbf{v},\mathbf{n}(y)\rangle\big(\mathbf{n}(y)-\mathbf{n}(z)\big)+\langle\mathbf{v},\mathbf{n}(x)\rangle\big(\mathbf{n}(x)-\mathbf{n}(z)\big)\\
&\qquad-\frac{1}{2}\langle\mathbf{v},\mathbf{n}(y)\rangle|\mathbf{n}(y)-\mathbf{n}(z)|^2\mathbf{n}(z)+\frac{1}{2}\langle\mathbf{v},\mathbf{n}(x)\rangle|\mathbf{n}(x)-\mathbf{n}(z)|^2\mathbf{n}(z).
\end{aligned}$$

Using the fact that $\langle\mathbf{v},\mathbf{n}(w)\rangle=0$, this can be written

$$\begin{aligned}
\pi_z(\pi_y-\pi_x)\pi_w\mathbf{v}=&-\langle\mathbf{v},\mathbf{n}(w)-\mathbf{n}(y)\rangle\big(\mathbf{n}(y)-\mathbf{n}(z)\big)\\
&+\langle\mathbf{v},\mathbf{n}(w)-\mathbf{n}(x)\rangle\big(\mathbf{n}(x)-\mathbf{n}(z)\big)\\
&-\frac{1}{2}\langle\mathbf{v},\mathbf{n}(w)-\mathbf{n}(y)\rangle|\mathbf{n}(y)-\mathbf{n}(z)|^2\mathbf{n}(z)\\
&+\frac{1}{2}\langle\mathbf{v},\mathbf{n}(w)-\mathbf{n}(x)\rangle|\mathbf{n}(x)-\mathbf{n}(z)|^2\mathbf{n}(z).
\end{aligned}$$

The desired estimate follows from (2.11) and the fact that

$$|\mathbf{n}(x)-\mathbf{n}(y)|^2\le\big(|\mathbf{n}(x)|+|\mathbf{n}(y)|\big)|\mathbf{n}(x)-\mathbf{n}(y)|\le 2K|x-y|. \qquad \square$$

2.2. Analytic preliminaries. Let T be a positive real number. We let $\mathrm{BV}([0,T];\mathbf{R})$ denote the set of functions $u:[0,T]\to\mathbf{R}$ of bounded variation, and $\mathrm{NBV}([0,T];\mathbf{R})\subset\mathrm{BV}([0,T];\mathbf{R})$ the subset consisting of functions that are right-continuous. By convention, we will consider each $u\in\mathrm{NBV}([0,T];\mathbf{R})$ to be a function defined on all of $\mathbf{R}$ by setting $u(t)=0$ for $t<0$ and $u(t)=u(T)$ for $t>T$; the extended function is still right-continuous and of bounded variation. With this understanding, we will follow the conventions of [F], and most of the properties of $\mathrm{NBV}([0,T];\mathbf{R})$ that we use can be found there.

It is easy to check that $\mathrm{NBV}([0,T];\mathbf{R})$ is closed under pointwise products and sums. Functions in $\mathrm{NBV}([0,T];\mathbf{R})$ have bounded images, at most countably many discontinuities, and well-defined left-hand limits at each discontinuity. In particular, they are examples of *càdlàg* functions (*continue à droite, limites à gauche*). (In fact, $\mathrm{NBV}([0,T];\mathbf{R})$ is exactly the set of *càdlàg* functions of bounded variation.) For any $u \in \mathrm{NBV}([0,T];\mathbf{R})$ and any $s \in [0,T]$, we set

$$u(s-) = \lim_{t \nearrow s} u(t),$$

and we define the *jump of u at s* to be

$$\Delta_s(u) = u(s) - u(s-).$$

Note that $u(0-) = 0$ and $\Delta_0(u) = u(0)$ by our conventions.

It follows from elementary measure theory that for each $u \in \mathrm{NBV}([0,T];\mathbf{R})$, there is a unique signed Borel measure du on $[0,T]$ characterized by

$$du((a,b]) = u(b) - u(a), \quad t \in [0,T].$$

Because this measure has atoms exactly at points $t \in [0,T]$ where u is discontinuous, we have to be careful to indicate whether endpoints are included or excluded in integrals. For example, we have the following versions of the fundamental theorem of calculus for $a, b \in [0,T]$:

$$\int_{(a,b]} du = u(b) - u(a); \qquad \int_{[a,b]} du = u(b) - u(a-);$$
$$\int_{(a,b)} du = u(b-) - u(a); \qquad \int_{[a,b)} du = u(b-) - u(a-).$$

The *total variation* of u, denoted by $\|du\|$, is given by either of two formulas:

$$\begin{aligned}\|du\| &= \sup\left\{\sum_{i=1}^{k} |u(x_i) - u(x_{i-1})| : 0 = x_0 < x_1 < \cdots < x_k = T\right\}\\ &= \int_{[0,T]} |du|.\end{aligned}$$

It follows from our conventions that $\|u\|_\infty \le \|du\|$ (where $\|\cdot\|_\infty$ is the usual L^∞ norm).

For $u \in \mathrm{NBV}([0,T];\mathbf{R})$, we will use the notation u_- to denote the function $u_-(t) = u(t-)$. Note that u_- has bounded variation, but is left-continuous rather than right-continuous.

LEMMA 2.3. *For any* $u, v \in \mathrm{NBV}([0,T];\mathbf{R})$ *and* $a, b \in [0,T]$, *the following integration by parts formula holds*:

$$\int_{(a,b]} u\,dv + \int_{(a,b]} v_-\,du = u(b)v(b) - u(a)v(a). \tag{2.12}$$

Proof. This follows as in [F, Thm. 3.36] by applying Fubini's theorem to the integral $\int_\Omega du \times dv$, where Ω is the triangle $\{(s,t) : a < s \le t \le b\}$. ☐

LEMMA 2.4. *The following product rules hold for* $u, v \in \mathrm{NBV}([0,T];\mathbf{R})$:

$$\begin{aligned} d(uv) &= u\,dv + v_-\,du \\ &= u_-\,dv + v\,du \\ &= u\,dv + v\,du - \sum_i \Delta_{s_i}(u)\Delta_{s_i}(v)\delta_{s_i}, \end{aligned}$$

where δ_{s_i} *is the Dirac mass at* s_i, *and the sum is over the* (*at most countably many*) *points* $s_i \in [0,T]$ *at which both* u *and* v *are discontinuous.*

Proof. The first two formulas follow immediately from (2.12) and the definition of $d(uv)$. For the third, we just note that the measure $(v - v_-)\,du$ is supported on the set of points where u and v are both discontinuous, and for each such point s_i,

$$\begin{aligned} \big(v(s_i) - v_-(s_i)\big)du(\{s_i\}) &= \big(v(s_i) - v(s_i-)\big)\big(u(s_i) - u(s_i-)\big) \\ &= \Delta_{s_i}(u)\Delta_{s_i}(v). \end{aligned}$$

☐

We will be interested primarily in vector-valued functions. We let $\mathrm{NBV}([0,T];\mathbf{R}^n)$ denote the set of functions $\mathbf{v} : [0,T] \to \mathbf{R}^n$ each of whose component functions is in $\mathrm{NBV}([0,T];\mathbf{R})$, and $\mathrm{NBV}([0,T];M) \subset \mathrm{NBV}([0,T];\mathbf{R}^n)$ the subset of functions taking their values in M. The considerations above apply equally well to such vector-valued functions, with obvious trivial modifications in notation. For example, if $\mathbf{v}, \mathbf{w} \in \mathrm{NBV}([0,T];\mathbf{R}^n)$, we consider $d\mathbf{v}$ and $d\mathbf{w}$ as $\mathbf{R}^n$-valued measures, and Lemma 2.3 implies that

$$\int_{(a,b]} \langle \mathbf{v}, d\mathbf{w}\rangle + \int_{(a,b]} \langle \mathbf{w}_-, d\mathbf{v}\rangle = \langle \mathbf{v}(b), \mathbf{w}(b)\rangle - \langle \mathbf{v}(a), \mathbf{w}(a)\rangle.$$

Suppose $\gamma \in \mathrm{NBV}([0,T];M)$ and $\mathbf{v}$ is an NBV vector field along γ. Note that the fact that γ takes its values in a bounded set, on which $\mathbf{n}$ is uniformly Lipschitz, guarantees that $\mathbf{n} \circ \gamma \in \mathrm{NBV}([0,T];\mathbf{R}^n)$.

We generalize the notion of covariant derivative for NBV vector fields by defining

$$\mathcal{D}\mathbf{v} = d\mathbf{v} + \langle \mathbf{v}_-, d(\mathbf{n} \circ \gamma)\rangle \mathbf{n} \circ \gamma.$$

One motivation for this definition is provided by the following lemma, which says that if $\mathbf{v}(0)$ is tangent to M and $\mathcal{D}\mathbf{v}$ is tangent to M on all of $[0,T]$, then $\mathbf{v}$ stays tangent to M.

LEMMA 2.5. *Suppose* $\gamma \in \mathrm{NBV}([0,T];M)$ *and* $\mathbf{v} \in \mathrm{NBV}([0,T];\mathbf{R}^n)$. *If* $\mathbf{v}(0) \in \mathcal{T}_{\gamma(0)}M$ *and* $\langle \mathcal{D}\mathbf{v}, \mathbf{n}\circ\gamma\rangle \equiv 0$, *then* $\mathbf{v}(t) \in \mathcal{T}_{\gamma(t)}M$ *for all* $t \in [0,T]$.

Proof. Using Lemma 2.4, we compute

$$\begin{aligned}
0 &= \langle \mathcal{D}\mathbf{v}, \mathbf{n}\circ\gamma\rangle \\
&= \langle d\mathbf{v}, \mathbf{n}\circ\gamma\rangle + \big\langle \langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle \mathbf{n}\circ\gamma, \mathbf{n}\circ\gamma\big\rangle \\
&= d\langle \mathbf{v}, \mathbf{n}\circ\gamma\rangle - \langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle + \langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle\langle \mathbf{n}\circ\gamma, \mathbf{n}\circ\gamma\rangle \\
&= d\langle \mathbf{v}, \mathbf{n}\circ\gamma\rangle.
\end{aligned}$$

Thus if $\langle \mathbf{v}(0), \mathbf{n}(\gamma(0))\rangle = 0$, we find by integration that $\langle \mathbf{v}(t), \mathbf{n}(\gamma(t))\rangle = 0$ for all t. $\square$

2.3. An existence and uniqueness theorem. The main purpose of this section is to prove the following theorem.

THEOREM 2.6. *Let* $M \subset \mathbf{R}^n$ *be a* C^2, *properly embedded hypersurface, and let* $\gamma \in \mathrm{NBV}([0,T];M)$. *For any* $\mathbf{v}_0 \in \mathcal{T}_{\gamma(0)}M$, *there exists a unique* NBV *vector field* $\mathbf{v}$ *along* γ *that is a solution to the following* (*measure-valued*) *ODE initial-value problem*:

$$\begin{aligned}
\mathcal{D}\mathbf{v} &= (\mathcal{S}\circ\gamma)\mathbf{v}\,dt, \\
\mathbf{v}(0) &= \mathbf{v}_0.
\end{aligned} \tag{2.13}$$

Before proving the theorem, we will establish some important preliminary results. We begin by dispensing with the uniqueness question.

LEMMA 2.7. *Let* $\gamma \in \mathrm{NBV}([0,T];M)$. *If* $\mathbf{v}, \widetilde{\mathbf{v}} \in \mathrm{NBV}([0,T];\mathbf{R}^n)$ *are both solutions to* (2.13) *with the same initial condition, they are equal.*

Proof. Suppose $\mathbf{v}$ is any solution to (2.13). Observe that Lemma 2.5 implies that $\mathbf{v}(t)$ is tangent to M for all t, so $\langle \mathbf{v}, \mathbf{n}\circ\gamma\rangle \equiv 0$. Let $R = \|\gamma\|_\infty$, so that γ takes its values in M_R. With K chosen as in Lemma 2.1, define $f \in \mathrm{NBV}([0,T];\mathbf{R})$ by $f(t) = e^{-2Kt}|\mathbf{v}(t)|^2$. Then Lemma 2.4 yields

$$\begin{aligned}
df &= e^{-2Kt}\bigg(-2K|\mathbf{v}|^2\,dt + 2\langle \mathbf{v}, d\mathbf{v}\rangle - \sum_i \langle \Delta_{s_i}\mathbf{v}, \Delta_{s_i}\mathbf{v}\rangle \delta_{s_i}\bigg) \\
&= e^{-2Kt}\bigg(-2K|\mathbf{v}|^2\,dt - 2\langle \mathbf{v}, \mathbf{n}\circ\gamma\rangle\langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle \\
&\qquad + 2\langle \mathbf{v}, (\mathcal{S}\circ\gamma)\mathbf{v}\rangle\,dt - \sum_i \langle \Delta_{s_i}\mathbf{v}, \Delta_{s_i}\mathbf{v}\rangle \delta_{s_i}\bigg) \\
&= e^{-2Kt}\bigg(2\big(\langle \mathbf{v}, (\mathcal{S}\circ\gamma)\mathbf{v}\rangle - K|\mathbf{v}|^2\big)\,dt - \sum_i |\Delta_{s_i}\mathbf{v}|^2 \delta_{s_i}\bigg).
\end{aligned}$$

Since (2.4) shows that $\langle \mathbf{v}, (\mathcal{S}\circ\gamma)\mathbf{v}\rangle \le K|\mathbf{v}|^2$, this last expression is a nonpositive measure on $[0,T]$. Integrating, we conclude that $f(t)\le f(0)$, or

$$|\mathbf{v}(t)|^2 \le e^{2Kt}|\mathbf{v}_0|^2.$$

In particular, the only solution with initial condition $\mathbf{v}_0 = 0$ is the zero solution. Because (2.13) is linear in $\mathbf{v}$, this suffices. □

To prove existence, we will work first with finite approximations. Define a *finite trajectory* in M to be a function $\gamma \in \mathrm{NBV}([0,T];M)$ that takes on only finitely many values. This means that there exists a partition $\{0 = t_0 < t_1 < \cdots < t_m = T\}$ of $[0,T]$ such that γ is constant on $[t_i, t_{i+1})$ for each i. For such a function, $d\gamma = \sum_{i=0}^m \Delta_{t_i}(\gamma)\delta_{t_i}$ and $\|d\gamma\| = \sum_{i=0}^m |\Delta_{t_i}(\gamma)|$.

Suppose γ is a finite trajectory in M and $\mathbf{v}_0 \in \mathcal{T}_{\gamma(0)}M$. Let $0 = t_0 < \cdots < t_m = T$ be a finite partition of $[0,T]$ including all of the discontinuities of γ, and write $x_i = \gamma(t_i)$. Define $\mathbf{v} : [0,T] \to \mathbf{R}^n$ by

$$\mathbf{v}(t) = e^{(t-t_k)\mathcal{S}_{x_k}}\pi_{x_k} e^{(t_k-t_{k-1})\mathcal{S}_{x_{k-1}}}\pi_{x_{k-1}}\cdots e^{(t_2-t_1)\mathcal{S}_{x_1}}\pi_{x_1} e^{(t_1-t_0)\mathcal{S}_{x_0}}\mathbf{v}_0, \tag{2.14}$$

where k is the largest index such that $t_k \le t$. Observe that the definition of $\mathbf{v}$ is unchanged if we insert more times t_i in the partition.

LEMMA 2.8. *Let $\gamma : [0,T] \to M$ be a finite trajectory. For and any $\mathbf{v}_0 \in \mathcal{T}_{\gamma(0)}M$, the map $\mathbf{v}$ defined by (2.14) is the unique solution to (2.13), and satisfies*

$$|\mathbf{v}(t)| \le e^{Ct}|\mathbf{v}_0|, \tag{2.15}$$

$$\|d\mathbf{v}\| \le C, \tag{2.16}$$

where C is a constant depending only on M, T, and $\|d\gamma\|$.

Proof. An easy computation shows that

$$\begin{aligned} d\mathbf{v} &= (\mathcal{S}\circ\gamma)\mathbf{v}\,dt + \sum_{i=0}^m \big(\pi_{x_i}\mathbf{v}(t_i-) - \mathbf{v}(t_i-)\big)\delta_{t_i} \\ &= (\mathcal{S}\circ\gamma)\mathbf{v}\,dt + \sum_{i=0}^m \langle \mathbf{v}(t_i-), \mathbf{n}(\gamma(t_i)) - \mathbf{n}(\gamma(t_i-))\rangle \mathbf{n}(\gamma(t_i))\delta_{t_i}, \end{aligned}$$

from which it follows that $\mathbf{v}$ solves (2.13).

To estimate $|\mathbf{v}(t)|$, observe first that the operator norm of each projection π_x is equal to one. Let K be the constant of Lemma 2.1 for $R = \|d\gamma\|$. Using (2.6), we have the following operator norm estimate for any finite collection of points $x_1,\dots,x_j \in M_R$ and real numbers $l_1,\dots,l_j \in [0,T]$:

$$\|e^{l_j\mathcal{S}_{x_j}}\circ\pi_{x_j}\circ\cdots\circ e^{l_1\mathcal{S}_{x_1}}\circ\pi_{x_1}\| \le e^{Kl_j}\cdots e^{Kl_1} = e^{K(l_j+\cdots+l_1)}. \tag{2.17}$$

Applying this to the definition of $\mathbf{v}$ proves (2.15). Then, using (2.15) and (2.11), we estimate

$$\begin{aligned}\|d\mathbf{v}\| &= \int_{[0,T]} |(\mathcal{S}\circ\gamma)\mathbf{v}|\,dt + \sum_{i=0}^{m} |\langle \mathbf{v}(t_i-), \mathbf{n}(\gamma(t_i)) - \mathbf{n}(\gamma(t_i-))\rangle \mathbf{n}(\gamma(t_i))| \\ &\le \int_{[0,T]} Ke^{Kt}\,dt + \sum_{i=0}^{m} e^{KT} K|\gamma(t_i) - \gamma(t_i-)| \\ &\le C(1+\|d\gamma\|). \end{aligned}$$

□

LEMMA 2.9. *Suppose γ and $\widetilde{\gamma}$ are any finite trajectories in M defined on $[0,T]$ and starting at the same point, and $\mathbf{v}$, $\widetilde{\mathbf{v}}$ are the corresponding solutions to (2.13). There is a constant C depending only on M, T, $\|\gamma\|_\infty$, and $\|\widetilde{\gamma}\|_\infty$ such that the following estimate holds*:

$$\|\mathbf{v} - \widetilde{\mathbf{v}}\|_\infty \le C(1 + \|d\gamma\| + \|d\widetilde{\gamma}\|)\|\gamma - \widetilde{\gamma}\|_\infty |\mathbf{v}_0|.$$

Proof. Lemma 2.8 shows that $\|\mathbf{v}\|_\infty$ and $\|\widetilde{\mathbf{v}}\|_\infty$ are both bounded by $C|\mathbf{v}_0|$ for some C depending only on M, T, $\|\gamma\|_\infty$, and $\|\widetilde{\gamma}\|_\infty$. Fix $t \in [0,T]$, and let $0 = t_0 < \cdots < t_k \le t$ denote a finite partition that includes all of the discontinuities of γ and $\widetilde{\gamma}$ in $[0,t]$. We introduce the following shorthand notations:

$$\begin{aligned} t_{k+1} &= t, & l_i &= t_{i+1} - t_i, \\ x_i &= \gamma(t_i), & \widetilde{x}_i &= \widetilde{\gamma}(t_i), \\ \mathcal{S}_i &= \mathcal{S}(x_i), & \widetilde{\mathcal{S}}_i &= \mathcal{S}(\widetilde{x}_i), \\ \pi_i &= \pi_{x_i}, & \widetilde{\pi}_i &= \pi_{\widetilde{x}_i}. \end{aligned}$$

Observing that $\pi_0\mathbf{v}_0 = \mathbf{v}_0$ and $\widetilde{\pi}_{k+1}\widetilde{\mathbf{v}}(t) = \widetilde{\mathbf{v}}(t)$, we can write $\mathbf{v}(t) - \widetilde{\mathbf{v}}(t)$ as a telescoping sum:

$$\begin{aligned}&\mathbf{v}(t) - \widetilde{\mathbf{v}}(t) \\ &\quad = \sum_{i=0}^{k} e^{l_k\mathcal{S}_k}\pi_k \cdots e^{l_{i+1}\mathcal{S}_{i+1}}\pi_{i+1}(e^{l_i\mathcal{S}_i}\pi_i - \widetilde{\pi}_{i+1}e^{l_i\widetilde{\mathcal{S}}_i})\widetilde{\pi}_i \cdots e^{l_1\widetilde{\mathcal{S}}_1}\widetilde{\pi}_1 e^{l_0\widetilde{\mathcal{S}}_0}\mathbf{v}_0.\end{aligned}$$

By (2.17), the compositions of operators before and after the parentheses in the summation above are uniformly bounded in operator norm by e^{KT}. Therefore,

$$|\mathbf{v}(t) - \widetilde{\mathbf{v}}(t)| \le e^{2KT} \sum_{i=0}^{k} \|\pi_{i+1}\circ(e^{l_i\mathcal{S}_i}\circ\pi_i - \widetilde{\pi}_{i+1}\circ e^{l_i\widetilde{\mathcal{S}}_i})\circ\widetilde{\pi}_i\| |\mathbf{v}_0|.$$

Using the fact that $\mathcal{S}_i$ and π_i commute, as do $\widetilde{\mathcal{S}}_i$ and $\widetilde{\pi}_i$, we decompose the middle factors as follows:

$$\begin{aligned}\pi_{i+1}\circ(e^{l_i\mathcal{S}_i}\circ\pi_i - \widetilde{\pi}_{i+1}\circ e^{l_i\widetilde{\mathcal{S}}_i})\circ\widetilde{\pi}_i &= \pi_{i+1}\circ\pi_i\circ(e^{l_i\mathcal{S}_i} - e^{l_i\widetilde{\mathcal{S}}_i})\circ\widetilde{\pi}_i \\ &\quad + \pi_{i+1}\circ(\pi_i - \widetilde{\pi}_{i+1})\circ\widetilde{\pi}_i\circ e^{l_i\widetilde{\mathcal{S}}_i}.\end{aligned}$$

We will deal with each of these terms separately.

For the first term, (2.9) implies

$$\|e^{l_i\mathcal{S}_i} - e^{l_i\widetilde{\mathcal{S}}_i}\| \le Kl_i|x_i - \widetilde{x}_i| \le Kl_i\|\gamma - \widetilde{\gamma}\|_\infty,$$

and after summing over i, we find that this is bounded by $KT\|\gamma - \widetilde{\gamma}\|_\infty$. For the second term, Lemma 2.2 allows us to conclude that

$$\begin{aligned}
&\|\pi_{i+1} \circ (\pi_i - \widetilde{\pi}_{i+1}) \circ \widetilde{\pi}_i \circ e^{l_i\widetilde{\mathcal{S}}_i}\| \\
&\quad \le C(|x_{i+1} - x_i||x_i - \widetilde{x}_i| + |x_{i+1} - \widetilde{x}_{i+1}||\widetilde{x}_{i+1} - \widetilde{x}_i|)\|e^{l_i\widetilde{\mathcal{S}}_i}\| \\
&\quad \le Ce^{KT}\|\gamma - \widetilde{\gamma}\|_\infty(|x_{i+1} - x_i| + |\widetilde{x}_{i+1} - \widetilde{x}_i|).
\end{aligned}$$

After summing, this is bounded by $Ce^{KT}\|\gamma - \widetilde{\gamma}\|_\infty(\|d\gamma\| + \|d\widetilde{\gamma}\|)$. This completes the proof. □

LEMMA 2.10. *Let $\gamma \in \mathrm{NBV}([0,T];M)$ be arbitrary. For any $\varepsilon > 0$, there exists a finite trajectory $\widetilde{\gamma}\colon [0,T] \to M$ such that $\|\gamma - \widetilde{\gamma}\|_\infty < \varepsilon$ and $\|d\widetilde{\gamma}\| \le \|d\gamma\|$.*

Proof. Let ε be given. Since γ is *càdlàg*, for each $a \in [0,T]$, there exists $\delta > 0$ such that for $t \in [0,T]$,

$$t \in [a, a+\delta) \implies |\gamma(t) - \gamma(a)| < \varepsilon, \tag{2.18}$$

$$t \in (a-\delta, a) \implies |\gamma(t) - \gamma(a-)| < \frac{\varepsilon}{2}. \tag{2.19}$$

By compactness, we can choose finitely many points $0 = a_0 < a_1 < \cdots < a_m = T$ and corresponding positive numbers $\delta_0, \dots, \delta_m$ so that $[0,T]$ is covered by the intervals $(a_i - \delta_i, a_i + \delta_i)$, $i = 1, \dots, m$. Because they are a cover, for each $i = 1, \dots, m$ we can choose b_i such that

$$b_i \in (a_{i-1}, a_{i-1} + \delta_{i-1}) \cap (a_i - \delta_i, a_i).$$

Now define a finite trajectory $\widetilde{\gamma}\colon [0,T] \to M$ by

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(a_{i-1}), & t \in [a_{i-1}, b_i), \\ \gamma(b_i), & t \in [b_i, a_i). \end{cases}$$

It is clear from the definition of the total variation that $\|d\widetilde{\gamma}\| \le \|d\gamma\|$. We will show that $\|\gamma - \widetilde{\gamma}\|_\infty < \varepsilon$.

Let $t \in [0,T]$ be arbitrary. For some i, either $t \in [a_{i-1}, b_i)$ or $t \in [b_i, a_i)$. In the first case, since $[a_{i-1}, b_i) \subset [a_{i-1}, a_{i-1} + \delta_{i-1})$ by construction, (2.18) yields

$$|\gamma(t) - \widetilde{\gamma}(t)| = |\gamma(t) - \gamma(a_{i-1})| < \varepsilon.$$

On the other hand, if $t \in [b_i, a_i) \subset (a_i - \delta_i, a_i)$, (2.19) yields

$$\begin{aligned}
|\gamma(t) - \widetilde{\gamma}(t)| &= |\gamma(t) - \gamma(b_i)| \le |\gamma(t) - \gamma(a_i-)| + |\gamma(a_i-) - \gamma(b_i)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2},
\end{aligned}$$

so we reach the same conclusion. □

LEMMA 2.11. *For any* $\gamma \in \mathrm{NBV}([0,T];M)$, *there exists a sequence of finite trajectories* $\gamma^{(k)} : [0,T] \to M$ *satisfying* $\|d\gamma^{(k)}\| \le \|d\gamma\|$ *and converging uniformly to* γ.

Proof. This is an immediate consequence of Lemma 2.10. □

Now we can prove the existence and uniqueness theorem.

Proof of Theorem 2.6. Given γ as in the statement of the theorem, let $\gamma^{(k)}$ be a sequence of finite trajectories converging uniformly to γ as guaranteed by Lemma 2.11. For each k, let $\mathbf{v}^{(k)}$ be the solution to (2.13) for $\gamma = \gamma^{(k)}$, as defined by (2.14). Then Lemma 2.9 guarantees that the sequence $\mathbf{v}^{(k)}$ is uniformly Cauchy, and hence there is a limit function $\mathbf{v} : [0,T] \to \mathbf{R}^n$ such that $\mathbf{v}^{(k)} \to \mathbf{v}$ uniformly. It is straightforward to check that $\mathbf{v} \in \mathrm{NBV}([0,T];\mathbf{R}^n)$. Moreover, since each $\mathbf{v}^{(k)}$ is tangent to M and $\mathbf{v}^{(k)} \to \mathbf{v}$ uniformly, it follows that $\mathbf{v}$ is also tangent to M.

We need to show that $\mathbf{v}$ solves (2.13) for γ. It suffices to show for any $\mathbf{w} \in \mathrm{NBV}([0,T];\mathbf{R}^n)$ that

$$\begin{aligned}\int_{[0,T]} \langle \mathbf{w}, d\mathbf{v}\rangle = {} & -\int_{[0,T]} \langle \mathbf{w}, \mathbf{n}\circ\gamma\rangle \langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle \\ & + \int_{[0,T]} \langle \mathbf{w}, (\mathcal{S}\circ\gamma)\mathbf{v}\rangle\, dt.\end{aligned}$$

If we write $\mathbf{w} = \mathbf{w}^\top + \mathbf{w}^\perp$, where $\mathbf{w}^\top$ is tangent to M and $\mathbf{w}^\perp$ is orthogonal to M, this is equivalent to the following two equations:

$$\int_{[0,T]} \langle \mathbf{w}^\perp, d\mathbf{v}\rangle = -\int_{[0,T]} \langle \mathbf{w}^\perp, \mathbf{n}\circ\gamma\rangle \langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle, \tag{2.20}$$

$$\int_{[0,T]} \langle \mathbf{w}^\top, d\mathbf{v}\rangle = \int_{[0,T]} \langle \mathbf{w}^\top, (\mathcal{S}\circ\gamma)\mathbf{v}\rangle\, dt. \tag{2.21}$$

Because $\mathbf{w}^\perp$ is proportional to $\mathbf{n}$, $\mathbf{w}^\perp = \langle \mathbf{w}^\perp, \mathbf{n}\circ\gamma\rangle \mathbf{n}\circ\gamma$. The fact that $\mathbf{v}$ is tangent to M means that $\langle \mathbf{n}\circ\gamma, \mathbf{v}\rangle \equiv 0$, from which we conclude

$$0 = d\langle \mathbf{n}\circ\gamma, \mathbf{v}\rangle = \langle \mathbf{n}\circ\gamma, d\mathbf{v}\rangle + \langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle.$$

Therefore,

$$\begin{aligned}\langle \mathbf{w}^\perp, d\mathbf{v}\rangle &= \big\langle \langle \mathbf{w}^\perp, \mathbf{n}\circ\gamma\rangle \mathbf{n}\circ\gamma, d\mathbf{v}\big\rangle \\ &= \langle \mathbf{w}^\perp, \mathbf{n}\circ\gamma\rangle \langle \mathbf{n}\circ\gamma, d\mathbf{v}\rangle \\ &= -\langle \mathbf{w}^\perp, \mathbf{n}\circ\gamma\rangle \langle \mathbf{v}_-, d(\mathbf{n}\circ\gamma)\rangle,\end{aligned}$$

from which (2.20) follows.

On the other hand, from Lemma 2.3 we conclude that

$$\begin{aligned}\int_{[0,T]} \langle \mathbf{w}^\top, d\mathbf{v}\rangle &= \langle \mathbf{w}^\top(T), \mathbf{v}(T)\rangle - \int_{[0,T]} \langle \mathbf{v}_-, d\mathbf{w}^\top\rangle \\ &= \lim_{k\to\infty}\left(\left\langle \mathbf{w}^\top(T), \mathbf{v}^{(k)}(T)\right\rangle - \int_{[0,T]} \left\langle \mathbf{v}_-^{(k)}, d\mathbf{w}^\top\right\rangle\right) \\ &= \lim_{k\to\infty}\int_{[0,T]} \left\langle \mathbf{w}^\top, d\mathbf{v}^{(k)}\right\rangle \\ &= \lim_{k\to\infty}\left(-\int_{[0,T]} \left\langle \mathbf{w}^\top, \mathbf{n}\circ\gamma^{(k)}\right\rangle\left\langle \mathbf{v}_-^{(k)}, d\big(\mathbf{n}\circ\gamma^{(k)}\big)\right\rangle \right.\\ &\qquad \left. + \int_{[0,T]} \left\langle \mathbf{w}^\top, \big(\mathcal{S}\circ\gamma^{(k)}\big)\mathbf{v}^{(k)}\right\rangle dt\right).\end{aligned}$$

Since $\langle \mathbf{w}^\top, \mathbf{n}\circ\gamma^{(k)}\rangle$ converges uniformly to $\langle \mathbf{w}^\top, \mathbf{n}\circ\gamma\rangle \equiv 0$, and the measures $\langle \mathbf{v}_-^{(k)}, d(\mathbf{n}\circ\gamma^{(k)})\rangle$ have uniformly bounded total variation, the first term above vanishes in the limit. Since both $\mathbf{v}^{(k)}$ and $\mathcal{S}\circ\gamma^{(k)}$ converge uniformly, the last term above converges to $\int_{[0,T]}\langle \mathbf{w}^\top, (\mathcal{S}\circ\gamma)\mathbf{v}\rangle\, dt$. This proves (2.21). $\square$

2.4. Stability. In this section, we wish to address the stability of the solution to (2.13) under perturbations of the trajectory γ. For applications to probability, we will need to consider perturbations in a weaker topology than the uniform one.

We define a metric d_S on $\mathrm{NBV}([0,T];\mathbf{R}^n)$, called the *Skorokhod metric*, by

$$d_S(\gamma,\widetilde{\gamma}) = \inf_{\lambda\in\Lambda} \max(\|\gamma - \widetilde{\gamma}\circ\lambda\|_\infty, \|\lambda - \mathrm{Id}\,\|_\infty),$$

where Λ is the set of increasing homeomorphisms $\lambda : [0,T]\to[0,T]$. We wish to show that the solution to (2.13) is continuous in the Skorokhod metric, as long as we stay within a set of trajectories with uniformly bounded total variation.

Because the Skorokhod metric is not homogeneous with respect to constant multiples, it will not be possible to bound $d_S(\mathbf{v},\widetilde{\mathbf{v}})$ directly in terms of $d_S(\gamma,\widetilde{\gamma})$. For this reason, we will work instead with the *solution operator*: for any NBV trajectory $\gamma : [0,T]\to M$, this is the endomorphism-valued function $\mathcal{A} : [0,T]\to\mathrm{End}(\mathbf{R}^n)$ defined by

$$\mathcal{A}(t)\mathbf{v}_0 = \mathbf{v}(t),$$

where $\mathbf{v}$ is the solution to (2.13) with initial value $\mathbf{v}_0$, and extended to an endomorphism of $\mathbf{R}^n$ by declaring $\mathcal{A}(t)\mathbf{n}_{\gamma(0)} = 0$. As before, $\|\mathcal{A}(t)\|$ will denote the operator norm of $\mathcal{A}(t)$, and we set

$$\begin{aligned}\|\mathcal{A}\|_\infty &= \sup\{\|\mathcal{A}(t)\| : t\in[0,T]\} \\ &= \sup\left\{\frac{|\mathbf{v}(t)|}{|\mathbf{v}_0|} : t\in[0,T],\ \mathbf{v}_0\in\mathcal{T}_{\gamma(0)}M,\ \mathbf{v}_0\neq 0\right\}.\end{aligned}$$

It follows easily from the results of the preceding section that for any $\gamma \in \mathrm{NBV}([0,T];M)$, the solution operator $\mathcal{A}$ is in $\mathrm{NBV}([0,T];\mathrm{End}(\mathbf{R}^n))$, and Lemma 2.9 translates immediately into the following estimate.

LEMMA 2.12. *Suppose γ and $\widetilde{\gamma}$ are any finite trajectories in M defined on $[0,T]$ and starting at the same point, and $\mathcal{A}$, $\widetilde{\mathcal{A}}$ are the corresponding solution operators. There is a constant C depending only on M, T, $\|d\gamma\|$, and $\|d\widetilde{\gamma}\|$ such that the following estimate holds*:

$$\|\mathcal{A} - \widetilde{\mathcal{A}}\|_\infty \le C\|\gamma - \widetilde{\gamma}\|_\infty.$$

Next, we need to examine the effect of a reparametrization on the solution associated with a finite trajectory.

LEMMA 2.13. *Let $\gamma : [0,T] \to M$ be a finite trajectory, let $\lambda : [0,T] \to [0,T]$ be an increasing homeomorphism, and let $\widetilde{\gamma} = \gamma \circ \lambda$. There is a constant C depending only on M, T, and $\|d\gamma\|$ such that the solutions $\mathbf{v}$ and $\widetilde{\mathbf{v}}$ to* (2.13) *associated to γ and $\widetilde{\gamma}$ with the same initial value $\mathbf{v}_0$ satisfy*

$$\|\mathbf{v} - \widetilde{\mathbf{v}} \circ \lambda\|_\infty \le C\|\lambda - \mathrm{Id}\,\|_\infty |\mathbf{v}_0|. \tag{2.22}$$

Proof. As in the proof of Lemma 2.9, fix $t \in [0,T]$ and let $0 = t_0 < t_1 < \cdots < t_k \le t$ be the points in $[0,t]$ at which γ is discontinuous. Set $t_{k+1} = t$, $x_i = \gamma(t_i)$, and $\widetilde{t}_i = \lambda(t_i)$, so that γ and $\widetilde{\gamma}$ are given by

$$\begin{aligned} \gamma(s) &= x_i \quad \text{if } t_i \le s < t_{i+1}, \\ \widetilde{\gamma}(s) &= x_i \quad \text{if } \widetilde{t}_i \le s < \widetilde{t}_{i+1}. \end{aligned}$$

We will also use the notations

$$\begin{aligned} l_i &= t_{i+1} - t_i, \\ \widetilde{l}_i &= \widetilde{t}_{i+1} - \widetilde{t}_i, \\ \mathcal{S}_i &= \mathcal{S}(x_i), \\ \pi_i &= \pi_{x_i}. \end{aligned}$$

We can write $\widetilde{\mathbf{v}}(\lambda(t)) - \mathbf{v}(t) = \widetilde{\mathbf{v}}(\widetilde{t}_{k+1}) - \mathbf{v}(t_{k+1})$ as a telescoping sum:

$$\begin{aligned} \widetilde{\mathbf{v}}(\lambda(t)) - \mathbf{v}(t) = {} & \big(\mathrm{Id} - e^{(t_{k+1} - \widetilde{t}_{k+1})\mathcal{S}_k}\big)\widetilde{\mathbf{v}}(\lambda(t)) \\ & + \sum_{i=1}^{k} e^{l_k \mathcal{S}_k}\pi_k \cdots e^{l_{i+1}\mathcal{S}_{i+1}}\pi_{i+1} \\ & \circ \big(e^{(t_{i+1} - \widetilde{t}_i)\mathcal{S}_i}\pi_i e^{\widetilde{l}_{i-1}\mathcal{S}_{i-1}} - e^{l_i \mathcal{S}_i}\pi_i e^{(t_i - \widetilde{t}_{i-1})\mathcal{S}_{i-1}}\big) \\ & \circ \pi_{i-1} e^{\widetilde{l}_{i-2}\mathcal{S}_{i-2}} \cdots e^{\widetilde{l}_1 \mathcal{S}_1}\pi_1 e^{\widetilde{l}_0 \mathcal{S}_0}\mathbf{v}_0. \end{aligned}$$

(To verify this equation, it is important to note that the exponential expression $e^{(t_{k+1} - \widetilde{t}_{k+1})\mathcal{S}_k}$ in the first term combines with the factor $e^{(\widetilde{t}_{k+1} - \widetilde{t}_k)\mathcal{S}_k}$ in the expansion of $\widetilde{\mathbf{v}}(\lambda(t))$ to yield a term that begins with $e^{(t_{k+1} - \widetilde{t}_k)\mathcal{S}_k}$ and cancels a similar expression in the $i=k$ term of the summation.) By virtue of (2.8), the first term is bounded by a constant multiple of $|t_{k+1}-\widetilde{t}_{k+1}||\mathbf{v}_0| \le \|\lambda - \mathrm{Id}\,\|_\infty |\mathbf{v}_0|$. As before, the compositions before and after the parentheses in the summation are uniformly bounded in operator norm, so we need only estimate the sum

$$\sum_{i=1}^{k} \big\| e^{(t_{i+1}-\widetilde{t}_i)\mathcal{S}_i} \circ \pi_i \circ e^{\widetilde{l}_{i-1}\mathcal{S}_{i-1}} - e^{l_i \mathcal{S}_i} \circ \pi_i \circ e^{(t_i - \widetilde{t}_{i-1})\mathcal{S}_{i-1}} \big\|.$$

Using the fact that π_i commutes with $\mathcal{S}_i$, we can rewrite the ith term in this sum as

$$\begin{aligned} &\big\| e^{l_i \mathcal{S}_i} \circ \pi_i \circ \big(e^{(t_i-\widetilde{t}_i)\mathcal{S}_i} - e^{(t_i-\widetilde{t}_i)\mathcal{S}_{i-1}} \big) e^{\widetilde{l}_{i-1}\mathcal{S}_{i-1}} \big\| \\ &\quad \le \| e^{l_i \mathcal{S}_i} \| \big\| e^{(t_i-\widetilde{t}_i)\mathcal{S}_i} - e^{(t_i-\widetilde{t}_i)\mathcal{S}_{i-1}} \big\| \| e^{\widetilde{l}_{i-1}\mathcal{S}_{i-1}} \|. \end{aligned}$$

From (2.6) and (2.9), this last expression is bounded by $C|t_i - \widetilde{t}_i||x_i - x_{i-1}|$. Summing over i, we conclude that this is bounded by $C\|\lambda - \mathrm{Id}\,\|_\infty \|d\gamma\|$. $\square$

LEMMA 2.14. *Suppose $\gamma, \widetilde{\gamma} : [0,T] \to M$ are finite trajectories starting at the same point, and let $\mathcal{A}$, $\widetilde{\mathcal{A}}$ be the corresponding solution operators. There exists a constant C depending only on M, T, $\|d\gamma\|$, and $\|d\widetilde{\gamma}\|$ such that*

$$d_S(\mathcal{A}, \widetilde{\mathcal{A}}) \le C d_S(\gamma, \widetilde{\gamma}). \tag{2.23}$$

Proof. Let $\delta = d_S(\gamma,\widetilde{\gamma})$ and let $\varepsilon > 0$ be arbitrary. By definition of the Skorokhod metric, there is an increasing homeomorphism $\lambda : [0,T] \to [0,T]$ such that $\|\gamma - \widetilde{\gamma}\circ\lambda\|_\infty \le \delta + \varepsilon$ and $\|\lambda - \mathrm{Id}\,\|_\infty \le \delta+\varepsilon$. Let $\mathcal{A}_1$ be the solution operator associated with $\widetilde{\gamma}\circ\lambda$. Then $\|\mathcal{A} - \mathcal{A}_1\|_\infty \le C(\delta+\varepsilon)$ by Lemma 2.12, and $\|\widetilde{\mathcal{A}} - \mathcal{A}_1 \circ \lambda\|_\infty \le C(\delta+\varepsilon)$ by Lemma 2.13. Thus by the triangle inequality,

$$\begin{aligned} d_S(\mathcal{A}, \widetilde{\mathcal{A}}) &\le d_S(\mathcal{A}, \mathcal{A}_1) + d_S(\mathcal{A}_1, \widetilde{\mathcal{A}}) \\ &\le \|\mathcal{A} - \mathcal{A}_1\|_\infty + \max(\|\widetilde{\mathcal{A}} - \mathcal{A}_1\circ\lambda\|_\infty, \|\lambda - \mathrm{Id}\,\|_\infty) \\ &\le C(\delta+\varepsilon) + \max\big(C(\delta+\varepsilon), \varepsilon\big). \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$d_S(\mathcal{A}, \widetilde{\mathcal{A}}) \le 2C d_S(\gamma, \widetilde{\gamma}).$$ $\square$

Here is our main stability result.

THEOREM 2.15. *Given positive constants R and T, there exists a constant C depending only on M, R, and T such that for any trajectories $\gamma, \widetilde{\gamma} \in \mathrm{NBV}([0,T]; M)$ starting at the same point and with total variation bounded by R, the corresponding solution operators $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ satisfy*

$$d_S(\mathcal{A}, \widetilde{\mathcal{A}}) \le C d_S(\gamma, \widetilde{\gamma}). \tag{2.24}$$

Proof. By the argument in the proof of Theorem 2.6, there exist sequences of finite trajectories converging uniformly to γ and $\widetilde{\gamma}$ whose solution operators converge uniformly to $\mathcal{A}$ and $\widetilde{\mathcal{A}}$, respectively. Thus for any $\varepsilon > 0$, we can choose finite trajectories γ' and $\widetilde{\gamma}'$, with corresponding solution operators $\mathcal{A}'$ and $\widetilde{\mathcal{A}}'$, such that

$$\begin{aligned} \|\gamma' - \gamma\|_\infty &< \varepsilon, & \|\widetilde{\gamma}' - \widetilde{\gamma}\|_\infty &< \varepsilon, \\ \|\mathcal{A}' - \mathcal{A}\|_\infty &< \varepsilon, & \|\widetilde{\mathcal{A}}' - \widetilde{\mathcal{A}}\|_\infty &< \varepsilon. \end{aligned}$$

Then by the triangle inequality,

$$d_S(\gamma', \widetilde{\gamma}') \le d_S(\gamma', \gamma) + d_S(\gamma, \widetilde{\gamma}) + d_S(\widetilde{\gamma}, \widetilde{\gamma}') < d_S(\gamma, \widetilde{\gamma}) + 2\varepsilon.$$

By Lemma 2.14, we have

$$d_S(\mathcal{A}', \widetilde{\mathcal{A}}') \le C d_S(\gamma', \widetilde{\gamma}') \le C d_S(\gamma, \widetilde{\gamma}) + 2C\varepsilon.$$

Thus by the triangle inequality once more,

$$\begin{aligned} d_S(\mathcal{A}, \widetilde{\mathcal{A}}) &\le d_S(\mathcal{A}, \mathcal{A}') + d_S(\mathcal{A}', \widetilde{\mathcal{A}}') + d_S(\widetilde{\mathcal{A}}', \widetilde{\mathcal{A}}) \\ &\le \varepsilon + \big(C d_S(\gamma, \widetilde{\gamma}) + 2C\varepsilon\big) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof. $\square$

2.5. Base trajectories of infinite variation. In the probabilistic context, we will have to analyze the situation when the base trajectory γ does not have finite variation on finite intervals. We will now present an example showing that some of the results proved in this section do not extend to (all) functions γ of infinite variation. Hence, arguments using piecewise-constant approximations in the probabilistic context will require some modification of our techniques.

EXAMPLE 2.16. Let $M \subset \mathbf{R}^2$ be the parabola $M = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 = x_1^2\}$, with the orientation of M chosen so that $\|\mathcal{S}_x\| < 1$ for all $x \in M$. Let $\gamma(t) = (0,0)$ for $t \in [0,1]$, and for even integers $j \ge 2$, let

$$\gamma_j(t) = \begin{cases} x_j := (j^{-1}, j^{-2}), & \text{for } t \in \big[2kj^{-3}, (2k+1)j^{-3}\big), \\ & \qquad k = 0, 1, \dots, j^3/2 - 1, \\ y_j := (-j^{-1}, j^{-2}), & \text{for } t \in \big[(2k+1)j^{-3}, (2k+2)j^{-3}\big), \\ & \qquad k = 0, 1, \dots, j^3/2 - 1, \\ (j^{-1}, j^{-2}), & \text{for } t = 1. \end{cases}$$

Clearly, $\gamma_j \to \gamma$ in the supremum norm on $[0,1]$, so $d_S(\gamma_j, \gamma) \to 0$. Let $\mathbf{v}_0 = (1,0)$ and let $\mathbf{v}_j(t)$ be defined as in (2.14), relative to γ_j. Similarly, let $\mathbf{v}(t)$ be defined by (2.14) relative to γ. We have $\mathbf{v}(1) = e^{\mathcal{S}_{(0,0)}}\mathbf{v}_0 \ne (0,0)$.

There exists $c_1 > 0$ such that for all $j \ge 2$, $\mathbf{z} \in \mathcal{T}_{x_j} M$, we have $|\pi_{y_j}\mathbf{z}| \le (1 - c_1 j^{-2})|\mathbf{z}|$, and similarly, $|\pi_{x_j}\mathbf{z}| \le (1 - c_1 j^{-2})|\mathbf{z}|$, for $\mathbf{z} \in \mathcal{T}_{y_j} M$. This implies that for some $c_2 < 1$, $|\mathbf{v}_j(1)| = |(\pi_{x_j} \circ \pi_{y_j})^{j^3/2}\mathbf{v}_0| \le c_2^j$. Hence, $\lim_{j\to\infty} \mathbf{v}_j(1) =$

$(0,0) \neq \mathbf{v}(1)$. This shows that results such as Lemma 2.9 do not hold for (some) functions γ which do not have bounded variation.

3. Multiplicative functional for reflected Brownian motion

Suppose $D \subset \mathbf{R}^n$, $n \geq 2$, is an open connected bounded set with C^2 boundary. We let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D$; because ∂D is of class C^2, it follows that $\mathbf{n}(x)$ is a C^1 function of x. Let B be standard d-dimensional Brownian motion, $x_* \in \overline{D}$, and consider the following Skorokhod equation,

$$X_t = x_* + B_t + \int_0^t \mathbf{n}(X_s)\, dL_s \qquad \text{for } t \geq 0. \tag{3.1}$$

Here L is the local time of X on ∂D. In other words, L is a nondecreasing continuous process which does not increase when X is in D, i.e., $\int_0^\infty \mathbf{1}_D(X_t)\, dL_t = 0$, a.s. Equation (3.1) has a unique pathwise solution (X, L) such that $X_t \in \overline{D}$ for all $t \geq 0$ (see [LS]).

We need an extra "cemetery point" Δ outside $\mathbf{R}^n$, so that we can send processes killed at a finite time to Δ. Excursions of X from ∂D will be denoted e or e_s, that is, if $s < u$, $X_s, X_u \in \partial D$, and $X_t \notin \partial D$ for $t \in (s,u)$ then $e_s = \{e_s(t) = X_{t+s}, t \in [0, u-s)\}$. Let $\zeta(e_s) = u - s$ be the lifetime of e_s. By convention, $e_s(t) = \Delta$ for $t \geq \zeta$, so $e_t \equiv \Delta$ if $\inf\{s > t : X_s \in \partial D\} = t$.

Let σ be the inverse local time, that is, $\sigma_t = \inf\{s \geq 0 : L_s \geq t\}$, and $\mathcal{E}_r = \{e_s : s < \sigma_r\}$. Fix some $r, \varepsilon > 0$ and let $\{e_{t_1}, e_{t_2}, \dots, e_{t_m}\}$ be the set of all excursions $e \in \mathcal{E}_r$ with $|e(0) - e(\zeta-)| \geq \varepsilon$. We assume that excursions are labeled so that $t_k < t_{k+1}$ for all k and we let $\ell_k = L_{t_k}$ for $k = 1, \dots, m$. We also let $t_0 = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0 = 0$, $\ell_{m+1} = r$, and $\Delta\ell_k = \ell_{k+1} - \ell_k$. Let $x_k = e_{t_k}(\zeta-)$ for $k = 1, \dots, m$, and let $x_0 = X_{t_0}$.

In this section, the boundary of D will play the role of the hypersurface M, that is, $M = \partial D$. Recall that $\mathcal{S}$ denotes the shape operator and π_x is the orthogonal projection on the tangent space $\mathcal{T}_x \partial D$, for $x \in \partial D$. For $\mathbf{v}_0 \in \mathbf{R}^n$, let

$$\mathbf{v}_{r,\varepsilon} = \exp(\Delta\ell_m \mathcal{S}(x_m))\pi_{x_m} \cdots \exp(\Delta\ell_1 \mathcal{S}(x_1))\pi_{x_1} \exp(\Delta\ell_0 \mathcal{S}(x_0))\pi_{x_0} \mathbf{v}_0. \tag{3.2}$$

Let $\mathcal{A}_{r,\varepsilon}$ be a linear mapping defined by $\mathbf{v}_{r,\varepsilon} = \mathcal{A}_{r,\varepsilon}\mathbf{v}_0$.

We point out that the "multiplicative functional" $\widetilde{\mathcal{A}}_t$ discussed in the Introduction is not the same as $\mathcal{A}_r$ defined in this section. Intuitively speaking, $\mathcal{A}_r = \widetilde{\mathcal{A}}_{\sigma_r}$, although we have not defined $\widetilde{\mathcal{A}}_t$ in a formal way.

Suppose that ∂D contains n nondegenerate $(n-1)$-dimensional spheres, such that vectors perpendicular to these spheres are orthogonal to each other. If the trajectory $\{X_t, 0 \leq t \leq r\}$ visits the n spheres and no other part of ∂D, then it is easy to see that $\mathcal{A}_{r,\varepsilon} = 0$ for small $\varepsilon > 0$. To avoid this uninteresting situation, we impose the following assumption on D.

ASSUMPTION 3.1. *For every* $x \in \partial D$, *the* $(n-1)$*-dimensional surface area measure of* $\{y \in \partial D : \langle \mathbf{n}(y), \mathbf{n}(x)\rangle = 0\}$ *is zero.*

THEOREM 3.2. *Suppose that Assumption 3.1 holds. With probability* 1, *for every* $r > 0$, *the limit* $\mathcal{A}_r := \lim_{\varepsilon\to 0} \mathcal{A}_{r,\varepsilon}$ *exists and it is a linear mapping of rank* $n-1$. *For any* $\mathbf{v}_0$, *with probability* 1, $\mathcal{A}_{r,\varepsilon}\mathbf{v}_0 \to \mathcal{A}_r\mathbf{v}_0$ *uniformly on compact sets.*

REMARK 3.3. Intuitively speaking, $\mathcal{A}_r\mathbf{v}_0$ represents the solution to the following ODE, similar to (2.13). Let $\gamma(t) = X(\sigma_t)$, and suppose that $\mathbf{v}_0 \in \mathbf{R}^n$. Consider the following ODE,

$$\mathcal{D}\mathbf{v} = (\mathcal{S} \circ \gamma)\mathbf{v}\, dt, \quad \mathbf{v}(0) = \pi_{x_0}\mathbf{v}_0.$$

Then $\mathcal{A}_r$ is defined by $\mathbf{v}(r) = \mathcal{A}_r\mathbf{v}_0$. We cannot use Theorem 2.6 to justify this definition of $\mathcal{A}_r$ because $\gamma \notin \mathrm{NBV}([0,r];\partial D)$. See [A], [IW1] or [H] for various versions of the above claim with rigorous proofs. Those papers also contain proofs of the fact that $\mathcal{A}_r$ is a multiplicative functional of reflected Brownian motion. This last claim follows directly from our definition of $\mathcal{A}_r$.

In the 2-dimensional case, and only in the 2-dimensional case, we have an alternative intuitive representation of $|\mathcal{A}_r\mathbf{v}_0|$. Let $\mu(x)$ be the signed curvature of ∂D at $x \in \partial D$ with respect to the inward normal; thus $\mu(x)$ is the eigenvalue of $\mathcal{S}(x)$. Then

$$|\mathcal{A}_r\mathbf{v}_0| = \exp\left(-\int_0^r \mu(X_{\sigma_t})\, dL_t\right) \prod_{e_s \in \mathcal{E}_r} |\langle \mathbf{n}(e_s(0)), \mathbf{n}(e_s(\zeta-))\rangle||\mathbf{v}_0|.$$

REMARK 3.4. Recall that B is standard d-dimensional Brownian motion and consider the following stochastic flow,

$$X_t^x = x + B_t + \int_0^t \mathbf{n}(X_s^x)\, dL_s^x \quad \text{for } t \ge 0, \tag{3.3}$$

where L^x is the local time of X^x on ∂D. The results in [LS] are deterministic in nature, so with probability 1, for all $x \in \overline{D}$ simultaneously, (3.3) has a unique pathwise solution (X^x, L^x). In [Bu2], it was proved that for every $r > 0$, a.s., $\lim_{\varepsilon\to 0} \sup_{\mathbf{v}:|\mathbf{v}|\le 1} |(X_{\sigma_r}^{x_0+\varepsilon\mathbf{v}} - X_{\sigma_r}^{x_0})/\varepsilon - \mathcal{A}_r\mathbf{v}| = 0$.

The rest of this section is devoted to the proof of Theorem 3.2. We precede the actual proof with a short review of the excursion theory. See, for example, [M] for the foundations of the theory in the abstract setting and [Bu1] for the special case of excursions of Brownian motion. Although [Bu1] does not discuss reflected Brownian motion, all results we need from that book readily apply in the present context.

An "exit system" for excursions of the reflected Brownian motion X from ∂D is a pair (L_t^*, H^x) consisting of a positive continuous additive functional L_t^* and a family of "excursion laws" $\{H^x\}_{x\in\partial D}$. In fact, $L_t^* = L_t$; see, for example, [BCJ]. Recall that Δ denotes the "cemetery" point outside $\mathbf{R}^n$ and let $\mathcal{C}$ be the space of all functions $f:[0,\infty)\to\mathbf{R}^n\cup\{\Delta\}$ which are continuous and take values in $\mathbf{R}^n$ on some interval $[0,\zeta)$, and are equal to Δ on $[\zeta,\infty)$. For $x\in\partial D$, the excursion law H^x is a σ-finite (positive) measure on $\mathcal{C}$, such that the canonical process is strong Markov on (t_0,∞), for every $t_0>0$, with transition probabilities of Brownian motion killed upon hitting ∂D. Moreover, H^x gives zero mass to paths which do not start from x. We will be concerned only with "standard" excursion laws; see Definition 3.2 of [Bu1]. For every $x\in\partial D$ there exists a unique standard excursion law H^x in D, up to a multiplicative constant.

Recall that excursions of X from ∂D are denoted e or e_s, that is, if $s<u$, $X_s,X_u\in\partial D$, and $X_t\notin\partial D$ for $t\in(s,u)$ then $e_s=\{e_s(t)=X_{t+s},t\in[0,u-s)\}$ and $\zeta(e_s)=u-s$. By convention, $e_s(t)=\Delta$ for $t\ge\zeta$, so $e_t\equiv\Delta$ if $\inf\{s>t: X_s\in\partial D\}=t$.

Recall that $\sigma_t=\inf\{s\ge 0: L_s\ge t\}$ and let I be the set of left endpoints of all connected components of $(0,\infty)\setminus\{t\ge 0: X_t\in\partial D\}$. The following is a special case of the exit system formula of [M],

$$\mathbf{E}\left[\sum_{t\in I}V_t\cdot f(e_t)\right]=\mathbf{E}\int_0^\infty V_{\sigma_s}H^{X(\sigma_s)}(f)\,ds=\mathbf{E}\int_0^\infty V_tH^{X_t}(f)\,dL_t, \tag{3.4}$$

where V_t is a predictable process and $f:\mathcal{C}\to[0,\infty)$ is a universally measurable function which vanishes on excursions e_t identically equal to Δ. Here and elsewhere $H^x(f)=\int_{\mathcal{C}}f\,dH^x$.

The normalization of the exit system is somewhat arbitrary, for example, if (L_t,H^x) is an exit system and $c\in(0,\infty)$ is a constant then $(cL_t,(1/c)H^x)$ is also an exit system. Let $\mathbf{P}_D^y$ denote the distribution of Brownian motion starting from y and killed upon exiting D. Theorem 7.2 of [Bu1] shows how to choose a "canonical" exit system; that theorem is stated for the usual planar Brownian motion but it is easy to check that both the statement and the proof apply to the reflected Brownian motion in $\mathbf{R}^n$. According to that result, we can take L_t^* to be the continuous additive functional whose Revuz measure is a constant multiple of the surface area measure on ∂D and H^x's to be standard excursion laws normalized so that

$$H^x(A)=\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbf{P}_D^{x+\delta\mathbf{n}(x)}(A), \tag{3.5}$$

for any event A in a σ-field generated by the process on an interval $[t_0,\infty)$, for any $t_0>0$. The Revuz measure of L is the measure $dx/(2|D|)$ on ∂D, that is, if the initial distribution of X is the uniform probability measure μ in D then $\mathbf{E}^\mu\int_0^1\mathbf{1}_A(X_s)\,dL_s=\int_A dx/(2|D|)$ for any Borel set $A\subset\partial D$, see Example 5.2.2 of [FOT]. It has been shown in [BCJ] that $(L_t^*,H^x)=(L_t,H^x)$ is an exit system for X in D, assuming the above normalization.

Proof of Theorem 3.2. The overall structure of our argument will be similar to that in the proof of Lemma 2.9.

We will first consider the case $r = 1$. Let $\varepsilon_j = 2^{-j}$, for $j \geq 1$. Fix some j for now and suppose that $\varepsilon' \in [\varepsilon_{j+1}, \varepsilon_j)$. Let

$$\{e_{t_1^j}, e_{t_2^j}, \dots, e_{t_{m_j}^j}\} = \{e \in \mathcal{E}_1 : |e(0) - e(\zeta-)| \geq \varepsilon_j\},$$
$$\{e_{t'_1}, e_{t'_2}, \dots, e_{t'_{m'}}\} = \{e \in \mathcal{E}_1 : |e(0) - e(\zeta-)| \geq \varepsilon'\}.$$

We label the excursions so that $t_k^j < t_{k+1}^j$ for all k and we let $\ell_k^j = L_{t_k^j}$ for $k = 1, \dots, m_j$. Similarly, $t'_k < t'_{k+1}$ for all k and $\ell'_k = L_{t'_k}$ for $k = 1, \dots, m'$. We also let $t_0^j = t'_0 = \inf\{t \geq 0 : X_t \in \partial D\}$, $\ell_0^j = \ell'_0 = 0$, $\ell_{m_j+1}^j = \ell'_{m'+1} = 1$, $\Delta\ell_k^j = \ell_{k+1}^j - \ell_k^j$, and $\Delta\ell'_k = \ell'_{k+1} - \ell'_k$. Let $x_k^j = e_{t_k^j}(\zeta-)$ for $k = 1, \dots, m_j$, and $x'_k = e_{t'_k}(\zeta-)$ for $k = 1, \dots, m'$. Let $x_0^j = X_{t_0^j}$, and $x'_0 = X_{t'_0}$.

Let $\gamma^j(s) = x_k^j$ for $s \in [\ell_k^j, \ell_{k+1}^j)$ and $k = 0, 1, \dots, m_j$, and $\gamma^j(1) = \gamma^j(\ell_{m_j}^j)$. Let $\gamma'(s) = x'_k$ for $s \in [\ell'_k, \ell'_{k+1})$ and $k = 0, 1, \dots, m'$, and $\gamma'(1) = \gamma'(\ell'_{m'})$.

For $\mathbf{v}_0 \in \mathbf{R}^n$, let

$$\mathbf{v}^j = \exp(\Delta\ell_{m_j}^j \mathcal{S}(x_{m_j}^j))\pi_{x_{m_j}^j} \cdots \exp(\Delta\ell_1^j \mathcal{S}(x_1^j))\pi_{x_1^j} \exp(\Delta\ell_0^j \mathcal{S}(x_0^j))\pi_{x_0^j}\mathbf{v}_0,$$
$$\mathbf{v}' = \exp(\Delta\ell'_{m'} \mathcal{S}(x'_{m'}))\pi_{x'_{m'}} \cdots \exp(\Delta\ell'_1 \mathcal{S}(x'_1))\pi_{x'_1} \exp(\Delta\ell'_0 \mathcal{S}(x'_0))\pi_{x'_0}\mathbf{v}_0.$$

Let $0 = \ell_0 < \cdots < \ell_{m+1} = 1$ denote the ordered set of all ℓ_k^j's, $0 \leq k \leq m_j + 1$, and ℓ'_k's, $0 \leq k \leq m' + 1$. In the definition of ℓ_k's, we followed the proof of Lemma 2.9 word by word, for conceptual consistency, although the set of ℓ_k's is the same as the set of ℓ'_k's.

We introduce the following shorthand notations, $\Delta_i = \ell_{i+1} - \ell_i$,

$$\begin{aligned} x_i &= \gamma^j(\ell_i), & \widetilde{x}_i &= \gamma'(\ell_i), \\ \mathcal{S}_i &= \mathcal{S}(x_i), & \widetilde{\mathcal{S}}_i &= \mathcal{S}(\widetilde{x}_i), \\ \pi_i &= \pi_{x_i}, & \widetilde{\pi}_i &= \pi_{\widetilde{x}_i}. \end{aligned}$$

Observing that $\pi_0\widetilde{\pi}_0\mathbf{v}_0 = \widetilde{\pi}_0\mathbf{v}_0$ and $\widetilde{\pi}_{m+1}\mathbf{v}' = \mathbf{v}'$, we can write $\mathbf{v}^j - \mathbf{v}'$ as a telescoping sum:

$$\begin{aligned} \mathbf{v}^j - \mathbf{v}' = \sum_{i=0}^m & e^{\Delta_m \mathcal{S}_m}\pi_m \cdots e^{\Delta_{i+1}\mathcal{S}_{i+1}} \\ & \times \pi_{i+1}(e^{\Delta_i \mathcal{S}_i}\pi_i - \widetilde{\pi}_{i+1}e^{\Delta_i \widetilde{\mathcal{S}}_i})\widetilde{\pi}_i \cdots e^{\Delta_1\widetilde{\mathcal{S}}_1}\widetilde{\pi}_1 e^{\Delta_0\widetilde{\mathcal{S}}_0}\widetilde{\pi}_0\mathbf{v}_0. \end{aligned}$$

By (2.17), the compositions of operators before and after the parentheses in the summation above are uniformly bounded in operator norm by a constant. Therefore, for some c_1 depending only on D,

$$|\mathbf{v}^j - \mathbf{v}'| \leq c_1 \sum_{i=0}^m \|\pi_{i+1} \circ (e^{\Delta_i \mathcal{S}_i} \circ \pi_i - \widetilde{\pi}_{i+1} \circ e^{\Delta_i\widetilde{\mathcal{S}}_i}) \circ \widetilde{\pi}_i\| |\mathbf{v}_0|. \tag{3.6}$$

Using the fact that $\mathcal{S}_i$ and π_i commute, as do $\widetilde{\mathcal{S}}_i$ and $\widetilde{\pi}_i$, we decompose the middle factors as follows:

$$\begin{aligned}\pi_{i+1}\circ(e^{\Delta_i\mathcal{S}_i}\circ\pi_i-\widetilde{\pi}_{i+1}\circ e^{\Delta_i\widetilde{\mathcal{S}}_i})\circ\widetilde{\pi}_i &\\ =\pi_{i+1}\circ\pi_i\circ(e^{\Delta_i\mathcal{S}_i}-e^{\Delta_i\widetilde{\mathcal{S}}_i})\circ\widetilde{\pi}_i &\\ +\pi_{i+1}\circ(\pi_i-\widetilde{\pi}_{i+1})\circ\widetilde{\pi}_i\circ e^{\Delta_i\widetilde{\mathcal{S}}_i}. &\end{aligned}\tag{3.7}$$

We will deal with each of these terms separately.

For the first term, we have by (2.9),

$$\|\pi_{i+1}\circ\pi_i\circ(e^{\Delta_i\mathcal{S}_i}-e^{\Delta_i\widetilde{\mathcal{S}}_i})\circ\widetilde{\pi}_i\|\le\|e^{\Delta_i\mathcal{S}_i}-e^{\Delta_i\widetilde{\mathcal{S}}_i}\|\le c_2\Delta_i|x_i-\widetilde{x}_i|.\tag{3.8}$$

For the second term, Lemma 2.2 and (2.6) allow us to conclude that

$$\begin{aligned}&\|\pi_{i+1}\circ(\pi_i-\widetilde{\pi}_{i+1})\circ\widetilde{\pi}_i\circ e^{\Delta_i\widetilde{\mathcal{S}}_i}\|\\ &\quad\le c_3(|x_{i+1}-x_i||x_i-\widetilde{x}_i|+|x_{i+1}-\widetilde{x}_{i+1}||\widetilde{x}_{i+1}-\widetilde{x}_i|)\|e^{\Delta_i\widetilde{\mathcal{S}}_i}\|\\ &\quad\le c_4(|x_{i+1}-x_i||x_i-\widetilde{x}_i|+|x_{i+1}-\widetilde{x}_{i+1}||\widetilde{x}_{i+1}-\widetilde{x}_i|).\end{aligned}\tag{3.9}$$

We will now estimate $\mathbf{E}\sup_{0\le i\le m}|x_i-\widetilde{x}_i|$. Suppose that $x_i\neq\widetilde{x}_i$ for some i. Then there exist k_1 and k_2 such that $\ell^j_{k_1}<\ell'_{k_2}<\ell^j_{k_1+1}$, $x_i=x^j_{k_1}$, and $\widetilde{x}_i=x'_{k_2}$. Hence,

$$\{|x_i-\widetilde{x}_i|>a\}\subset\bigcup_k\Big\{\sup_{t^j_k+\zeta(e^j_k)<t<t^j_{k+1},X_t\in\partial D}|x^j_k-X_t|>a\Big\}.\tag{3.10}$$

Intuitively speaking, the last condition means that the process X deviates by more than a units from $x^j_{k_1}$ (the right endpoint of an excursion $e_{t^j_{k_1}}$), when X is on the boundary of D, at some time between the lifetime of this excursion and the start of the next excursion in this family, $e_{t^j_{k_1+1}}$.

Since ∂D is C^2, standards estimates (see, e.g., [Bu1]) show that for some $a_0,c_5>0$, all $x\in\partial D$ and $a\in(0,a_0)$,

$$1/(c_5a)\le H^x\big(|e(\zeta-)-x|>a\big)\le c_5/a.\tag{3.11}$$

It follows from this and (3.4) that there exists c_6 so large that for any stopping time T and $a\in(0,a_0)$,

$$\mathbf{P}\big(\exists e_s:|e_s(\zeta-)-e_s(0)|>a,s\in\big(T,\sigma(L_T+c_6a)\big)\big)\ge 3/4.\tag{3.12}$$

Let $\tau_{\mathcal{B}(x,a)}$ be the exit time of X from the ball $\mathcal{B}(x,a)$ in $\mathbf{R}^n$ with center x and radius a. Routine estimates show that for some $c_7,a_1>0$, and all $a\in(0,a_1)$ and $x\in\partial D$,

$$\mathbf{P}^x\big(L\big(\tau_{\mathcal{B}(x,c_7a)}\big)>c_6a\big)>3/4.\tag{3.13}$$

Let $T^j_{k,0}=t^j_k$, and

$$T^j_{k,i+1}=\inf\{t\ge T^j_{k,i}:X(t)\in\partial D,|X(t)-X(T^j_{k,i})|\ge c_7\varepsilon_j\},\quad i\ge 0.$$

According to (3.13), the amount of local time generated on $(T^j_{k,0}, T^j_{k,1})$ will be greater than $c_6\varepsilon_j$ with probability greater than $3/4$. This and (3.12) imply that there exists an excursion e_s with $|e_s(\zeta-) - e_s(0)| > \varepsilon_j$ and $s \in (T^j_{k,0}, T^j_{k,1})$, with probability greater than $1/2$. By the strong Markov property, if there does not exist an excursion e_s with $|e_s(\zeta-) - e_s(0)| > \varepsilon_j$ and $s \in (T^j_{k,0}, T^j_{k,i})$ then there exists an excursion e_s with $|e_s(\zeta-) - e_s(0)| > \varepsilon_j$ and $s \in (T^j_{k,i}, T^j_{k,i+1})$, with probability greater than $1/2$. Let M^j_k be the smallest i with the property that there exists an excursion e_s with $|e_s(\zeta-) - e_s(0)| > \varepsilon_j$ and $s \in (T^j_{k,i}, T^j_{k,i+1})$. We see that M^j_k is majorized by a geometric random variable $\widetilde{M}^j_k$ with mean 2. Note that

$$|X(T^j_{k,i+1}) - X(T^j_{k,i})| \le (c_7 + 1)\varepsilon_j = c_8\varepsilon_j,$$

for $i < M^j_k$. Therefore,

$$\sup_{t^j_k + \zeta(e^j_k) < t < t^j_{k+1}, X_t \in \partial D} |x^j_k - X_t| \le c_8 M^j_k \varepsilon_j. \tag{3.14}$$

It is easy to see, using the strong Markov property at the stopping times t^j_k, that we can assume that all $\{\widetilde{M}^j_k, k \ge 0\}$ are independent.

Consider an arbitrary $\beta_1 < -1$ and let $n_j = \varepsilon_j^{\beta_1}$. For some $c_9 > 0$, not depending on j,

$$\begin{aligned}\mathbf{P}\Big(\max_{1\le k\le n_j} c_8\widetilde{M}^j_k\varepsilon_j \ge c_8 i\varepsilon_j\Big) &= 1 - \big(1 - (1/2)^i\big)^{n_j} \\ &\le \begin{cases} 1, & \text{if } i \le \beta_1 j, \\ c_9 n_j (1/2)^i, & \text{if } i > \beta_1 j.\end{cases}\end{aligned} \tag{3.15}$$

Let ρ_0 be the diameter of D and j_1 be the largest integer smaller than $\log\rho_0$. By (3.15), for any $\beta_2 < 1$, some $c_{12} < \infty$, and all $j \ge j_1$,

$$\begin{aligned}\mathbf{E}\Big(\max_{1\le k\le n_j} c_8 M^j_k\varepsilon_j\Big) &\le \mathbf{E}\Big(\max_{1\le k\le n_j} c_8\widetilde{M}^j_k\varepsilon_j\Big) \\ &\le \sum_{i\le\beta_1 j} c_8 i\varepsilon_j + \sum_{i>\beta_1 j} c_8 i\varepsilon_j c_9 n_j (1/2)^i \\ &\le c_{10}\varepsilon_j(\log\varepsilon_j)^2 + c_{11}\varepsilon_j|\log\varepsilon_j| \le c_{12}\varepsilon_j^{\beta_2}.\end{aligned} \tag{3.16}$$

Let N_ε be the number of excursions e_s with $s \le \sigma_1$ and $|e_s(0) - e_s(\zeta-)| \ge \varepsilon$. For $\varepsilon = \varepsilon_j$, $N_\varepsilon = m_j$. Then (3.4) and (3.11) imply that N_ε is stochastically majorized by a Poisson random variable $\widetilde{N}_\varepsilon$ with mean c_{13}/ε, where $c_{13} < \infty$ does not depend on $\varepsilon > 0$. We have $\mathbf{E}\exp(\widetilde{N}_\varepsilon) = \exp(c_{13}\varepsilon^{-1}(e-1))$, so for any $a > 0$,

$$\mathbf{P}(N_\varepsilon \ge a) \le \mathbf{P}(\widetilde{N}_\varepsilon \ge a) = \mathbf{P}\big(\exp(\widetilde{N}_\varepsilon) \ge \exp(a)\big) \le \exp(c_{14}\varepsilon^{-1} - a).$$

Standard calculations yield the following estimates. For any $\beta_3 < -1$, $\beta_4 < 0$, $\delta_1 > 0$, some $\delta_2 \in (0, \delta_1)$, and all $\delta_3, \delta_4 \in (0, \delta_2)$,

$$\mathbf{P}(N_{\delta_3} \geq \delta_3^{\beta_3}) \leq \delta_3^2, \tag{3.17}$$

and

$$\sup_{\delta_4 \leq \delta \leq \delta_1} \mathbf{E}\big(N_\delta \mathbf{1}_{\{N_\delta \geq \delta^{\beta_3} \delta_4^{\beta_4}\}}\big) \leq \delta_4^2. \tag{3.18}$$

It follows from (3.14), (3.17) and (3.17) that, for any $\beta_2 < 1$, some c_{16}, and $j \geq j_1$,

$$\begin{aligned} &\mathbf{E}\Big(\max_{0 \leq k \leq m_j} \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t|\Big) \\ &\quad \leq \mathbf{E}\Big(\max_{0 \leq k \leq n_j} \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t|\Big) + \rho_0 \mathbf{P}(m_j \geq n_j) \\ &\quad \leq \mathbf{E}\Big(\max_{0 \leq k \leq n_j} c_8 M_k^j \varepsilon_j\Big) + c_{15}\varepsilon_j^2 \\ &\quad \leq c_{12}\varepsilon_j^{\beta_2} + c_{15}\varepsilon_j^2 \leq c_{16}\varepsilon_j^{\beta_2}. \end{aligned} \tag{3.19}$$

Note that $\sum_{i=0}^m \Delta_i = 1$. This, (3.19), (3.8) and (3.10) imply that,

$$\begin{aligned} &\mathbf{E}\left(\sup_{\varepsilon_{j+1} \leq \varepsilon' < \varepsilon_j} \sum_{i=0}^m \|\pi_{i+1} \circ \pi_i \circ (e^{\Delta_i \mathcal{S}_i} - e^{\Delta_i \widetilde{\mathcal{S}}_i}) \circ \widetilde{\pi}_i\|\right) \\ &\quad \leq \mathbf{E}\left(\sup_{\varepsilon_{j+1} \leq \varepsilon' < \varepsilon_j} \sum_{i=0}^m c_2 \Delta_i |x_i - \widetilde{x}_i|\right) \\ &\quad \leq \mathbf{E}\left(\sup_{\varepsilon_{j+1} \leq \varepsilon' < \varepsilon_j} \max_{0 \leq i \leq m} |x_i - \widetilde{x}_i| \sum_{i=0}^m c_2 \Delta_i\right) \\ &\quad = c_2 \mathbf{E}\Big(\sup_{\varepsilon_{j+1} \leq \varepsilon' < \varepsilon_j} \max_{0 \leq i \leq m} |x_i - \widetilde{x}_i|\Big) \\ &\quad \leq c_2 \mathbf{E}\Big(\max_{0 \leq k \leq m_j} \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t|\Big) \leq c_{16}\varepsilon_j^{\beta_2}. \end{aligned} \tag{3.20}$$

We will now estimate the right-hand side of (3.10). We start with an observation similar to (3.10). Suppose that $x_i \neq x_{i+1}$ for some i. Then there exists k_1 such that $x_i = x_{k_1}^j$, and $x_{i+1} = x_{k_1+1}^j$. Note that k_1's corresponding to distinct i's are distinct. Hence,

$$\begin{aligned} &\{|x_i - x_{i+1}| > a\} \\ &\quad \subset \bigcup_k \Big\{\sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t| > a/2\Big\} \\ &\qquad \cup \{|e_{t_{k+1}^j}(0) - e_{t_{k+1}^j}(\zeta-)| > a/2\} \end{aligned} \tag{3.21}$$

$$\subset \bigcup_k \Big\{ \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t| > a/2 \Big\}$$
$$\cup \bigcup_k \{|e_{t_k^{j+1}}(0) - e_{t_k^{j+1}}(\zeta-)| > a/2\}.$$

Similarly, suppose that $\widetilde{x}_i \neq \widetilde{x}_{i+1}$ for some i. Then there exists k_2 such that $\widetilde{x}_i = x'_{k_2}$, and $\widetilde{x}_{i+1} = x'_{k_2+1}$. Again, k_2's corresponding to distinct i's are distinct. Hence,

$$\{|\widetilde{x}_i - \widetilde{x}_{i+1}| > a\} \subset \bigcup_k \Big\{ \sup_{t'_k + \zeta(e'_k) < t < t'_{k+1}, X_t \in \partial D} |x'_k - X_t| > a/2 \Big\}$$
$$\cup \{|e_{t'_{k+1}}(0), e_{t'_{k+1}}(\zeta-)| > a/2\}.$$

Since $\varepsilon_{j+1} \le \varepsilon' < \varepsilon_j$, this implies that,

$$\begin{aligned}
&\{|\widetilde{x}_i - \widetilde{x}_{i+1}| > a\} \\
&\quad \subset \bigcup_{0 \le k \le m_j} \Big\{ \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t| > a/2 \Big\} \\
&\quad\quad \cup \bigcup_{0 \le k \le m_{j+1}} \{|e_{t_k^{j+1}}(0) - e_{t_k^{j+1}}(\zeta-)| > a/2\}.
\end{aligned} \tag{3.22}$$

It follows from (3.10), (3.21) and (3.22) that

$$\begin{aligned}
&\sup_{\varepsilon_{j+1} \le \varepsilon' < \varepsilon_j} \sum_{0 \le i \le m} (|x_{i+1} - x_i||x_i - \widetilde{x}_i| + |x_{i+1} - \widetilde{x}_{i+1}||\widetilde{x}_{i+1} - \widetilde{x}_i|) \\
&\quad \le 4 \sum_{0 \le i \le m} \Big(\max_{0 \le k \le m_j} \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t| \Big)^2 \\
&\quad\quad + 8 \Big(\max_{0 \le k \le m_j} \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t| \Big) \\
&\quad\quad \times \left(\sum_{0 \le k \le m_{j+1}} |e_{t_k^{j+1}}(0) - e_{t_k^{j+1}}(\zeta-)| \right) \\
&\quad = 4(m+1) \Big(\max_{0 \le k \le m_j} \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t| \Big)^2 \\
&\quad\quad + 8 \Big(\max_{0 \le k \le m_j} \sup_{t_k^j + \zeta(e_k^j) < t < t_{k+1}^j, X_t \in \partial D} |x_k^j - X_t| \Big) \\
&\quad\quad \times \left(\sum_{0 \le k \le m_{j+1}} |e_{t_k^{j+1}}(0) - e_{t_k^{j+1}}(\zeta-)| \right).
\end{aligned} \tag{3.23}$$

We have the following estimate, similar to (3.17). For any $\beta_5 < 2$, some $c_{19} < \infty$, and $j \geq j_1$,

$$\begin{aligned}
\mathbf{E}\Big(\max_{1\leq k\leq n_j} c_8 M^j_k \varepsilon_j\Big)^2 &\leq \mathbf{E}\Big(\max_{1\leq k\leq n_j} c_8 \widetilde{M}^j_k \varepsilon_j\Big)^2 \\
&\leq \sum_{i\leq \beta_1 j} (c_8 i\varepsilon_j)^2 + \sum_{i>\beta_1 j} (c_8 i \varepsilon_j)^2 c_9 n_j (1/2)^i \\
&\leq c_{17}\varepsilon_j^2 |\log\varepsilon_j|^3 + c_{18}\varepsilon_j^2(\log\varepsilon_j)^2 \leq c_{19}\varepsilon_j^{\beta_5}.
\end{aligned}\tag{3.24}$$

We now proceed as in (3.19). For any $\beta_5 < 2$ and $j \geq j_1$,

$$\begin{aligned}
&\mathbf{E}\Big(\max_{0\leq k\leq m_j} \sup_{t^j_k+\zeta(e^j_k)<t<t^j_{k+1}, X_t\in\partial D} |x^j_k - X_t|\Big)^2 \\
&\quad\leq \mathbf{E}\Big(\max_{0\leq k\leq n_j} \sup_{t^j_k+\zeta(e^j_k)<t<t^j_{k+1}, X_t\in\partial D} |x^j_k - X_t|\Big)^2 \\
&\qquad + \rho_0^2 \mathbf{P}(m_j \geq n_j) \\
&\quad\leq \mathbf{E}\Big(\max_{0\leq k\leq n_j} c_8 M^j_k\varepsilon_j\Big)^2 + c_{20}\varepsilon_j^2 \\
&\quad\leq c_{19}\varepsilon_j^{\beta_5} + c_{20}\varepsilon_j^2 \leq c_{21}\varepsilon_j^{\beta_5}.
\end{aligned}\tag{3.25}$$

Recall that m is random and note that $m \leq m_{j+1}$. We obtain the following from (3.17) and (3.26), for any $\beta_7 < 1$, by choosing appropriate $\beta_5 < 2$ and $\beta_6 < -1$,

$$\begin{aligned}
&\mathbf{E}\Big((m+1)\Big(\max_{0\leq k\leq m_j} \sup_{t^j_k+\zeta(e^j_k)<t<t^j_{k+1}, X_t\in\partial D} |x^j_k - X_t|\Big)^2\Big) \\
&\quad\leq \mathbf{E}\Big(\varepsilon_j^{\beta_6}\Big(\max_{0\leq k\leq m_j} \sup_{t^j_k+\zeta(e^j_k)<t<t^j_{k+1}, X_t\in\partial D} |x^j_k - X_t|\Big)^2\Big) \\
&\qquad + \rho_0^2\mathbf{P}(m+1 \geq \varepsilon_j^{\beta_6}) \\
&\quad\leq c_{21}\varepsilon_j^{\beta_6+\beta_5} + c_{22}\varepsilon_j^2 \leq c_{23}\varepsilon_j^{\beta_7}.
\end{aligned}\tag{3.26}$$

Next we estimate the second term on the right-hand side of (3.23) as follows. The number of excursions $e_{t^{j+1}_k}$ with $|e_{t^{j+1}_k}(0) - e_{t^{j+1}_k}(\zeta-)| \in [\varepsilon_{i+1}, \varepsilon_i]$ is bounded by m_{i+1}, so

$$\sum_{0\leq k\leq m_{j+1}} |e_{t^{j+1}_k}(0) - e_{t^{j+1}_k}(\zeta-)| \leq \sum_{i=j_1}^{j+1} m_i\varepsilon_{i-1}.$$

Hence, for any $\beta_9 < 0$, we can choose $\beta_8 < 0$, $\beta_1 < -1$ and $c_{23} < \infty$ so that for all $j \geq j_1$,

$$\begin{aligned}
&\Bigl(\max_{0\leq k\leq m_j} \sup_{t_k^j+\zeta(e_k^j)<t<t_{k+1}^j, X_t\in\partial D} |x_k^j - X_t|\Bigr)\Bigl(\sum_{0\leq k\leq m_{j+1}} |e_{t_k^{j+1}}(0) - e_{t_k^{j+1}}(\zeta-)|\Bigr)\\
&\quad\leq \Bigl(\max_{0\leq k\leq m_j} \sup_{t_k^j+\zeta(e_k^j)<t<t_{k+1}^j, X_t\in\partial D} |x_k^j - X_t|\Bigr)\sum_{i=j_1}^{j+1} m_i\varepsilon_{i-1}\\
&\quad\leq \Bigl(\max_{0\leq k\leq m_j} \sup_{t_k^j+\zeta(e_k^j)<t<t_{k+1}^j, X_t\in\partial D} |x_k^j - X_t|\Bigr)\\
&\qquad\times \sum_{i=j_1}^{j+1} \varepsilon_j^{\beta_8} n_i 2\varepsilon_i + \rho_0 \sum_{i=j_1}^{j+1} m_i \mathbf{1}_{\{m_i\geq n_i\varepsilon_j^{\beta_8}\}} 2\varepsilon_i\\
&\quad\leq \Bigl(\max_{0\leq k\leq m_j} \sup_{t_k^j+\zeta(e_k^j)<t<t_{k+1}^j, X_t\in\partial D} |x_k^j - X_t|\Bigr) 2(j-j_1)\varepsilon_j^{\beta_8+\beta_1+1}\\
&\qquad + 2\rho_0 \sum_{i=j_1}^{j+1} m_i \mathbf{1}_{\{m_i\geq n_i\varepsilon_j^{\beta_8}\}}\varepsilon_i\\
&\quad\leq c_{23}\varepsilon_j^{\beta_9}\Bigl(\max_{0\leq k\leq m_j} \sup_{t_k^j+\zeta(e_k^j)<t<t_{k+1}^j, X_t\in\partial D} |x_k^j - X_t|\Bigr)\\
&\qquad + 2\rho_0 \sum_{i=j_1}^{j+1} m_i \mathbf{1}_{\{m_i\geq n_i\varepsilon_j^{\beta_8}\}}\varepsilon_i.
\end{aligned}$$

This, (3.19) and (3.18) imply that for any $\beta_{10} < 1$, by choosing an appropriate $\beta_2 < 1$ and $\beta_8, \beta_9 < 0$, we obtain for some $c_{26} < \infty$ and $j \geq j_1$,

$$\begin{aligned}
&\mathbf{E}\Bigl(\max_{0\leq k\leq m_j} \sup_{t_k^j+\zeta(e_k^j)<t<t_{k+1}^j, X_t\in\partial D} |x_k^j - X_t|\Bigr)\\
&\qquad\times\Bigl(\sum_{0\leq k\leq m_{j+1}} |e_{t_k^{j+1}}(0) - e_{t_k^{j+1}}(\zeta-)|\Bigr)\\
&\quad\leq c_{23}\varepsilon_j^{\beta_9}\mathbf{E}\Bigl(\max_{0\leq k\leq m_j} \sup_{t_k^j+\zeta(e_k^j)<t<t_{k+1}^j, X_t\in\partial D} |x_k^j - X_t|\Bigr)\\
&\qquad + 2\rho_0\mathbf{E}\Biggl(\sum_{i=j_1}^{j+1} m_i \mathbf{1}_{\{m_i\geq n_i\varepsilon_j^{\beta_8}\}}\varepsilon_i\Biggr)\\
&\quad\leq c_{24}\varepsilon_j^{\beta_9}\varepsilon_j^{\beta_2} + c_{25}\sum_{i=j_1}^{j+1}\varepsilon_j^2\varepsilon_i \leq c_{26}\varepsilon_j^{\beta_{10}}.
\end{aligned}\tag{3.27}$$

We combine (3.23), (3.26) and (3.27) to see that for any $\beta_{10} < 1$, some $c_{27} < \infty$ and all $j \ge j_1$,

$$\mathbf{E}\biggl(\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}\sum_{0\le i\le m}(|x_{i+1}-x_i||x_i-\widetilde{x}_i| + |x_{i+1}-\widetilde{x}_{i+1}||\widetilde{x}_{i+1}-\widetilde{x}_i|)\biggr) \le c_{27}\varepsilon_j^{\beta_{10}}.$$

We use this estimate and (3.10) to see that for any $\beta_{10} < 1$, some $c_{27} < \infty$ and all $j \ge j_1$,

$$\begin{aligned}\mathbf{E}\biggl(\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}\sum_{i=0}^{m}\|\pi_{i+1}\circ(\pi_i-\widetilde{\pi}_{i+1})\circ\widetilde{\pi}_i\circ e^{\Delta_i\widetilde{\mathcal{S}}_i}\|\biggr) &\le \mathbf{E}\biggl(\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}\sum_{i=0}^{m}c_4(|x_{i+1}-x_i||x_i-\widetilde{x}_i| \\ &\qquad + |x_{i+1}-\widetilde{x}_{i+1}||\widetilde{x}_{i+1}-\widetilde{x}_i|)\biggr) \\ &\le c_{27}\varepsilon_j^{\beta_{10}}.\end{aligned} \tag{3.28}$$

It follows (3.6), (3.7), (3.20), and (3.28) that for any $\beta_{10} < 1$, some $c_{28} < \infty$ and all $j \ge j_1$,

$$\mathbf{E}\Bigl(\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}|\mathbf{v}^j-\mathbf{v}'|\Bigr) \le c_{28}\varepsilon_j^{\beta_{10}}|\mathbf{v}_0| = c_{28}2^{-\beta_{10}j}|\mathbf{v}_0|.$$

This implies that $\sum_{j\ge j_1}\mathbf{E}(\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}|\mathbf{v}^j-\mathbf{v}'|) < \infty$, and, therefore, a.s.,

$$\sum_{j\ge j_1}\Bigl(\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}|\mathbf{v}^j-\mathbf{v}'|\Bigr) < \infty. \tag{3.29}$$

Note that the definition of $\mathbf{v}'$ given at the beginning of the proof applies in its current form not only to ε' in the range $[\varepsilon_{j+1}, \varepsilon_j)$ but to all $\varepsilon' > 0$. It is elementary to see that (3.29) implies that $\mathbf{v}_1 := \lim_{\varepsilon'\downarrow 0}\mathbf{v}'$ exists. For every $\varepsilon' > 0$, the mapping $\mathbf{v}_0 \to \mathbf{v}'$ is linear, so the same can be said about the mapping $\mathbf{v}_0 \to \mathbf{v}_1 := \mathcal{A}_1\mathbf{v}_0$.

Note that the right-hand side of (3.6) corresponding to $r \in [0, 1)$ is less than or equal to the right-hand side of (3.6) in the case $r = 1$. Hence, we can strengthen (3.29) to the claim that a.s.,

$$\sum_{j\ge j_1}\Bigl(\sup_{0\le r\le 1}\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}|\mathbf{v}_r^j-\mathbf{v}_r'|\Bigr) < \infty,$$

where $\mathbf{v}_r^j$ and $\mathbf{v}_r'$ are defined in a way analogous to $\mathbf{v}^j$ and $\mathbf{v}'$, relative to $r \in [0, 1]$. The analogous argument shows that for any integer $r_0 > 0$, a.s.,

$$\sum_{j\ge j_1}\Bigl(\sup_{0\le r\le r_0}\sup_{\varepsilon_{j+1}\le\varepsilon'<\varepsilon_j}|\mathbf{v}_r^j-\mathbf{v}_r'|\Bigr) < \infty.$$

We use the same argument as above to conclude that for any $\mathbf{v}_0$, with probability 1, $\mathcal{A}_{r,\varepsilon}\mathbf{v}_0 \to \mathcal{A}_r\mathbf{v}_0$ uniformly on compact sets.

It remains to show that $\mathcal{A}_r$ has rank $n-1$. Without loss of generality, we will consider only $r=1$. Recall definition (3.2) of $\mathbf{v}_r$ and note that $\pi_{x_0}\mathbf{v}_0 \in \mathcal{T}_{x_0}\partial D$. It will suffice to show that for any $\mathbf{w} \in \mathcal{T}_{x_0}D$ such that $\mathbf{w} \neq 0$, we have $\mathcal{A}_1\mathbf{w} \neq 0$.

Recall the definition of x_k^j's and related notation from the beginning of the proof. It follows from (2.7) that for some $c_{29} < \infty$ depending only on D, all $x \in \partial D$, $\mathbf{z} \in \mathbf{R}^n$, and all $t \geq 0$, we have $|e^{t\mathcal{S}(x)}\mathbf{z}| \geq e^{-c_{29}t}|\mathbf{z}|$. Therefore, for any $\mathbf{w} \in \mathcal{T}_{x_0}D$,

$$\begin{aligned}|\mathbf{v}^j| &= |\exp(\Delta\ell_{m_j}^j\mathcal{S}(x_{m_j}^j))\pi_{x_{m_j}^j}\cdots\exp(\Delta\ell_1^j\mathcal{S}(x_1^j))\pi_{x_1^j}\exp(\Delta\ell_0^j\mathcal{S}(x_0^j))\pi_{x_0^j}\mathbf{w}|\\ &\geq \exp\left(-c_{29}\sum_{i=0}^{m_j}\Delta\ell_i\right)|\pi_{x_{m_j}^j}\pi_{x_{m_j-1}^j}\cdots\pi_{x_1^j}\pi_{x_0^j}\mathbf{w}|\\ &= c_{30}|\pi_{x_{m_j}^j}\pi_{x_{m_j-1}^j}\cdots\pi_{x_1^j}\pi_{x_0^j}\mathbf{w}|.\end{aligned}$$

It follows that

$$\frac{|\mathbf{v}^j|}{|\mathbf{w}|} = \prod_{k=1}^{m_j}\frac{|\pi_{x_k^j}\cdots\pi_{x_1^j}\pi_{x_0^j}\mathbf{w}|}{|\pi_{x_{k-1}^j}\cdots\pi_{x_1^j}\pi_{x_0^j}\mathbf{w}|},$$

and, therefore,

$$\log|\mathbf{v}^j| = \log|\mathbf{w}| + \sum_{k=1}^{m_j}(\log|\pi_{x_k^j}\cdots\pi_{x_2^j}\pi_{x_1^j}\mathbf{w}| - \log|\pi_{x_{k-1}^j}\cdots\pi_{x_2^j}\pi_{x_1^j}\mathbf{w}|).$$

By the Pythagorean theorem, $|\mathbf{z}|^2 = |\pi_x\mathbf{z}|^2 + \langle\mathbf{z}/|\mathbf{z}|, \mathbf{n}(x)\rangle^2|\mathbf{z}|^2$. This implies that for some $c_{31} < \infty$, if $\mathbf{z} \in \mathcal{T}_y\partial D$ then

$$|\pi_x\mathbf{z}| \geq (1 - c_{31}|x-y|^2)|\mathbf{z}|.$$

Thus we can find $\rho_1 > 0$ so small that for some c_{32} and all $|x-y| \leq \rho_1$ and $\mathbf{z} \in \mathcal{T}_y\partial D$,

$$\log|\pi_x\mathbf{z}| \geq \log|\mathbf{z}| - c_{32}|x-y|^2.$$

Therefore,

$$\begin{aligned}\log|\mathbf{v}^j| \geq{}& \log|\mathbf{w}| - c_{32}\sum_{k=1}^{m_j}|x_k^j - x_{k+1}^j|^2\mathbf{1}_{\{|x_k^j - x_{k+1}^j| \leq \rho_1\}}\\ &+ \sum_{k=1}^{m_j}\left(\mathbf{1}_{\{|x_k^j - x_{k+1}^j| > \rho_1\}}\log\frac{|\pi_{x_k^j}\cdots\pi_{x_1^j}\pi_{x_0^j}\mathbf{w}|}{|\pi_{x_{k-1}^j}\cdots\pi_{x_1^j}\pi_{x_0^j}\mathbf{w}|}\right).\end{aligned} \tag{3.30}$$

We make ρ_1 smaller, if necessary, so that $\rho_1/2 = \varepsilon_{j_2}$ for some integer j_2. Note that the set of excursions $e_{t_k^{j_2}}$ is finite, with cardinality m_{j_2}.

The hitting distribution of ∂D for any excursion law H^x is absolutely continuous with respect to the surface area measure on ∂D, because the same is true for Brownian motion. This, (3.4) and Assumption 3.1 imply that with probability 1, for all $k = 1, 2, \ldots, m_{j_2}$, we have $|\langle \mathbf{n}(e_{t_k^{j_2}}(0)), \mathbf{n}(e_{t_k^{j_2}}(\zeta-))\rangle| > \delta$, for some random $\delta > 0$. For large j, because of continuity of reflected Brownian motion paths, and because excursions are dense in the trajectory, the only points x^j_{k+1} such that $|x^j_k - x^j_{k+1}| > \rho_1$ can be the endpoints of excursions $e_{t_i^{j_2}}$, $i = 1, 2, \ldots, m_{j_2}$.

Fix a point $e_{t_i^{j_2}}$ and let $k(j)$ be such that $x^j_{k(j)} = e_{t_i^{j_2}}$. Then $x^j_{k(j)-1} \to x^j_{k(j)}$ as $j \to \infty$, again by the continuity of reflected Brownian motion, and because excursions are dense in the trajectory. It follows that for large j, for all pairs (x^j_k, x^j_{k+1}) with $|x^j_k - x^j_{k+1}| > \rho_1$, we have $|\langle \mathbf{n}(x^j_k), \mathbf{n}(x^j_{k+1})\rangle| > \delta/2$. This implies that, a.s., for some random $U > -\infty$, and all sufficiently large j,

$$\sum_{k=1}^{m_j} \left(\mathbf{1}_{\{|x^j_k - x^j_{k+1}| > \rho_1\}} \log \frac{|\pi_{x^j_k} \cdots \pi_{x^j_1} \pi_{x^j_0} \mathbf{w}|}{|\pi_{x^j_{k-1}} \cdots \pi_{x^j_1} \pi_{x^j_0} \mathbf{w}|} \right) > U. \tag{3.31}$$

In view of (3.21) and (3.26), for any $\beta_7 < 1$,

$$\begin{aligned} &\mathbf{E}\left(\sum_{i=0}^{m_j} |x^j_i - x^j_{i+1}|^2 \right) \\ &\quad \le 8\mathbf{E}\Big(m_j \Big(\max_{0 \le k \le m_j} \sup_{t^j_k + \zeta(e^j_k) < t < t^j_{k+1}, X_t \in \partial D} |x^j_k - X_t| \Big)^2 \Big) \\ &\qquad + 8\mathbf{E}\left(\sum_{k=1}^{m_j} |e_{t^j_k}(0) - e_{t^j_k}(\zeta-)|^2 \right) \\ &\quad \le c_{23} \varepsilon_j^{\beta_7} + 8\mathbf{E}\left(\sum_{k=1}^{m_j} |e_{t^j_k}(0) - e_{t^j_k}(\zeta-)|^2 \right). \end{aligned} \tag{3.32}$$

By (3.4) and (3.11), the expected number of excursions e_s with $|e_s(\zeta-) - e_s(0)| \in [2^{-i-1}, 2^{-i}]$ and $s \in [0,1]$ is bounded by $c_{33} 2^i$. It follows that for some $c_{34} < \infty$, not depending on j,

$$\mathbf{E}\left(\sum_{k=1}^{m_j} |e_{t^j_k}(0) - e_{t^j_k}(\zeta-)|^2 \right) \le \sum_{i=j_1}^{j} c_{34} 2^{-2i} 2^i < c_{35} < \infty,$$

and this combined with (3.32) yields

$$\sup_{j \ge j_1} \mathbf{E}\left(\sum_{i=0}^{m_j} |x^j_i - x^j_{i+1}|^2 \right) < \infty.$$

In view of (3.30) and (3.31),

$$\liminf_{j\to\infty} \mathbf{E}(\log|\mathbf{v}^j| - U) \geq \log|\mathbf{w}| - \limsup_{j\to\infty} \mathbf{E}\left(c_{32} \sum_{k=1}^{m_j} |x_k^j - x_{k+1}^j|^2 \right) > -\infty,$$

so, with probability 1, $\liminf_{j\to\infty} |\mathbf{v}^j| > 0$, and, therefore, $|\mathbf{v}_1| \neq 0$. □

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