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INVARIANT SUBLATTICES

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ABSTRACT. This paper is concerned with the problem of existence of invariant sublattices for a positive matrix or a positive operator on $L_p(\mu)$. Common invariant sublattices for certain semigroups of positive operators are constructed. The paper also provides extensions of Perron–Frobenius theorem.

0. Preliminaries and notation

This paper investigates a special case of the *invariant subspace problem.* Suppose that X is a Banach space and T is a (bounded linear) operator on X, that is, $T \in L(X)$. A closed subspace Y of X is said to be *invariant* under T if $T(Y) \subseteq Y$. To make exposition simpler, whenever we mention an invariant subspace, we will always assume that it is nonzero and proper unless we specify otherwise. It is known [Enf76], [Enf87], [Rea84] that there exist operators on some infinite-dimensional Banach spaces with no invariant closed subspaces. On the other hand, since every matrix has a complex eigenvalue, it follows that if $\dim X < \infty$ and either X is complex or X is real with $\dim X > 2$, then every operator on X has an invariant subspace.

In this paper, the symbol X will usually stand for a real Banach lattice. Given an operator T on X, one can investigate the existence of invariant subspaces of T satisfying some additional conditions related to the order structure of X. In particular, one may be concerned with the existence of ideals or sublattices of X invariant under T. Recall that a subspace E of X is a *sublattice* if it is closed under the lattice operations. That is, for any $x, y \in E$, we have $x \wedge y$ and $x \vee y$ belong to E. It follows that x^+, x^-, and $|x|$ are in E. A subspace E of a Banach lattice X is an *ideal* if $x \in E$ and $|y| \leq |x|$ imply $y \in E$.

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It is easy to see that every ideal is a sublattice. In ℓ_p with $1 \le p < \infty$, the closed ideals are exactly the subspaces of the form $[e_i]_{i \in A}$ where $A \subseteq \mathbb{N}$ (here (e_i) stands for the canonical basis of ℓ_p; throughout the paper, $[x_i]_{i \in A}$ stands for the closed linear span of $\{x_i\}_{i \in A}$). In particular, this gives a complete description of ideals in $\mathbb{R}^n$.

In Section 1, we discuss characterizations of operators with no invariant ideals. We then proceed to describe among operators with no invariant ideals those which have invariant sublattices.

Recall that an operator T on a Banach lattice is said to be positive if $Tx \ge 0$ whenever $x \ge 0$. For matrices, this is equivalent to all the entries of the matrix being nonnegative. This is also true for operators on ℓ_p or c_0. Indeed, if T is an operator on ℓ_p or c_0, we can view it as an infinite matrix with entries $t_{ij} = (Te_j)_i$. In this case, again, $T \ge 0$ if and only if $t_{ij} \ge 0$ for all $i, j \in \mathbb{N}$.

Note that every positive (finite) matrix has an invariant sublattice. Indeed, suppose that A is a positive matrix in $M_n(\mathbb{R})$. If it has an invariant ideal, we are done. Otherwise, the Perron–Frobenius theorem (see, e.g., Corollary 5.2.3 in [RR00]) guarantees that A has a unique positive eigenvector x, and the corresponding eigenvalue equals the spectral radius $r(A)$ of A. Then $[x]$ is a one-dimensional A-invariant sublattice.

The situation is similar for a positive compact operator T on an arbitrary Banach lattice. Indeed, if T is quasinilpotent, then it has an invariant closed ideal; see [dP86]. On the other hand, if T is not quasinilpotent, then its spectral radius is a positive real number, and is actually an eigenvalue corresponding to a positive eigenvector, see [KR48]. In either case, T has an invariant sublattice.

The situation is quite different if we drop either the condition that T is positive or the condition that T is compact. In a recent paper [KW], the authors present several examples of positive operators on Banach lattices with no invariant sublattices. In Section 2, we present a few examples of nonpositive operators with no invariant sublattices.

In Section 3, we provide a complete characterization of matrices that have invariant sublattices. In Section 4, we investigate the structure of semigroups of positive matrices with no invariant ideals. In particular, Theorem 4.7 provides sufficient conditions for such a semigroup to have an invariant sublattice. In Section 5, we present infinite-dimensional versions of the results of the preceding sections in ℓ_p, c_0, and $L_p(\mu)$.

For a vector x in $\mathbb{R}^n$, ℓ_p, or c_0, we will write $\operatorname{supp} x$ for the support of x, i.e., $\operatorname{supp} x = \{i \in \mathbb{N} \mid x_i \ne 0\}$. We say that x is strictly positive if $x_n > 0$ for every n. For an element x in a Banach lattice, we write $x > 0$ if $x \ge 0$ but $x \ne 0$; we follow the same convention for matrices and operators. Following [RR00], whenever we consider $L_p(\mu)$, μ will stand for a σ-finite regular Borel measure on a Hausdorff–Lindelöf space, and $1 \le p < \infty$. A *projection* is an idempotent operator. A collection $\mathcal{A}$ of matrices is said to satisfy some property *up to a permutation* if there exists a permutation matrix P such that $P^{-1}\mathcal{A}P$ satisfies the required property.

For convenience of the reader, in the rest of this section, we collect a few known results that we will be using throughout the paper. We give corresponding theorem numbers in [RR00]. We should warn the reader that our notation differs considerably from the one used in [RR00], as we use terminology commonly accepted in the literature on Banach lattices.

THEOREM 0.1 (Perron–Frobenius, 5.2.13). *If A is a positive matrix in $M_n(\mathbb{R})$ then its spectral radius $r(A)$ is an eigenvalue corresponding to a positive eigenvector. Moreover, if A has no invariant ideals, then this eigenvector is strictly positive and unique up to scaling.*

LEMMA 0.2 ([Mar99], [Mar02], 5.1.5, 8.7.6). *For a semigroup $\mathcal{S}$ of positive operators in $M_n(\mathbb{R})$ or on $L_p(\mu)$, the following statements are equivalent:*

(i) *$\mathcal{S}$ has an invariant closed ideals;*
(ii) *$A\mathcal{S}B=\{0\}$ for some nonzero positive operators A and B;*
(iii) *Some nonzero semigroup ideal in $\mathcal{S}$ has an invariant closed ideals.*

THEOREM 0.3 ([Mar99], [Mar02], 5.1.13, 8.7.27). *Every semigroup of positive projections of finite rank on $L_p(\mu)$ with minimal rank greater than one has a closed invariant ideal.*

THEOREM 0.4 ([Zho93], [Mar99], [Mar02], 5.1.9, 8.7.12, 9.4.10). *Let $\mathcal{S}$ be a semigroup of positive projections in $M_n(\mathbb{R})$ of constant rank r. Then, up to a permutation, $\mathcal{S}$ has a common block form, so that all the non-zero $A\in\mathcal{S}$ are of one of the following forms:*

$$A=\begin{bmatrix}0 & YE & YEX\\ 0 & E & EX\\ 0 & 0 & 0\end{bmatrix},\quad or\quad A=\begin{bmatrix}0 & YE\\ 0 & E\end{bmatrix},\quad or\quad A=\begin{bmatrix}E & EX\\ 0 & 0\end{bmatrix},\quad or$$
$$A=E,$$

where X and Y are some matrices and

$$E=\begin{bmatrix}E_1 & & \\ & \ddots & \\ & & E_k\end{bmatrix},$$

with each E_i of the form $x_i\otimes y_i$ where x_i and y_i are strictly positive vectors with $\langle x_i,y_i\rangle=1$. Moreover, if no row or column is zero in all the matrices in $\mathcal{S}$, then we have the case $A=E$.

We say that $\mathcal{S}$ a semigroup of positive operators on a Banach lattice is $\mathbb{R}^+$*-closed* if $\mathcal{S}$ is norm closed and $\alpha A\in\mathcal{S}$ whenever $A\in\mathcal{S}$ and $\alpha\in\mathbb{R}_+$. If P and Q are two projections in $\mathcal{S}$, we write $P\le Q$ if $\operatorname{Range}P\subseteq\operatorname{Range}Q$ and

$\ker P \supseteq \ker Q$. A non-zero projection P in $\mathcal{S}$ is said to be *minimal* if it is minimal with respect to this order or, equivalently, if $PE = EP = E$ implies $E = P$ for every nonzero projection in $\mathcal{S}$.

THEOREM 0.5 ([Rad99], 5.2.2, 8.7.17, 5.2.6, 8.7.20). *Let $\mathcal{S}$ be an $\mathbb{R}^+$-closed semigroup of positive compact operators on $L_p(\mu)$ or of positive matrices in $M_n(\mathbb{R})$ such that $\mathcal{S}$ has no invariant closed ideals. Then:*

(i) (a) *The minimal rank r of operators in $\mathcal{S}$ is finite;*
  (b) *A projection P in $\mathcal{S}$ is minimal iff* $\operatorname{rank} P = r$*;*
  (c) *For each $A \in \mathcal{S}$ of* $\operatorname{rank} A = r$ *there exists a minimal projection P in $\mathcal{S}$ such that $PA = A$.*

(ii) *For each vector $x > 0$ there exists a minimal projection P in $\mathcal{S}$ such that $Px \neq 0$; for each functional $\phi > 0$ there exists a minimal projection $P \in \mathcal{S}$ such that ϕ in non-zero on* $\operatorname{Range} P$*. In particular, in the finite-dimensional case, with $\mathcal{S}$ viewed as a subset of $M_n(\mathbb{R})$, for each i there exists a minimal projection in $\mathcal{S}$ whose ith row is nonzero; same for the columns.*

(iii) *If all the minimal projections in $\mathcal{S}$ have the same range, then there exists an almost everywhere positive vector x such that $Ax = r(A)x$ for all $A \in \mathcal{S}$. This vector is unique up to scaling.*

Theorem 0.5(iii) yields the following extension of Theorem 0.1.

COROLLARY 0.6 ([Rad99], 8.7.23). *Let $\mathcal{S}$ be a commutative semigroup of positive compact operators on $L_p(\mu)$ such that $\mathcal{S}$ has no invariant closed ideals. Then there exists an almost everywhere positive vector x such that $Ax = r(A)x$ for all $A \in \mathcal{S}$. This vector is unique up to scaling.*

THEOREM 0.7 ([Rad85], 7.4.5). *Let $\mathcal{S}$ be an $\mathbb{R}^+$-closed semigroup of compact operators on a Banach space. If $\mathcal{S}$ contains a nonquasinilpotent operator, then $\mathcal{S}$ contains a non-zero finite-rank operator which is either a projection or a nilpotent operator of index 2.*

THEOREM 0.8 ([Tur99], 8.1.11). *Every semigroup of compact quasinilpotent operators on a Banach space has an invariant closed subspace.*

THEOREM 0.9 ([Drn01], 8.7.9). *Every semigroup of positive compact quasinilpotent operators on $L_p(\mu)$ has an invariant closed ideal.*

1. Invariant ideals

It is easy to construct an operator with no invariant ideals, even in the finite-dimensional case. For example, it is trivial that every matrix $A \in M_n(\mathbb{R})$ such that $a_{ij} > 0$ for every i, j has no invariant ideals. Similarly, suppose that E is a Banach lattice with a strong order unit u, and f is a strictly positive functional in E^*, i.e., $f(x) > 0$ whenever $x > 0$; then the rank-one operator $f \otimes u$ has no invariant ideals.

The following well-known fact follows immediately from the preceding description of ideals in $\mathbb{R}^n$.

PROPOSITION 1.1. *A positive matrix A has an invariant ideal if and only if, up to a permutation, A is of the form $\left[\begin{smallmatrix} B & C \\ 0 & D \end{smallmatrix}\right]$, where B and D are square matrices.*

It is easy to see that Proposition 1.1 remains valid for positive operators on ℓ_p spaces. Namely, such an operator has an invariant closed ideal if, after a permutation of the basis, it can be written in the block form $\left[\begin{smallmatrix} B & C \\ 0 & D \end{smallmatrix}\right]$.

We proceed to another characterization of positive operators on ℓ_p spaces with no invariant closed ideals. Let T be an operator on ℓ_p, $1 \leq p < \infty$ and let (t_{ij}) be the infinite matrix of T, i.e., $t_{ij} = (Te_j)_i$. Fix two positive integers i and j. We say that there is an *arc* from i to j (write $i \to j$), if $(Te_i)_j = t_{ji} \neq 0$. We say that there is a *path* of length n from i to j if there is a sequence of n arcs, going from i to j:

$$i = k_0 \to k_1 \to \cdots \to k_n = j,$$

or, equivalently, if $(T^n e_i)_j \neq 0$. In other words, after applying T n times some weight from the ith coordinate ends up at the jth coordinate.

PROPOSITION 1.2. *Let T be a positive operator on ℓ_p, $1 \leq p < \infty$ or c_0. Then T has no closed invariant ideals if and only if for every two different positive integers i and j there is a finite path from i to j.*

Proof. Suppose that there is a finite path between every two indices. Let V be a closed T-invariant ideal, then there is a positive vector x in V. Since $x_i > 0$ for some index i, we have $e_i \in V$. By hypothesis, there is a finite path form i to j for every index j, so that $(T^n e_i)_j > 0$. But since V is T-invariant we have $T^n e_i \in V$, and we conclude that $e_j \in V$ for every index j. Thus, V is the whole space.

Now, assume that there is no finite path from i to j. We would like to find an invariant closed ideal. If $Te_i = 0$ then $[e_i]$ is such an ideal. Suppose $Te_i \neq 0$, then the set of all positive integers k such that there is a path from i to k is nonempty; call this set A. Then $j \notin A$, so that $V = [e_k]_{k \in A}$ is a proper nontrivial closed ideal in ℓ_p. Finally, show that V is T-invariant. Let $k \in A$, so that there is a finite path from i to k. Notice that if $(Te_k)_m \neq 0$ for some m then there is an arc from k to m. It follows that there is a finite path from i to m and, therefore, $m \in A$. Therefore, $Te_k \in V$, hence $T(V) \subseteq V$. □

A review of results about the existence of closed invariant ideals for positive quasinilpotent operators on Banach lattices can be found in [AA02]. In particular, every positive quasinilpotent operator on ℓ_p with $1 \leq p < \infty$ has a closed invariant ideal.

2. Special classes of operators with no invariant sublattices

In this section, we present a few simple examples of operators which have no invariant subspaces containing positive vectors. It should be clear that such operators have no invariant sublattices.

PROPOSITION 2.1. *For every $n \geq 2$, there exists a matrix $A \in M_n(\mathbb{R})$ such that no proper invariant subspace of A contains a positive vector.*

Proof. Let N be the nilpotent forward shift operator, that is,

$$Ne_i = \begin{cases} e_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases}$$

Let U be a unitary matrix such that $Ue_1 = (e_1 + \cdots + e_n)/\sqrt{n}$. Clearly, such a matrix exists. Since N and, therefore, UNU^{-1} is nilpotent of order n, it is unicellular (that is, the set of its invariant subspaces is totally ordered by inclusion). Therefore, all the invariant subspaces of UNU^{-1} are contained in $\operatorname{Range} UNU^{-1}$. Since U^{-1} is a bijection, we have

$$M := \operatorname{Range} UNU^{-1} = \operatorname{Range} UN = U(\operatorname{Range} N) = U[e_i]_{i=2}^n = (Ue_1)^\perp.$$

It follows that M is a 1-codimensional subspace consisting exactly of the vectors whose coordinates sum up to zero. In particular, M contains no positive vectors. $\square$

PROPOSITION 2.2. *There exist operators in L_2 and in ℓ_2 none of whose proper invariant subspaces contain a positive vectors.*

Proof. Let T be the adjoint Donahue operator on ℓ_2 given by $Te_n = e_{n+1}/n$ for all $n \in \mathbb{N}$. Then T is quasinilpotent, and $\operatorname{Lat} T$ consists exactly of the subspaces of the form $[e_i]_{i=m}^\infty$, for some $m \in \mathbb{N}$; see [RR03] for details. Let $M = [e_i]_{i=2}^\infty$. Then M is of codimension one and is the greatest proper invariant subspace of T.

Let f be a strictly positive functional on L_2 or on ℓ_2, and let $N = \ker f$. Then N is a subspace of codimension one, containing no positive vectors. There is a unitary U from L_2 to ℓ_2 or from ℓ_2 to ℓ_2, respectively, such that $N = U^{-1}MU$. It follows that N is the greatest proper invariant subspace of $U^{-1}TU$, so that no invariant subspace of $U^{-1}TU$ contains positive vectors. $\square$

Next, we consider signed cyclic permutation. By a *signed cyclic permutation* on $\mathbb{R}^n$, we mean an operator A of the form $Ae_i = \pm e_{\sigma(i)}$, where σ is a permutation of $\{1, \ldots, n\}$ of order n. It is easy to see that up to a permutation corresponding to σ, we have $Ae_i = \pm e_{(i+1) \bmod n}$ as $1 \leq i \leq n$.

Proposition 2.3. *Suppose that $A \in M_n$ is a matrix given by*

$$Ae_i = \begin{cases} e_{i+1} & \text{if } 1 \le i \le k, \\ -e_{(i+1) \bmod n} & \text{if } k < i \le n, \end{cases}$$

for some $k < n$. Then A has no invariant sublattices.

Proof. Note that $|A|$ is the forward shift operator given by $Se_i = e_{(i+1) \bmod n}$ for all $i \le n$. Suppose that M is a nontrivial invariant sublattice for A. We claim that $\mathbb{1}$ is in M. Indeed, M contains a positive vector, say $x = (\alpha_1, \dots, \alpha_n)^T$. Let $h = \sum_{i=0}^{n-1} |A^i x|$; then h is in M. It follows from $|A^j x| = S^j x$ that every coordinate of h equals $\alpha_1 + \dots + \alpha_n$, hence h is a nonzero multiple of $\mathbb{1}$, so that $\mathbb{1} \in M$.

Let $z_1 = \mathbb{1}$. By the preceding argument, we have $z_1 \in M$. Then

$$Az_1 = \big(-1, \underbrace{1, \dots, 1}_{k}, -1, \dots, -1\big)^T.$$

Let $z_2 = (Az_1)^+$, then $z_2 \in M$,

$$Az_2 = \big(0, 0, \underbrace{1, \dots, 1}_{k-1}, -1, 0, \dots, 0\big)^T.$$

Put $z_3 = (Az_2)^+$. Proceeding like this, we get $z_k = e_k \in M$. It follows immediately that $e_i \in M$ for all i, so that $M = X$. □

3. Matrices with no invariant sublattices

Recall that in Section 1 we described the matrices that have invariant ideals. In this section, we provide a complete characterization of matrices that have no invariant ideals, but have invariant sublattices. We start by characterizing the sublattices of $\mathbb{R}^n$.

Lemma 3.1. *Every sublattice of $\mathbb{R}^n$ has a basis consisting of positive pairwise disjoint vectors.*

Proof. Suppose that L is a sublattice of $\mathbb{R}^n$ and let $\{z_1, \dots, z_m\}$ be a set of pairwise disjoint positive vectors in L of maximal cardinality. We will show that $L = [z_k]_{k=1}^m$. Suppose not, and take any $y \in L \setminus [z_k]_{k=1}^m$. Then either y^+ or y^- fails to be in $[z_k]_{k=1}^m$, so that we can assume that $y \ge 0$. Notice that $\operatorname{supp} y \subseteq \bigcup_{k=1}^m \operatorname{supp} z_k$, as, otherwise, $(y - \lambda(z_1 + \dots + z_m))^+$ is a nonzero element of L for every $\lambda \in \mathbb{R}_+$; but it is disjoint from every z_k for a sufficiently large $\lambda \in \mathbb{R}_+$. This would contradict the maximality of $\{z_1, \dots, z_m\}$.

It follows from $\operatorname{supp} y \subseteq \bigcup_{k=1}^m \operatorname{supp} z_k$ that $y = P_1 y + \dots + P_m y$ where P_k is the standard projection onto $\operatorname{supp} z_k$. Then $P_k y$ is not a multiple of z_k for some $k \le m$. Without loss of generality, $k = m$. For a sufficiently large $\lambda \in \mathbb{R}_+$ the support of $x = (y - \lambda(z_1 + \dots + z_{m-1}))^+$ is contained in $\operatorname{supp} z_m$. Since $x = P_m x = P_m y$, it follows that x is not a multiple of z_m. Then we can find a real $\mu \geq 0$ such that if we put $u = (x - \mu z_m)^+$ then $u \neq 0$ and $\operatorname{supp} u \subsetneq \operatorname{supp} z_m$. It follows that $u \perp (z_m - \nu u)^+$ for a sufficiently large $\nu \in \mathbb{R}_+$. Put $v = (z_m - \nu u)^+$; then $v \perp u$ and $\operatorname{supp} v \subset \operatorname{supp} z_m$. Then $\{z_1, \dots, z_{m-1}, u, v\}$ is a set of pairwise disjoint positive vectors in L; a contradiction. □

Note that this lemma may be viewed as a special case of [AB99, Theorem 12.11].

Suppose that L is a sublattice of $\mathbb{R}^n$ and $z_1, \dots, z_m$ are as in Lemma 3.1. We can find a permutation matrix P such that $Pz_1, \dots, Pz_n$ have consecutive supports, i.e., $\min \operatorname{supp} z_1 = 1$ and $1 + \max \operatorname{supp} z_k = \min \operatorname{supp} z_{k+1}$ for all $k < m$. Recall that a vector v is a *component of* $\mathbb{1}$ if $v_i \in \{0, 1\}$ for all i. We can find a diagonal matrix D with all diagonal entries strictly positive such that DPz_k is a component of $\mathbb{1}$ for every $k = 1, \dots, m$. Thus, we have the following characterization.

COROLLARY 3.2. *If L is a sublattice of $\mathbb{R}^n$, then up to a permutation and a positive diagonal similarity, L has a basis $(z_k)_{k=1}^m$ such that z_k's are components of $\mathbb{1}$ with successive supports.*

COROLLARY 3.3. *Every sublattice L of $\mathbb{R}^n$ is, up to a permutation and a positive diagonal similarity, the range of a projection of the following form:*

$$E = \widetilde{E} \quad or \quad \begin{bmatrix} \widetilde{E} & 0 \\ 0 & 0 \end{bmatrix}, \quad such\ that \quad \widetilde{E} = \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_m \end{bmatrix},$$

where

$$K_k = \frac{1}{l_k} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

and l_k is the size of K_k.

Proof. Find a basis of L as in Corollary 3.2, and take l_k to be the cardinality of $\operatorname{supp} z_k$ □

The following lemma is standard.

LEMMA 3.4. *Let X be a Banach space, P a projection on X, and A any operator on X. Then* $\operatorname{Range} P$ *is invariant under A iff $AE = EAE$.*

We say that a matrix is *constant-row* if the entries of every row add up to the same number. We say that the matrix is *zero-row* if this number is zero (for all the rows). If this number is 1 and the matrix is nonnegative, we say that the matrix is *row-stochastic.*

REMARK 3.5. Next, we describe the general structure of a matrix with an invariant sublattice. Suppose that $A \in M_n$ and L is a proper invariant sublattice. Let E, $\widetilde{E}$, and $K_1, \dots, K_m$ be as in Corollary 3.3. By Lemma 3.4, we have $AE = EAE$. Write A in block form $A = (A_{i,j})$ matching the block form of E given in Corollary 3.3. This block form is $m \times m$ if $E = \widetilde{E}$ and $(m+1) \times (m+1)$ if $E = \left[\begin{smallmatrix} \widetilde{E} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Expanding $AE = EAE$, we get $K_i A_{i,j} K_j = A_{i,j} K_j$ for all $i, j \le m$. Recalling the structure of K_i and K_j, it follows that all the rows of $A_{i,j}$ have identical averages, hence $A_{i,j}$ is a constant-row matrix. Furthermore, if the block form is $(m+1) \times (m+1)$, then we also have $A_{m+1,j} K_j = 0$ for all $j \le m$, hence all the rows of $A_{m+1,j}$ have zero sums, so that $A_{m+1,j}$ is zero-row. Summarizing, after a permutation and a diagonal similarity, one can write A in a block form such that $A_{i,j}$ is constant-row for all $i, j \le m$, and, if the block form is $(m+1) \times (m+1)$, then all $A_{m+1,j}$ are zero-row for all $j \le m$. Conversely, it is easy to see that if A is a matrix with such structure then $AE = EAE$ and, therefore, L is invariant under A. Hence, we obtained a characterization of matrices leaving L invariant. The trivial case when all the blocks are 1×1 corresponds to the situation when L is the entire space.

Now suppose that, in addition, A is positive and have no invariant ideals. It follows that the block form above is $m \times m$ because otherwise $A_{m+1,j} = 0$ for each $1 \le j \le m$, but then A would have an invariant ideal by Proposition 1.1. Hence, we end up with the following simpler characterization.

THEOREM 3.6. *Suppose that $A \in M_n$ is positive and has no invariant ideals. Then A has an invariant sublattice if and only if A can be written, up to a permutation and a positive diagonal similarity, in a block form where each block is a constant-row matrix, the diagonal blocks are square, and not all the blocks are* 1×1.

Recall that a positive matrix with no invariant ideals must always have at least one invariant sublattice; namely, the one-dimensional sublattice spanned by the positive eigenvector, whose existence is guaranteed by Theorem 0.1.

THEOREM 3.7. *Suppose A is a positive matrix with no invariant ideals. Then A has a one-dimensional invariant sublattice which is contained in every invariant sublattice. Furthermore, if the spectral radius of A is 1, then the block form of A given by Theorem 3.6 can be chosen to be row-stochastic.*

Proof. Theorem 0.1 guarantees that A has a strictly positive eigenvector h, corresponding to $r(A)$, unique up to scaling. Hence, h spans a one-dimensional invariant sublattice. We will show next that this sublattice is contained in every invariant sublattice. Indeed, let L be an invariant sublattice for A. Find a basis $(z_k)_{k=1}^m$ as in Corollary 3.2 (after a permutation and a positive diagonal similarity). Then $\bigvee_{k=1}^m z_k = \mathbb{1}$ as, otherwise, the ideal generated by

L would be proper and invariant under A. As in the proof of Theorem 3.6, we conclude that the blocks of A corresponding to the supports of $z_1,\dots,z_m$ are constant-row.

Let A_0 be the matrix of the restriction of A to L with respect to $(z_k)_{k=1}^m$; then $A_0 \geq 0$. Note that the restriction has no invariant ideals in L as such an ideal would generate an A-invariant ideal in the entire space. Again, Theorem 0.1 yields that A_0 and, therefore, A have a strictly positive eigenvector in L. Since h is the unique (up to scaling) strictly positive eigenvector of A, it follows that $h \in L$. Thus, h is contained in every A-invariant sublattice.

It also follows that $h = \sum_{i=1}^m \gamma_i z_i$ for some positive $\gamma_1,\dots,\gamma_m$. Let

$$D = \begin{bmatrix} \gamma_1 I_1 & & & \\ & \gamma_2 I_2 & & \\ & & \ddots & \\ & & & \gamma_m I_m \end{bmatrix},$$

where I_k is the identity matrix of dimension $\#(\operatorname{supp} z_k)$ for $k = 1,\dots,m$. Then $D\mathbb{1} = h$. If the spectral radius of A is one, then $Ah = h$, so that $D^{-1}AD\mathbb{1} = \mathbb{1}$, hence $D^{-1}AD$ is row-stochastic. It should be clear that the block structure of $D^{-1}AD$ is the same as that of A, and the blocks of $D^{-1}AD$ are still constant-row. □

THEOREM 3.8. *There is a positive matrix with no invariant ideals and exactly one invariant sublattice.*

Proof. Let $0 < r_1 < \cdots < r_n < \frac{1}{2}$. For $i = 1,\dots,n$ choose $R_i > 0$ so that the matrix

$$A = \begin{bmatrix} R_1 & r_1 & r_1^2 & \cdots & r_1^{n-1} \\ R_2 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ \vdots & & & & \vdots \\ R_n & r_n & r_n^2 & \cdots & r_n^{n-1} \end{bmatrix}$$

is row stochastic. Clearly, A has no invariant ideals and $\mathbb{1}$ is the Perron–Frobenius positive eigenvector of A. We will use Theorems 3.6 and 3.7 to show that the span of $\mathbb{1}$ is the only invariant sublattice of A. Indeed, suppose that L is an invariant sublattice. Theorem 3.7 yields $\mathbb{1} \in L$. It follows that there is a basis of L consisting of components of $\mathbb{1}$, so that we can write A in a constant-row block form as in Theorems 3.6 and 3.7 using only a permutation, without a diagonal similarity. If there is only one $n \times n$ block then we are done. Suppose that there is a proper constant-row block made up of the entries in rows $i_1 < \cdots < i_m$ and columns $j_1 < \cdots < j_k$ of the original matrix. Without loss of generality, we can assume that $m > 1$ and $j_1 \neq 1$; otherwise consider another block in the same row of blocks. Then the sums of the first two rows of this block satisfy

$$r_{i_1}^{j_1-1} + \cdots + r_{i_1}^{j_k-1} < r_{i_2}^{j_1-1} + \cdots + r_{i_2}^{j_k-1},$$

which contradicts the block being constant-row. □

Next, we will present another corollary of Remark 3.5. An $n \times n$ matrix A is a *signed permutation matrix* if $Ae_i = \pm e_{\sigma(i)}$, where σ is a permutation of $\{1, \dots, n\}$.

COROLLARY 3.9. *Suppose that A is a signed permutation matrix with no invariant ideals. Then A has an invariant sublattice if and only if A is of the following block form up to a permutation*

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & \pm S \\ \pm I & 0 & \dots & 0 & 0 \\ 0 & \pm I & \ddots & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & & \pm I & 0 \end{bmatrix}, \quad \text{where } S = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}, \tag{1}$$

where all the blocks are square and are of the same size greater than one.

Proof. Suppose that $Ae_i = \pm e_{\sigma(i)}$, where σ is a permutation of $\{1, \dots, n\}$. Since A has no invariant ideals, σ is of order n, so that A is a signed cyclic permutation. Suppose that A has an invariant sublattice; then we can apply Remark 3.5. Since, even after a permutation and a positive diagonal similarity, A will still have exactly one nonzero entry in every row, the block form in Remark 3.5 has to be $m \times m$. Furthermore, only one block in every row of blocks is nonzero. For the same reason, there cannot be any block with its vertical dimension exceeding its horizontal dimension. But this implies that all the blocks are square. It follows, furthermore, that all the blocks must have the same dimension, say k, and $k > 1$. Then n has to be divisible by k; let $m = \frac{n}{k}$. Thus, the pattern of nonzero blocks in A follows an $m \times m$ permutation matrix P. Again, since A has no invariant ideals, P must be given by a permutation of order m. In other words, up to another permutation, we may assume that $P = S$, the forward shift matrix, so that

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_{1,m} \\ A_{2,1} & 0 & & \vdots & 0 \\ 0 & A_{3,2} & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & A_{m,m-1} & 0 \end{bmatrix}.$$

It is clear that the positive diagonal similarity involved in the beginning of the proof could have been chosen to be the identity, so that each nonzero block of A is either a permutation matrix or the negation of a permutation matrix. Therefore, up to yet another permutation, A is of the form (1) with block size greater than one.

On the other hand, if after a permutation A can be written as in (1) then A has an invariant sublattice by Remark 3.5. $\square$

It follows that under the hypotheses of Corollary 3.9 if A has an invariant sublattice then the number m of minus signs in A has a common divisor with n. This observation leads to the following question: is the condition $\gcd(m,n)>1$ sufficient for A having an invariant sublattice?

4. Semigroups of positive matrices

We start by extending Theorem 0.3 to semigroups in $M_n(\mathbb{R})$ containing zero.

LEMMA 4.1. *Let $\mathcal{S}$ be a semigroup of positive projections in M_n. If $PQ=0$ for some non-zero $P,Q\in\mathcal{S}$ then $\mathcal{S}$ has a closed invariant ideal.*

Proof. Assume that there exist nonzero $P,Q\in\mathcal{S}$ such that $PQ=0$. For every $A\in\mathcal{S}$, we have $(QAP)^2=0$, so QAP is a nilpotent projection, hence $QAP=0$. Thus, $Q\mathcal{S}P=0$, so that $\mathcal{S}$ has an invariant ideal by Lemma 0.2. $\square$

REMARK 4.2. In a similar fashion, one can show that the preceding lemma remains valid for a semigroup of positive projection on $L_p(\mu)$.

LEMMA 4.3. *Let $\mathcal{S}$ be a semigroup of positive projection in M_n such that the minimal rank of nonzero members of $\mathcal{S}$ is greater than one. Then $\mathcal{S}$ has an invariant ideal.*

Proof. If $PQ=0$ for some nonzero $P,Q\in\mathcal{S}$ then $\mathcal{S}$ has an invariant ideal by Lemma 4.1. Otherwise, the result follows from Theorem 0.3 applied to $\mathcal{S}\setminus\{0\}$. $\square$

REMARK 4.4. Suppose that P is a projection in M_n written in a block upper-triangular form; then $\operatorname{rank}P=\operatorname{trace}P$ implies that $\operatorname{rank}P$ is the sum of the ranks of the diagonal blocks.

We now extend Theorem 0.4 to semigroups containing zero.

THEOREM 4.5. *Let $\mathcal{S}$ be a semigroup of positive projections in $M_n(\mathbb{R})$ such that for any P and Q in $\mathcal{S}$ either $PQ=0$ or P and Q have the same rank r. Then there exists s such that, up to a permutation of the basis, $\mathcal{S}$ has a common block form, so that all the nonzero $A\in\mathcal{S}$ are of one of the following forms:*

$$A=\begin{bmatrix}0 & YE & YEX\\ 0 & E & EX\\ 0 & 0 & 0\end{bmatrix},\quad \textit{or}\quad A=\begin{bmatrix}0 & YE\\ 0 & E\end{bmatrix},\quad \textit{or}\quad A=\begin{bmatrix}E & EX\\ 0 & 0\end{bmatrix},\quad \textit{or}$$
$$A=E,$$

where X and Y are some matrices and E has the following description:

$$E = \begin{bmatrix} E_1 & & \\ & \ddots & \\ & & E_{rs} \end{bmatrix},$$

and there exists $0 \le t < s$ such that $E_{tr+1}, \dots, E_{(t+1)r}$ are of rank one and are the only nonzero blocks of E. Moreover, if no row or column is zero in all the matrices in $\mathcal{S}$, then we have the case $A = E$.

Proof. Take a maximal chain of invariant ideals of $\mathcal{S}$, and consider the block upper-triangular form of $\mathcal{S}$ corresponding to this chain. Let J be the difference between two consecutive ideals in the chain. For $P \in \mathcal{S}$, write P_J for the compression of P to J, i.e., for the diagonal block of P corresponding to J. Let $\mathcal{S}_J = \{P_J \mid P \in \mathcal{S}\}$. Then $\mathcal{S}_J$ can be viewed as a semigroup of positive projections on J with no invariant ideals. Since every projection in $\mathcal{S}$ is in block upper-triangular form relative to the chain then $(PQ)_J = P_J Q_J$ for every $P, Q \in \mathcal{S}$. Lemma 4.1 yields that

$$\forall P, Q \in \mathcal{S} \quad \text{if } P_J \neq 0 \text{ and } Q_J \neq 0 \text{ then } (PQ)_J \neq 0. \tag{2}$$

It follows from Lemma 4.3 that there exists $P \in \mathcal{S}$ such that $\operatorname{rank} P_J = 1$. In fact, the rank of every nonzero element of $\mathcal{S}_J$ is one. Indeed, suppose that $Q \in \mathcal{S}$ and $\operatorname{rank} Q_J > 1$. It follows from (2) that $PQ \neq 0$, hence $\operatorname{rank} PQ = r$. But Remark 4.4 yields that the sum of the ranks of the remaining diagonal blocks of Q, and hence, of PQ, is less than $r-1$. Since $\operatorname{rank}(PQ)_J \le \operatorname{rank} P_J = 1$ it follows that the total rank of PQ would be less than r; a contradiction. It follows that the diagonal blocks are of rank zero or one. Remark 4.4 yields that every nonzero matrix in $\mathcal{S}$ has exactly r non-zero diagonal blocks.

Let $P, Q \in \mathcal{S} \setminus \{0\}$. It follows from (2) that their sets of nonzero diagonal blocks must either be the same or disjoint, as, otherwise, $0 \neq \operatorname{rank} PQ < r$. Therefore, we have s pairwise disjoint groupings $A_1, \dots, A_s$ of r diagonal blocks in each, such that for each projection P in $\mathcal{S}$ its set of non-zero diagonal blocks is exactly one of the A_k's. In this case we will write $P \in \mathcal{S}_k$.

Now let $\mathcal{T}$ be the set of all expressions of the form $P_1 + \cdots + P_s$ such that $P_k \in \mathcal{S}_k$ for each $1 \le k \le s$. Note that if $k \neq m$ then $P_k P_m$ has zero diagonal blocks, and is hence nilpotent, but $P_k P_m \in \mathcal{S}$ is a projection, so that $P_k P_m = 0$. It follows that every element of $\mathcal{T}$ is a projection of rank rs, and that $\mathcal{T}$ is a semigroup. Now the result follows from Theorem 0.4. Note that if no row or column is zero in all the matrices in $\mathcal{S}$, then the same is true for $\mathcal{T}$, so that we have $A = E$ for each $A \in \mathcal{S}$ as in the last clause of Theorem 0.4. $\square$

The following result extends Theorem 3.7 to semigroups of operators in the same way as Theorem 0.5(iii) extends Theorem 0.1.

THEOREM 4.6. *Let $\mathcal{S} \subseteq M_n(\mathbb{R})$ and $x \in \mathbb{R}^n$ be as in Theorem 0.5*(iii)*. Then x is contained in every $\mathcal{S}$-invariant sublattice, hence* span x *is the unique minimal invariant sublattice of $\mathcal{S}$.*

Proof. Let L be an $\mathcal{S}$-invariant sublattice of $x \in \mathbb{R}^n$. By Lemma 3.1, it has a basis $(z_k)_{k=1}^m$ consisting of positive pairwise disjoint vectors. Since $\mathcal{S}$ has no invariant ideals, Lemma 0.2 implies that for every $i, j \leq n$ there exists a matrix $A^{(i,j)} \in \mathcal{S}$ such that its (i,j)th entry is nonzero. Let $A = \sum_{i,j=1}^n A^{(i,j)}$, then all the entries of A are positive, hence A has no invariant ideals. It follows from Theorem 0.1 that x is the unique (up to scaling) positive eigenvector of A. On the other hand, L is invariant under A, and the matrix of $A_{|L}$ in the basis $(z_k)_{k=1}^m$ is positive, hence Theorem 0.1 yields that A has a positive eigenvector in L. It follows that $x \in L$. □

Theorem 0.5(iii) asserts that under certain conditions a semigroup $\mathcal{S}$ of positive operators has a common positive eigenvector, hence a one-dimensional invariant sublattice. We will show that under certain weaker conditions we can still guarantee that $\mathcal{S}$ has an invariant sublattice, though not necessarily one-dimensional.

THEOREM 4.7. *Let $\mathcal{S}$ be an $\mathbb{R}^+$-closed semigroup of positive matrices in M_n such that $\mathcal{S}$ has no invariant ideals. Let $\mathcal{S}_0$ be the set of all minimal projection in $\mathcal{S}$. Suppose that for every $P, Q \in \mathcal{S}_0$ either $PQ = 0$ or* $\operatorname{Range} P = \operatorname{Range} Q$. *Then:*

(i) $\operatorname{Range} P$ *is a sublattice of X for every $P \in \mathcal{S}_0$.*
(ii) *The linear span of the ranges of all members of $\mathcal{S}_0$ is an $\mathcal{S}$-invariant (not necessarily proper) sublattice of $\mathbb{R}^n$.*
(iii) *There is a maximal set $(P_k)_{k=1}^s$ of projections in $\mathcal{S}$ with pairwise disjoint ranges, and vectors $x_k \in \operatorname{Range} P_k$ such that $G = \operatorname{span}_{1 \leq k \leq s} x_k$ is an $\mathcal{S}$-invariant (not necessarily proper) sublattice of $\mathbb{R}^n$.*
(iv) *G is a minimal $\mathcal{S}$-invariant sublattice, i.e., it is contained in every $\mathcal{S}$-invariant sublattice.*
(v) *G is proper unless $\mathcal{S}$ contains all the rank one tensors $e_i \otimes e_j^*$.*

Proof. (i) Let r be the rank of the minimal projections in $\mathcal{S}$. By Theorem 0.5(ii), we know that $\mathcal{S}_0$ is nontrivial. For every $P, Q \in \mathcal{S}_0$, we either have $PQ = 0$ or $PQ = Q$, hence $\mathcal{S}_0 \cup \{0\}$ is a semigroup. It follows that after a permutation of basis, $\mathcal{S}_0$ can be written in a block form as in Theorem 4.5. Furthermore, since $\mathcal{S}$ has no invariant ideals, Theorem 0.5(ii) asserts that out of the four cases described in Theorem 4.5 we can only have $A = E$. That is, the block form of every $E \in \mathcal{S}_0$ is diagonal with diagonal blocks $E_1, \ldots, E_{rs}$ of rank zero or one, such that the only nonzero blocks are $E_{rk+1}, \ldots, E_{r(k+1)}$ for some $0 \leq k < s$. Again, in this case, we will write $E \in \mathcal{S}_k$. Since E_i is of rank one when $rk+1 \leq i < r(k+1)$, we deduce that $Y_i := \operatorname{Range} E_i$ is spanned by a positive vector y_i. By the assumption of the theorem, Y_i does not depend on the particular choice of $E\in\mathcal{S}_k$. It follows that $\operatorname{Range} E = L_k$ where $L_k=\operatorname{span}\{y_{rk+1},\dots,y_{r(k+1)}\}$. Since all the y_i's have disjoint supports, L_k is a sublattice of $\mathbb{R}^n$ for each $k=1,\dots,s$.

(ii) The vectors $y_1,\dots,y_{rs}$ are pairwise disjoint and have consecutive supports. Let $L=\operatorname{span}\{y_1,\dots,y_{rs}\}$. Then L is a sublattice of $\mathbb{R}^n$. Clearly, L is exactly the span of the ranges of all members of $\mathcal{S}_0$. We will show that L is invariant under $\mathcal{S}$. Let $A\in\mathcal{S}$.

Case 1. Suppose first that $\operatorname{rank} A = r$. Then Theorem 0.5(i) implies that there is $P\in\mathcal{S}_0$ such that $PA=A$. Then $P\in\mathcal{S}_k$ for some k. It follows that for each $x\in L$ we have

$$Ax = PAx\in\operatorname{Range} P=\operatorname{span}\{y_{rk+1},\dots,y_{r(t+1)}\}\subseteq L.$$

Case 2. Now consider the general case. Again, let $x\in L$. Pick $E_k\in\mathcal{S}_k$ for each $1\le k<s$. Then we can write $x=\alpha_1 z_1+\cdots+\alpha_s z_s$ where $z_k\in\operatorname{span}\{y_{rk+1},\dots,y_{r(t+1)}\}=\operatorname{Range} E_k$ as $k=1,\dots,s$. Note that $\operatorname{rank} AE_k\le r$ so that $AE_k x\in L$ by Case 1 for every k. It follows that

$$Ax=\alpha_1 AE_1 x+\cdots+\alpha_s AE_s x\in L.$$

(iii) Fix $k\le s$ and let $E\in\mathcal{S}_k$. We claim that the restriction of the semigroup $E\mathcal{S}E$ to L_k has no invariant ideals in L_k. Indeed, if there were such an ideal, then Lemma 0.2 would guarantee the existence of $A,B>0$ such that $AE\mathcal{S}EB=\{0\}$, but this would yield that $\mathcal{S}$ has an invariant ideal, a contradiction. It follows now from Theorem 0.5(iii) that there exists a strictly positive vector $x_k\in L_k$ such that x_k is a unique (up to scaling) positive eigenvector of $E\mathcal{S}E$. If P is another projection in $\mathcal{S}_k$ and $A\in\mathcal{S}$ is arbitrary, then $PAPx_k=EPAPEx_k$ is a multiple of x_k, hence x_k does not depend on the choice of $E\in\mathcal{S}_k$:

We will show that

(3) if $P\in\mathcal{S}_k$, $Q\in\mathcal{S}_m$, and $A\in\mathcal{S}$ then $QAPx_k$ is a multiple of x_m.

Put $y:=QAPx_\alpha\in L_m$ and suppose that $y\neq 0$. Since $P\mathcal{S}Q\neq\{0\}$ by Lemma 0.2, there exists $B\in\mathcal{S}$ such that $PBQ\neq 0$. Then $\operatorname{rank} PBQ=r$, so that PBQ is injective as a map from L_m to L_k. It follows that

$$(PBQ)y=P(BQA)Px_k=\lambda x_k\quad\text{for some }\lambda>0. \tag{4}$$

Let $C\in\mathcal{S}$, then

$$(PBQ)(QCQ)y=P(BQCQA)Px_k=\mu x_k\quad\text{for some }\mu. \tag{5}$$

Since PBQ is injective on L_k, (4) and (5) imply that $(QCQ)y$ is a multiple of y. Thus, y is a positive eigenvector of $Q\mathcal{S}Q$, hence y is a multiple of x_m. This proves (3).

Let $A\in\mathcal{S}$ and $k\le s$. Pick any $P\in\mathcal{S}_k$, then $AP\in\mathcal{S}_1$, hence there exists $Q\in\mathcal{S}_0$ such that $AP=QAP$. Then $Q\in\mathcal{S}_m$ for some $m\le s$. Therefore,

$Ax_k = APx_k = QAPx_k$ is a multiple of x_m by (3). It follows that G defined by $G = \operatorname{span}_{1 \le k \le s} x_k$ is invariant under $\mathcal{S}$. Since all the x_k's have pairwise disjoint supports, G is a sublattice of $\mathbb{R}^n$.

(iv) Let M be an $\mathcal{S}$-invariant sublattice of X. Pick any nonzero $x \in M$. Then there exists $P \in \mathcal{S}_0$ such that $Px \neq 0$. Since $\operatorname{Range} P = L_k$ for some $k \le s$, we conclude that $M \cap L_k \neq \varnothing$. Hence, $M \cap L_k$ is a nonzero (not necessarily proper) sublattice of L_k invariant under $P\mathcal{S}P$. As in the proof of (iii), the restriction of $P\mathcal{S}P$ to L_k has no invariant ideals, and x_k is a positive eigenvector for $P\mathcal{S}P_{|L_k}$. It follows from Theorem 4.6 that $x_k \in M$.

We will show that $x_m \in M$ for every $m \le s$, hence $G \subseteq M$. Take any $m \le s$ and $Q \in \mathcal{S}_m$. Since $Q\mathcal{S}P \neq \{0\}$ by Lemma 0.2, there exists $A \in \mathcal{S}$ such that $QAP \neq 0$. It follows from (3) that $QAPx_k$ is a multiple of x_m, and, clearly, $QAPx_k \in M$. It remains to show that $QAPx_k \neq 0$.

Recall that $\{y_{rk+1}, \dots, y_{r(k+1)}\}$ is a basis of L_k consisting of positive vectors with consecutive supports. Since x_k is strictly positive, its expansion with respect to this basis has strictly positive coefficients. Since $QAP \neq 0$, there exists a nonzero $x \in L_k$ such that $QAPx \neq 0$. By replacing x with x^+ or x^- we may assume that $x > 0$. In particular, the expansion of x with respect to the basis $\{y_{rk+1}, \dots, y_{r(k+1)}\}$ has nonnegative coefficients. It follows that $0 < x \le \lambda x_k$ for some $\lambda \in \mathbb{R}_+$, so that $QAPx_k \ge QAPx > 0$.

(v) Finally, suppose that $G = X$. It follows, in particular, that $G = L$, hence $r = 1$. Then X is the closed span of pairwise disjoint one-dimensional ranges of the minimal projections in $\mathcal{S}$. This implies that $\mathcal{S}_0$ contains all the rank-one tensors $e_i \otimes e_j^*$. □

EXAMPLE 4.8. Let $\mathcal{S}$ be the $\mathbb{R}^+$-closed semigroup of M_n generated by the following block matrices

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 \\ K & 0 & 0 & \cdots & & 0 \\ 0 & K & 0 & \cdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & \cdots & K & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & K & 0 & \cdots & 0 \\ 0 & 0 & K & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & K \\ 0 & & & \cdots & 0 \end{bmatrix},$$

where K is an $m \times m$ block of ones for some m. Then $\mathcal{S}$ has no invariant ideals and no common eigenvectors. However, $\mathcal{S}$ has an invariant sublattice $L = \operatorname{span}\{x_1, \dots, x_k\}$ where $k = \frac{n}{m}$ and

$$x_i = \bigl(0, \dots, 0, \underbrace{1, \dots, 1}_{mi+1, \dots, m(i+1)}, 0, \dots, 0\bigr).$$

5. Operators on infinite-dimensional spaces

In this section, we extend some of the preceding results to ℓ_p, c_0, and $L_p(\mu)$.

We start by characterizing the closed sublattices of ℓ_p with $1\le p<\infty$ or c_0. Namely, we will show that every such sublattice is the closed span of a finite or infinite sequence of pairwise disjoint positive vectors. This fact can be deduced from Ando's theorem (see [LT79, Theorem 1.b.8]) combined with [LT77, Theorem 2.a.4]. We present a short direct proof of this fact here.

If $E\subseteq\mathbb{N}$ and x is a vector in ℓ_p or c_0, let Ex be the vector defined as follows: $(Ex)_i=x_i$ if $i\in E$ and $(Ex)_i=0$ otherwise.

LEMMA 5.1. *Let L be a closed sublattice in ℓ_p with $1\le p<\infty$ or c_0, and let $x,y\in L_+$. Then $Ex\in L$ where $E=(\operatorname{supp} y)^C$.*

Proof. Let $\varepsilon>0$. Choose $n\in\mathbb{N}$ such that $\|Qx\|<\varepsilon$ where $Q=\{n+1,\dots\}$. Put $P=\{1,\dots,n\}$. Since P is finite, we can find $\lambda>0$ such that $(x-\lambda y)^+$ vanishes on $P\cap\operatorname{supp} y$. Let $h=(x-\lambda y)^+$, then $h\in L$ and $EPx=Ph$. Therefore, $Ex=E(Px+Qx)=Ph+EQx$. It follows from $0\le h\le x$ that

$$\|Ex-h\|\le\|Ex-Ph\|+\|Qh\|\le\|EQx\|+\|Qx\|<2\varepsilon.$$

Therefore, $Ex\in L$. $\square$

THEOREM 5.2. *Every closed sublattice of ℓ_p with $1\le p<\infty$ or c_0 is the closed span of a finite or infinite sequence of disjoint positive vectors.*

Proof. Let X be ℓ_p with $1\le p<\infty$ or c_0, and let L be a closed sublattice of X. We will show that we can assume that there exists $x\in L_+$ with $\operatorname{supp} x=\mathbb{N}$. Indeed, we can assume that the ideal generated by L in X is all of X as, otherwise, we can replace X with this ideal. Thus, for every $i\in\mathbb{N}$ there exists $x_i\in L_+$ such that $i\in\operatorname{supp} x_i$. Put $x=\sum_{i=1}^\infty\frac{x_i}{2^i\|x_i\|}$, then $x\in L_+$ and $\operatorname{supp} x=\mathbb{N}$.

For $i,j\in\mathbb{N}$, we write $i\approx j$ if $\frac{y_i}{x_i}=\frac{y_j}{x_j}$ for every $y\in L$. Clearly, this is an equivalence relation on $\mathbb{N}$. Let $E_1,E_2,\dots$ be the equivalence classes of this relation. Note that there may be finitely or countably many such classes, and each E_k may be finite or infinite. Let $n_k=\min E_k$. Without loss of generality, $n_1<n_2<\cdots$.

Since $n_1\not\approx n_2$, there exists $y\in L$ such that $\frac{y_{n_1}}{x_{n_1}}\ne\frac{y_{n_2}}{x_{n_2}}$. Put $z=|y-\frac{y_{n_1}}{x_{n_1}}x|$. Then $z\in L^+$, $z_{n_1}=0$, and $z_{n_2}\ne 0$. It follows that z vanishes on E_1 and $E_2\subseteq\operatorname{supp} z$. Repeating the same argument as above with 2 replaced with any $k>1$ we obtain $z^{(k)}\in L_+$ which vanishes on E_1 and $E_k\subseteq\operatorname{supp} z^{(k)}$. Let $F_k=(\operatorname{supp} z^{(k)})^C$, then $E_1\subseteq F_k$ and $E_k\perp F_k$.

Given any $u\in L_+$, put $h^{(1)}=u$ and $h^{(k)}=F_kh^{(k-1)}$ for $k>1$. Then $h^{(k)}\in L$ by Lemma 5.1. It follows from $E_1\subseteq F_k$ that $E_1u\le h^{(k)}$ for every k. Also, note that $h^{(k)}$ is decreasing; it follows that $h=\lim_k h^{(k)}=\inf_k h^{(k)}$ exists, so that $h\in L$. Since $E_1u\le h^{(k)}$ for every k, we have $E_1u\le h$. On the other hand, suppose that $i\notin E_1$. Then $i\in E_k$ for some $k>1$, so that $h_i^{(k)}=0$. It follows that $h_i=0$, so that $\operatorname{supp} h\subseteq E_1$. Therefore ,$E_1u=h\in L$. Thus,

$E_1 u \in L$ for every $u \in L_+$ and, therefore, for every $u \in L$. A similar argument shows that $E_k u \in L$ for every $k \in \mathbb{N}$ and every $u \in L$.

Let $a_k = E_k x$. Then (a_k) is a (finite or infinite) sequence of disjoint positive vectors in L. Let $y \in L$. Then $y = \sum_k E_k y$. For each $k \in \mathbb{N}$, we have $E_k y \in L$ and $\operatorname{supp} E_k y \subseteq E_k$. By our definition of E_k, it follows that $E_k y$ is a scalar multiple of a_k. It follows that $y \in [a_k]$. $\square$

COROLLARY 5.3. *Every infinite-dimensional closed sublattice of ℓ_p with $1 \le p < \infty$ or c_0 is order isometric to ℓ_p or c_0 respectively. Every finite-dimensional sublattice of ℓ_p or c_0 is order isometric to ℓ_p^n or ℓ_∞^n, respectively.*

EXAMPLE 5.4. The following example shows that Theorem 5.2 fails for ℓ_∞. Let $x^{(n)}$ be the sequence in ℓ_∞ given by

$$x_i^{(n)} = \begin{cases} 1 & \text{if } i \text{ is a multiple of } 2^n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $L = [x^{(n)}]$. It is easy to see that $\operatorname{span}\{x^{(n)}\}$ is a lattice, hence its closure L is a lattice. However, L has no atoms.

REMARK 5.5. The following observation is an infinite-dimensional analogue of Corollary 3.3. Theorem 1.b.8 of [LT79] implies that every sublattice of ℓ_p is the range of a positive contractive projection. Using Theorem 5.2, we can now construct such a projection explicitly. Suppose that L is a closed sublattice of ℓ_p with $1 \le p < \infty$ or c_0. Then Theorem 5.2 says that $L = [x_i]$, where (x_i) is a disjoint sequence of positive vectors. We can assume that $\|x_i\| = 1$. Choose a sequence of positive functionals (x_i^*) such that $\operatorname{supp} x_i^* \subseteq \operatorname{supp} x_i$ and $\|x_i^*\| = x_i^*(x_i) = 1$ for every i, and let $P = \sum_i x_i \otimes x_i^*$. Then P is a positive contractive projection with $\operatorname{Range} P = L$.

REMARK 5.6. It is easy to see that Theorem 5.2 also provides infinite-dimensional analogues of Remark 3.5 and Theorem 3.6. That is, suppose that T is an operator on ℓ_p with $1 \le p < \infty$ or c_0 which has no invariant closed ideals, but has an invariant closed sublattice L. Let (x_i) be the sequence given by Theorem 5.2. Then T can be written in the block form (T_{ij}) corresponding to the supports of x_i's. Of course, now it makes no sense to talk about stochastic or constant-row blocks. However, we still have that $T_{ij} x_j$ must be a multiple of x_i for any i and j. In particular, x_i is an eigenvector of T_{ii}.

Next, we present an infinite-dimensional version of Theorem 3.7. Suppose that T is a positive compact operator on a Banach lattice. If T is quasinilpotent, then [dP86] guarantees that T has an invariant closed ideal. On the other hand, if T is not quasinilpotent, then it was shown in [KR48] that its spectral radius is a positive real number, and is actually an eigenvalue corresponding to a positive eigenvector. Furthermore, if T is a positive compact operator on a Banach lattice and T has no invariant closed ideals, then the

Jentzsch–Perron theorem (see, e.g., Corollary 4.2.14 in [MN91]) asserts that the positive eigenvector corresponding to $r(T)$ is unique up to scaling.

Theorem 5.7. *Let X be ℓ_p with $1 \le p < \infty$ or c_0, and let T be a compact positive operator on X such that T has no invariant closed ideals. Then T has a unique (up to scaling) positive eigenvector, and this eigenvector is contained in every T-invariant closed sublattice.*

Proof. By the preceding comments, T is not quasinilpotent and there is a unique (up to scaling) positive vector h of T corresponding to $r(T)$. Without loss of generality, $r(T)=1$. Suppose that L is a closed sublattice invariant under T. Theorem 5.2 implies that $L=[x_i]$ where (x_i) is a finite or infinite sequence of pairwise disjoint positive vectors in X. We can assume that $\|x_i\|=1$ for all i. Suppose first that (x_i) is an infinite sequence. Define an operator $U: X \to L$ by $Ue_i = x_i$. It is easy to see that U is a surjective positive isometry. Then $U^{-1}TU: X \to X$ is again a compact positive operator. Since $r(U^{-1}TU)=r(T)=1$, there exists a positive vector $z \in X$ such that $U^{-1}TUx = x$. It follows that $T(Uz)=Uz$, so that $Uz=h$. It follows that $h \in L$. If the sequence (x_i) is of finite length n, then we can use a similar argument with U defined on ℓ_p^n or ℓ_∞^n instead of ℓ_p or c_0, respectively.

Now suppose that x is another positive eigenvector of T (even corresponding to a possibly different eigenvalue), then $\operatorname{span}\{x\}$ is an invariant closed sublattice, so that x is a multiple of h. $\square$

Next, we extend Theorem 3.8 to the infinite-dimensional case.

Theorem 5.8. *There is a positive compact operator on ℓ_p with $1 < p < \infty$ with no invariant closed ideals and exactly one closed invariant sublattice.*

Proof. Let $\frac{1}{2} > r_i \downarrow 0$. For each $i \ge 1$ chose R_i so that $R_i + r_i + r_i^2 + r_i^3 + \dots = 1$. Define an operator K via $k_{i1} = \frac{1}{i}R_i$ and $k_{ij} = \frac{j}{i}r_i^{j-1}$ if $j>1$.

It is easy to see that K can be chosen to be compact as an operator from ℓ_p to ℓ_p. Indeed, for every $n \in \mathbb{N}$ let K_n be defined as follows: $K_n e_j = Ke_j$ if $j \le n$ and $K_n e_j = 0$ otherwise. Then K_n is of finite rank. Estimating the ℓ_q-norms of the rows of $K - K_n$ we observe that provided that (r_i) is decreasing sufficiently rapidly, the nuclear norm of $K-K_n$ tends to zero as $n \to \infty$. It follows that $\|K-K_n\| \to 0$, so that K is compact as the limit of a sequence of finite-rank operators.

Let $x = (1, \frac{1}{2}, \frac{1}{3}, \dots)$. It is easy to check that $Kx = x$. Hence, x is the unique positive eigenvector of K. Suppose that L is a closed invariant sublattice of K such that $L \ne \operatorname{span} x$. Since $x \in L$, it follows that $\dim L > 1$. We know that L is spanned by a finite or infinite positive disjoint sequence (x_i). For every $i \ge 1$, let P_i be the coordinate projection on $\operatorname{supp} x_i$. It follows from $x \in L$ that $P_i x$ is a multiple of x_i. Without loss of generality, we can assume that $P_i x = x_i$.

Fix i and j so that $1 \notin \operatorname{supp} x_j$ and $\operatorname{supp} x_i$ has cardinality greater than one. Let $\operatorname{supp} x_j = \{n_k\}$ and $1 < n_1 < n_2 < \cdots$. Then $x_j = \sum_k \frac{1}{n_k} e_{n_k}$. Since L is invariant under K, it follows that $P_i K x_j = \lambda x_i$ for some $\lambda > 0$. Let $m \in \mathbb{N}$. Now

$$(Kx_j)_m = \frac{n_1}{m} r_m^{n_1 - 1} \cdot \frac{1}{n_1} + \frac{n_2}{m} r_m^{n_2 - 1} \cdot \frac{1}{n_2} + \frac{n_3}{m} r_m^{n_3 - 1} \cdot \frac{1}{n_3} + \cdots = \frac{1}{m} \alpha_m,$$

where $\alpha_m = r_m^{n_1 - 1} + r_m^{n_2 - 1} + \cdots$. It follows from $P_i K x_j = \lambda x_i$ that $\lambda \frac{1}{m} = \frac{1}{m} \alpha_m$ for every $m \in \operatorname{supp} x_i$, so that α_m is constant on $\operatorname{supp} x_i$. But this is a contradiction because (α_m) is a strictly decreasing function of m. □

Several examples of positive operators on Banach lattices with no invariant closed sublattices were presented in [KW]. We will show that one of these examples can be easily verified using Theorem 5.2.

Proposition 5.9. *Suppose that Q is an operator on ℓ_p, $1 \le p < \infty$ or c_0 of the following form:*

$$Q = \begin{bmatrix} 0 & * & 0 & 0 & 0 & \dots \\ * & 0 & * & 0 & 0 & \\ 0 & * & 0 & * & 0 & \\ 0 & 0 & * & 0 & * & \\ \vdots & & & \ddots & & \ddots \end{bmatrix},$$

where the stars correspond to positive reals. Then Q has no invariant closed infinite-dimensional sublattices.

Proof. Note that Q has no invariant closed ideals by Proposition 1.2. Suppose that Q has an invariant closed infinite-dimensional sublattice L. Theorem 5.2 implies that $L = [x_n]_{n=1}^{\infty}$ where (x_n) is a disjoint sequence of positive vectors. The union of the supports of x_n's is all of $\mathbb{N}$, as, otherwise, the closed ideal generated by L would be proper and invariant under Q. Let (q_{ij}) be the matrix of Q, and let (A_{mn}) be the block form of Q with respect to the supports of x_n's as $n \in \mathbb{N}$. Note that generally the blocks might be of infinite size. Note also that if an A_{mn} has a zero row then A_{mn} must be entirely zero as $A_{mn} x_n$ is a multiple of x_m. By the symmetrical structure of the matrix, the same is true for columns: if A_{mn} has a zero column then $A_{mn} = 0$.

Without loss of generality (up to a permutation of x_n's), we have $1 \in \operatorname{supp} x_1$. We claim that $2 \notin \operatorname{supp} x_1$. Indeed, suppose that $2 \in \operatorname{supp} x_1$. Then q_{21} is in A_{11}. Since q_{21} is the only nonzero entry in the first column of Q, it follows that for every $m > 1$ the first column of A_{m1} is zero, hence $A_{m1} = 0$. But then the ideal generated by the support of x_1 is invariant; a contradiction.

So, without loss of generality, $2 \in \operatorname{supp} x_2$. We will show inductively that we can assume that $n \in \operatorname{supp} x_n$ for every n. This would imply that $\operatorname{supp} x_n = \{n\}$, hence L is the whole space.

Suppose that we already have $k \in \operatorname{supp} x_k$ for all $1 \le k \le n$. It suffices to show that $n+1 \notin \bigcup_{k=1}^n \operatorname{supp} x_k$; then by renumbering x_k's for $k > n$ we can assume that $n+1 \in \operatorname{supp} x_{n+1}$. Suppose that $n+1 \in \operatorname{supp} x_r$ for some $r \le n$. Let $1 \le k < n$. Recall that the only nonzero entries of the kth column of Q are $q_{k-1,k}$ and $q_{k+1,k}$. It follows that they are located in $A_{k-1,k}$ and $A_{k+1,k}$, respectively. Therefore, for every $m > k+1$, the column of A_{mk} corresponding to the kth column in Q is zero, hence $A_{mk} = 0$. In particular, $A_{mk} = 0$ whenever $m > n$ and $k < n$. Also, the only nonzero entries of the nth column of Q are $q_{n-1,k}$ and $q_{n+1,k}$, and they are located in $A_{n-1,k}$ and $A_{r,k}$, respectively. As before, this yields $A_{mn} = 0$ whenever $m > n$. It follows that the closed ideal generated by $\bigcup_{k=1}^n \operatorname{supp} x_k$ is invariant; a contradiction. □

The following result was proved in [KW] for ℓ_p. We can now deduce it from Proposition 5.9.

COROLLARY 5.10. *The operator*

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \\ \vdots & & & \ddots & & \ddots \end{bmatrix}$$

on ℓ_p $(1 \le p < \infty)$ or c_0 has no invariant closed sublattices.

Proof. By Proposition 5.9 it suffices to show that Q has no finite-dimensional invariant sublattices. Suppose that L is such a sublattice. Then $L + iL$ is an invariant closed finite-dimensional subspace of the complexification of Q, so that Q viewed as an operator on the complex ℓ_p or c_0 has an eigenvector. Suppose that a nonzero complex vector $x = (x_1, x_2, \dots)$ satisfies $Qx = \lambda x$ for some $\lambda \in \mathbb{C}$. It follows that $x_{n+1} = \lambda x_n - x_{n-1}$ for each $n \ge 1$ (assuming $x_0 = 0$). If $x_1 = 0$ then $x = 0$; a contradiction. So, we can assume without loss of generality that $x_1 = 1$. It can be easily verified that $x_n = \frac{\mu_1^n - \mu_2^n}{\mu_1 - \mu_2}$ where μ_1 and μ_2 are the roots of the quadratic $z^2 - \lambda z + 1 = 0$. It follows that $\mu_1\mu_2 = 1$. If either $|\mu_1| > 1$ or $|\mu_2| > 1$ then (x_n) diverges; a contradiction. Hence, $|\mu_1| = |\mu_2| = 1$. It is easy to see now that (x_n) contains a subsequence that does not converge to zero; a contradiction. □

REMARK 5.11. Recall that an operator on a Banach lattice is said to be *strictly positive* if $Tx > 0$ whenever $x > 0$. We would also like to mention here that *the range of a strictly positive projection is a sublattice*. Indeed, suppose that E is a strictly positive projection. Clearly, $\operatorname{Range} E$ is invariant under E. We claim that $\operatorname{Range} E$ is a sublattice and $E(x \vee y) = x \vee y$ for all $x, y \in \operatorname{Range} E$. Suppose that $x, y \in \operatorname{Range} E$; we will show that $x \vee y \in \operatorname{Range} E$. Notice that $E(x \vee y) \ge (Ex) \vee (Ey) = x \vee y$. If $E(x \vee y) = x \vee y$, we are done. Suppose that $E(x \vee y) > x \vee y$, put $z = E(x \vee y) - x \vee y$, then $z > 0$ and $Ez = E(x \vee y) - E(x \vee y) = 0$, but this contradicts strict positivity of E. Further details on this subject can be found in [dJ82]. It follows, in particular, from Lemma 3.4 that if $PAP = AP$ for some operator A, then $\operatorname{Range} P$ is an invariant sublattice of A.

Finally, we present the infinite-dimensional analogue of Theorem 4.7.

THEOREM 5.12. *Let $X = L_p(\mu)$ for $1 \leq p < \infty$; suppose that $\mathcal{S}$ is an $\mathbb{R}^+$-closed semigroup of positive operators on X, and let $\mathcal{S}_0$ be the set of minimal projections in $\mathcal{S}$. Suppose that $\mathcal{S}$ has no closed invariant ideals and contains a nonzero compact operator. Suppose also that for every $P, Q \in \mathcal{S}_0$ either $PQ = 0$ or* $\operatorname{Range} P = \operatorname{Range} Q$. *Then:*

(i) *$\mathcal{S}_0 \cup \{0\}$ is a nontrivial semigroup; all the projections in $\mathcal{S}_0$ have the same finite rank.*
(ii) $\operatorname{Range} P$ *is a sublattice of X for every $P \in \mathcal{S}_0$.*
(iii) *For every $P, Q \in \mathcal{S}_0$, either* $\operatorname{Range} P = \operatorname{Range} Q$ *or* $\operatorname{Range} P \perp \operatorname{Range} Q$.
(iv) *The closed linear span of the ranges of all members of $\mathcal{S}_0$ is an $\mathcal{S}$-invariant (not necessarily proper) sublattice of X.*
(v) *There is a maximal set $(P_\alpha) \subseteq \mathcal{S}_0$ with pairwise disjoint ranges, and vectors (x_α) with $x_\alpha \in \operatorname{Range} P_\alpha$ such that $G = [x_\alpha]$ is a (not necessarily proper) $\mathcal{S}$-invariant sublattice.*
(vi) *G is a minimal $\mathcal{S}$-invariant sublattice, i.e., it is contained in every closed $\mathcal{S}$-invariant sublattice.*
(vii) *Furthermore, G is proper unless μ is discrete and $\mathcal{S}$ contains all the rank one tensors $e_\gamma \otimes e_\delta^*$, where e_γ and e_δ are discrete elements of $L_p(\mu)$.*

Proof. (i) Suppose that $K \in \mathcal{S}$ is a nonzero compact operator. Let $\mathcal{S}_1$ be the norm closure (or, equivalently, the $\mathbb{R}^+$-closure) of $\mathcal{S}K\mathcal{S}$. Then $\mathcal{S}_1$ is a semigroup ideal in $\mathcal{S}$. It follows from Lemma 0.2 that $\mathcal{S}_1$ has no invariant closed ideals. Applying Theorem 0.5(i) to $\mathcal{S}_1$, we conclude that $\mathcal{S}_1$ and, therefore, $\mathcal{S}$ contains nonzero finite rank operators. Therefore, we can assume that K is of finite rank and, moreover, that K is an operator of the minimal nonzero rank in $\mathcal{S}$; say, $\operatorname{rank} K = r$. Since the set of all operators of rank at most r is closed then every nonzero operator in $\mathcal{S}_1$ is of rank r.

CLAIM. *$\mathcal{S}_0 \subseteq \mathcal{S}_1$. Suppose that $P \in \mathcal{S}_0$. Then $P\mathcal{S}_1P$ is a subsemigroup of $\mathcal{S}_1$. We will show that $P\mathcal{S}_1P$ is closed, hence $\mathbb{R}^+$-closed. Indeed, suppose that $PA_nP \to B$ for some sequence (A_n) in $\mathcal{S}_1$. Since $PA_nP \in \mathcal{S}_1$, we have $B \in \mathcal{S}_1$. On the other hand, $B = PBP$, so that $B \in P\mathcal{S}_1P$. Therefore, $P\mathcal{S}_1P$ is $\mathbb{R}^+$-closed. Also, $P\mathcal{S}_1P \neq \{0\}$ by Lemma 0.2 and all nonzero operators in $P\mathcal{S}_1P$ are of rank r.*

We will show that $P\mathcal{S}_1P$ contains a nonnilpotent operator. Indeed, suppose that every operator in $P\mathcal{S}_1P$ is nilpotent. Then for every $A, B \in \mathcal{S}_1$ we have

$$r(APB) = r(BAP) = r(BAP^2) = r(PBAP) = 0,$$

so that $\mathcal{S}_1 P \mathcal{S}_1$ consists of nilpotent operators. Theorem 0.9 yields that $\mathcal{S}_1 P \mathcal{S}_1$ has a closed invariant ideal. On the other hand, $\mathcal{S}_1 P \mathcal{S}_1$ is a semigroup ideal in $\mathcal{S}_1$, hence it has no closed invariant ideals by Lemma 0.2; a contradiction.

Thus, there exists a nonnilpotent operator A in $P\mathcal{S}_1 P$. Now Theorem 0.7 yields that there exists a nonzero operator E in the $\mathbb{R}^+$-closed semigroup generated by A such that E is either a projection or nilpotent. We will show that E cannot be nilpotent. Indeed, suppose that $E^m \neq 0$ but $E^{m+1} = 0$ for some m. Replacing E with E^m if necessary, we can assume that $m = 1$. Let $Y = \operatorname{Range} A$. Observe that $\dim Y = r$. Since A is not nilpotent, then $\operatorname{rank} A^n = r$ for every n, so that the restriction of A to Y is invertible. Note that $E = \lim_i \alpha_i A^{n_i}$ for some (n_i) and a sequence (α_i) of positive reals, it follows that $\operatorname{Range} E \subseteq Y$. Also, $E \neq 0$ yields $\operatorname{rank} E = r$, hence $Y = \operatorname{Range} E$. It follows from $E^2 = 0$ that E vanishes on Y. Let $y \in Y$, then $y = Ex$ for some x, so that $Ay = EAx \in E(Y) = 0$. This contradicts A being invertible on Y.

Thus, E is a projection. Then $E \in P\mathcal{S}_1 P$ implies $PE = EP = E$. The minimality of P yields that $P = E \in P\mathcal{S}_1 P \subseteq \mathcal{S}_1$. This completes the proof of the *Claim*.

Since $\mathcal{S}_0 \subseteq \mathcal{S}_1$, then $\mathcal{S}_0$ is the set of minimal projections in $\mathcal{S}_1$. It follows immediately from the hypotheses that for every $P, Q \in \mathcal{S}_0$ we either have $PQ = 0$ or $PQ = Q$, hence $\mathcal{S}_0$ is a semigroup. Applying Theorem 0.5(i) to $\mathcal{S}_1$ we conclude that $\mathcal{S}_0$ is non-trivial and consists of projections of rank r. It follows that $\operatorname{Range} P$ is r-dimensional for every $P \in \mathcal{S}_0$. By Theorem 0.5(i,ii), for every positive $x \in X$, there exists $P \in \mathcal{S}_0$ such that $Px \neq 0$, and that for every $A \in \mathcal{S}_1$ with $\operatorname{rank} A = r$, there exists $P \in \mathcal{S}_0$ such that $PA = A$.

(ii) Pick $P \in \mathcal{S}_0$ and $0 \leq a, b \in \operatorname{Range} P$; it suffices to show that $a \vee b \in \operatorname{Range} P$ as well. It follows from $a \leq a \vee b$ that $a = Pa \leq P(a \vee b)$. Similarly, $b \leq P(a \vee b)$, so that $a \vee b \leq P(a \vee b)$. Let $z = P(a \vee b) - (a \vee b)$, then $z \geq 0$. It suffices to show that $z = 0$. Suppose $z \neq 0$. Then there exists $Q \in \mathcal{S}_0$ such that $Qz \neq 0$. In the case when $QP = 0$, we have

$$0 \leq Qz = QP(a \vee b) - Q(a \vee b) = -Q(a \vee b) \leq 0,$$

so that $Qz = 0$; a contradiction. On the other hand, if $\operatorname{Range} Q = \operatorname{Range} P$, then

$$Qz = P(a \vee b) - Q(a \vee b) \leq P(a \vee b) - (Qa) \vee (Qb) = P(a \vee b) - a \vee b = z,$$

so that $0 \leq Qz = PQz \leq Pz = 0$; a contradiction.

(iii) Suppose that $P, Q \in \mathcal{S}_0$ such that $PQ = 0$, and suppose that $0 \leq a \in \operatorname{Range} P$ and $0 \leq b \in \operatorname{Range} Q$. It suffices to show that $a \wedge b = 0$. Suppose that $a \wedge b > 0$, then $E(a \wedge b) > 0$ for some $E \in \mathcal{S}_0$. Since $\operatorname{Range} P \neq \operatorname{Range} Q$, we either have $\operatorname{Range} E \neq \operatorname{Range} P$ or $\operatorname{Range} E \neq \operatorname{Range} Q$. Suppose the former, then $EP = 0$ so that $Ea = EPa = 0$; then $0 < E(a \wedge b) \leq (Ea) \wedge (Eb) = 0$; a contradiction.

(iv) Let L_0 be the linear span of all the ranges of the members of $\mathcal{S}_0$. Since every two of these ranges are either identical or disjoint sublattices of X, it follows that L_0 is itself a sublattice. It suffices to show that L_0 is invariant under $\mathcal{S}$. Let $A \in \mathcal{S}$ and $x \in L_0$, we will show that $Ax \in L_0$. We may assume that $x \in \operatorname{Range} P$ for some $P \in \mathcal{S}_0$. It follows from $\mathcal{S}_0 \subseteq \mathcal{S}_1$ that $AP \in \mathcal{S}_1$, so that Theorem 0.5(i) applied to $\mathcal{S}_1$ there exists $Q \in \mathcal{S}_0$ such that $QAP = AP$. It follows that $Ax = APx = QAPx \in \operatorname{Range} Q \subseteq L_0$.

(v) Let $\Lambda = \{\operatorname{Range} P | P \in \mathcal{S}_0\}$; then Λ consists of pairwise disjoint r-dimensional sublattices of X. For each $\alpha \in \Lambda$, let $\mathcal{S}_\alpha = \{P \in \mathcal{S}_0 | \operatorname{Range} P = \alpha\}$.

Let $\alpha \in \Lambda$ and $P \in \mathcal{S}_\alpha$. We claim that the restriction of $P\mathcal{S}P$ to α has no invariant ideals. Indeed, suppose that $J \subsetneq \alpha$ is an ideal invariant under $P\mathcal{S}P$. Let R be the natural positive projection from α to the disjoint complement of J in α. Then $RP\mathcal{S}P = \{0\}$, so that $\mathcal{S}$ has an invariant closed ideal by Lemma 0.2; a contradiction. Thus, $P\mathcal{S}P$ has no invariant ideals. All the minimal projections in $P\mathcal{S}P_{|\alpha}$ have the same range α, so that $P\mathcal{S}P$ has a positive common eigenvector x in α by Theorem 0.5(iii), which is unique up to scaling. Observe that x does not depend on the choice of P in $\mathcal{S}_\alpha$. Indeed, if $P' \in \mathcal{S}_\alpha$ and $A \in \mathcal{S}$, then $P'AP'x = PP'AP'Px$ is a multiple of x; hence, x is a common eigenvector of $P'\mathcal{S}P'$. We will denote x by x_α.

Next, we will show that

$$\text{if } P \in \mathcal{S}_\alpha,\ Q \in \mathcal{S}_\beta, \text{ and } A \in \mathcal{S} \text{ then } QAPx_\alpha \text{ is a multiple of } x_\beta. \tag{6}$$

Suppose that $y := QAPx_\alpha$ is nonzero. Clearly, $y \in \beta$. By Lemma 0.2, we have $P\mathcal{S}Q \neq \{0\}$, so that there exists $B \in \mathcal{S}$ with $PBQ \neq 0$. Then $\operatorname{rank} PBQ = r$, so that PBQ takes β to α injectively. It follows that

$$(PBQ)y = P(BQA)Px_\alpha = \lambda x_\alpha \quad \text{for some } \lambda > 0. \tag{7}$$

For any $C \in \mathcal{S}$, we have

$$(PBQ)(QCQ)y = P(BQCQA)Px_\alpha = \mu x_\alpha \quad \text{for some } \mu. \tag{8}$$

Injectivity of PBQ together with (7) and (8) implies that $(QCQ)y$ is a multiple of y. Thus, y is a positive eigenvector of $Q\mathcal{S}Q$, so that y is a multiple of x_β. This proves (6).

Let $A \in \mathcal{S}$ and $\alpha \in \Lambda$. We will show that Ax_α is a multiple of x_β for some $\beta \in \Lambda$. Indeed, pick any $P \in \mathcal{S}_\alpha$, then $AP \in \mathcal{S}_1$, hence there exists $Q \in \mathcal{S}_0$ such that $AP = QAP$. Let $\beta = \operatorname{Range} Q$, then $Ax_\alpha = APx_\alpha = QAPx_\alpha$ is a multiple of x_β by (6).

Let $G = [x_\alpha]_{\alpha \in \Lambda}$. Since all the x_α's are pairwise disjoint, G is a sublattice of X. By the preceding paragraph, G is invariant under $\mathcal{S}$.

(vi) Let F be a closed $\mathcal{S}$-invariant sublattice of X; take any nonzero $x \in F$. Then $Px \neq 0$ for some $P \in \mathcal{S}_0$. It follows that $F \cap \alpha \neq \varnothing$ where $\alpha = \operatorname{Range} P$. Hence, $F \cap \alpha$ is a nonzero (not necessarily proper) sublattice of α invariant under $P\mathcal{S}P$. As in the proof of (v), the restriction of $P\mathcal{S}P$ to α has no invariant ideals, and x_α is a positive eigenvector for $P\mathcal{S}P_{|\alpha}$. It follows from Theorem 4.6 that $x_\alpha \in F$.

We will show that $x_\beta \in F$ for every $\beta \in \Lambda$, and hence $G \subseteq F$. Take any $\beta \in \Lambda$ and $Q \in \mathcal{S}_\beta$. Since $Q\mathcal{S}P \neq \{0\}$ by Lemma 0.2, there exists $A \in \mathcal{S}$ such that $QAP \neq 0$. It follows from (6) that $QAPx_\alpha$ is a multiple of x_β, and, clearly, $QAPx_\alpha \in F$. It remains to show that $QAPx_\alpha \neq 0$.

Since α is a finite-dimensional sublattice of X, it has a basis of pairwise disjoint positive vectors $(z_k)_{k=1}^r$ by [LT79, Corollary 1.b.4]. It follows from Theorem 0.1 that x_α is strictly positive with respect to this basis. Since $QAP \neq 0$, there exists a non-zero $x \in \alpha$ such that $QAPx \neq 0$. By replacing x with x^+ or x^- we may assume that $x > 0$. In particular, the expansion of x with respect to the basis $(z_k)_{k=1}^r$ has nonnegative coefficients. It follows that $0 < x \leq \lambda x_\alpha$ for some $\lambda \in \mathbb{R}_+$, so that $QAPx_\alpha \geq QAPx > 0$.

(vii) Finally, suppose that $G = X$. It follows, in particular, that $G = L$, and hence $r = 1$. Then X is the closed span of pairwise disjoint one-dimensional ranges of the minimal projections in $\mathcal{S}$. This implies that μ is discrete and $\mathcal{S}_0$ contains all the rank-one tensors $e_\gamma \otimes e_\delta^*$, where e_i are the discrete elements of $L_p(\mu)$. $\square$

COROLLARY 5.13. *Suppose that $\mathcal{S}$ is an $\mathbb{R}^+$-closed semigroup of positive operators on $L_p(\mu)$ where $1 \leq p < \infty$ and μ is not discrete. Suppose that $\mathcal{S}$ contains a nonzero compact operator and for every two minimal projections P and Q in $\mathcal{S}$ either $PQ = 0$ or* $\operatorname{Range} P = \operatorname{Range} Q$. *Then $\mathcal{S}$ has a closed invariant sublattice.*

REMARK 5.14. Note that the hypothesis in Theorem 5.12 and Corollary 5.13 that $PQ = 0$ or $\operatorname{Range} P = \operatorname{Range} Q$ for every two minimal projections is automatically satisfied when the semigroup of minimal projections in $\mathcal{S}$ is commutative.

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