Illinois Journal of Mathematics
Volume 52, Number 4, Winter 2008, Pages 1259–1276
S 0019-2082

INSTABILITY OF STANDING WAVES TO THE INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL

JIANQING CHEN AND YUE LIU

Abstract. We study the instability of standing-wave solutions $e^{i\omega t}\phi_\omega(x)$ to the inhomogeneous nonlinear Schrödinger equation

$$i\varphi_t = -\triangle\varphi + |x|^2\varphi - |x|^b|\varphi|^{p-1}\varphi, \quad x \in \mathbb{R}^N,$$

where $b > 0$ and ϕ_ω is a ground-state solution. The results of the instability of standing-wave solutions reveal a balance between the frequency ω of wave and the power of nonlinearity p for any fixed $b > 0$.

1. Introduction

Considered here is the nonlinear Schrödinger equation with harmonic potential and inhomogeneous nonlinearity (INLS-equation henceforth)

$$i\varphi_t = -\triangle\varphi + |x|^2\varphi - |x|^b|\varphi|^{p-1}\varphi, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

with the initial profile

$$\varphi(x,0) = \varphi_0(x), \tag{1.2}$$

where $\varphi = \varphi(x,t) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{C}$ is a complex-valued function, $N \geq 2$, $\triangle = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$, $p > 1$ and $b \geq 0$.

Equation (1.1) is a model from various physical contexts in the description of nonlinear waves such as propagation of a laser beam and plasma waves. For example, when $b = 0$, it models the magnetic confining trap in the study of the Bose–Einstein condensates (BEC). With $b > 0$ on the inhomogeneous nonlinearity, it can be thought of as modeling inhomogeneities in the medium.

Received November 18, 2007; received in final form August 22, 2008.

J. Chen was supported by Youth Foundation of NSFC (10501006) and by the program for NCETFJ.

2000 *Mathematics Subject Classification.* 35Q55, 35B35, 35A15, 35J20.

The nonlinearity enters due to the effect of changes in the field intensity on the wave propagation characteristics of the medium and the nonlinear weight can be looked as the proportional to the electron density (see [1], [16], [19], [26], [27]). The nonlinearity in the inhomogeneous medium usually can be considered in the form of $f(x,|\varphi|^2)\varphi$ in general, where $f(x,|\varphi|^2)$ is the nonlinear index of refraction which depends on the medium. Berge [4] also studied formally the stability condition for soliton solutions depending on the shape of $f(x,|\varphi|^2)$. In our case, we assume that the preliminary laser beam creates a situation that the nonlinear index of refraction has the form $V(|x|)|\varphi|^{p-1}$ with $V(|x|)$ could be unbounded. In particular, $V(|x|)=|x|^b$ with $b>0$.

The goal of this paper is to derive conditions on ω, p, and b for orbital instability of standing-wave solutions $e^{i\omega t}\phi_\omega(x)$. This type of problem goes back to the works [3], [7], [28] which were concerned only with autonomous versions of Equation (1.1). The autonomous cases are simpler because of the possibility to use dilation invariances. Subsequently, there had been several works for the NLS-equation with harmonic potential (for example, see [13], [24]) with the autonomous nonlinearity and also constant potential but nonautonomous (inhomogeneous in this paper) nonlinearities (see [10], [12], [14], [15], [18], [20]). The present paper is the first one to combine these two cases.

A crucial ingredient to obtain the instability result is the use of a new Gagliardo–Nirenberg type inequality in Lemma 1.1, and it is possible to have more applications to other more general nonlinearities.

To study the INLS-equation, an important issue that is often related to whether or not global existence obtains for arbitrary classes of initial data is the stability of the standing-wave solutions $e^{i\omega t}\phi(x)$ of Equation (1.1), where the localized functions ϕ are called ground-state solutions of Equation (1.6). The orbital and asymptotic stability of these special solutions have been a central theme of development for more than three decades (cf. [2], [5], [7], [17], [25], [28], etc.). For example, when $b=0$, the Cauchy problem (1.1)–(1.2) and the issue of stability of standing waves of the INLS-equation have been studied extensively (cf. [6], [13], [22], [24]). For inhomogeneous nonlinear Schrödinger equation without potential, i.e.,

$$i\varphi_t + \triangle\varphi + K(x)|\varphi|^{p-1}\varphi = 0, \quad x\in\mathbb{R}^N, \tag{1.3}$$

Fibich, Liu, and Wang ([12], [20]) have proved the stability and instability of standing waves of Equation (1.3) for $p\geq 1+4/N$ and $K(\varepsilon|x|)$ with ε small and $K\in C^4(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Merle [21] also showed the existence and nonexistence of blow-up solutions of (1.3) for the critical power $p=1+\frac{4}{N}$. On the other hand, based on Hardy inequality, Fukuizumi and Ohta [14] obtained the result of the instability of standing-wave solutions $e^{i\omega t}\phi_\omega(x)$ of Equation (1.3) for a small $\omega>0$ when the inhomogeneity K of nonlinearity behaves like $|x|^{-b}$ at infinity with $0<b<2$. It was shown in [14] that if

$1+(4-2b)/N<p<1+(4-2b)/(N-2), N\geq 3$, then the standing-wave solutions $e^{i\omega t}\phi_\omega(x)$ of Equation (1.3) are orbital unstable for a sufficiently small $\omega>0$.

More recently, for K that decays at infinity like $|x|^{-b}$ for some $b\in(0,2)$, de Bouard and Fukuizumi [10] use minimization on the Nehari manifold and Jeanjean and Le Coz [18] use a version of the mountain pass theorem to established the stability of the standing-wave solutions $e^{i\omega t}\phi_\omega(x)$ of Equation (1.3) for a small $\omega>0$ when $1<p<1+(4-2b)/N$. These stability and instability results ([10], [14], [18]) are also obtained and improved by Genoud and Stuart [15] through an implicit function theorem to obtain the continuous dependence of the solution ϕ_ω on the small $\omega>0$. However, little is known for the inhomogeneous nonlinear Schrödinger equation (1.1) mainly due to the unbounded coefficient $|x|^b, b>0$ in the nonlinearity.

Recently, we established an improved inequality of Gagliardo–Nirenberg type interpolation (see [9, Theorem 2.3]) to study the Cauchy problem and the existence of standing-wave solutions of the INLS-equation in the case of $b>0$. More precisely, let $N\geq 2$ and $H_r^1(\mathbb{R}^N)=\{u\in H^1(\mathbb{R}^N);\ u(x)=u(|x|)\}$. Define

$$\Sigma=\left\{u\in H^1(\mathbb{R}^N); u(x)=u(|x|), \int |x|^2|u(x)|^2<+\infty\right\}.$$

Then Σ endowed with the inner product

$$\langle u,v\rangle_\Sigma=\mathcal{R}\int(\nabla u\nabla\bar{v}+|x|^2u\bar{v}+u\bar{v})$$

is a Hilbert space whose norm is denoted by

$$\|u\|_\Sigma^2=\int(|\nabla u|^2+|x|^2|u|^2+|u|^2).$$

Define

$$\tilde{p}=\begin{cases}\frac{N+2}{N-2}+\frac{2b}{N-1}, & \text{if } N\geq 3,\\ +\infty, & \text{if } N=2.\end{cases}$$

The following improved inequality of Gagliardo–Nirenberg type interpolation is crucial to establish the existence of the standing-wave solutions for Equation (1.1). We leave the proof of this inequality in the Appendix.

Lemma 1.1. *Assume that $N\geq 2$, $b\geq 0$ and $1+\frac{2b}{N-1}<p<\tilde{p}$. Then there is a constant $C>0$ depending only on $N,p,$ and b such that for any $u\in H_r^1(\mathbb{R}^N)$,*

$$\int |x|^b|u|^{p+1}\leq C\left(\int|\nabla u|^2\right)^{\frac{N(p-1)-2b}{4}}\left(\int|u|^2\right)^{\frac{2(p+1)-(N(p-1)-2b)}{4}}.$$

Using this inequality, one can also obtain [9, Proposition 3.1] the local existence of the Cauchy problem (1.1) and (1.2) in Σ.

PROPOSITION 1.2. *Let $N \geq 2$, $b \geq 0$ and $1 + 2b/(N-1) < p < \tilde{p}$. For any $\varphi_0 \in \Sigma$, there is a $T = T(\|\varphi_0\|_\Sigma) > 0$ and a unique solution φ of (1.1) with $\varphi \in C([0,T),\Sigma)$ and $\varphi(0) = \varphi_0$. Moreover, we have the conserved particle number*

$$\int |\varphi|^2 \equiv \int |\varphi_0|^2 \tag{1.4}$$

and the conserved energy

$$E(\varphi) = \frac{1}{2}\int (|\nabla\varphi|^2 + |x|^2|\varphi|^2) - \frac{1}{p+1}\int |x|^b|\varphi|^{p+1} \equiv E(\varphi_0) \tag{1.5}$$

for all $t \in [0,T)$, where either $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T^-} \|\varphi\|_\Sigma = +\infty$.

The main purpose of the present paper is to determine ω, b, and p such that the standing-wave solutions $e^{i\omega t}\phi(x)$ of INLS-equation are unstable in Σ. Our results reveal that there is a balance among the frequency ω, parameter b related to the unbounded inhomogeneity and the power of nonlinearity p when the instability of standing waves is concerned. We emphasize that the arguments used in [13], [14] cannot be used here due to the unbounded coefficient $|x|^b$ in the nonlinearity.

By a standing wave, we mean a solution of (1.1) with the form

$$\varphi(x,t) = e^{i\omega t}\phi_\omega(x),$$

where $\omega \in \mathbb{R}$ is a given parameter and ϕ_ω is a ground-state solution of the following stationary problem

$$\begin{cases} -\triangle\phi + \omega\phi + |x|^2\phi = |x|^b|\phi|^{p-1}\phi, \\ x \in \mathbb{R}^N, \qquad \phi \in \Sigma, \qquad \phi \not\equiv 0. \end{cases} \tag{1.6}$$

Before stating the main results, we introduce several notations:

$$L_\omega(u) = \frac{1}{2}\int (|\nabla u|^2 + |x|^2|u|^2 + |u|^2) - \frac{1}{p+1}\int |x|^b|u|^{p+1}, \tag{1.7}$$

$$I_\omega(u) = \int (|\nabla u|^2 + |x|^2|u|^2 + \omega|u|^2 - |x|^b|u|^{p+1}), \tag{1.8}$$

$$S_\omega = \{u \in \Sigma; u \neq 0, -\triangle u + \omega u + |x|^2 u = |x|^b|u|^{p-1}u\} \tag{1.9}$$

and

$$G_\omega = \{u \in S_\omega; L_\omega(u) \leq L_\omega(v) \text{ for all } v \in S_\omega\}. \tag{1.10}$$

An element in G_ω is often referred to as a ground state of (1.6), since it minimizes the action $L_\omega(u)$ on S_ω. Please note that with the help of Lemma 1.1, the functional L_ω and I_ω are well defined on Σ. It is shown [9, Theorem 4.2] by Lemma 1.1 that G_ω is not empty for any $\omega > 0$. More precisely, defining the following minimization problem

$$d(\omega) = \inf\{L_\omega(u); u \neq 0, u \in \Sigma,\ I_\omega(u) = 0\},$$

then we have the following lemma.

LEMMA 1.3 ([9, Theorem 4.2]). *Assume* $\omega > 0$, $N \geq 2$ *and* $b \geq 0$. *If* $1 + 2b/(N-1) < p < \tilde{p}$, *then* $d(\omega) > 0$ *and* $d(\omega)$ *is achieved by a ground-state solution* ϕ *of Equation* (1.6).

REMARK. Note that the issue of uniqueness of a ground state is open for this equation in the case of $b > 0$. The frequencies ω could be negative due to the harmonic potential [13] as far as the existence of ground state is concerned, see also [11] for related result.

DEFINITION 1.4. We say that the standing wave $e^{i\omega t}\phi_\omega$ is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $\varphi_0 \in \Sigma$ and $\|\varphi_0 - \phi_\omega\|_\Sigma < \delta$, then

$$\sup_{0 \leq t < \infty} \inf_{\theta \in \mathbb{R}} \|\varphi(t) - e^{i\theta}\phi_\omega\|_\Sigma < \varepsilon,$$

where $\varphi(t)$ is a solution of (1.1) with $\varphi(0) = \varphi_0$. Otherwise, $e^{i\omega t}\phi_\omega$ is said to be unstable.

The main results of the present paper may now be enunciated.

THEOREM 1.5. *Let* $N \geq 2$, $b > 0$ *and* $p_0(N) = (N^2 + 2Nb + 4 + 4\sqrt{N^2 + Nb + 1})/N^2$. *If* $1 + 2b/(N-1) < p < \tilde{p}$ *and* $p_0(N) \leq p < \tilde{p}$, *then the standing wave* $e^{i\omega t}\phi_\omega$ *is unstable for all* $\omega \in (0, +\infty)$, *where* ϕ_ω *is the ground state solution of Equation* (1.6).

Note that $p_0(N)$ does not seem optimal, since $p_0(N) > 1 + (4 + 2b)/N$. In fact, without harmonic potential, the exponent $1 + (4 + 2b)/N$ is optimal [8]. We next have the following theorem.

THEOREM 1.6. *Let* $N \geq 2$, $b > 0$ *and* $\max\{1 + 2b/(N-1), 1 + (4+2b)/N\} < p < \tilde{p}$. *There is* $\omega_* > 0$ *such that for any* $\omega \in (\omega_*, +\infty)$, *the standing wave* $e^{i\omega t}\phi_\omega$ *is unstable for all* $\omega \in (\omega_*, +\infty)$, *where* ϕ_ω *is the ground state solution of Equation* (1.6).

The plan of this paper is as follows. In Section 2, the properties of the ground-state solutions ϕ_ω are described. The main results of the paper are also stated to give focus to the technical developments which follow the general idea of [23] in Section 3 and Section 4, where instabilities are established.

NOTATION. As above and henceforth, we denote the norm of the space $L^q(\mathbb{R}^n)$ by $|\cdot|_q, 1 \leq q \leq \infty$ and denote the integral $\int_{\mathbb{R}^N} dx$ simply by $\int$ unless stated otherwise. We also denote various positive constants by C.

2. Ground-state solutions

In this section, we will give some properties of the ground-state solutions ϕ_ω of Equation (1.6).

LEMMA 2.1. *Let* $\phi_\omega \in G_\omega$, $N \geq 2$, $b > 0$ *and* $\max\{1+2b/(N-1), 1+(4+2b)/N\} < p < \tilde{p}$. *Then we have that*

$$(1) \qquad \int |x|^b |\phi_\omega|^{p+1} = \inf\left\{\int |x|^b |v|^{p+1}; v \neq 0, v \in \Sigma, I_\omega(v) = 0\right\}$$
$$= \inf\left\{\int |x|^b |v|^{p+1}; v \neq 0, v \in \Sigma, I_\omega(v) \leq 0\right\};$$

$$(2) \qquad L_\omega(\phi_\omega) = \inf\left\{L_\omega(v);\, v \in \Sigma, \int |x|^b |v|^{p+1} = \int |x|^b |\phi_\omega|^{p+1}\right\}.$$

REMARK. When $b = 0$, similar results have been proved in [13]. But when $b > 0$, the use of Lemma 1.1 (the improved inequality of Gagliardo–Nirenberg interpolation) is essential. Without Lemma 1.1, the functionals in Lemma 2.1 may not be well defined.

Proof of Lemma 2.1. (1) Since $L_\omega(v) = \frac{1}{2} I_\omega(v) + \frac{p-1}{2(p+1)} \int |x|^b |v|^{p+1}$, we see that

$$d(\omega) = \inf\{L_\omega(v); v \neq 0, v \in \Sigma, I_\omega(v) = 0\}$$
$$= \inf\left\{\frac{p-1}{2(p+1)} \int |x|^b |v|^{p+1}; v \neq 0, v \in \Sigma, I_\omega(v) = 0\right\}.$$

Since $\phi_\omega \in S_\omega$, we know from Equation (1.6) that $I_\omega(\phi_\omega) = 0$. That is

$$\int (|\nabla \phi_\omega|^2 + |x|^2 |\phi_\omega|^2 + \omega |\phi_\omega|^2) = \int |x|^b |\phi_\omega|^{p+1}.$$

Lemma 1.3 implies that $d(\omega) = L_\omega(\phi_\omega)$. Therefore,

$$d(\omega) = L_\omega(\phi_\omega) = \frac{p-1}{2(p+1)} \int |x|^b |\phi_\omega|^{p+1}.$$

Define

$$d_1(\omega) = \inf\left\{\frac{p-1}{2(p+1)} \int |x|^b |v|^{p+1};\ v \neq 0, v \in \Sigma, I_\omega(v) \leq 0\right\}.$$

Clearly, $d_1(\omega) \leq d(\omega)$. It remains to prove that $d(\omega) \leq d_1(\omega)$. Indeed, for any $v \neq 0$ and $I_\omega(v) < 0$, $\lambda > 0$, we have that

$$I(\lambda v) = \lambda^2 \left(\int (|\nabla v|^2 + |x|^2 |v|^2 + \omega |v|^2) - \lambda^{p-1} \int |x|^b |v|^{p+1}\right).$$

Since $p > 1$, we have that $I(\lambda v) \to I(v) < 0$ as $\lambda \to 1$ and $I(\lambda v) > 0$ for $\lambda > 0$ and λ sufficiently small. Therefore, there is $\lambda_0 \in (0,1)$ such that $I_\omega(\lambda_0 v) = 0$. Hence,

$$d(\omega) \leq L_\omega(\lambda_0 v) = \frac{p-1}{2(p+1)} \lambda_0^{p+1} \int |x|^b |v|^{p+1} < \frac{p-1}{2(p+1)} \int |x|^b |v|^{p+1}.$$

It then follows that $d(\omega) \leq d_1(\omega)$. This completes the proof of (1).

(2) We also define

$$d_2(\omega) = \inf\left\{ L_\omega(v);\, v \in \Sigma, \int |x|^b |v|^{p+1} = \int |x|^b |\phi_\omega|^{p+1} \right\}.$$

Since $d_2(\omega) \le L_\omega(\phi_\omega)$, it suffices to prove that $L_\omega(\phi_\omega) \le d_2(\omega)$. For any v satisfying $\int |x|^b|v|^{p+1} = \int |x|^b |\phi_\omega|^{p+1}$, we claim that $I_\omega(v) \ge 0$. Indeed, since $\int |x|^b|v|^{p+1} = \int |x|^b|\phi_\omega|^{p+1}$, $v \ne 0$. If $I_\omega(v) < 0$, then similar to those in the proof of (1), we obtain a $\lambda_1 \in (0,1)$ such that $I_\omega(\lambda_1 v) = 0$. Using the first equality of (1), we have that

$$\int |x|^b|\phi_\omega|^{p+1} \le \int |x|^b |\lambda_1 v|^{p+1} = \lambda_1^{p+1} \int |x|^b|v|^{p+1} = \lambda_1^{p+1} \int |x|^b |\phi_\omega|^{p+1}.$$

This is impossible because of $\lambda_1 \in (0,1)$ and $p > 1$. The claim is proved. Therefore,

$$\begin{aligned} L_\omega(v) &= \frac{1}{2} I_\omega(v) + \frac{p-1}{2(p+1)} \int |x|^b|v|^{p+1} \\ &\ge \frac{p-1}{2(p+1)} \int |x|^b |v|^{p+1} \\ &= \frac{p-1}{2(p+1)} \int |x|^b|\phi_\omega|^{p+1} = L_\omega(\phi_\omega), \end{aligned}$$

which implies that

$$d_2(\omega) \ge L_\omega(\phi_\omega).$$

This completes the proof of (2). □

As in [13], [14], [23], we introduce the following notations.

$$\begin{aligned} v^\lambda(x) &= \lambda^{\frac{N}{2}} v(\lambda x), \quad \lambda > 0,\, v \in \Sigma, \\ \mathcal{N}_\delta(\phi_\omega) &= \left\{ v; \inf\{\|v - e^{i\theta}\phi_\omega\|_\Sigma;\ \theta \in \mathbb{R}\} < \delta \right\}, \quad \delta > 0, \end{aligned}$$

and

$$Q(v) = \int \left(|\nabla v|^2 - |x|^2|v|^2 - \frac{N(p-1) - 2b}{2(p+1)} |x|^b |v|^{p+1} \right), \quad v \in \Sigma.$$

In dealing with the instability issues just raised, the following two lemmas will be useful.

LEMMA 2.2. *If $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then there is positive constants ε_1 and δ_1 with the following property: for any $v \in \mathcal{N}_{\delta_1}(\phi_\omega)$ satisfying $\|v\|_2^2 = \|\phi_\omega\|_2^2$, there exists $\lambda(v) \in (1-\varepsilon_1, 1+\varepsilon_1)$ such that*

$$E(\phi_\omega) \le E(v) + \big(\lambda(v) - 1\big) Q(v).$$

Proof. From the assumption $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ and the continuity of $\partial_\lambda^2 E(v^\lambda)$ in λ and v, there exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that $\partial_\lambda^2 E(v^\lambda) < 0$

for any $\lambda \in (1-\varepsilon_1, 1+\varepsilon_1)$ and $v \in \mathcal{N}_{\delta_1}(\phi_\omega)$. Since $\partial_\lambda E(v^\lambda)|_{\lambda=1} = Q(v)$, the Taylor expansion at $\lambda = 1$ gives

$$E(v^\lambda) \le E(v) + (\lambda - 1)Q(v), \quad \lambda \in (1-\varepsilon_1, 1+\varepsilon_1), v \in \mathcal{N}_{\delta_1}(\phi_\omega).$$

For any $v \in \mathcal{N}_{\delta_1}(\phi_\omega) = \{v; \inf\{\|v - e^{i\theta}\phi_\omega\|_\Sigma; \theta \in \mathbb{R}\} < \delta_1\}$, we put

$$\lambda(v) = \left(\frac{\int |x|^b |\phi_\omega|^{p+1}}{\int |x|^b |v|^{p+1}}\right)^{\frac{2}{N(p-1)-2b}}.$$

Then we have $\int |x|^b |v^{\lambda(v)}|^{p+1} = \int |x|^b |\phi_\omega|^{p+1}$, and we can take δ_1 small enough such that $\lambda(v) \in (1-\varepsilon_1, 1+\varepsilon_1)$. Furthermore, in view of (2) of Lemma 2.1, if $\|v\|_2^2 = \|\phi_\omega\|_2^2$, we have

$$\begin{aligned} E\big(v^{\lambda(v)}\big) &= L_\omega\big(v^{\lambda(v)}\big) - \frac{\omega}{2}\big\|v^{\lambda(v)}\big\|_2^2 \\ &\ge L_\omega(\phi_\omega) - \frac{\omega}{2}\|\phi_\omega\|_2^2 = E(\phi_\omega). \end{aligned}$$

Consequently, we have

$$E(\phi_\omega) \le E(v) + \big(\lambda(v) - 1\big)Q(v)$$

for any $v \in \mathcal{N}_{\delta_1}(\phi_\omega)$ satisfying $\|v\|_2^2 = \|\phi_\omega\|_2^2$. □

Definition. Let δ_1 be the positive constant in Lemma 2.2 and let

$$\mathcal{A} = \{v \in \mathcal{N}_{\delta_1}(\phi_\omega); E(v) < E(\phi_\omega),\ \|v\|_2^2 = \|\phi_\omega\|_2^2, Q(v) < 0\}.$$

For any $\varphi_0 \in \mathcal{N}_{\delta_1}(\phi_\omega)$, we define the exist time from $\mathcal{N}_{\delta_1}(\phi_\omega)$ by

$$T(\varphi_0) = \sup\{T > 0; \varphi(t) \in \mathcal{N}_{\delta_1}(\phi_\omega), 0 \le t \le T\},$$

where $\varphi(t)$ is a solution of (1.1) with $\varphi(0) = \varphi_0$.

Lemma 2.3. *If $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then for any $\varphi_0 \in \mathcal{A}$, there exists $\varepsilon_0 = \varepsilon_0(\varphi_0) > 0$ such that $Q(\varphi(t)) \le -\varepsilon_0$ for $0 \le t < T(\varphi_0)$.*

Proof. Take $\varphi_0 \in \mathcal{A}$ and put $\varepsilon_2 = E(\phi_\omega) - E(\varphi_0) > 0$. In view of Lemma 2.2 and the conserved identities, we have

$$\varepsilon_2 \le \big(\lambda(\varphi(t)) - 1\big)Q(\varphi(t)), \quad 0 \le t < T(\varphi_0). \tag{2.1}$$

Therefore, we see that $Q(\varphi(t)) \ne 0$ for $0 \le t < T(\varphi_0)$. Since the function $t \mapsto Q(\varphi(t))$ is continuous and $Q(\varphi_0) < 0$, we have $Q(\varphi(t)) < 0$ for $0 \le t < T(\varphi_0)$. Now using Lemmas 2.2 and (2.1), we obtain that

$$-Q(\varphi(t)) \ge \frac{\varepsilon_2}{1 - \lambda(\varphi(t))} \ge \frac{\varepsilon_2}{\varepsilon_1}, \quad 0 \le t < T(\varphi_0).$$

So putting $\varepsilon_0 = \varepsilon_2/\varepsilon_1$, we have $Q(\varphi(t)) < -\varepsilon_0$ for $0 \le t < T(\varphi_0)$. The proof of Lemma 2.3 is complete. □

3. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5. The idea is originated from Ohta [23]. More precisely, we will determine ω and p such that $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$, where ϕ_ω is the ground state solution of (1.6). The proof of Theorem 1.5 is approached via the following two lemmas.

LEMMA 3.1. *Let ϕ_ω be the ground-state solution of* (1.6). *If $N \geq 2$, $b > 0$ and* $\max\{1 + 2b/(N-1), 1 + (4+2b)/N\} < p < \tilde{p}$, *then*

$$\int |\nabla\phi_\omega|^2 - \frac{2(p+1) + (N(p-1) - 2b)}{2(p+1) - (N(p-1) - 2b)} \int |x|^2 |\phi_\omega|^2 > 0. \tag{3.1}$$

Proof. Since ϕ_ω is the ground-state solution of (1.6), $I_\omega(\phi_\omega) = 0$ and $Q(\phi_\omega) = 0$. From $Q(\phi_\omega) = 0$, we get that

$$\int |x|^b |\phi_\omega|^{p+1} = \frac{2(p+1)}{N(p-1) - 2b} \int (|\nabla\phi_\omega|^2 - |x|^2|\phi_\omega|^2). \tag{3.2}$$

It is deduced from (3.2) and $I_\omega(\phi_\omega) = 0$ that

$$\left(1 - \frac{2(p+1)}{N(p-1) - 2b}\right) \int |\nabla\phi_\omega|^2 + \left(1 + \frac{2(p+1)}{N(p-1) - 2b}\right) \int |x|^2|\phi_\omega|^2 + \omega \int |\phi_\omega|^2 = 0. \tag{3.3}$$

As $\omega > 0$, the assumption on p implies that

$$\int |\nabla\phi_\omega|^2 - \frac{2(p+1) + (N(p-1) - 2b)}{2(p+1) - (N(p-1) - 2b)} \int |x|^2 |\phi_\omega|^2 > 0.$$

This completes the proof of Lemma 3.1. □

LEMMA 3.2. *Let $N \geq 2$ and $b > 0$. If $p_0(N) \leq p < \tilde{p}$ and $1 + 2b/(N-1) < p < \tilde{p}$, then for any $\omega > 0$,*

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0.$$

Proof. Denote $R(\phi_\omega) = \partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1}$. Then

$$R(\phi_\omega) = \int \left(|\nabla\phi_\omega|^2 + 3|x|^2|\phi_\omega|^2 - \frac{(N(p-1) - 2b)(N(p-1) - 2b - 2)}{4(p+1)} |x|^b |\phi_\omega|^{p+1} \right). \tag{3.4}$$

Since $Q(\phi_\omega^\lambda) = 0$, (3.2) and (3.4) yield that

$$R(\phi_\omega) = \left(1 - \frac{N(p-1) - 2b - 2}{2}\right) \int |\nabla\phi_\omega|^2 + \left(3 + \frac{N(p-1) - 2b - 2}{2}\right) \int |x|^2|\phi_\omega|^2. \tag{3.5}$$

Or, what is the same,

$$R(\phi_\omega) = \frac{4-(N(p-1)-2b)}{2} \times \int \left(|\nabla\phi_\omega|^2 - \frac{N(p-1)-2b+4}{N(p-1)-2b-4}|x|^2|\phi_\omega|^2 \right). \tag{3.6}$$

As $p \geq p_0(N) > 1+(4+2b)/N$, an elementary computation yields

$$\frac{N(p-1)-2b+4}{N(p-1)-2b-4} \leq \frac{2(p+1)+(N(p-1)-2b)}{2(p+1)-(N(p-1)-2b)}. \tag{3.7}$$

It now follows from (3.1) and the inequality $4-(N(p-1)-2b)<0$ that

$$R(\phi_\omega) = \partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$$

as claimed. □

Proof of Theorem 1.5. Since ϕ_ω is the ground state solution of (1.6), it is found from Lemma 3.2 that $Q(\phi_\omega) = \partial_\lambda E(\phi_\omega^\lambda)|_{\lambda=1} = 0$, $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$. Since $Q(\phi_\omega) = \lambda\partial_\lambda E(\phi_\omega^\lambda)$, we have $E(\phi_\omega^\lambda) < E(\phi_\omega)$ and $Q(\phi_\omega^\lambda) < 0$ for $\lambda > 1$ sufficiently close to 1. Furthermore, since $\|\phi_\omega^\lambda\|_2^2 = \|\phi_\omega\|_2^2$ and $\lim_{\lambda\to 1}\|\phi_\omega^\lambda - \phi_\omega\|_\Sigma = 0$, we have $\phi_\omega^\lambda \in \mathcal{A}$ for $\lambda > 1$ sufficiently close to 1. Let $\varphi_\lambda(t)$ be the solution of Equation (1.1) with $\varphi_\lambda(0) = \phi_\omega^\lambda$. Note that $|x|\phi_\omega^\lambda(x) \in L^2(\mathbb{R}^N)$, we obtain from Proposition A.3 (see the Appendix) that

$$\frac{d^2}{dt^2}\|x\varphi_\lambda(t)\|_2^2 = 8Q(\varphi_\lambda(t)), \quad 0 \leq t < T(\phi_\omega^\lambda), \tag{3.8}$$

It then follows from Lemma 2.3 that there is $\varepsilon_\lambda > 0$ such that

$$Q(\varphi_\lambda(t)) \leq -\varepsilon_\lambda, \quad 0 \leq t < T(\phi_\omega^\lambda). \tag{3.9}$$

It is now concluded from (3.8) and (3.9) that $T(\phi_\omega^\lambda) < +\infty$. The proof of Theorem 1.5 is complete. □

4. Proof of Theorem 1.6

Theorem 1.5 seems satisfactory since we got the instability of standing-wave solutions for any $\omega > 0$. However, we need the nonlinear growth slightly large, i.e., $p_0(N) \leq p < \tilde{p}$. As we have pointed out in the introduction, $p_0(N) > 1+(4+2b)/N$ and $p_0(N)$ does not seem optimal. On the other hand, for the inhomogeneous nonlinear Schrödinger equation without potential

$$i\varphi_t = -\triangle\varphi - |x|^b|\varphi|^{p-1}\varphi, \quad x \in \mathbb{R}^N, \tag{4.1}$$

it is known that $1+(4+2b)/N$ is optimal. Namely, for all $\omega > 0$, if $1 < p < 1+(4+2b)/N$, then all the standing waves are stable; and if $p \geq 1+(4+2b)/N$, then all the standing waves are unstable, see [8]. To obtain the optimal result of the instability of standing waves $e^{i\omega t}\phi_\omega(x)$ of Equation (1.1), we need to find a balance between the frequency ω and the nonlinear growth p for any fixed $b > 0$. Our next purpose is to prove that if $1+(4+2b)/N < p < \tilde{p}$, then a sufficient large $\omega > 0$, the standing waves $e^{i\omega t}\phi_\omega(x)$ are unstable, where $\phi_\omega(x)$ are the ground-state solutions of (1.6). Define the rescaled function $\tilde{\phi}_\omega(x)$ as follows:

$$\phi_\omega(x) = \omega^{\frac{2+b}{2(p-1)}}\tilde{\phi}_\omega(\sqrt{\omega}x), \quad \omega > 0. \tag{4.2}$$

Then $\tilde{\phi}_\omega(x)$ satisfies

$$-\triangle\phi + \phi + \omega^{-2}|x|^2\phi = |x|^b|\phi|^{p-1}\phi, \quad \phi \in H_r^1(\mathbb{R}^N). \tag{4.3}$$

Moreover, $\tilde{\phi}_\omega(x)$ are the ground-state solutions of Equation (4.3).

Let $\psi_1(x)$ be the ground-state solution of

$$-\triangle\phi + \phi = |x|^b|\phi|^{p-1}\phi, \quad \phi \in H_r^1(\mathbb{R}^N). \tag{4.4}$$

Define

$$\tilde{I}_\omega(v) = \int(|\nabla v|^2 + \omega^{-2}|x|^2|v|^2 + |v|^2 - |x|^b|v|^{p+1})$$

and

$$I_1^0(v) = \int(|\nabla v|^2 + |v|^2 - |x|^b|v|^{p+1}).$$

It is observed from Lemma 1.1 that $\tilde{I}_\omega$ is well defined on Σ and I_1^0 is well defined on $H_r^1(\mathbb{R}^N)$. Moreover, we have the following lemma.

LEMMA 4.1. *Let $N \geq 2$, $b > 0$ and $1+2b/(N-1) < p < \tilde{p}$. Assume $\phi_\omega \in G_\omega$, $\tilde{\phi}_\omega(x)$ is the rescaled function defined by* (4.2) *and $\psi_1(x)$ is the ground-state solution of Equation* (4.4). *Then we have*

$$\text{(i)} \quad \lim_{\omega\to\infty}\int|x|^b|\tilde{\phi}_\omega|^{p+1} = \int|x|^b|\psi_1|^{p+1},$$

$$\text{(ii)} \quad \lim_{\omega\to\infty} I_1^0(\tilde{\phi}_\omega) = 0,$$

$$\text{(iii)} \quad \lim_{\omega\to\infty}\|\tilde{\phi}_\omega\|_{H_r^1}^2 = \|\psi_1\|_{H_r^1}^2, \quad \textit{and}$$

$$\text{(iv)} \quad \lim_{\omega\to\infty}\omega^{-2}\int|x|^2|\tilde{\phi}_\omega|^2 = 0.$$

Proof. (i) First, we claim that for any $\mu > 1$ there exists $\omega(\mu) > 0$ such that $\tilde{I}_\omega(\mu\psi_1) < 0$ and $I_1^0(\mu\tilde{\phi}_\omega) < 0$ hold for any $\omega \in (\omega(\mu), +\infty)$. Indeed, from $\tilde{I}_\omega(\tilde{\phi}_\omega) = 0$, that is

$$\int(|\nabla\tilde{\phi}_\omega|^2 + \omega^{-2}|x|^2|\tilde{\phi}_\omega|^2 + |\tilde{\phi}_\omega|^2 - |x|^b|\tilde{\phi}_\omega|^{p+1}) = 0,$$

we have that

$$\begin{aligned}\mu^{-2}I_1^0(\mu\tilde{\phi}_\omega) &= \int(|\nabla\tilde{\phi}_\omega|^2 + |\tilde{\phi}_\omega|^2) - \mu^{p-1}\int|x|^b|\tilde{\phi}_\omega|^{p+1} \\ &= -\omega^{-2}\int|x|^2|\tilde{\phi}_\omega|^2 - (\mu^{p-1}-1)\int|x|^b|\tilde{\phi}_\omega|^{p+1} < 0\end{aligned} \tag{4.5}$$

for any $\mu > 1$ and $\omega > 0$. Next, from $I_1^0(\psi_1) = 0$, i.e.,

$$\int (|\nabla \psi_1|^2 + |\psi_1|^2) = \int |x|^b |\psi_1|^{p+1},$$

we know that for any $\mu > 1$,

$$\begin{aligned} \mu^{-2} \tilde{I}_\omega(\mu\psi_1) &= \int (|\nabla \psi_1|^2 + \omega^{-2}|x|^2|\psi_1|^2 + |\psi_1|^2 - \mu^{p-1}|x|^b|\psi_1|^{p+1}) \\ &= \omega^{-2} \int |x|^2|\psi_1|^2 - (\mu^{p-1} - 1) \int |x|^b |\psi_1|^{p+1}. \end{aligned} \tag{4.6}$$

Since ψ_1 is exponentially decay at infinity, we have that

$$\omega^{-2} \int |x|^2 |\psi_1|^2 \to 0 \quad \text{as } \omega \to +\infty.$$

Thus for any $\mu > 1$, there is $\omega(\mu) > 0$ such that for any $\omega \in (\omega(\mu), +\infty)$

$$\tilde{I}_\omega(\mu\psi_1) < 0.$$

This completes the proof of the claim.

Secondly, from the proof of Lemma 2.1, we know that $\tilde{\phi}_\omega(x)$ is a minimizer of

$$\inf \Big\{ \int |x|^b |v|^{p+1}; v \neq 0, v \in \Sigma, \tilde{I}_\omega(v) \leq 0 \Big\} \tag{4.7}$$

and $\psi_1(x)$ is a minimizer of

$$\inf \Big\{ \int |x|^b |v|^{p+1}; v \neq 0, v \in H_r^1(\mathbb{R}^N), I_1^0(v) \leq 0 \Big\}. \tag{4.8}$$

It then follows from (4.7) and (4.8) that

$$\int |x|^b |\tilde{\phi}_\omega|^{p+1} \leq \int |x|^b |\mu\psi_1|^{p+1} = \mu^{p+1} \int |x|^b |\psi_1|^{p+1}$$

and

$$\int |x|^b |\psi_1|^{p+1} \leq \int |x|^b |\mu\tilde{\phi}_\omega|^{p+1} = \mu^{p+1} \int |x|^b |\tilde{\phi}_\omega|^{p+1},$$

which imply that

$$\frac{1}{\mu^{p+1}} \int |x|^b |\psi_1|^{p+1} \leq \int |x|^b |\tilde{\phi}_\omega|^{p+1} \leq \mu^{p+1} \int |x|^b |\psi_1|^{p+1}, \quad \omega \in (\omega(\mu), +\infty).$$

Since $\mu > 1$ is arbitrary, we conclude (i).

For (ii), by (4.5) with $\mu = 1$ and (i), we have that

$$I_1^0(\tilde{\phi}_\omega) = -\omega^{-2} \int |x|^2 |\tilde{\phi}_\omega|^2 < 0 \quad \text{for any } \omega > 0.$$

Since

$$I_1^0(\lambda\tilde{\phi}_\omega) = \lambda^2 \int (|\nabla \tilde{\phi}_\omega|^2 + |\tilde{\phi}_\omega|^2) - \lambda^{p+1} \int |x|^b |\tilde{\phi}_\omega|^{p+1} > 0$$

for $\lambda > 0$, sufficiently small, it follows that for any $\omega > 0$ there is $\mu(\omega) \in (0,1)$ such that

$$I_1^0(\mu(\omega)\tilde{\phi}_\omega) = 0.$$

In particular,

$$\limsup_{\omega\to+\infty} \mu(\omega) \le 1.$$

On the other hand, it is found from the proof of Lemma 2.1 that

$$\int |x|^b|\psi_1|^{p+1} \le \int |x|^b|\mu(\omega)\tilde{\phi}_\omega|^{p+1} = \mu(\omega)^{p+1}\int |x|^b|\tilde{\phi}_\omega|^{p+1},$$

which together with (i) implies that

$$\liminf_{\omega\to\infty} \mu(\omega) \ge \liminf_{\omega\to\infty} \frac{\int |x|^b|\psi_1|^{p+1}}{\int |x|^b|\tilde{\phi}_\omega|^{p+1}} = 1.$$

This in turn transpires that $\lim_{\omega\to+\infty}\mu(\omega) = 1$. It is now deduced from $I_1^0(\mu(\omega)\tilde{\phi}_\omega) = 0$ that

$$\lim_{\omega\to\infty} I_1^0(\tilde{\phi}_\omega) = 0.$$

Thus, we conclude (ii).

Next, from (i), (ii), and $I_1^0(\psi_1) = 0$, we have that

$$\lim_{\omega\to\infty} \|\tilde{\phi}_\omega\|^2_{H^1_r} = \lim_{\omega\to\infty}\int |x|^b|\tilde{\phi}_\omega|^{p+1} = \int |x|^b|\psi_1|^{p+1} = \|\psi_1\|^2_{H^1_r},$$

which yields (iii).

Finally, it follows from (ii) and $\tilde{I}_\omega(\tilde{\phi}_\omega) = 0$ that

$$\lim_{\omega\to\infty} \omega^{-2}\int |x|^2|\tilde{\phi}_\omega|^2 = 0,$$

which proves (iv). The proof of the lemma is complete. □

Proof of Theorem 1.6. In view of the proof of Theorem 1.5, it suffices to prove that there exists $\omega_* > 0$, such that for any $\omega \in (\omega_*, +\infty)$, $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$. To this end, we firstly apply (3.4) and $Q(\phi_\omega) = 0$ to obtain that

$$R(\phi_\omega) = 4\int |x|^2|\phi_\omega|^2 + \frac{(N(p-1)-2b)}{2(p+1)} \times \left(1 - \frac{(N(p-1)-2b-2)}{2}\right)\int |x|^b|\phi_\omega|^{p+1}. \tag{4.9}$$

Since

$$\int |x|^2|\phi_\omega|^2 = \omega^{\frac{2+b}{p-1}-\frac{N}{2}-1}\int |x|^2|\tilde{\phi}_\omega|^2$$

and

$$\int |x|^b|\phi_\omega|^{p+1} = \omega^{\frac{(2+b)(p+1)}{2(p-1)}-\frac{N}{2}-\frac{b}{2}}\int |x|^b|\tilde{\phi}_\omega|^{p+1},$$

it follows that

$$\frac{\int |x|^2|\phi_\omega|^2}{\int |x|^b|\phi_\omega|^{p+1}} = \frac{\omega^{-2}\int |x|^2|\tilde{\phi}_\omega|^2}{\int |x|^b|\tilde{\phi}_\omega|^{p+1}}.$$

In view of (i) and (iv) of Lemma 4.1, we deduce that

$$\lim_{\omega\to+\infty} \frac{\int |x|^2|\phi_\omega|^2}{\int |x|^b|\phi_\omega|^{p+1}} = 0.$$

In consequence, there is $\omega_* > 0$ such that for all $\omega \in (\omega_*, +\infty)$,

$$R(\phi_\omega) < 0$$

because of $1 - (N(p-1) - 2b - 2)/2 < 0$. That is for all $\omega \in (\omega_*, +\infty)$,

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0.$$

This completes the proof of Theorem 1.6. □

Appendix

In this section, we first give a detailed proof of Lemma 1.1. The following two lemmas are useful.

Lemma A.1 (Strauss' inequality). *Let $N \geq 2$. For any $u \in H_r^1$, there is a constant $C_N > 0$ such that*

$$|x|^{\frac{N-1}{2}}|u(x)| \leq C_N \left(\int |u|^2\right)^{\frac{1}{4}} \left(\int |\nabla u|^2\right)^{\frac{1}{4}} \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{A.1}$$

Lemma A.2 (Gagliardo–Nirenberg inequality). *Let $1 < q < q^*$, where $q^* = \frac{N+2}{N-2}$ when $N \geq 3$ and $q^* = +\infty$ when $N = 2$. Then there is a positive constant C such that for any $u \in H^1(\mathbb{R}^N)$,*

$$\int |u|^{q+1} \leq C\left(\int |\nabla u|^2\right)^{\frac{N(q-1)}{4}} \left(\int |u|^2\right)^{\frac{2(q+1)-N(q-1)}{4}}. \tag{A.2}$$

Proof of Lemma 1.1. By Lemma A.1, we have

$$\begin{aligned} \int |x|^b|u|^{p+1} &= \int \left(|x|^{\frac{N-1}{2}}|u(x)|\right)^{\frac{2b}{N-1}} |u|^{(p+1)-\frac{2b}{N-1}} \\ &\leq C_N \left(\int |u|^2\right)^{\frac{b}{2(N-1)}} \left(\int |\nabla u|^2\right)^{\frac{b}{2(N-1)}} \\ &\quad \times \int |u|^{(p+1)-\frac{2b}{N-1}}. \end{aligned} \tag{A.3}$$

On the other hand, since $1+\frac{2b}{N-1}<p<\tilde{p}$, we have that $1<p-\frac{2b}{N-1}<q^*$. It is deduced from Lemma A.2 that

$$\begin{aligned}(A.4)\quad &\int |u|^{(p+1)-\frac{2b}{N-1}}\\ &\le C\Big(\int|\nabla u|^2\Big)^{\frac{N(p-\frac{2b}{N-1}-1)}{4}}\Big(\int|u|^2\Big)^{\frac{2(p-\frac{2b}{N-1}+1)-N(p-\frac{2b}{N-1}-1)}{4}}.\end{aligned}$$

Since

$$\frac{b}{2(N-1)}+\frac{N(p-\frac{2b}{N-1}-1)}{4}=\frac{N(p-1)-2b}{4}$$

and

$$\begin{aligned}&\frac{b}{2(N-1)}+\frac{2(p-\frac{2b}{N-1}+1)-N(p-\frac{2b}{N-1}-1)}{4}\\ &\quad=\frac{2(p+1)-(N(p-1)-2b)}{4},\end{aligned}$$

we obtain from (A.3) and (A.4) that

$$\int |x|^b|u|^{p+1}\le C\Big(\int|\nabla u|^2\Big)^{\frac{N(p-1)-2b}{4}}\Big(\int|u|^2\Big)^{\frac{2(p+1)-(N(p-1)-2b)}{4}}. \qquad \square$$

Now, we are going to prove the following virial identity.

PROPOSITION A.3. *Let $\varphi(t)\in C^1([0,T(\varphi_0)),\Sigma)$ be a solution of Equation (1.1) with initial value $\varphi(0)=\varphi_0(x)\in\Sigma$. Then one has*

$$(A.5)\qquad \frac{d^2}{dt^2}\|x\varphi(t)\|_2^2=8Q(\varphi(t)),\quad 0\le t<T(\varphi_0).$$

Proof. We only prove Equation (A.5) formally. Since φ satisfies Equation (1.1), we have that

$$\varphi_t=i(\triangle\varphi-|x|^2\varphi+|x|^b|\varphi|^{p-1}\varphi).$$

Therefore,

$$\frac{d}{dt}\|x\varphi(t)\|_2^2=2\,\mathrm{Re}\int|x|^2\bar\varphi\varphi_t=4\,\mathrm{Im}\int\bar\varphi x\nabla\varphi$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\|x\varphi(t)\|_2^2&=4\,\mathrm{Im}\int(\bar\varphi_t x\nabla\varphi+\bar\varphi x\nabla\varphi_t)\\ &=4\,\mathrm{Im}\int\bar\varphi_t x\nabla\varphi-4\,\mathrm{Im}\int\varphi_t(N\bar\varphi+x\nabla\bar\varphi)\end{aligned}$$

$$
\begin{aligned}
&= -4\,\mathrm{Im}\int \varphi_t(N\bar{\varphi}+2x\nabla\bar{\varphi})\\
&= -4\,\mathrm{Re}\int (N\bar{\varphi}+2x\nabla\bar{\varphi})(\triangle\varphi-|x|^2\varphi+|x|^b|\varphi|^{p-1}\varphi).
\end{aligned}
$$

Direct computations show that

$$
\mathrm{Re}\int (N\bar{\varphi}+2x\nabla\bar{\varphi})\triangle\varphi = -2\int|\nabla\varphi|^2;
$$

$$
\mathrm{Re}\int (N\bar{\varphi}+2x\nabla\bar{\varphi})|x|^2\varphi = -2\int|x|^2|\varphi|^2;
$$

$$
\begin{aligned}
&\mathrm{Re}\int (N\bar{\varphi}+2x\nabla\bar{\varphi})|x|^b|\varphi|^{p-1}\varphi\\
&= N\int_{\mathbb{R}^N}\int |x|^b|\varphi|^{p+1} + \mathrm{Re}\int 2x|x|^b|\varphi|^{p-1}\varphi\nabla\bar{\varphi}\\
&= N\int_{\mathbb{R}^N}\int |x|^b|\varphi|^{p+1} + \frac{2}{p+1}\int x|x|^b\nabla(|\varphi|^{p+1})\\
&= N\int_{\mathbb{R}^N}\int |x|^b|\varphi|^{p+1} - \frac{2}{p+1}\int |\varphi|^{p+1}(N|x|^b+b|x|^b)\\
&= \frac{N(p-1)-2b}{p+1}\int |x|^b|\varphi|^{p+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d^2}{dt^2}\|x\varphi(t)\|_2^2 &= 8\left(\int(|\nabla\varphi|^2-|x|^2|\varphi|^2) - \frac{N(p-1)-2b}{2(p+1)}\int|x|^b|\varphi|^{p+1}\right)\\
&= 8Q(\varphi(t)). \qquad \square
\end{aligned}
$$

Acknowledgments. The authors thank the referees for valuable suggestions and comments.

References

[1] G. Baym and C. J. Pethick, *Ground state properties of magnetically trapped Bose–Einstein condensate rubidium gas*, Phys. Rev. Lett. **76** (1996), 6–9.

[2] T. B. Benjamin, *The stability of solitary waves*, Proc. R. Soc. Lond. Ser. A **328** (1972), 153–183. MR 0338584

[3] H. Berestycki and T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein–Gordon non linéarires*, C. R. Acad. Sci. Paris I **293** (1981), 489–492. MR 0646873

[4] L. Berge, *Soliton stability versus collapse*, Phys. Rev. E **62** (2000), R3071–R3074. MR 1788946

[5] J. L. Bona, *On the stability theory of solitary waves*, Proc. R. Soc. Lond. Ser. A **344** (1975), 363–374. MR 0386438

[6] T. Cazenave, *An introduction to nonlinear Schrödinger equations*, Textos de metodos matematicos, vol. 22, Instituto de Matemática, UFRJ, 1996.

[7] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561. MR 0677997

[8] J. Chen and B. Guo, *Sharp global existence and blowing up results for inhomogeneous Schrödinger equations*, Discrete Contin. Dyn. Syst. Ser. B **8** (2007), 357–367. MR 2317813

[9] J. Chen and B. Guo, *An interpolation inequality and applications to the inhomogeneous nonlinear Schrödinger equation with harmonic potential*, submitted.

[10] A. de Bouard and F. Fukuizumi, *Stability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities*, Ann. Henri Poincaré **6** (2005), 1157–1177. MR 2189380

[11] W. Ding and W.-M. Ni, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Ration. Mech. Anal. **91** (1986), 283–308. MR 0807816

[12] G. Fibich and X. P. Wang, *Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearity*, Phys. D **175** (2003), 96–108. MR 1957907

[13] R. Fukuizumi, *Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential*, Discrete Contin. Dyn. Syst. **7** (2001), 525–544. MR 1815766

[14] R. Fukuizumi and M. Ohta, *Instability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities*, J. Math. Kyoto Univ. **45** (2005), 145–158. MR 2138804

[15] F. Genoud and C. Stuart, *Schrödinger equations with a spatially decaying nonlinearity: Existence and stability of standing waves*, preprint. MR 2379460

[16] T. S. Gill, *Optical guiding of laser beam in nonuniform plasma*, Pramana Journal of Physics **55** (2000), 845–852.

[17] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry, I*, J. Funct. Anal. **74** (1987), 160–197. MR 0901236

[18] L. Jeanjean and S. Le Coz, *An existence and stability result for standing waves of nonlinear Schrödinger equations*, Adv. Differential Equations **11** (2006), 813–840. MR 2236583

[19] C. S. Liu and V. K. Tripathi, *Laser guiding in an axially nonuniform plasma channel*, Phys. Plasmas **1** (1994), 3100–3103.

[20] Y. Liu, X. P. Wang and K. Wang, *Instability of standing waves of the Schrödinger equations with inhomogeneous nonlinearity*, Trans. Amer. Math. Soc. **358** (2006), 2105–2122. MR 2197450

[21] F. Merle, *Nonexistence of minimal blow up solutions of equations* $iu_t = -\triangle u - K(x)|u|^{4/N}u$ *in* $\mathbb{R}^N$, Ann. Inst. H. Poincaré Phys. Théor. **64** (1996), 33–85. MR 1378233

[22] Y.-G. Oh, *Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equation with potentials*, J. Differential Equations **81** (1989), 255–274. MR 1016082

[23] M. Ohta, *Instability of standing waves for the generalized Davey–Stewartson system*, Ann. Inst. H. Poincaré Phys. Théor. **62** (1995), 69–80. MR 1313361

[24] H. A. Rose and M. I. Weinstein, *On the bound states of the nonlinear Schrödinger equation with linear potential*, Phys. D **30** (1988), 207–218. MR 0939275

[25] J. Shatah and W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), 173–190. MR 0804458

[26] T. Tsurumi and M. Waditi, *Collapses of wave functions in multidimensional nonlinear Schrödinger equations under harmonic potential*, J. Phys. Soc. Japan **66** (1997), 3031–3034.

[27] T. Tsurumi and M. Waditi, *Instability of the Bose–Einstein condensate under magnetic trap*, J. Phys. Soc. Japan **66** (1997), 3035–3039.

[28] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1983), 567–576. MR 0691044

Jianqing Chen, School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, P.R. China

Yue Liu, Department of Mathematics, University of Texas, Arlington, TX 76019, USA

E-mail address: yliu@uta.edu