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Research Article

The Existence and Uniqueness of Intuitionistic Fuzzy Solutions for Intuitionistic Fuzzy Partial Functional Differential Equations

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In this paper, we consider intuitionistic fuzzy partial functional differential equations with local and nonlocal initial conditions using the Banach fixed point theorem. A new complete intuitionistic fuzzy metric space is proposed to investigate the existence and uniqueness of intuitionistic fuzzy solutions for these problems. We use the level-set representation of intuitionistic fuzzy functions and define the solution to an intuitionistic fuzzy partial functional differential equation problem through a corresponding parametric problem and further develop theoretical results on the existence and uniqueness of the solution. An example is presented to illustrate the results with some numerical simulation for α-cuts of the intuitionistic fuzzy solutions: we give the representation of the surface of intuitionistic fuzzy solutions.

1. Introduction

Among several generalizations of fuzzy set theory [1] is the concept of intuitionistic fuzzy sets (IFSs). In the 1980s Atanassov generalized fuzzy sets by introducing the idea of intuitionistic fuzzy sets [2–4]. Further improvements of IFS theory [5] include the exploration of intuitionistic fuzzy geometry, intuitionistic fuzzy logic, intuitionistic fuzzy topology, an intuitionistic fuzzy approach to artificial intelligence, and intuitionistic fuzzy generalized nets may be found in [6–8]. The concept of intuitionistic fuzzy differential equations was first introduced by S. Melliani and L. S. Chadli [9]. The first step, which included using applicable definitions of intuitionistic fuzzy derivative and the intuitionistic fuzzy integral, was followed by introducing intuitionistic fuzzy differential equations and establishing sufficient conditions for the existence of unique solutions to these equations [10–12]. The study of numerical methods for solving intuitionistic fuzzy differential equations has been rapidly growing in recent years. It is difficult to obtain exact solutions for intuitionistic fuzzy DEs, and hence some numerical methods presented in [13–17].

Differential equations (DEs) with state-dependent delays, known as functional, are used to model a large number of natural or physical phenomena. There exists extensive literature dealing with functional DEs and their applications, and for this reason the study of such equations has received great attention in recent years. The literature related to partial functional differential equations with state-dependent delay is limited, but these equations provide more realistic models for various phenomena, and they have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, and population dynamics. There is still a lack of qualitative and quantitative researches for fuzzy partial differential equations (PDEs), which were first introduced by Buckley and Feuring [18] in 1999. The available theoretical results obtained up to now for such equations may be found in [19–21]. The papers [22–24] considered the existence and uniqueness of fuzzy solutions for some fuzzy functional partial differential equations with different types of initial conditions for some classes of hyperbolic equations; almost all of the previous results were based on the Banach fixed point theorem. However, rarely have we seen such results for intuitionistic fuzzy partial functional differential equations (PDEs), since the first and only publication on intuitionistic fuzzy partial differential equations was [25]. To the best of our knowledge, we are for the first time employing intuitionistic fuzzy mathematics to partial functional differential equations by using the Banach fixed point theorem.

Motivated and inspired by the above works, in this paper a new complete intuitionistic fuzzy metric space [7] is proposed to investigate the existence and uniqueness of intuitionistic fuzzy solutions by using the Banach fixed point theorem for the following intuitionistic fuzzy partial functional differential equations with local and nonlocal conditions:

$$
\begin{aligned}
&\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s} = G\left(t,s,\langle w,z\rangle_{(t,s)}\right), \\
&\qquad\qquad (t,s) \in I_a = [0,a] \times I_b = [0,b] \\
&\langle w,z\rangle (t,s) = \psi (t,s), \quad (t,s) \in [-r,0] \times [-r,0], \\
&\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s} = \left(q(t,s)\langle w,z\rangle (t,s)\right)_s \\
&\qquad + G\left(t,s,\langle w,z\rangle_{(t,s)}\right), \\
&\qquad\qquad (t,s) \in I_a = [0,a] \times I_b = [0,b] \\
&\langle w,z\rangle (t,s) = \psi (t,s), \quad (t,s) \in [-r,0] \times [-r,0]
\end{aligned} \tag{1}
$$

where $G : I_a \times I_b \times C([-r,0] \times [-r,0], \mathrm{IF}_n) \longrightarrow \mathrm{IF}_n$ is continuous, and $q : I_a \times I_b \longrightarrow \mathbb{R}$. Also we propose a method of steps which can be useful to solve intuitionistic fuzzy partial functional differential equations.

The main contributions of this paper include introducing intuitionistic fuzzy partial functional differential equations with local and nonlocal conditions and defining their solution; further developing theoretical results on the existence and uniqueness of the solution; using the level-set representation of intuitionistic fuzzy functions and defining the solution to an intuitionistic fuzzy partial functional differential equations problem through a corresponding parametric problem; and developing numerical examples for computing intuitionistic fuzzy quantities, taking into account the full interaction between intuitionistic fuzzy variables. The development of some efficient examples for solving intuitionistic fuzzy partial functional differential equations is the subject of our further work and will be presented elsewhere.

The remainder of the paper is arranged as follows. In Section 2, we give some basic preliminaries which will be used throughout this paper. In Sections 3 and 4 we discuss intuitionistic fuzzy partial functional differential equations with local and nonlocal conditions. Under suitable conditions we prove the existence and uniqueness of the solution to intuitionistic fuzzy PFDEs. In Section 5 we propose a method of steps to solve Section 6 to illustrate these results with some numerical simulation for α-cuts of the intuitionistic fuzzy solutions, and finally some conclusions and future works are discussed in Section 7.

2. Preliminaries

We consider

$$
\begin{aligned}
\mathrm{IF}_n = \mathrm{IF}(\mathbb{R}^n) = \Big\{&\langle w,z\rangle : \mathbb{R}^n \longrightarrow [0,1]^2, \mid \forall x \in \mathbb{R}^n 0 \\
&\leq w(x) + z(x) \leq 1\Big\}
\end{aligned} \tag{2}
$$

By an intuitionistic fuzzy number we mean any element $\langle w,z\rangle$ of IF_n verifying the following statements:

(a) $\langle w,z\rangle$ is normal i.e $\exists x_0, x_1 \in \mathbb{R}^n$ such that $w(x_0) = 1$ and $z(x_1) = 1$.

(b) w is convex and z is concave in the fuzzy sense.

(c) w and z are upper and lower semicontinuous, respectively.

(d) $\operatorname{supp}\langle w,z\rangle = \operatorname{cl}\overline{\{t \in \mathbb{R}^n :\mid z(t) < 1\}}$ is bounded.

IF_n denotes the set of all intuitionistic fuzzy numbers.

For $\langle w,z\rangle \in \mathrm{IF}_n$, we define the upper and lower α-cuts of $\langle w,z\rangle$ as

$$
\begin{aligned}
[\langle w,z\rangle]^\alpha &= \{t \in \mathbb{R}^n : z(t) \leq 1-\alpha\}, \\
[\langle w,z\rangle]_\alpha &= \{t \in \mathbb{R}^n : w(t) \geq \alpha\}
\end{aligned} \tag{3}
$$

Let $\langle w,z\rangle, \langle w',z'\rangle \in \mathrm{IF}_n$ and $\gamma \in \mathbb{R}$; we define

$$
\begin{aligned}
&\left(\langle w,z\rangle \oplus \left\langle w',z'\right\rangle\right)(k) = \bigg(\sup_{k=t+s} \min\left(w(t), w'(s)\right), \\
&\quad \inf_{k=t+s} \max\left(z(t), z'(s)\right)\bigg) \\
&\gamma \langle w,z\rangle = \begin{cases} \langle \gamma w, \gamma z\rangle & \text{if } \gamma \neq 0 \\ 0_{(1,0)} & \text{if } \gamma = 0 \end{cases}
\end{aligned} \tag{4}
$$

For $\langle w,z\rangle$, $\langle w',z'\rangle \in \mathrm{IF}_n$ and $\gamma \in \mathbb{R}$. The addition and scalar-multiplication are defined by

$$
\begin{aligned}
\left[\langle w,z\rangle \oplus \left\langle w',z'\right\rangle\right]^\alpha &= [\langle w,z\rangle]^\alpha + \left[\left\langle w',z'\right\rangle\right]^\alpha, \\
[\gamma \langle z,w\rangle]^\alpha &= \gamma \left[\left\langle w',z'\right\rangle\right]^\alpha \\
\left[\langle w,z\rangle \oplus \left\langle w',z'\right\rangle\right]_\alpha &= [\langle w,z\rangle]_\alpha + \left[\left\langle w',z'\right\rangle\right]_\alpha, \\
[\gamma \langle z,w\rangle]_\alpha &= \gamma [\langle w,z\rangle]_\alpha
\end{aligned} \tag{5}
$$

Definition 1. For $\langle w,z\rangle \in IF_n$ and $\alpha \in [0,1]$, define

$$
\begin{aligned}
[\langle w,z\rangle]_l^+(\alpha) &= \inf\{t \in \mathbb{R}^n \mid w(t) \geq \alpha\}, \\
[\langle w,z\rangle]_r^+(\alpha) &= \sup\{t \in \mathbb{R}^n \mid w(t) \geq \alpha\} \\
[\langle w,z\rangle]_l^-(\alpha) &= \inf\{t \in \mathbb{R}^n \mid z(t) \leq 1-\alpha\}, \\
[\langle w,z\rangle]_r^-(\alpha) &= \sup\{t \in \mathbb{R}^n \mid z(t) \leq 1-\alpha\}
\end{aligned} \tag{6}
$$

We denote

$$[\langle w,z\rangle]_\alpha = [[\langle w,z\rangle]_l^+(\alpha), [\langle w,z\rangle]_r^+(\alpha)]$$
$$[\langle w,z\rangle]^\alpha = [[\langle w,z\rangle]_l^-(\alpha), [\langle w,z\rangle]_r^-(\alpha)] \quad (7)$$

Proposition 2 (see [7]). *For all $\alpha, \beta \in [0,1]$ and $\langle w,z\rangle \in \mathrm{IF}_n$*

(a) $[\langle w,z\rangle]_\alpha \subset [\langle w,z\rangle]^\alpha$

(b) $[\langle w,z\rangle]_\alpha$ *and* $[\langle w,z\rangle]^\alpha$ *are nonempty compact and convex subsets in* $\mathbb{R}^n$

(c) *if* $\alpha \le \beta$ *then* $[\langle w,z\rangle]_\beta \subset [\langle w,z\rangle]_\alpha$ *and* $[\langle w,z\rangle]^\beta \subset [\langle w,z\rangle]^\alpha$

(d) *If* $\alpha_n \nearrow \alpha$ *then* $[\langle w,z\rangle]_\alpha = \bigcap_n [\langle w,z\rangle]_{\alpha_n}$ *and* $[\langle w,z\rangle]^\alpha = \bigcap_n [\langle w,z\rangle]^{\alpha_n}$

Let M a subset of $\mathbb{R}^n$ and $\alpha \in [0,1]$; we denote by

$$M_\alpha = \{t \in \mathbb{R}^n : w(t) \ge \alpha\},$$
$$M^\alpha = \{t \in \mathbb{R}^n : z(t) \le 1-\alpha\} \quad (8)$$

Lemma 3 (see [7]). *If the two families $\{M_\alpha,\ \alpha \in [0,1]\}$ and $\{M^\alpha,\ \alpha \in [0,1]\}$ verify (a)–(d) in Proposition 2, and w and z are defined by*

$$w(t) = \begin{cases} 0 & \text{if } t \notin M_0 \\ \sup\{\alpha \in [0,1] : t \in M_\alpha\} & \text{if } t \in M_0 \end{cases}$$
$$z(t) = \begin{cases} 1 & \text{if } t \notin M^0 \\ 1 - \sup\{\alpha \in [0,1] : t \in M^\alpha\} & \text{if } t \in M^0 \end{cases} \quad (9)$$

then $\langle w,z\rangle \in IF_n$.

The following application is a metric on IF_n:

$$\begin{aligned} d_\infty^n(\langle w,z\rangle, \langle u,v\rangle) &= \frac{1}{4}\sup_{0<\alpha\le 1}\left\|[\langle w,z\rangle]_r^+(\alpha) - [\langle u,v\rangle]_r^+(\alpha)\right\| \\ &+ \frac{1}{4}\sup_{0<\alpha\le 1}\left\|[\langle w,z\rangle]_l^+(\alpha) - [\langle u,v\rangle]_l^+(\alpha)\right\| \\ &+ \frac{1}{4}\sup_{0<\alpha\le 1}\left\|[\langle w,z\rangle]_r^-(\alpha) - [\langle u,v\rangle]_r^-(\alpha)\right\| \\ &+ \frac{1}{4}\sup_{0<\alpha\le 1}\left\|[\langle w,z\rangle]_l^-(\alpha) - [\langle u,v\rangle]_l^-(\alpha)\right\| \end{aligned} \quad (10)$$

where $\|.\|$ is the Euclidean norm in $\mathbb{R}^n$.

Theorem 4 (see [7]). *(IF_n, d_∞^n) is a complete metric space.*

$\mathscr{C}(I_a \times I_b, \mathrm{IF}_n)$ denotes the space of all continuous mappings on $I_a \times I_b$ into IF_n.

A standard proof applies to show that the metric space $(\mathscr{C}(I_a \times I_b, \mathrm{IF}_n), D)$ is complete, where the supremum metric D on $\mathscr{C}(I_a \times I_b, \mathrm{IF}_n)$ is defined by

$$D(\langle u,v\rangle, \langle u',v'\rangle) = \sup_{(x,y)\in I_a\times I_b} d_\infty^n(\langle u,v\rangle(x,y), \langle u',v'\rangle(x,y)) \quad (11)$$

Definition 5. A mapping $G : I_a \times I_b \longrightarrow IF_n$ is called continuous at point $(t_0,s_0) \in I_a \times I_b$ provided for any $\varepsilon > 0$, there is an $\delta(\varepsilon)$ such that

$$d_\infty^n(G(t_1,s_1), G(t_0,s_0)) < \varepsilon \quad (12)$$

whenever $\max\{|t_1-t_0|, |s_1-s_0|\} < \delta(\varepsilon)$ for all $(t_1,s_1) \in I_a\times I_b$.

Definition 6. A mapping $G : I_a \times I_b \times IF_n \longrightarrow IF_n$ is called continuous at point $(t_0, s_0, \langle w,z\rangle_0) \in I_a \times I_b \times IF_n$ provided for any $\varepsilon > 0$, there is an $\delta(\varepsilon)$ such that

$$d_\infty^n(G(t_1,s_1,\langle w,z\rangle), G(t_0,s_0,\langle w,z\rangle_0)) < \varepsilon \quad (13)$$

whenever $\max\{|t_1-t_0|, |s_1-s_0|\} < \delta(\varepsilon)$ and $d_\infty^n(\langle w,z\rangle, \langle w,z\rangle_0) < \delta(\varepsilon)$ for all $(t_1,s_1) \in I_a \times I_b$, $\langle w,z\rangle \in IF_n$.

Definition 7. The mapping $G : I_a \times I_b \longrightarrow IF_n$ is said to be strongly measurable if the set-valued mappings $G_\alpha : I_a \times I_b \longrightarrow P_k(\mathbb{R}^n)$ defined by $G_\alpha(t,s) = [G(t,s)]_\alpha$ and $G^\alpha : I_a \times I_b \longrightarrow P_k(\mathbb{R}^n)$ defined by $G^\alpha(t,s) = [G(t,s)]^\alpha$ are (Lebesgue) measurable for all $\alpha \in [0,1]$.

Definition 8. $G : I_a \times I_b \longrightarrow IF_n$ is said to be integrably bounded if there is an integrable mapping $h : I_a \times I_b \longrightarrow \mathbb{R}^n$ such that $\|x\| \le h(s)$ holds for any $x \in \mathrm{supp}(G(y,z))$, $(y,z) \in I_a \times I_b$.

Definition 9. Suppose $G : I_a \times I_b \longrightarrow IF_n$ is integrable for each $\alpha \in (0,1]$; write

$$\left[\int_0^a\int_0^b G(x,y)\,dy\,dx\right]_\alpha = \int_0^a\int_0^b [G(x,y)]_\alpha\,dy\,dx$$

$$= \left\{ \int_0^a \int_0^b g(x,y)\,dydx \mid g : I_a \times I_b \longrightarrow \mathbb{R}^n \textit{ is a measurable selection for } G_\alpha \right\}.$$

$$\left[\int_0^a \int_0^b G(x,y)\,dsdt \right]^\alpha = \int_0^a \int_0^b [G(x,y)]^\alpha\,dydx = \left\{ \int_0^a \int_0^b g(x,y)\,dydx \mid g : I_a \times I_b \longrightarrow \mathbb{R}^n \textit{ is a measurable selection for } G^\alpha \right\}. \quad (14)$$

Let $\langle u,z\rangle$ and $\langle v,w\rangle \in IF_n$; the Hukuhara difference is the intuitionistic fuzzy number $\langle k,l\rangle \in IF_n$, if it exists such that

$$\langle u,z\rangle - \langle v,w\rangle = \langle k,l\rangle \Longleftrightarrow \langle u,z\rangle = \langle v,w\rangle + \langle k,l\rangle \quad (15)$$

Definition 10. Let $G : I_a \times I_b \longrightarrow IF_n$. The intuitionistic fuzzy partial derivative of G w.r.t. x at $(t_0,s_0) \in I_a \times I_b$ is the intuitionistic fuzzy quantity $\partial G(t,s)/\partial x \in IF_n$ if there exists, such that for all $h > 0$ very small, the H-difference $G(t_0+h,s_0) - G(t_0,s_0)$ exists in IF_n and the limit

$$\frac{\partial G(t,s)}{\partial x} = \lim_{h\longrightarrow 0^+} \frac{G(t_0+h,s_0) - G(t_0,s_0)}{h} \quad (16)$$

The intuitionistic fuzzy partial derivative of G w.r.t. y at $(t_0,s_0) \in I_a \times I_b$ is defined analogously.

3. Intuitionistic Fuzzy Partial Functional Differential Equations with Local Conditions

The first part of the paper we provide an existence and uniqueness result for the intuitionistic fuzzy PFDEs in the following form:

$$\begin{aligned}
&\frac{\partial^2 \langle w,z\rangle(x,y)}{\partial x \partial y} = G\left(x,y,\langle w,z\rangle_{(x,y)}\right), \quad (x,y) \in I_a \times I_b := [0,a]\times[0,b]\\
&\langle w,z\rangle(x,y) = \psi(x,y), \quad (x,y) \in [-r,0]\times[-r,0]\\
&\langle w,z\rangle(x,0) = \eta_1(x), \quad x \in I_a\\
&\langle w,z\rangle(0,y) = \eta_2(y), \quad y \in I_b\\
&\psi(0,0) = \eta_1(0) = \eta_2(0)
\end{aligned} \quad (17)$$

where $G : I_a \times I_b \times C([-r,0]\times[-r,0],\mathrm{IF}_n) \longrightarrow \mathrm{IF}_n$, $\psi : [-r,0]\times[-r,0] \longrightarrow \mathrm{IF}_n$, $\eta_1 \in \mathscr{C}(I_a,\mathrm{IF}_n)$, $\eta_2 \in \mathscr{C}(I_b,\mathrm{IF}_n)$ are given functions, and $\langle w,z\rangle_{(x,y)}(t,s)$ is defined by

$$\langle w,z\rangle_{(x,y)}(t,s) = \langle w,z\rangle(x+t,y+s), \quad (t,s) \in [-r,0]\times[-r,0] \quad (18)$$

In the second part of this section we consider the intuitionistic fuzzy PFDEs in more general form

$$\begin{aligned}
&\frac{\partial^2 \langle w,z\rangle(x,y)}{\partial x \partial y} = \left(q(x,y)\langle w,z\rangle_{(x,y)}\right)_y + G\left(x,y,\langle w,z\rangle_{(x,y)}\right), \quad (x,y) \in I_a \times I_b := [0,a]\times[0,b]\\
&\langle w,z\rangle(x,y) = \psi(x,y), \quad (x,y) \in [-r,0]\times[-r,0]\\
&\langle w,z\rangle(x,0) = \eta_1(x), \quad x \in I_a\\
&\langle w,z\rangle(0,y) = \eta_2(y), \quad y \in I_b\\
&\psi(0,0) = \eta_1(0) = \eta_2(0)
\end{aligned} \quad (19)$$

where G,ψ,η_1,η_2 are as in problem (17) and $q : I_a \times I_b \times \longrightarrow \mathbb{R}$.

Definition 11. A function $\langle w,z\rangle \in C([-r,a]\times[-r,b],IF_n)$ is called an intuitionistic fuzzy solution of the problem (17) if it satisfies the equation $\partial^2\langle w,z\rangle(x,y)/\partial x\partial y = G(x,y,\langle w,z\rangle_{(x,y)})$ and boundary conditions.

Theorem 12. *Assume that*

(1) *a mapping $G : I_a \times I_b \times IF_n \longrightarrow IF_n$ is continuous,*

(2) *for any pair $(x,y) \in I_a \times I_b$, $\langle w,z\rangle, \langle w',z'\rangle \in C([-r,a],IF_n) \times C([-r,b],IF_n)$ and $\theta_1,\theta_2 \in [-r,0]$, we have*

$$d_\infty^n\left(G\left(x,y,\langle w,z\rangle_{(x,y)}\right), G\left(x,y,\left\langle w',z'\right\rangle_{(x,y)}\right)\right) \le K d_\infty^n\left(\langle w,z\rangle(x+\theta_1,y+\theta_2), \left\langle w',z'\right\rangle \cdot (x+\theta_1,y+\theta_2)\right) \quad (20)$$

where $K > 0$ is a given constant,

moreover, if $Kab < 1$, then problem (17) has an unique intuitionistic fuzzy solution.

Proof. Transform problem (17) into a fixed point problem. It is clear that the solutions of problem (17) are fixed points of the operator $T : \mathscr{C}([-r,a]\times[-r,b],IF_n) \longrightarrow \mathscr{C}([-r,a]\times[-r,b],IF_n)$ defined by

$$T(\langle w,z\rangle)(x,y) := \begin{cases} \psi(x,y), & (x,y)\in[-r,0]\times[-r,0] \\ \eta_1(x)+\eta_2(y)-\psi(0,0)+\int_0^x\int_0^y G(t,s,\langle w,z\rangle_{(t,s)})\,dsdt, & if\ \ (x,y)\in I_a\times I_b \end{cases} \tag{21}$$

We shall show that T is a contraction operator. Indeed, consider $\langle w,z\rangle, \langle w',z'\rangle \in \mathscr{C}([-r,a]\times[-r,b],IF_n)$ and $\alpha\in(0,1]$, then

$$T(\langle w,z\rangle)(x,y) := \eta_1(x)+\eta_2(y)-\psi(0,0) + \int_0^x\int_0^y G(t,s,\langle w,z\rangle_{(t,s)})\,dsdt : \tag{22}$$

and

$$\begin{aligned} &T(\langle w',z'\rangle)(x,y) := \eta_1(x)+\eta_2(y)-\psi(0,0) + \int_0^x\int_0^y G\left(t,s,\langle w',z'\rangle_{(t,s)}\right)dsdt : \\ &d_\infty^n\left(T(\langle w,z\rangle)(t,s), T(\langle w',z'\rangle)(t,s)\right) \\ &= d_\infty^n\Big(\eta_1(x)+\eta_2(y)-\psi(0,0)+\int_0^x\int_0^y G(t,s,\langle w,z\rangle_{(t,s)})\,dsdt, \\ &\eta_1(x)+\eta_2(y)-\psi(0,0)+\int_0^x\int_0^y G\left(t,s,\langle w',z'\rangle_{(t,s)}\right)dsdt\Big) \\ &= d_\infty^n\Big(\eta_1(x)+\eta_2(y)-\psi(0,0)+\int_0^x\int_0^y G(t,s,\langle w,z\rangle(t+\theta_1,s+\theta_2))\,dsdt, \\ &\eta_1(x)+\eta_2(y)-\psi(0,0)+\int_0^x\int_0^y G\left(t,s,\langle w',z'\rangle(t+\theta_1,s+\theta_2)\right)dsdt\Big) \\ &\le \int_0^x\int_0^y d_\infty^n\left(G(t,s,\langle w,z\rangle(t+\theta_1,s+\theta_2)), G\left(t,s,\langle w',z'\rangle(t+\theta_1,s+\theta_2)\right)\right)dsdt \\ &\le K\int_0^x\int_0^y d_\infty^n\left(\langle w,z\rangle(t+\theta_1,s+\theta_2), \langle w',z'\rangle(t+\theta_1,s+\theta_2)\right)dsdt \\ &\le K\int_0^a\int_0^b \sup_{(t,s)\in I_a\times I_b} d_\infty^n\left(\langle w,z\rangle(t+\theta_1,s+\theta_2), \langle w',z'\rangle(t+\theta_1,s+\theta_2)\right)dsdt \\ &\le KabD\left(\langle w,z\rangle, \langle w',z'\rangle\right) \end{aligned} \tag{23}$$

Hence for each $(t,s)\in I_a\times I_b$

$$D\left(T(\langle w,z\rangle), T\left(\langle w',z'\rangle\right)\right) \le KabD\left(\langle w,z\rangle, \langle w',z'\rangle\right) \tag{24}$$

□

So, T is a contraction and thus, by Banach fixed point theorem; T has an unique fixed point, which is solution to (17).

Definition 13. A function $\langle w,z\rangle \in C([-r,a]\times[-r,b],IF_n)$ is called an intuitionistic fuzzy solution to problem (19) if it satisfies the equation $\partial^2\langle w,z\rangle(x,y)/\partial x\partial y = (q(x,y)\langle w,z\rangle_{(x,y)})_y + G(x,y,\langle w,z\rangle_{(x,y)})$, $(x,y)\in I_a\times I_b$ and boundary conditions.

Theorem 14. *Assume that*

(1) *a mapping $G : I_a\times I_b\times IF_n \longrightarrow IF_n$ is continuous,*

(2) *for any pair $(x,y)\in I_a\times I_b$, $\langle w,z\rangle, \langle w',z'\rangle \in C([-r,a],IF_n)\times C([-r,b],IF_n)$ and $\theta_1,\theta_2\in[-r,0]$, we have*

$$\begin{aligned} &d_\infty^n\left(G\left(x,y,\langle w,z\rangle_{(x,y)}\right), G\left(x,y,\langle w',z'\rangle_{(x,y)}\right)\right) \\ &\le Kd_\infty^n\left(\langle w,z\rangle(x+\theta_1,y+\theta_2), \langle w',z'\rangle \cdot (x+\theta_1,y+\theta_2)\right) \end{aligned} \tag{25}$$

where $K>0$ is a given constant. Moreover, if

$$a\sup_{(t,s)\in I_a\times I_b}\left|q(t,s)\right| + Kab < 1, \tag{26}$$

then problem (19) has an unique intuitionistic fuzzy solution.

Proof. Transform problem (19) into a fixed point problem. Clearly the solutions of problem (19) are fixed points of the operator $T : \mathscr{C}([-r,a]\times[-r,b],IF_n) \longrightarrow \mathscr{C}([-r,a]\times[-r,b],IF_n)$ defined by

$$T(\langle w,z\rangle)(x,y) := \begin{cases} \psi(x,y), & (x,y) \in [-r,0]\times[-r,0] \\ p(x,y) + \int_0^x q(s,y)\langle w,z\rangle_{(s,y)}\,ds + \int_0^x\int_0^y G(t,s,\langle w,z\rangle_{(t,s)})\,dsdt, & (x,y)\in I_a\times I_b \end{cases} \tag{27}$$

where $p(x,y) = \eta_1(x) + \eta_2(y) - \psi(0,0) - \int_0^x q(s,0)\eta_1(s)ds$.

We shall show that T is a contraction operator. Indeed, consider $\langle w,z\rangle, \langle w',z'\rangle \in \mathscr{C}([-r,a]\times[-r,b], IF_n)$ and $\alpha \in (0,1]$, then

$$\begin{aligned} &T(\langle w,z\rangle)(x,y) \\ &\quad := p(x,y) + \int_0^x q(s,y)\langle w,z\rangle_{(s,y)}\,ds \\ &\qquad + \int_0^x\int_0^y G(t,s,\langle w,z\rangle_{(t,s)})\,dsdt, \\ &T(\langle w',z'\rangle)(x,y) \\ &\quad := p(x,y) + \int_0^x q(s,y)\langle w',z'\rangle_{(s,y)}\,ds \\ &\qquad + \int_0^x\int_0^y G(t,s,\langle w',z'\rangle_{(t,s)})\,dsdt \end{aligned} \tag{28}$$

Then

$$\begin{aligned} &d_\infty^n\left(T(\langle w,z\rangle)(t,s), T(\langle w',z'\rangle)(t,s)\right) = d_\infty^n\Big(p(x, \\ &\quad y) + \int_0^x q(s,y)\langle w,z\rangle_{(s,y)}\,ds \\ &\quad + \int_0^x\int_0^y G(t,s,\langle w,z\rangle_{(t,s)})\,dsdt, p(x,y) \\ &\quad + \int_0^x q(s,y)\langle w',z'\rangle_{(t,s)}\,ds \\ &\quad + \int_0^x\int_0^y G(t,s,\langle w',z'\rangle_{(t,s)})\,dsdt\Big) \\ &\quad \le d_\infty^n\Big(\int_0^x q(s,y)\langle w,z\rangle_{(t,s)}\,ds \\ &\quad + \int_0^x\int_0^y G(t,s,\langle w,z\rangle_{(t,s)})\,dsdt, \int_0^x q(s,y) \\ &\quad \cdot \langle w',z'\rangle_{(t,s)}\,ds \\ &\quad + \int_0^x\int_0^y G(t,s,\langle w',z'\rangle_{(t,s)})\,dsdt\Big) \\ &\quad \le \sup_{(t,s)\in I_a\times I_b} |q(t,s)| \int_0^a d_\infty^n\big(\langle w,z\rangle \\ &\quad \cdot (t+\theta_1, s+\theta_2), \langle w',z'\rangle(t+\theta_1,s+\theta_2)\big)\,ds \\ &\quad + K\int_0^x\int_0^y d_\infty^n\big(\langle u,v\rangle(t+\theta_1,s+\theta_2), \langle w',z'\rangle \\ &\quad \cdot (t+\theta_1,s+\theta_2)\big)\,dsdt \le \sup_{(t,s)\in I_a\times I_b}|q(t,s)| \\ &\quad \cdot \int_0^a \sup_{(t,s)\in I_a\times I_b} d_\infty^n\big(\langle w,z\rangle(t+\theta_1,s+\theta_2), \langle w',z'\rangle \\ &\quad \cdot (t+\theta_1,s+\theta_2)\big)\,ds + K\int_0^a\int_0^b \sup_{(t,s)\in I_a\times I_b} d_\infty^n \\ &\quad \cdot \big(\langle w,z\rangle(t+\theta_1,s+\theta_2), \langle w',z'\rangle \\ &\quad \cdot (t+\theta_1,s+\theta_2)\big)\,dsdt \le \Big(a \\ &\quad \cdot \sup_{(t,s)\in I_a\times I_b}|q(t,s)| + Kab\Big) D\left(\langle w,z\rangle, \langle w',z'\rangle\right) \end{aligned} \tag{29}$$

Hence for each $(t,s)\in I_a\times I_b$

$$\begin{aligned} &D\left(T(\langle w,z\rangle), T(\langle w',z'\rangle)\right) \\ &\quad \le \Big(a\sup_{(t,s)\in I_a\times I_b}|q(t,s)| + Kab\Big) D\left(\langle w,z\rangle, \langle w',z'\rangle\right) \end{aligned} \tag{30}$$

□

Then, T is a contraction and by Banach fixed point theorem, T has a unique fixed point, which is intuitionistic fuzzy solution to (19).

4. Intuitionistic Fuzzy Partial Functional Differential Equations with Nonlocal Conditions

Now we extend equation (17) into nonlocal problems. More precisely, we consider the following nonlocal problem:

$$\frac{\partial^2\langle w,z\rangle(x,y)}{\partial x\partial y} = G\left(x,y,\langle w,z\rangle_{(x,y)}\right), \quad (x,y)\in I_a\times I_b \tag{31}$$

$$\langle w,z\rangle(x,y)+\sum_{i=1}^{p}f_i(x)\langle w,z\rangle(x,b_i+y)=\varphi(x,y),\quad (x,y)\in I_a\times[-r,0],\ i=1,\ldots,p \tag{32}$$

$$\langle w,z\rangle(x,y)+\sum_{j=1}^{r}g_j(y)\langle w,z\rangle(a_j+x,y)=\psi(x,y),\quad (x,y)\in[-r,0]\times I_b,\ j=1,\ldots,r \tag{33}$$

where $\varphi:[-r,0]\times[-r,0]\longrightarrow IF_n$, $\psi:[-r,0]\times[-r,0]\longrightarrow IF_n$, $f_i\in\mathscr{C}(I_a,\mathbb{R}^n)$, $i=1,\ldots,p$, $g_j\in\mathscr{C}(I_b,\mathbb{R}^n)$, $j=1,\ldots,r$, $0<a_1<a_2<\ldots<a_p<a$, and $0<b_1<b_2<\ldots<b_r<b$.

Definition 15. By an intuitionistic fuzzy solution of (31)–(33) we mean a function $\langle w,z\rangle\in\mathscr{C}(I_a\times I_b,IF_n)$ which satisfies (31)–(33).

Theorem 16. *Assume that*

(1) *a mapping $G:I_a\times I_b\times IF_n\longrightarrow IF_n$ is continuous,*

(2) *for any pair $(x,y)\in I_a\times I_b$, $\langle w,z\rangle,\langle w',z'\rangle\in C([-r,a],IF_n)\times C([-r,b],IF_n)$ and $\theta_1,\theta_2\in[-r,0]$, we have*

$$d_\infty^n\left(G\left(x,y,\langle w,z\rangle_{(x,y)}\right),G\left(x,y,\left\langle w',z'\right\rangle_{(x,y)}\right)\right)\le Kd_\infty^n\left(\langle w,z\rangle(x+\theta_1,y+\theta_2),\left\langle w',z'\right\rangle\cdot(x+\theta_1,y+\theta_2)\right) \tag{34}$$

where $K>0$ is a given constant. Moreover, if

$$\sum_{i=1}^{p}\sup_{x\in I_a}\left|f_i(x)\right|+\sum_{j=1}^{r}\sup_{y\in I_b}\left|g_j(y)\right|+Kab<1, \tag{35}$$

then problem (31)–(33) has an unique intuitionistic fuzzy solution on $[-r,a]\times[-r,b]$.

Proof. Transform problem (31)–(33) into a fixed point problem. Clearly the solutions of problem (31)–(33) are fixed points of the operator $T:\mathscr{C}([-r,a]\times[-r,b],IF_n)\longrightarrow\mathscr{C}([-r,a]\times[-r,b],IF_n)$ defined by

$$T(\langle w,z\rangle)(x,y):=\begin{cases}\varphi(x,y)-\sum_{i=1}^{p}f_i(x)\langle w,z\rangle(x,b_i+y), & (x,y)\in I_a\times[-r,0]\\ \psi(x,y)-\sum_{j=1}^{r}g_j(y)\langle w,z\rangle(a_j+x,y), & (x,y)\in[-r,0]\times I_b\\ \varphi(x,0)+\psi(0,y)-\varphi(0,0)+\int_0^x\int_0^y G\left(t,s,\langle w,z\rangle_{(t,s)}\right)dsdt, & if\ \ (x,y)\in I_a\times I_b\end{cases} \tag{36}$$

The reasoning used in the proof of Theorem 14 shows that T is a contraction operator, and hence, it has an unique fixed point, which is an intuitionistic fuzzy solution of (31)–(33) □

Remark 17. The arguments used in Theorem 14 can be applied to obtain a uniqueness result for the following intuitionistic fuzzy problem with nonlocal condition:

$$\frac{\partial^2\langle w,z\rangle(x,y)}{\partial x\partial y}=\left(q(x,y)\langle w,z\rangle_{(x,y)}\right)_y+G\left(x,y,\langle w,z\rangle_{(x,y)}\right),\quad (x,y)\in I_a\times I_b$$

$$\langle w,z\rangle(x,y)+\sum_{i=1}^{p}f_i(x)\langle w,z\rangle(x,b_i+y)=\varphi(x,y),\quad (x,y)\in I_a\times[-r,0],\ i=1,\ldots,p \tag{37}$$

$$\langle w,z\rangle(x,y)+\sum_{j=1}^{r}g_j(y)\langle w,z\rangle(a_j+x,y)=\psi(x,y),\quad (x,y)\in[-r,0]\times I_b,\ j=1,\ldots,r$$

5. Solving Intuitionistic Fuzzy Partial Functional Differential Equation

We propose a procedure to solve the following intuitionistic fuzzy partial functional differential equation:

$$\begin{gathered}\frac{\partial^2\langle w,z\rangle(t,s)}{\partial x\partial y}=G\left(t,s,\langle w,z\rangle_{(t,s)}\right),\quad (t,s)\in I_a\times I_b:=[0,a]\times[0,b]\\ \langle w,z\rangle(t,s)=\psi(t,s),\quad (t,s)\in[-r,0]\times[-r,0]\\ \langle w,z\rangle(t,0)=\eta_1(t),\quad t\in I_a\\ \langle w,z\rangle(0,s)=\eta_2(s),\quad s\in I_b\\ \psi(0,0)=\eta_1(0)=\eta_2(0)\end{gathered} \tag{38}$$

where $G:I_a\times I_b\times\mathrm{IF}_n\longrightarrow\mathrm{IF}_n$ is obtained by extension principle from a continuous function $g:I_a\times I_b\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$. Since

$$\begin{gathered}[G(t,s,\langle w,z\rangle)]_\alpha=G(t,s,[\langle w,z\rangle)]_\alpha)\\ [G(t,s,\langle w,z\rangle)]^\alpha=G(t,s,[\langle w,z\rangle)]^\alpha)\end{gathered} \tag{39}$$

for all $\alpha \in [0,1]$ and $\langle w,z\rangle \in IF_n$, we denote

$$\begin{aligned}
&[\langle w,z\rangle (t,s)]_\alpha = [[\langle w,z\rangle (t,s)]_l^+ (\alpha), [\langle w,z\rangle (t,s)]_r^+ \\
&\cdot (\alpha)], [\langle w,z\rangle (t,s)]^\alpha = [[\langle w,z\rangle (t,s)]_l^- (\alpha), \\
&[\langle w,z\rangle (t,s)]_r^- (\alpha)] \\
&\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_\alpha = \left[\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_l^+ (\alpha), \right. \\
&\left.\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_r^+ (\alpha)\right] \\
&\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]^\alpha = \left[\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_l^- (\alpha), \right. \\
&\left.\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_r^- (\alpha)\right] \\
&[\langle w,z\rangle (0,0)]_\alpha = \left[\left[\left\langle u_0, v_0\right\rangle\right]_l^+ (\alpha), \left[\left\langle u_0, v_0\right\rangle\right]_r^+ (\alpha)\right], \\
&[\langle w,z\rangle (0,0)]^\alpha = \left[\left[\left\langle u_0, v_0\right\rangle\right]_l^- (\alpha), \left[\left\langle u_0, v_0\right\rangle\right]_r^- (\alpha)\right] \\
&[\eta_1 (t)]_\alpha = \left[[\eta_1 (t)]_l^+ (\alpha), [\eta_1 (t)]_r^+ (\alpha)\right], \\
&[\eta_1 (t)]^\alpha = \left[[\eta_1 (t)]_l^- (\alpha), [\eta_1 (t)]_r^- (\alpha)\right] \\
&[\eta_2 (s)]_\alpha = \left[[\eta_2 (s)]_l^+ (\alpha), [\eta_2 (s)]_r^+ (\alpha)\right], \\
&[\eta_2 (s)]^\alpha = \left[[\eta_2 (s)]_l^- (\alpha), [\eta_2 (s)]_r^- (\alpha)\right] \\
&[\psi (t,s)]_\alpha = \left[[\psi (t,s)]_l^+ (\alpha), [\psi (t,s)]_r^+ (\alpha)\right], \\
&[\psi (t,s)]^\alpha = \left[[\psi (t,s)]_l^- (\alpha), [\psi (t,s)]_r^- (\alpha)\right], \\
&[G (t,s,\langle w,z\rangle)]_\alpha \\
&= \left[G_l^+ \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^+ (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^+ (\alpha)\right), \right. \\
&\left. G_r^+ \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^+ (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^+ (\alpha)\right)\right] \\
&[G (t,s,\langle w,z\rangle)]^\alpha \\
&= \left[G_l^- \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^- (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^- (\alpha)\right), \right. \\
&\left. G_r^- \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^- (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^- (\alpha)\right)\right]
\end{aligned} \tag{40}$$

Then, with this notation, problem (38) is transformed into the following parametrized partial differential system:

$$\begin{aligned}
&\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_l^+ (\alpha) \\
&= G_l^+ \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^+ (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^+ (\alpha)\right), \\
&\qquad (t,s) \in I_a \times I_b \\
&\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_r^+ (\alpha) \\
&= G_r^+ \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^+ (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^+ (\alpha)\right), \\
&\qquad (t,s) \in I_a \times I_b \\
&\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_l^- (\alpha) \\
&= G_l^- \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^- (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^- (\alpha)\right), \\
&\qquad (t,s) \in I_a \times I_b \\
&\left[\frac{\partial^2 \langle w,z\rangle (t,s)}{\partial t \partial s}\right]_r^- (\alpha) \\
&= G_r^- \left(t,s,\left[\langle w,z\rangle_{(t,s)}\right]_l^- (\alpha), \left[\langle w,z\rangle_{(t,s)}\right]_r^- (\alpha)\right), \\
&\qquad (t,s) \in I_a \times I_b
\end{aligned} \tag{41}$$

with initial conditions

$$\begin{aligned}
&[\langle w,z\rangle (t,s)]_l^+ (\alpha) = \left[\left\langle \psi (t,s)\right\rangle\right]_l^+ (\alpha), \\
&\qquad (t,s) \in [-r,0] \times [-r,0] \\
&[\langle w,z\rangle (t,s)]_r^+ (\alpha) = \left[\left\langle \psi (t,s)\right\rangle\right]_r^+ (\alpha), \\
&\qquad (t,s) \in [-r,0] \times [-r,0] \\
&[\langle w,z\rangle (t,s)]_l^- (\alpha) = \left[\left\langle \psi (t,s)\right\rangle\right]_l^- (\alpha), \\
&\qquad (t,s) \in [-r,0] \times [-r,0] \\
&[\langle w,z\rangle (t,s)]_r^- (\alpha) = \left[\left\langle \psi (t,s)\right\rangle\right]_r^- (\alpha), \\
&\qquad (t,s) \in [-r,0] \times [-r,0] \\
&[\langle w,z\rangle (0,0)]_l^+ (\alpha) = \left[\left\langle u_0, v_0\right\rangle\right]_l^+ (\alpha) \\
&[\langle w,z\rangle (0,0)]_r^+ (\alpha) = \left[\left\langle u_0, v_0\right\rangle\right]_r^+ (\alpha) \\
&[\langle w,z\rangle (0,0)]_l^- (\alpha) = \left[\left\langle u_0, v_0\right\rangle\right]_l^- (\alpha) \\
&[\langle w,z\rangle (0,0)]_r^- (\alpha) = \left[\left\langle u_0, v_0\right\rangle\right]_r^- (\alpha) \\
&[\langle w,z\rangle (t,0)]_l^+ (\alpha) = \left[\eta_1 (t)\right]_l^+ (\alpha), \quad t \in I_a \\
&[\langle w,z\rangle (t,0)]_r^+ (\alpha) = \left[\eta_1 (t)\right]_r^+ (\alpha), \quad t \in I_a \\
&[\langle w,z\rangle (t,0)]_l^- (\alpha) = \left[\eta_1 (t)\right]_l^- (\alpha), \quad t \in I_a \\
&[\langle w,z\rangle (t,0)]_r^- (\alpha) = \left[\eta_1 (t)\right]_r^- (\alpha), \quad t \in I_a \\
&[\langle w,z\rangle (0,s)]_l^+ (\alpha) = \left[\eta_2 (s)\right]_l^+ (\alpha), \quad s \in I_b
\end{aligned}$$

$$[\langle w,z\rangle(0,s)]_r^+(\alpha)=[\eta_2(s)]_r^+(\alpha),\quad s\in I_b$$
$$[\langle w,z\rangle(0,s)]_l^-(\alpha)=[\eta_2(s)]_l^-(\alpha),\quad s\in I_b$$
$$[\langle w,z\rangle(0,s)]_r^-(\alpha)=[\eta_2(s)]_r^-(\alpha),\quad s\in I_b \tag{42}$$

(1) We solve system (41)-(42).

(2) If $[\langle w,z\rangle(t,s)]_l^+(\alpha)$, $[\langle w,z\rangle(t,s)]_r^+(\alpha)$, $[\langle w,z\rangle(t,s)]_l^-(\alpha)$, $[\langle w,z\rangle(t,s)]_r^-(\alpha)$ is the solution of system (41)-(42), then denote

$$\begin{aligned}
&[[\langle w,z\rangle(t,s)]_l^+(\alpha),[\langle w,z\rangle(t,s)]_r^+(\alpha)]=M_\alpha,\\
&[[\langle w,z\rangle(t,s)]_l^-(\alpha),[\langle w,z\rangle(t,s)]_r^-(\alpha)]=M^\alpha,\\
&\left[\left[\frac{\partial^2\langle w,z\rangle(t,s)}{\partial t\partial s}\right]_l^+(\alpha),\left[\frac{\partial^2\langle w,z\rangle(t,s)}{\partial t\partial s}\right]_r^+(\alpha)\right]\\
&\quad=M'_\alpha,\\
&\left[\left[\frac{\partial^2\langle w,z\rangle(t,s)}{\partial t\partial s}\right]_l^-(\alpha),\left[\frac{\partial^2\langle w,z\rangle(t,s)}{\partial t\partial s}\right]_r^-(\alpha)\right]\\
&\quad=M'^\alpha
\end{aligned}\tag{43}$$

ensure that (M_α,M^α) and (M'_α,M'^α) satisfying (a)-(d) of Proposition 2.

(3) By using the Lemma 3 we construct the intuitionistic fuzzy solution $\langle w,z\rangle(t,s)\in IF_n$ for (38) such that

$$\begin{aligned}
&[\langle w,z\rangle(t,s)]_\alpha\\
&\quad=[[\langle w,z\rangle(t,s)]_l^+(\alpha),[\langle w,z\rangle(t,s)]_r^+(\alpha)],\\
&[\langle w,z\rangle(t,s)]^\alpha\\
&\quad=[[\langle w,z\rangle(t,s)]_l^-(\alpha),[\langle w,z\rangle(t,s)]_r^-(\alpha)]
\end{aligned}\tag{44}$$

for all $\alpha\in[0,1]$

The same method of step is used to this equation with nonlocal condition.

Remark 18. The same procedure can be used to this equation $\partial^2\langle w,z\rangle(x,y)/\partial x\partial y=(q(x,y)\langle w,z\rangle_{(x,y)})_y+G(x,y,\langle w,z\rangle_{(x,y)})$, $(x,y)\in I_a=[0,a]\times I_b=[0,b]$ with local and nonlocal conditions.

6. Application

Consider the following intuitionistic fuzzy partial functional differential equation

$$\begin{aligned}
\frac{\partial^2\langle w,z\rangle(x,y)}{\partial x\partial y}&=-\frac{1}{3}\left(y\langle w,z\rangle(x,y)\right)_y\\
&\quad+\frac{1}{3}\langle w,z\rangle_{(x,y)}\left(-\frac{1}{3},-\frac{1}{3}\right)+\frac{1}{9}C
\end{aligned}\tag{45}$$

for $(x,y)\in[0,1]\times[0,1]$, with the local conditions

$$\langle w,z\rangle(0,0)=C, \tag{46}$$
$$\langle w,z\rangle(x,0)=Cx+C, \tag{47}$$
$$\langle w,z\rangle(0,y)=C. \tag{48}$$

where $C\in IF_1$ is a triangular intuitionistic fuzzy number (TIFN) and

$$\begin{aligned}
&\langle w,z\rangle(x,y)=Cxy+Cx+1,\\
&\quad\text{for }(x,y)\in\left[-\frac{1}{3},1\right]\times\left[-\frac{1}{3},1\right]\setminus(0,1]\times(0,1]
\end{aligned}\tag{49}$$

From (45), we have the function $G:[0,1]\times[0,1]\times C([-1/3,0]\times[-1/3,0],\mathrm{IF}_1)\longrightarrow\mathrm{IF}_1$ defined by

$$G\left(x,y,\langle w,z\rangle_{(x,y)}\right)=\frac{1}{3}\langle w,z\rangle\left(x-\frac{1}{3},y-\frac{1}{3}\right)+\frac{1}{9}C \tag{50}$$

satisfies assumptions (1) and (2) of the Theorem 14. Indeed, is easy to see that G is continuous.

Thus

$$\begin{aligned}
&d_\infty\left(G\left(x,y,\langle w,z\rangle_{(x,y)}\right),G\left(x,y,\langle w',z'\rangle_{(x,y)}\right)\right)\\
&\quad\le\frac{1}{3}d_\infty\left(\langle w,z\rangle\left(x-\frac{1}{3},y-\frac{1}{3}\right),\langle w',z'\rangle\right.\\
&\quad\left.\cdot\left(x-\frac{1}{3},y-\frac{1}{3}\right)\right)
\end{aligned}\tag{51}$$

and so F satisfy (2).

It is clear that the hypotheses are satisfied with an positive number $K=1/3$, $a=b=1$, and $q(x,y)=-(1/3)y$; we have $\sup_{(x,y)\in[0,1]\times[0,1]}|q(x,y)|=1/3$. That follows all the conditions in the Theorem 14 hold. Therefore there exists a unique intuitionistic fuzzy solution of this problem.

We will find an intuitionistic fuzzy solution of this equation by using the method of step in Section 3. The deterministic solution of the crisp equation is

$$u=cx+c \tag{52}$$

We apply the fuzzification in c and supposed that the parametric form of corresponding intuitionistic fuzzy number C is

$$\begin{aligned}
&[C]_\alpha=[C_l^+(\alpha),C_r^+(\alpha)]\\
&[C]^\alpha=[C_l^-(\alpha),C_r^-(\alpha)]
\end{aligned}\tag{53}$$

where is verify the conditions of Lemma 3.

Then the function $G:[0,1]\times[0,1]\times IF_1\longrightarrow IF_1$ defined by $G(x,y,\langle w,z\rangle_{(x,y)})=(1/3)\langle w,z\rangle(x-1/3,y-1/3)+(1/9)C$ is obtained by extension principle from the function

$g(x, y, u) = (1/3)u(x - 1/3, y - 1/3) + (1/9)c$, $(x, y, c) \in [0, 1] \times [0, 1] \times \mathbb{R}$

$$\begin{aligned} [G]_\alpha &= \left[\frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_l^+(\alpha) \right. \\ &\quad + \frac{1}{9}C_l^+(\alpha), \frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_r^+(\alpha) \\ &\quad \left. + \frac{1}{9}C_r^+(\alpha)\right] \\ [G]^\alpha &= \left[\frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_l^-(\alpha) \right. \\ &\quad + \frac{1}{9}C_l^-(\alpha), \frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_r^-(\alpha) \\ &\quad \left. + \frac{1}{9}C_r^-(\alpha)\right] \end{aligned} \tag{54}$$

If

$$\begin{aligned} &[\langle w, z\rangle(x, y)]_\alpha \\ &\quad = \left[[\langle w, z\rangle(x, y)]_l^+(\alpha), [\langle w, z\rangle(x, y)]_r^+(\alpha)\right] \\ &[\langle w, z\rangle(x, y)]^\alpha \\ &\quad = \left[[\langle w, z\rangle(x, y)]_l^-(\alpha), [\langle w, z\rangle(x, y)]_r^-(\alpha)\right] \end{aligned} \tag{55}$$

then

$$\begin{aligned} \left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_\alpha &= \left[\left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_l^+ \right. \\ &\quad \left. \cdot(\alpha), \left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_r^+(\alpha)\right] \\ \left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]^\alpha &= \left[\left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_l^- \right. \\ &\quad \left. \cdot(\alpha), \left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_r^-(\alpha)\right] \end{aligned} \tag{56}$$

Then, we have to solve the following PDEs:

$$\begin{aligned} &\left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_l^+(\alpha) \\ &\quad = -\frac{1}{3}\left[\left(y\langle w, z\rangle(x, y)\right)_y\right]_l^+(\alpha) \\ &\qquad + \frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_l^+(\alpha) + \frac{1}{9}C_l^+(\alpha), \\ &\qquad (x, y) \in [0, 1] \times [0, 1] \\ &\left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_r^+(\alpha) \\ &\quad = -\frac{1}{3}\left[\left(y\langle w, z\rangle(x, y)\right)_y\right]_r^+(\alpha) \\ &\qquad + \frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_r^+(\alpha) + \frac{1}{9}C_r^+(\alpha), \\ &\qquad (x, y) \in [0, 1] \times [0, 1] \\ &\left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_l^-(\alpha) \\ &\quad = -\frac{1}{3}\left[\left(y\langle w, z\rangle(x, y)\right)_y\right]_l^-(\alpha) \\ &\qquad + \frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_l^-(\alpha) + \frac{1}{9}C_l^-(\alpha), \\ &\qquad (x, y) \in [0, 1] \times [0, 1] \\ &\left[\frac{\partial^2\langle w, z\rangle(x, y)}{\partial x\partial y}\right]_r^-(\alpha) \\ &\quad = -\frac{1}{3}\left[\left(y\langle w, z\rangle(x, y)\right)_y\right]_r^-(\alpha) \\ &\qquad + \frac{1}{3}\left[\langle w, z\rangle\left(x - \frac{1}{3}, y - \frac{1}{3}\right)\right]_r^-(\alpha) + \frac{1}{9}C_r^-(\alpha), \\ &\qquad (x, y) \in [0, 1] \times [0, 1] \end{aligned} \tag{57}$$

with initial conditions

$$\begin{aligned} &[\langle w, z\rangle(x, y)]_l^+(\alpha) = xyC_l^+(\alpha) + xC_l^+(\alpha) + 1, \\ &\qquad (x, y) \in \left[-\frac{1}{3}, 1\right] \times \left[-\frac{1}{3}, 1\right] \setminus (0, 1] \times (0, 1] \\ &[\langle w, z\rangle(x, y)]_r^+(\alpha) = xyC_r^+(\alpha) + xC_r^+(\alpha) + 1, \\ &\qquad (x, y) \in \left[-\frac{1}{3}, 1\right] \times \left[-\frac{1}{3}, 1\right] \setminus (0, 1] \times (0, 1] \\ &[\langle w, z\rangle(x, y)]_l^-(\alpha) = xyC_l^-(\alpha) + xC_l^-(\alpha) + 1, \\ &\qquad (x, y) \in \left[-\frac{1}{3}, 1\right] \times \left[-\frac{1}{3}, 1\right] \setminus (0, 1] \times (0, 1] \\ &[\langle w, z\rangle(x, y)]_r^-(\alpha) = xyC_r^-(\alpha) + xC_r^-(\alpha) + 1, \\ &\qquad (x, y) \in \left[-\frac{1}{3}, 1\right] \times \left[-\frac{1}{3}, 1\right] \setminus (0, 1] \times (0, 1] \\ &[\langle w, z\rangle(0, 0)]_l^+(\alpha) = C_l^+(\alpha) \\ &[\langle w, z\rangle(0, 0)]_r^+(\alpha) = C_r^+(\alpha) \\ &[\langle w, z\rangle(0, 0)]_l^-(\alpha) = C_l^-(\alpha) \\ &[\langle w, z\rangle(0, 0)]_r^-(\alpha) = C_r^-(\alpha) \end{aligned}$$

$$\begin{aligned}
&[\langle w,z\rangle (x,0)]_l^+(\alpha) = xC_l^+(\alpha) + C_l^+(\alpha), \quad x \in [0,1] \\
&[\langle w,z\rangle (x,0)]_r^+(\alpha) = xC_r^+(\alpha) + C_r^+(\alpha), \quad x \in [0,1] \\
&[\langle w,z\rangle (x,0)]_l^-(\alpha) = xC_l^-(\alpha) + C_l^-(\alpha), \quad x \in [0,1] \\
&[\langle w,z\rangle (x,0)]_r^-(\alpha) = xC_r^-(\alpha) + C_r^-(\alpha), \quad x \in [0,1] \\
&[\langle w,z\rangle (0,y)]_l^+(\alpha) = C_l^+(\alpha), \quad y \in [0,1] \\
&[\langle w,z\rangle (0,y)]_r^+(\alpha) = C_r^+(\alpha), \quad y \in [0,1] \\
&[\langle w,z\rangle (0,y)]_l^-(\alpha) = C_l^-(\alpha), \quad y \in [0,1] \\
&[\langle w,z\rangle (0,y)]_r^-(\alpha) = C_r^-(\alpha), \quad y \in [0,1]
\end{aligned} \tag{58}$$

We get

$$\begin{aligned}
&[\langle w,z\rangle (x,y)]_l^+(\alpha) = xC_l^+(\alpha) + C_l^+(\alpha), \\
&\qquad (x,y) \in [0,1]\times[0,1] \\
&[\langle w,z\rangle (x,y)]_r^+(\alpha) = xC_r^+(\alpha) + C_r^+(\alpha), \\
&\qquad (x,y) \in [0,1]\times[0,1] \\
&[\langle w,z\rangle (x,y)]_l^-(\alpha) = xC_l^-(\alpha) + C_l^-(\alpha), \\
&\qquad (x,y) \in [0,1]\times[0,1] \\
&[\langle w,z\rangle (x,y)]_r^-(\alpha) = xC_r^-(\alpha) + C_r^-(\alpha), \\
&\qquad (x,y) \in [0,1]\times[0,1]
\end{aligned} \tag{59}$$

Therefore

$$\begin{aligned}
&[\langle w,z\rangle (x,y)]_\alpha \\
&\quad = [xC_l^+(\alpha) + C_l^+(\alpha), xC_r^+(\alpha) + C_r^+(\alpha)] \\
&[\langle w,z\rangle (x,y)]^\alpha \\
&\quad = [xC_l^-(\alpha) + C_l^-(\alpha), xC_r^-(\alpha) + C_r^-(\alpha)]
\end{aligned} \tag{60}$$

Now we denote

$$\begin{aligned}
&[xC_l^+(\alpha) + C_l^+(\alpha), xC_r^+(\alpha) + C_r^+(\alpha)] = M_\alpha, \\
&[xC_l^-(\alpha) + C_l^-(\alpha), xC_r^-(\alpha) + C_r^-(\alpha)] = M^\alpha
\end{aligned} \tag{61}$$

It easy to see that (M_α, M^α) satisfy (a)-(d) of Proposition 2 and by the Lemma 3 we can construct the intuitionistic fuzzy solution $\langle w,z\rangle(x,y) \in IF_1$ for (45)-(48) by the following form:

$$\begin{aligned}
&[\langle w,z\rangle (x,y)]_\alpha \\
&\quad = [xC_l^+(\alpha) + C_l^+(\alpha), xC_r^+(\alpha) + C_r^+(\alpha)] \\
&[\langle w,z\rangle (x,y)]^\alpha \\
&\quad = [xC_l^-(\alpha) + C_l^-(\alpha), xC_r^-(\alpha) + C_r^-(\alpha)]
\end{aligned} \tag{62}$$

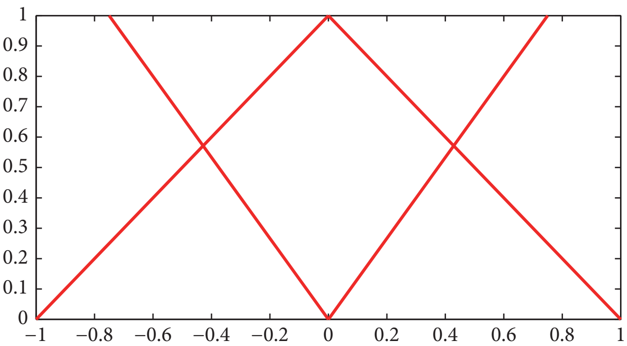

Figure 1: C=(-1,0,1;-0.75,0,0.75).

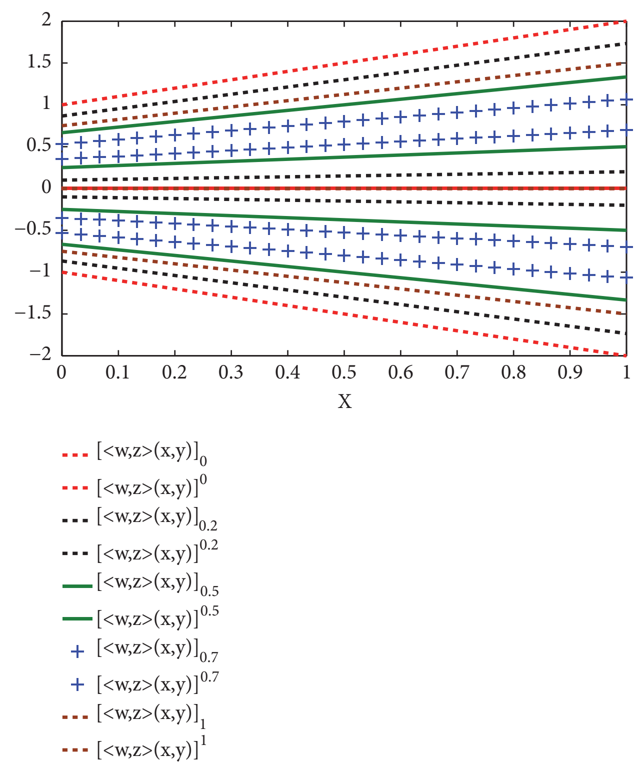

Figure 2: The 2D graphs of some α-cuts of intuitionistic fuzzy solutions.

for every $\alpha \in [0,1]$.

Therefore, $\langle w,z\rangle(x,y)$ is an intuitionistic fuzzy solution which also satisfies the initial conditions (46)-(48). This solution can be expressed by

$$\langle w,z\rangle (x,y) = Cx + C \tag{63}$$

Numerical simulations are used to obtain a graphical representation of the intuitionistic fuzzy solution. The membership function and nonmembership mapping of TIFN C=(-1,0,1;-0.75,0,0.75) in Figure 1.

Figures 2 and 3 show the α-cuts of intuitionistic fuzzy solutions in 2D and 3D. Concretely, the curves of the surface correspond to the lower bounded and the upper bounded solutions of $\langle w,z\rangle(x,y)$. They are the curves of level sets $[\langle w,z\rangle(x,y)]_\alpha$ and $[\langle w,z\rangle(x,y)]_\alpha$; here we choose $\alpha = 0.2$,

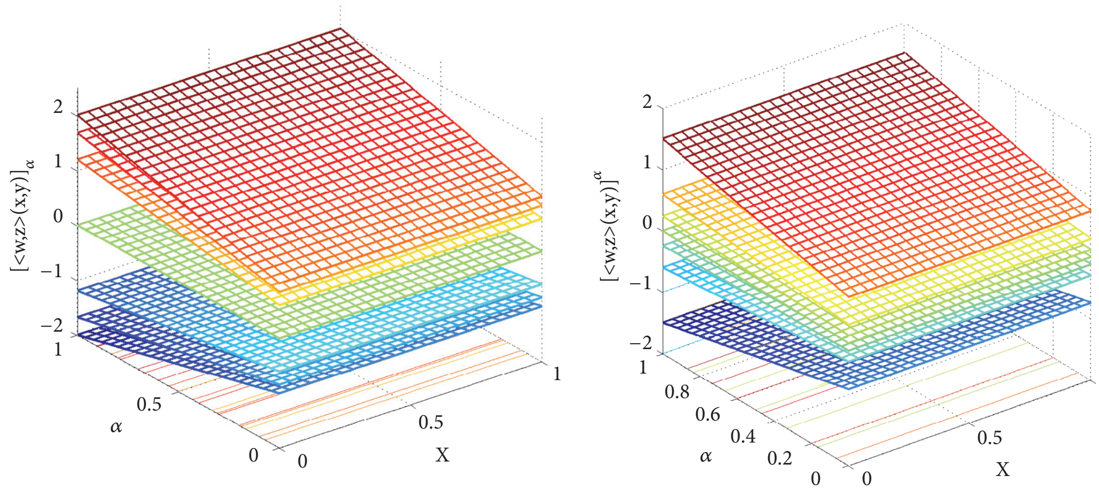

Figure 3: The 3D graphs of some α-cuts of intuitionistic fuzzy solutions.

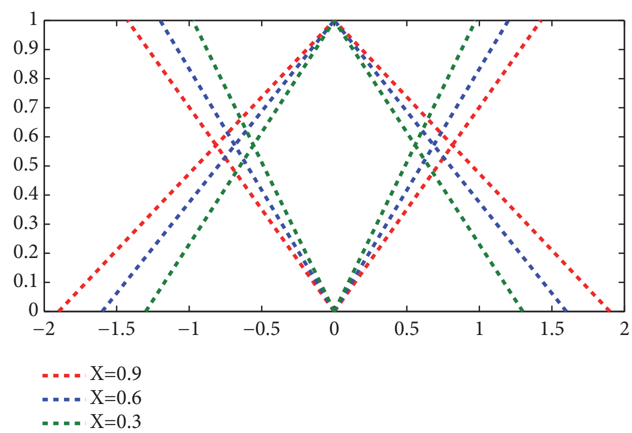

Figure 4: The curve of intuitionistic fuzzy solutions $\langle w, z\rangle(x, y)$ at different values of (x, y).

$\alpha = 0.5$, and $\alpha = 0.7$ for demonstrating upper-lower bounded solutions. The curves in the middle of the surface correspond to the crisp solutions $[\langle w, z\rangle(x, y)]_1$ and $[\langle w, z\rangle(x, y)]^0$.

The intuitionistic fuzzy solutions $\langle w, z\rangle(x, y)$ at different values of (x, y) in Figure 4.

In Figure 5 we present the surface of intuitionistic fuzzy solution with triangular intuitionistic fuzzy number C=(-1,0,1;-0.75,0,0.75).

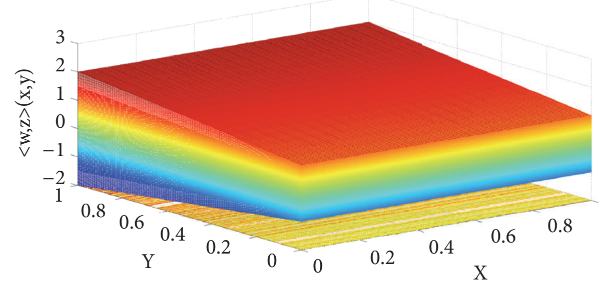

Figure 5: The surface of intuitionistic fuzzy solution.

7. Conclusion

This paper investigates the existence and uniqueness of intuitionistic fuzzy solutions for partial functional differential equations with local and nonlocal conditions. We have achieved these goals by using the Banach fixed point theorem. As far as we know, the deterministic class of partial functional differential equations was well studied by many authors but this is the first paper that addresses intuitionistic fuzzy cases. We have used the level-set representation of intuitionistic fuzzy functions and have defined the solution to an intuitionistic fuzzy partial functional differential equation problem through a corresponding parametric problem and further develop theoretical results on the existence and uniqueness of the solution. These results are illustrated by a computational example. For our future research, we will develop the proposed approach for studying different types of intuitionistic fuzzy partial differential equations in an ordered generalized metric space [7] with the use of fixed point principles under some conditions which are weaker than Lipschitz.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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