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Research Article

High-Speed Transmission in Long-Haul Electrical Systems

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We study the equations governing the high-speed transmission in long-haul electrical systems $i\partial_t u - (1/3)|\partial_x|^3 u = i\lambda\partial_x(|u|^2 u)$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $u(0,x) = u_0(x)$, $x \in \mathbb{R}$, where $\lambda \in \mathbb{R}$, $|\partial_x|^\alpha = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}$, and $\mathcal{F}$ is the Fourier transformation. Our purpose in this paper is to obtain the large time asymptotics for the solutions under the nonzero mass condition $\int u_0(x)dx \neq 0$.

1. Introduction

We study the equations governing the high-speed transmission in long-haul electrical systems

$$\begin{aligned} i\partial_t u - \frac{1}{3}\left|\partial_x\right|^3 u &= i\lambda\partial_x\left(|u|^2 u\right), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{R}$, $|\partial_x|^3 = \mathcal{F}^{-1}|\xi|^3\mathcal{F}$, and $\mathcal{F}$ is the Fourier transformation defined by $\mathcal{F}\phi = (1/\sqrt{2\pi})\int_{\mathbb{R}} e^{-ix\xi}\phi\,dx$. Note that we have the relation $u(-t,x) = \overline{u}(t,-x)$, so we can only consider the case $t > 0$. For the regular solution of (1) we have the conservation law $\|u(t)\|_{\mathbf{L}^2} = \|u_0\|_{\mathbf{L}^2}$. We are interested in the case of nonzero mass condition $\int_{\mathbb{R}} u_0(x)dx \neq 0$. By (1) we get the conservation of the mass $\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx \neq 0$ for all $t > 0$.

This equation arises in the context of high-speed soliton transmission in long-haul optical communication system [1]. Also it can be considered as a particular form of the higher order nonlinear Schrödinger equation introduced by [2] to describe the nonlinear propagation of pulses through optical fibers. This equation also represents the propagation of pulses by taking higher dispersion effects into account than those given by the Schrödinger equation (see [3–11]).

The higher order nonlinear Schrödinger equations have been widely studied recently. For the local and global well-posedness of the Cauchy problem we refer to [12–14] and references cited therein. The dispersive blow-up was obtained in [15]. The existence and uniqueness of solutions to (1) were proved in [16–25] and the smoothing properties of solutions were studied in [18–21, 24, 26–31]. The blow-up effect for a special class of slowly decaying solutions of Cauchy problem (1) was found in [32].

As far as we know the question of the large time asymptotics for solutions to Cauchy problem (1) is an open problem. We develop here the factorization technique originated in our previous papers [33–38].

We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = (\int |\phi(x)|^p dx)^{1/p}$ for $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbb{R}}|\phi(x)|$. The weighted Sobolev space is $\mathbf{H}_p^{m,s} = \{\varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{m,s}} = \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^p} < \infty\}$, where $m, s \in \mathbb{R}$, $1 \leq p \leq \infty$, $\langle x\rangle = \sqrt{1+x^2}$, and $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C. We denote by $\mathcal{F}\phi$ or $\widehat{\phi}(\xi) = (1/\sqrt{2\pi})\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx$ the Fourier transform of the function ϕ, then the inverse Fourier transformation is given by $\mathscr{F}^{-1}\phi = (1/\sqrt{2\pi})\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi$.

We are now in a position to state our result.

Theorem 1. *Assume that the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$ have a sufficiently small norm $\|u_0\|_{\mathbf{H}^1\cap\mathbf{H}^{0,1}} \le \varepsilon$. Then there exists a unique global solution $\mathscr{F}e^{-(it/3)|\partial_x|^3}u \in \mathbf{C}([0,\infty);\mathbf{L}^\infty \cap \mathbf{H}^{0,1})$ of Cauchy problem (1). Furthermore the estimate*

$$\begin{aligned}\sup_{t>0}\Big(&\left\|\mathscr{F}e^{-(it/3)|\partial_x|^3}u(t)\right\|_{\mathbf{L}^\infty}\\&+\langle t\rangle^{-1/6}\left\|xe^{-(it/3)|\partial_x|^3}u(t)\right\|_{\mathbf{L}^2}\\&+\langle t\rangle^{(1/3)(1-1/p)}\|u(t)\|_{\mathbf{L}^p}\Big)\le C\varepsilon\end{aligned} \tag{2}$$

is true, where $p > 4$.

Next we prove the existence of the self-similar solutions $v_m(t,x) = t^{-1/3}f_m(xt^{-1/3})$.

Theorem 2. *There exists a unique solution of Cauchy problem (1) in the self-similar form $v_m(t,x) = t^{-1/3}f_m(xt^{-1/3})$, such that*

$$m = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} v_m(t,x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f_m(x)\,dx \ne 0, \tag{3}$$

where m is sufficiently small number and

$$\begin{aligned}&\mathscr{F}e^{-(it/3)|\partial_x|^3}v_m \in \mathbf{C}\left([1,\infty);\mathbf{L}^\infty\right),\\&xe^{-(it/3)|\partial_x|^3}v_m \in \mathbf{C}\left([1,\infty);\mathbf{L}^2\right).\end{aligned} \tag{4}$$

Furthermore the estimate

$$\begin{aligned}\sup_{t>1}\Big(&\left\|\mathscr{F}e^{-(it/3)|\partial_x|^3}v_m(t)\right\|_{\mathbf{L}^\infty}\\&+t^{-1/6}\left\|xe^{-(it/3)|\partial_x|^3}v_m(t)\right\|_{\mathbf{L}^2}\\&+t^{(1/3)(1-1/p)}\left\|v_m(t)\right\|_{\mathbf{L}^p}\Big)\le C|m|\end{aligned} \tag{5}$$

is true, where $p > 4$.

Now we state the stability of solutions to Cauchy problem (1) in the neighborhood of the self-similar solution $v_m(t,x)$.

Theorem 3. *Suppose that*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f_m(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} u_0(x)\,dx = m \ne 0. \tag{6}$$

Let $u(t,x)$ and $v_m(t,x)$ be the solutions constructed in Theorems 1 and 2, respectively. Then there exists small $\gamma > 0$ such that the asymptotics

$$\left\|u(t)-v_m(t)\right\|_{\mathbf{L}^\infty} \le Ct^{-1/2+\gamma} \tag{7}$$

are true for $t \ge 1$.

Our approach is based on the factorization techniques. Define the free evolution group $\mathscr{U}(t) = \mathscr{F}^{-1}e^{-(it/3)|\xi|^3}\mathscr{F}$ and write

$$\mathscr{U}(t)\mathscr{F}^{-1}\phi = \mathscr{D}_t\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{it(x\xi-(1/3)|\xi|^3)}\phi(\xi)\,d\xi, \tag{8}$$

where $\mathscr{D}_t\phi = |t|^{-1/2}\phi(x/t)$ is the dilation operator. There is a unique stationary point $\xi = \mu(x) \equiv (x/|x|)\sqrt{|x|}$ in the integral $\int_{\mathbb{R}} e^{it(x\xi-(1/3)|\xi|^3)}\phi(\xi)d\xi$, which is defined as the root of the equation $\xi|\xi| = x$ for all $x \in \mathbb{R}$. Define the scaling operator $(\mathscr{B}\phi)(x) = \phi(\mu(x))$. Hence we find the following decomposition $\mathscr{U}(t)\mathscr{F}^{-1}\phi = \mathscr{D}_t\mathscr{B}M\mathscr{V}\phi$, where the multiplication factor $M = e^{(2it/3)|\eta|^3}$ and the deformation operator

$$\mathscr{V}(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi, \tag{9}$$

where the phase function $S(\xi,\eta) = (1/3)|\xi|^3 - (1/3)|\eta|^3 - \eta|\eta|(\xi-\eta)$. Denote $\mathscr{A}_k = \overline{M}^k(1/2t|\eta|)\partial_\eta M^k$, $k = 0,1$. We have $\mathscr{A}_1 = \mathscr{A}_0 + i\eta$, and also $\mathscr{A}_1\mathscr{V} = \mathscr{V}i\xi$, $[i\eta,\mathscr{V}] = -\mathscr{A}_0\mathscr{V}$; therefore we obtain the commutator $\partial_\eta\mathscr{V} = -2t|\eta|[i\eta,\mathscr{V}]$. Since $\partial_\xi S(\xi,\eta) = \xi|\xi| - \eta|\eta|$, then we get $it[\eta|\eta|,\mathscr{V}]\phi = -\mathscr{V}\partial_\xi\phi$. Also we need the representation for the inverse evolution group $\mathscr{F}\mathscr{U}(-t)\phi = \mathscr{V}^*\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}$, where the inverse dilation operator $\mathscr{D}_t^{-1}\phi = |t|^{1/2}\phi(xt)$, the inverse scaling operator $(\mathscr{B}^{-1}\phi)(\eta) = \phi(\eta|\eta|)$, and the inverse deformation operator

$$\mathscr{V}^*(t)\phi = \sqrt{\frac{2|t|}{\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)|\eta|\,d\eta. \tag{10}$$

We have $i\xi\mathscr{V}^*\phi = \mathscr{V}^*\mathscr{A}_1\phi$. Hence the commutator $[i\xi,\mathscr{V}^*] = \mathscr{V}^*\mathscr{A}_0$. Define the new dependent variable $\widehat{\varphi} = \mathscr{F}\mathscr{U}(-t)u(t)$. Since $\mathscr{F}\mathscr{U}(-t)\mathscr{L} = \partial_t\mathscr{F}\mathscr{U}(-t)$ with $\mathscr{L} = \partial_t + (i/3)|\partial_x|^3$, applying the operator $\mathscr{F}\mathscr{U}(-t)$ to (1), substituting $u(t) = \mathscr{U}(t)\mathscr{F}^{-1}\widehat{\varphi} = \mathscr{D}_t\mathscr{B}M\mathscr{V}\widehat{\varphi}$, and using the factorization techniques, we get

$$\begin{aligned}\partial_t\widehat{\varphi} &= \mathscr{F}\mathscr{U}(-t)\mathscr{L}u = i\lambda\xi\mathscr{F}\mathscr{U}(-t)\left(|u|^2u\right)\\&= i\lambda\xi\mathscr{V}^*\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\left(\left|\mathscr{D}_t\mathscr{B}M\mathscr{V}\widehat{\varphi}\right|^2\mathscr{D}_t\mathscr{B}M\mathscr{V}\widehat{\varphi}\right)\\&= i\lambda\xi t^{-1}\mathscr{V}^*\overline{M}\mathscr{B}^{-1}\left(\left|\mathscr{B}M\mathscr{V}\widehat{\varphi}\right|^2\mathscr{B}M\mathscr{V}\widehat{\varphi}\right)\\&= i\lambda\xi t^{-1}\mathscr{V}^*\overline{M}\left(\left|M\mathscr{V}\widehat{\varphi}\right|^2M\mathscr{V}\widehat{\varphi}\right)\\&= i\lambda\xi t^{-1}\mathscr{V}^*\left(\left|\mathscr{V}\phi\right|^2\mathscr{V}\phi\right)\end{aligned} \tag{11}$$

since the nonlinearity is gauge invariant. Finally we mention some important identities. The operator $\mathscr{J} = \mathscr{U}(t)x\mathscr{U}(-t) = x + it\partial_x|\partial_x|$ plays a crucial role in the large time asymptotic estimates. Note that $\mathscr{J}$ commutes with $\mathscr{L}$, that is, $[\mathscr{J},\mathscr{L}] = 0$. To avoid the derivative loss we also use the operator $\mathscr{P} = 3t\partial_t + \partial_x x$. Note the commutator relation $[\widehat{\mathscr{P}}, e^{-(it/3)|\xi|^3}] = 0$

with $\widehat{\mathscr{P}} = 3t\partial_t - \xi\partial_\xi$. Thus using $u(t) = \mathscr{U}(t)\mathscr{F}^{-1}\widehat{\varphi} = \mathscr{D}_t M\mathscr{V}\widehat{\varphi} = \mathscr{F}^{-1}e^{-(it/3)|\xi|^3}\widehat{\varphi}$, we get $\mathscr{P}u = \mathscr{U}(t)\mathscr{F}^{-1}\widehat{\mathscr{P}}\widehat{\varphi}$. Also we have the identity $\mathscr{J} = \partial_x^{-1}\mathscr{P} - 3t\partial_x^{-1}\mathscr{L}$ and $[\mathscr{L},\mathscr{P}] = 3\mathscr{L}$ holds.

2. Estimates in the Uniform Norm

2.1. Kernels. Define the kernel

$$A_j(t,\eta) = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\Theta\left(\xi\eta^{-1}\right)\xi^j d\xi \tag{12}$$

for $\eta \neq 0$, where the cutoff function $\Theta(z) \in \mathbf{C}^2(\mathbb{R})$ is such that $\Theta(z) = 0$ for $z \leq 1/3$ or $z \geq 3$ and $\Theta(z) = 1$ for $2/3 \leq z \leq 3/2$. We change $\xi = \eta y$, then we get $A_j(t,\eta) = |\eta|\eta^j\sqrt{|t|/2\pi}\int_{1/3}^{3} e^{-it|\eta|^3 G(y)}\Theta(y)y^j dy$, where $S(\eta y,\eta) = |\eta|^3 G(y)$ and $G(y) = (1/3)(y+2)(y-1)^2$, $y > 0$. To compute the asymptotics of the kernel $A_j(t,\eta)$ for large t we apply the stationary phase method (see [39], p. 110)

$$\begin{aligned}\int_{\mathbb{R}} e^{izg(y)} f(y)\,dy &= e^{izg(y_0)} f(y_0)\sqrt{\frac{2\pi}{z|g''(y_0)|}}e^{i(\pi/4)\operatorname{sgn} g''(y_0)} \\ &\quad + O\left(z^{-3/2}\right)\end{aligned} \tag{13}$$

for $z \to +\infty$, where the stationary point y_0 is defined by the equation $g'(y_0) = 0$. By virtue of formula (13) with $g(y) = -G(y)$, $f(y) = \Theta(y)y^j$, and $y_0 = 1$, we get

$$A_j(t,\eta) = \frac{t^{1/2}|\eta|\eta^j}{\sqrt{2i\langle t\eta^3\rangle}} + O\left(t^{1/2}\eta^{1+j}\left\langle t\eta^3\right\rangle^{-1}\right) \tag{14}$$

for $t\eta^3 \to \infty$. Also we have the estimate $|A_j(t,\eta)| \leq Ct^{1/2}|\eta|^{j+1}\langle t\eta^3\rangle^{-1/2}$.

In the same manner changing $\eta = \xi y$, we get for the kernel

$$\begin{aligned}A^*(t,\xi) &= \sqrt{\frac{2|t|}{\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\Theta\left(\eta\xi^{-1}\right)|\eta|\,d\eta \\ &= \xi^2\sqrt{\frac{2|t|}{\pi}}\int_{1/3}^{3} e^{it|\xi|^3\widetilde{G}(y)}\Theta(y)|y|\,dy\end{aligned} \tag{15}$$

for $\xi \neq 0$, where $S(\xi,\xi y) = |\xi|^3\widetilde{G}(y)$ with $\widetilde{G}(y) = (1/3)(2y+1)(y-1)^2$, $y > 0$. Then by virtue of formula (13) with $g(y) = \widetilde{G}(y)$, $f(y) = \Theta(y)|y|$, and $y_0 = 1$, we obtain

$$A^*(t,\xi) = 2t^{1/2}\xi^2\sqrt{\frac{2i}{\langle t\xi^3\rangle}} + O\left(t^{1/2}\xi^2\left\langle t\xi^3\right\rangle^{-1}\right) \tag{16}$$

for $t\xi^3 \to \infty$. Also we have the estimate $|A^*(t,\xi)| \leq Ct^{1/2}\xi^2\langle t\xi^3\rangle^{-1/2}$.

2.2. Asymptotics for the Operator $\mathscr{V}$. In the next lemma we estimate the operator $\mathscr{V}$ in the uniform norm. Define the cutoff function $\chi_1(z) \in \mathbf{C}^2(\mathbb{R})$ such that $\chi_1(z) = 0$ for $|z| \geq 3$ and $\chi_1(z) = 1$ for $|z| \leq 2$ and $\chi_2(z) = 1 - \chi_1(z)$. Consider two operators

$$\mathscr{V}_j(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi)\chi_j\left(\xi\eta^{-1}\right)d\xi, \tag{17}$$

so that we have $\mathscr{V}(t)\phi = \mathscr{V}_1(t)\phi + \mathscr{V}_2(t)\phi$ for $\eta \neq 0$. Define the norm $\|\phi\|_{\mathbf{Y}} = \|\phi\|_{\mathbf{L}^\infty} + t^{-1/6}\|\partial_\xi\phi\|_{\mathbf{L}^2}$.

Lemma 4. *The following estimates $|\mathscr{V}_1\xi^j\phi - A_j(t,\eta)\phi(\eta)| \leq Ct^{1/2}|\eta|^{j+1}\langle t^{1/3}\eta\rangle^{-7/4}\|\phi\|_{\mathbf{Y}}$ if $j \geq 0$ and $|\mathscr{V}_2\xi^j\phi| \leq Ct^{1/6-j/3}\langle t^{1/3}\eta\rangle^{j-3/2}\|\phi\|_{\mathbf{Y}}$ if $j = 0,1$ are valid for all $t \geq 1$, $\eta \neq 0$.*

Proof. We write

$$\begin{aligned}\mathscr{V}_1\xi^j\phi - A_j\phi &= \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\left(\phi(\xi) - \phi(\eta)\right) \\ &\quad \cdot \Theta\left(\xi\eta^{-1}\right)\xi^j d\xi + \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi) \\ &\quad \cdot \left(\chi_1\left(\xi\eta^{-1}\right) - \Theta\left(\xi\eta^{-1}\right)\right)\xi^j d\xi = I_1 + I_2\end{aligned} \tag{18}$$

for $\eta \neq 0$. For the summand I_1 we integrate by parts via identity

$$e^{-itS(\xi,\eta)} = H_1\partial_\xi\left((\xi-\eta)e^{-itS(\xi,\eta)}\right) \tag{19}$$

with $H_1 = (1 - it(\xi-\eta)\partial_\xi S(\xi,\eta))^{-1}$, to get

$$\begin{aligned}I_1 &= Ct^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\left(\phi(\xi) - \phi(\eta)\right)(\xi-\eta) \\ &\quad \cdot \partial_\xi\left(H_1\Theta\left(\xi\eta^{-1}\right)\xi^j\right)d\xi \\ &\quad + Ct^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}(\xi-\eta)H_1\Theta\left(\xi\eta^{-1}\right) \\ &\quad \cdot \xi^j\partial_\xi\phi(\xi)\,d\xi.\end{aligned} \tag{20}$$

We find the estimates

$$\begin{aligned}&\left|H_1\Theta\left(\xi\eta^{-1}\right)\xi^j\right| + \left|(\xi-\eta)\partial_\xi\left(H_1\Theta\left(\xi\eta^{-1}\right)\xi^j\right)\right| \\ &\quad \leq \frac{C|\eta|^j}{1 + t|\eta|(\xi-\eta)^2}\end{aligned} \tag{21}$$

in the domain $1/3 < \xi/\eta < 3$. Therefore we obtain

$$\begin{aligned}|I_1| &\leq Ct^{1/2}|\eta|^j \\ &\quad \cdot \int_{1/3<\xi/\eta<3}\frac{|\phi(\xi) - \phi(\eta)|}{|\xi-\eta|}\frac{|\xi-\eta|\,d\xi}{1 + t|\eta|(\xi-\eta)^2} \\ &\quad + Ct^{1/2}|\eta|^j\int_{1/3<\xi/\eta<3}\frac{|\xi-\eta|\left|\partial_\xi\phi(\xi)\right|d\xi}{1 + t|\eta|(\xi-\eta)^2}.\end{aligned} \tag{22}$$

By the Hardy inequality $\int_{1/3<\xi/\eta<3}(|\phi(\xi)-\phi(\eta)|^2/|\xi-\eta|^2)d\xi \le C\|\partial_\xi\phi\|^2_{\mathbf{L}^2}$ and by the Cauchy-Schwarz inequality, changing $\xi = \eta y$ we find

$$\begin{aligned}|I_1| &\le Ct^{1/2}|\eta|^j\|\partial_\xi\phi\|_{\mathbf{L}^2}\\ &\quad\cdot\left(\int_{1/3<\xi/\eta<3}\frac{(\xi-\eta)^2 d\xi}{(1+t|\eta|(\xi-\eta)^2)^2}\right)^{1/2}\\ &\le Ct^{1/2}|\eta|^{j+3/2}\|\partial_\xi\phi\|_{\mathbf{L}^2}\\ &\quad\cdot\left(\int_{1/3}^{3}\frac{(y-1)^2 dy}{(1+|t||\eta|^3(y-1)^2)^2}\right)^{1/2}\\ &\le Ct^{1/2}\|\partial_\xi\phi\|_{\mathbf{L}^2}|\eta|^{j+3/2}\langle t\eta^3\rangle^{-3/4}.\end{aligned}\tag{23}$$

To estimate the integral I_2 we integrate by parts via the identity

$$e^{-itS(\xi,\eta)} = H_2\partial_\xi\left(\xi e^{-itS(\xi,\eta)}\right)\tag{24}$$

with $H_2 = (1-it\xi\partial_\xi S(\xi,\eta))^{-1}$, to get

$$\begin{aligned}I_2 &= Ct^{1/2}\phi(0)\\ &\quad\cdot\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\xi\partial_\xi\left(\left(\chi_1(\xi\eta^{-1})-\Theta(\xi\eta^{-1})\right)H_2\xi^j\right)d\xi\\ &\quad+Ct^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\\ &\quad\cdot\frac{\phi(\xi)-\phi(0)}{\xi}\xi^2\partial_\xi\left(\left(\chi_1(\xi\eta^{-1})-\Theta(\xi\eta^{-1})\right)\right.\\ &\quad\left.\cdot H_2\xi^j\right)d\xi+Ct^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\left(\chi_1(\xi\eta^{-1})\right.\\ &\quad\left.-\Theta(\xi\eta^{-1})\right)H_2\xi^{j+1}\phi_\xi(\xi)\,d\xi.\end{aligned}\tag{25}$$

We find the estimates $|\xi\partial_\xi((\chi_1(\xi\eta^{-1})-\Theta(\xi\eta^{-1}))H_2\xi^j)| \le C|\xi|^j/(1+t|\xi|\eta^2)$ and

$$\begin{aligned}&\left|\left(\chi_1(\xi\eta^{-1})-\Theta(\xi\eta^{-1})\right)H_2\xi^{j+1}\right|\\ &\quad+\left|\xi^2\partial_\xi\left(\left(\chi_1(\xi\eta^{-1})-\Theta(\xi\eta^{-1})\right)H_2\xi^j\right)\right|\\ &\le\frac{C|\xi|^{1+j}}{1+t|\xi|\eta^2}.\end{aligned}\tag{26}$$

Then by the Hardy inequality we obtain

$$\begin{aligned}|I_2| &\le Ct^{1/2}|\phi(0)|\int_{|\xi|\le 3|\eta|}\frac{|\xi|^j d\xi}{1+t|\xi|\eta^2}\\ &\quad+Ct^{1/2}\|\partial_\xi\phi\|_{\mathbf{L}^2}\left(\int_{|\xi|\le 3|\eta|}\frac{|\xi|^{2+2j}d\xi}{(1+t\xi\eta^2)^2}\right)^{1/2}.\end{aligned}\tag{27}$$

We have $\int_{|\xi|\le3|\eta|}(|\xi|^j d\xi/(1+t|\xi|\eta^2)) \le C|\eta|^{j+1}\int_0^3(y^j dy/(1+t|\eta|^3y)) \le C|\eta|^{j+1}\langle t\eta^3\rangle^{-1}\log\langle t\eta^3\rangle$ and $\int_{|\xi|\le3|\eta|}(|\xi|^{2+2j}d\xi/(1+t|\xi|\eta^2)^2) \le C|\eta|^{2j+3}\int_0^3(y^{2j+2}dy/(1+t|\eta|^3y)^2) \le C|\eta|^{2j+3}\langle t\eta^3\rangle^{-2}$. Thus we have $|I_2| \le Ct^{1/2}|\phi(0)||\eta|^{j+1}\langle t\eta^3\rangle^{-3/4}+Ct^{1/2}\|\partial_\xi\phi\|_{\mathbf{L}^2}|\eta|^{j+3/2}\langle t\eta^3\rangle^{-1}$ for all $\ge 1,\eta\ne 0$.

To estimate $\mathscr{V}_2\xi^j\phi$ we integrate by parts via identity (24)

$$\begin{aligned}\mathscr{V}_2\xi^j\phi &= Ct^{1/2}\phi(0)\\ &\quad\cdot\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\xi\partial_\xi\left(H_2\chi_2(\xi\eta^{-1})\xi^j\right)d\xi\\ &\quad+Ct^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\\ &\quad\cdot\frac{\phi(\xi)-\phi(0)}{\xi}\xi^2\partial_\xi\left(H_2\chi_2(\xi\eta^{-1})\xi^j\right)d\xi\\ &\quad+Ct^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\xi^{1+j}\chi_2(\xi\eta^{-1})H_2\phi_\xi(\xi)\,d\xi.\end{aligned}\tag{28}$$

We find the estimates $|\xi\partial_\xi(H_2\chi_2(\xi\eta^{-1})\xi^j)| \le C|\xi|^j/(1+t|\xi|^3)$ and $|\xi^2\partial_\xi(H_2\chi_2(\xi\eta^{-1})\xi^j)|+|\xi^{1+j}\chi_2(\xi\eta^{-1})H_2| \le C|\xi|^{1+j}/(1+t|\xi|^3)$ in the domain $|\xi| \ge (3/2)|\eta|$. Then by the Hardy inequality we obtain

$$\begin{aligned}&\left|\mathscr{V}_2\xi^j\phi\right|\\ &\le Ct^{1/2}|\phi(0)|\int_{|\xi|\ge(3/2)|\eta|}\frac{|\xi|^j d\xi}{1+t|\xi|^3}\\ &\quad+Ct^{1/2}\|\partial_\xi\phi\|_{\mathbf{L}^2}\left(\int_{|\xi|\ge(3/2)|\eta|}\frac{|\xi|^{2+2j}d\xi}{(1+t|\xi|^3)^2}\right)^{1/2}.\end{aligned}\tag{29}$$

We have $\int_{|\xi|\ge(3/2)|\eta|}(|\xi|^j d\xi/(1+t|\xi|^3)) \le C|\eta|^{1+j}\int_{3/2}^\infty(y^j dy/(1+t|\eta|^3y^3)) \le Ct^{-(j+1)/3}\langle t\eta^3\rangle^{(j-2)/3}$ and $\int_{|\xi|\ge(3/2)|\eta|}(|\xi|^{2+2j}d\xi/(1+t|\xi|^3)^2) \le C|\eta|^{3+2j}\int_{3/2}^\infty(y^{2+2j}dy/(1+t|\eta|^3y^3)^2) \le Ct^{-2j/3-1}\langle t\eta^3\rangle^{2j/3-1}$ if $j=0,1$. Thus we have $|\mathscr{V}_2\xi^j\phi| \le C|\phi(0)|t^{1/6-j/3}\langle t\eta^3\rangle^{(j-2)/3}+C\|\partial_\xi\phi\|_{\mathbf{L}^2}t^{-j/3}\langle t\eta^3\rangle^{j/3-1/2}$ for all $t\ge1$ if $j=0,1$. Lemma 13 is proved. □

2.3. Asymptotics for the Operator $\mathscr{V}^*$. We next consider the operator $\mathscr{V}^*$. Since $\|\mathscr{V}^*\phi\|_{\mathbf{L}^\infty} \le C|t|^{1/2}\|\eta\phi\|_{\mathbf{L}^1}$ and $\|\mathscr{V}^*\phi\|_{\mathbf{L}^2} \le C\|\sqrt{|\eta|}\phi\|_{\mathbf{L}^2}$, then by the Riesz interpolation theorem (see [40], p. 52) we have

$$\|\mathscr{V}^*\phi\|_{\mathbf{L}^p} \le C|t|^{1/2-1/p}\left\||\eta|^{1-1/p}\phi\right\|_{\mathbf{L}^{p/(p-1)}}\tag{30}$$

for $2 \le p \le \infty$. In the next lemma we find the asymptotics of $\mathscr{V}^*$. Denote $\hat{\xi} = \xi t^{1/3}$. Also define the norm $\|\phi\|_{\mathbf{I}_{\alpha,\beta}} = \||\eta|^\alpha\langle\hat{\eta}\rangle^{-\beta}\partial_\eta\phi\|_{\mathbf{L}^2}+\||\eta|^{\alpha-1}\langle\hat{\eta}\rangle^{-\beta}\phi\|_{\mathbf{L}^2}$.

Lemma 5. *Let $1/4 + 2\beta \le \alpha < 5/2 - 2\beta$, $0 \le \beta < 1/2$. Then the estimate $\|\langle\xi\rangle^{\beta}(\mathscr{V}^{*}\phi - A^{*}\phi)\|_{\mathbf{L}^{\infty}} \le Ct^{(\alpha-1)/3}\|\phi\|_{\mathbf{I}_{\alpha,\beta}}$ is valid for all $t \ge 1$.*

Proof. We write

$$\begin{aligned}
&\mathscr{V}^{*}\phi - A^{*}\phi \\
&= \sqrt{\frac{2|t|}{\pi}}\int_{-\infty}^{\infty} e^{itS(\xi,\eta)}(\phi(\eta)-\phi(\xi))\Theta(\eta\xi^{-1})|\eta|\,d\eta \\
&+ \sqrt{\frac{2|t|}{\pi}}\int_{-\infty}^{\infty} e^{itS(\xi,\eta)}\phi(\eta)(1-\Theta(\eta\xi^{-1}))|\eta|\,d\eta \\
&= I_1 + I_2
\end{aligned} \tag{31}$$

for $\xi \ne 0$. In the integral I_1 we use the identity

$$e^{itS(\xi,\eta)} = H_3\partial_\eta\left((\eta-\xi)e^{itS(\xi,\eta)}\right) \tag{32}$$

with $H_3 = (1 + it(\eta-\xi)\partial_\eta S(\xi,\eta))^{-1}$, $\partial_\eta S(\xi,\eta) = -2|\eta|(\xi-\eta)$, and integrate by parts

$$\begin{aligned}
I_1 &= Ct^{1/2}\int_{-\infty}^{\infty} e^{itS(\xi,\eta)}\frac{\phi(\eta)-\phi(\xi)}{\eta-\xi}|\eta|^{\alpha}\langle\hat{\eta}\rangle^{-\beta}(\eta-\xi)^2 \\
&\cdot\partial_\eta\left(H_3|\eta|^{1-\alpha}\langle\hat{\eta}\rangle^{\beta}\Theta(\eta\xi^{-1})\right)d\eta \\
&+ Ct^{1/2}\int_{-\infty}^{\infty} e^{itS(\xi,\eta)}(\eta-\xi)H_3|\eta|^{1-\alpha}\langle\hat{\eta}\rangle^{\beta}\Theta(\eta\xi^{-1}) \\
&\cdot\partial_\eta\left(|\eta|^{\alpha}\langle\hat{\eta}\rangle^{-\beta}\phi(\eta)\right)d\eta.
\end{aligned} \tag{33}$$

Then apply the estimates $|(\eta-\xi)H_3|\eta|^{1-\alpha}\langle\hat{\eta}\rangle^{\beta}\Theta(\eta\xi^{-1})| + |(\eta-\xi)^2\langle\hat{\eta}\rangle^{\beta}\partial_\eta(H_3|\eta|^{1-\alpha}\Theta(\eta\xi^{-1}))| \le C|\xi|^{1-\alpha}\langle\hat{\xi}\rangle^{\beta}|\eta-\xi|/(1+t|\xi|(\eta-\xi)^2)$ in the domain $1/3 \le \eta/\xi \le 3$. If $|\kappa(x)| \le C|\kappa(yx)|$ for all $y \in (0,1)$ then we find the Hardy inequality

$$\begin{aligned}
&\left\|\frac{\kappa(x)}{x}(\phi(x)-\phi(0))\right\|_{\mathbf{L}^2} = \left\|\frac{\kappa(x)}{x}\int_0^x\phi'(y)\,dty\right\|_{\mathbf{L}^2} \\
&= \left\|\kappa(x)\int_0^1\phi'(xy)\,dy\right\|_{\mathbf{L}^2} \\
&\le C\left\|\int_0^1\kappa(xy)\phi'(xy)\,dy\right\|_{\mathbf{L}^2} \\
&\le \int_0^1\|\kappa(xy)\phi'(xy)\|_{\mathbf{L}^2}\,dt \le \int_0^1 y^{-1/2}dy\,\|\kappa\phi'\|_{\mathbf{L}^2} \\
&\le 2\|\kappa\phi'\|_{\mathbf{L}^2}.
\end{aligned} \tag{34}$$

Hence $\int_{1/3\le\eta/\xi\le3}(|\phi(\eta)-\phi(\xi)|^2/(\eta-\xi)^2)|\eta|^{2\alpha}\langle\hat{\eta}\rangle^{-2\beta}d\eta \le C\|\phi\|^2_{\mathbf{I}_{\alpha,\beta}}$. Therefore changing $\eta = y\xi$, we have

$$\begin{aligned}
&\langle\hat{\xi}\rangle^{\beta}|I_1| \le Ct^{1/2}|\xi|^{1-\alpha}\langle\hat{\xi}\rangle^{2\beta}\|\phi\|_{\mathbf{I}_{\alpha,\beta}} \\
&\cdot\left(\int_{1/3\le\eta/\xi\le3}\frac{(\eta-\xi)^2\,d\eta}{(1+t|\xi|(\eta-\xi)^2)^2}\right)^{1/2} \\
&\le Ct^{1/2}|\xi|^{1-\alpha}\langle\hat{\xi}\rangle^{2\beta}\|\phi\|_{\mathbf{I}_{\alpha,\beta}} \\
&\cdot\left(\int_{1/3}^{3}\frac{|\xi|^3(1-y)^2\,dy}{(1+t|\xi|^3(1-y)^2)^2}\right)^{1/2} \le Ct^{1/2}|\xi|^{5/2-\alpha} \\
&\cdot\langle\hat{\xi}\rangle^{2\beta-9/4}\|\phi\|_{\mathbf{I}_{\alpha,\beta}} \le Ct^{(\alpha-1)/3}\|\phi\|_{\mathbf{I}_{\alpha,\beta}}
\end{aligned} \tag{35}$$

if $1/4 + 2\beta \le \alpha \le 5/2$. In the integral I_2 using the identity

$$e^{itS(\xi,\eta)} = H_4\partial_\eta\left(\eta e^{itS(\xi,\eta)}\right) \tag{36}$$

with $H_4 = (1 + it\eta\partial_\eta S(\xi,\eta))^{-1}$, $\partial_\eta S(\xi,\eta) = -2|\eta|(\xi-\eta)$, we integrate by parts

$$\begin{aligned}
I_2 &= Ct^{1/2}\int_{-\infty}^{\infty} e^{itS(\xi,\eta)} \\
&\cdot\frac{|\eta|^{\alpha}\phi(\eta)}{\eta\langle\hat{\eta}\rangle^{\beta}}\eta^2\partial_\eta\left(H_4(1-\Theta(\eta\xi^{-1}))|\eta|^{1-\alpha}\right. \\
&\cdot\left.\langle\hat{\eta}\rangle^{\beta}\right)d\eta + Ct^{1/2}\int_{-\infty}^{\infty} e^{itS(\xi,\eta)}\eta H_4\left(1\right. \\
&\left.-\Theta(\eta\xi^{-1})\right)|\eta|^{1-\alpha}\langle\hat{\eta}\rangle^{\beta}\partial_\eta\left(|\eta|^{\alpha}\langle\hat{\eta}\rangle^{-\beta}\phi(\eta)\right)d\eta.
\end{aligned} \tag{37}$$

Then using the estimates $\partial_\eta S(\xi,\eta) = O(|\eta|(\xi+|\eta|))$ in the domains $\eta/\xi < 1/3$ and $\eta/\xi \ge 3$, we get

$$\begin{aligned}
&\left|\eta^2\partial_\eta\left(H_4(1-\Theta(\eta\xi^{-1}))|\eta|^{1-\alpha}\langle\hat{\eta}\rangle^{\beta}\right)\right| \\
&\quad + \left|\eta H_4(1-\Theta(\eta\xi^{-1}))|\eta|^{1-\alpha}\langle\hat{\eta}\rangle^{\beta}\right| \\
&\le \frac{C|\eta|^{2-\alpha}\langle\hat{\eta}\rangle^{\beta}}{1+t\eta^2(|\xi|+|\eta|)}.
\end{aligned} \tag{38}$$

Therefore by the Hardy inequality

$$\begin{aligned}
&\langle\hat{\xi}\rangle^{\beta}|I_2| \le Ct^{1/2}\langle\hat{\xi}\rangle^{\beta}\left\|\partial_\eta\left(|\eta|^{\alpha}\langle\hat{\eta}\rangle^{-\beta}\phi(\eta)\right)\right\|_{\mathbf{L}^2} \\
&\cdot\left(\int_{-\infty}^{\infty}\frac{\langle\hat{\eta}\rangle^{2\beta}|\eta|^{4-2\alpha}\,d\eta}{(1+t\eta^2(|\xi|+|\eta|))^2}\right)^{1/2} \le Ct^{1/2}\|\phi\|_{\mathbf{I}_{\alpha,\beta}} \\
&\cdot\left(\int_{-\infty}^{\infty}\frac{\langle\hat{\xi}\rangle^{2\beta}\langle\hat{\eta}\rangle^{2\beta}|\eta|^{4-2\alpha}\,d\eta}{(1+t\eta^2(|\xi|+|\eta|))^2}\right)^{1/2}.
\end{aligned} \tag{39}$$

We have

$$\int_{-\infty}^{\infty}\frac{\langle\widehat{\xi}\rangle^{2\beta}\langle\widehat{\eta}\rangle^{2\beta}|\eta|^{4-2\alpha}d\eta}{(1+t\eta^2(|\xi|+|\eta|))^2}\le C\int_0^1\frac{\langle\widehat{\xi}\rangle^{2\beta}\eta^{4-2\alpha}\langle\widehat{\eta}\rangle^{2\beta}d\eta}{(1+t\eta^2(|\xi|+|\eta|))^2}+Ct^{2\beta/3-2}\int_1^{\infty}\langle\widehat{\xi}\rangle^{2\beta}\frac{\eta^{2\beta-2\alpha}d\eta}{(|\xi|+|\eta|)^2}\le C\int_0^1\eta^{4-2\alpha}\langle\widehat{\eta}\rangle^{2\beta-6}d\eta+Ct^{-4\beta/3}\int_0^1\eta^{4-4\beta-2\alpha}\langle\widehat{\eta}\rangle^{6\beta-6}d\eta+Ct^{4\beta/3-2}\int_1^{\infty}\eta^{4\beta-2\alpha-2}d\eta\le Ct^{(2\alpha-5)/3}\int_0^{t^{1/3}}\eta^{4-2\alpha}\langle\eta\rangle^{2\beta-6}d\eta+Ct^{(2\alpha-5)/3}\int_0^{t^{1/3}}\eta^{4-4\beta-2\alpha}\langle\eta\rangle^{6\beta-6}d\eta+Ct^{4\beta/3-2}\le Ct^{(2\alpha-5)/3}\tag{40}$$

if $1/4+2\beta\le\alpha<5/2-2\beta$, $0\le\beta<1/2$. Therefore we get $\langle\widehat{\xi}\rangle^{\beta}|I_2|\le Ct^{(\alpha-1)/3}\|\phi\|_{\mathbf{I}_{\alpha,\beta}}$. Lemma 5 is proved. □

3. Commutators with $\mathscr{V}$

First we estimate the Fourier type integral

$$\mathscr{W}\phi=t^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}q(t,\xi,\eta)\phi(\xi)\,d\xi\tag{41}$$

in the $\mathbf{L}^2$-norm. In the particular factorized case $q(t,\xi,\eta)=q_1(\xi)q_2(\eta)$, with estimate $|q_2(\mu(x))|\le|\mu(x)|^{1/2}$, we find

$$\|\mathscr{W}\phi\|_{\mathbf{L}^2}\le t^{1/2}\left\|\overline{M}q_2(\eta)\cdot\int_{\mathbb{R}}e^{it\eta^2\xi}e^{-(it/3)|\xi|^3}q_1(\xi)\phi(\xi)\,d\xi\right\|_{\mathbf{L}^2}\le t^{1/2}\left\|q_2(\mu(x))|\mu(x)|^{-1/2}\cdot\int_{\mathbb{R}}e^{itx\xi}e^{-(it/3)|\xi|^3}q_1(\xi)\phi(\xi)\,d\xi\right\|_{\mathbf{L}^2}\le C\left\|e^{-(it/3)|\xi|^3}q_1\phi\right\|_{\mathbf{L}^2}=C\|q_1\phi\|_{\mathbf{L}^2}.\tag{42}$$

Next we obtain a more general result.

Lemma 6. *Suppose that $|(\eta\partial_\eta)^kq(t,\xi,\eta)|\le C\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}$ for all $\xi,\eta\in\mathbb{R}$, $k=0,1,2$, where $\nu\in(0,1)$. Then the estimate $\||\eta|^{1/2}\mathscr{W}\phi\|_{\mathbf{L}^2}\le C\|\phi\|_{\mathbf{L}^2}$ is true for all $t\ge1$.*

Proof. We write

$$\left\||\eta|^{1/2}\mathscr{W}\phi\right\|_{\mathbf{L}^2}^2=Ct\int_{\mathbb{R}}d\eta\,|\eta|\int_{\mathbb{R}}d\xi\cdot\int_{\mathbb{R}}d\zeta e^{it(S(\zeta,\eta)-S(\xi,\eta))}q(t,\xi,\eta)\phi(\xi)\overline{q(t,\zeta,\eta)}\overline{\phi(\zeta)}=Ct\int_{\mathbb{R}}d\xi e^{-(it/3)|\xi|^3}\phi(\xi)\cdot\int_{\mathbb{R}}d\zeta e^{(it/3)|\zeta|^3}\overline{\phi(\zeta)}K(t,\xi,\zeta),\tag{43}$$

where the kernel $K(t,\xi,\zeta)=\int_{\mathbb{R}}d\eta|\eta|e^{it\eta|\eta|(\xi-\zeta)}q(t,\xi,\eta)\overline{q(t,\zeta,\eta)}$. Integrating two times by parts via the identity $e^{it\eta|\eta|(\xi-\zeta)}=H\partial_\eta(\eta e^{it\eta|\eta|(\xi-\zeta)})$, with $H=(1+2it\eta|\eta|(\xi-\zeta))^{-1}$ we get

$$K(t,\xi,\zeta)=\int_{\mathbb{R}}e^{it\eta^2(\xi-\zeta)}\eta\partial_\eta\left(H\eta\partial_\eta\left(H|\eta|q(t,\xi,\eta)\cdot\overline{q(t,\zeta,\eta)}\right)\right)d\eta.\tag{44}$$

Since $|\eta\partial_\eta(H\eta\partial_\eta(H|\eta|q(t,\xi,\eta)\overline{q(t,\zeta,\eta)}))|\le C|\eta|\{\widehat{\eta}\}^{2\nu}\langle\widehat{\eta}\rangle^{-2\nu}/(1+t\eta^2|\xi-\zeta|)^2$ we get

$$|K(t,\xi,\zeta)|\le C\int_0^1\frac{\{\widehat{\eta}\}^{2\nu}\langle\widehat{\eta}\rangle^{-2\nu}|\eta|\,d\eta}{(1+t\eta^2|\xi-\zeta|)^2}+C\int_1^{\infty}\frac{\langle\widehat{\eta}\rangle^{-2\nu}|\eta|\,d\eta}{(1+t\eta^2|\xi-\zeta|)^2}\le Ct^{-2/3}\int_0^1\frac{\eta^{1+2\nu}d\eta}{(1+\eta^2t^{1/3}|\xi-\zeta|)^2}+Ct^{-2/3}\int_1^{\infty}\frac{\eta^{1-2\nu}d\eta}{(1+t^{1/3}\eta^2|\xi-\zeta|)^2}+Ct^{-2\nu/3}(|\xi-\zeta|t)^{\nu-1}\int_{t^{1/2}|\xi-\zeta|^{1/2}}^{\infty}\eta^{1-2\nu}\langle\eta\rangle^{-4}d\eta\le Ct^{-2/3}\left\langle(\xi-\zeta)t^{1/3}\right\rangle^{-1-\nu}+Ct^{-2/3}\left(|\xi-\zeta|t^{1/3}\right)^{\nu-1}\left\langle(\xi-\zeta)t^{1/3}\right\rangle^{-1}+Ct^{-2\nu/3}(|\xi-\zeta|t)^{\nu-1}\langle(\xi-\zeta)t\rangle^{-1-\nu}.\tag{45}$$

Then by the Cauchy-Schwarz inequality and Young inequality we obtain

$$\left\||\eta|^{1/2}\mathscr{W}\phi\right\|_{\mathbf{L}^2}^2\le Ct^{1/3}\|\phi\|_{\mathbf{L}^2}\cdot\left\|\int\left\langle(\xi-\zeta)t^{1/3}\right\rangle^{-1-\nu}|\phi(\zeta)|\,d\zeta\right\|_{\mathbf{L}^2}+Ct^{1/3}\|\phi\|_{\mathbf{L}^2}\cdot\left\|\int\left(|\xi-\zeta|t^{1/3}\right)^{\nu-1}\left\langle(\xi-\zeta)t^{1/3}\right\rangle^{-1}|\phi(\zeta)|\,d\zeta\right\|_{\mathbf{L}^2}+Ct^{1-2\nu/3}\|\phi\|_{\mathbf{L}^2}$$

$$\begin{aligned}
&\cdot \left\| \int \left(\left| \xi - \zeta \right| t \right)^{\nu - 1} \left\langle (\xi - \zeta) t \right\rangle^{-1-\nu} \left| \phi (\zeta) \right| d\zeta \right\|_{\mathbf{L}^2} \\
&\leq C t^{1/3} \left\| \left\langle \xi t^{1/3} \right\rangle^{-1-\nu} \right\|_{\mathbf{L}^1} \left\| \phi \right\|_{\mathbf{L}^2}^2 \\
&+ C t^{1/3} \left\| \left(\left| \xi \right| t^{1/3} \right)^{\nu - 1} \left\langle \xi t^{1/3} \right\rangle^{-1} \right\|_{\mathbf{L}^1} \left\| \phi \right\|_{\mathbf{L}^2}^2 \\
&+ C t^{1-2\nu/3} \left\| \left(\left| \xi \right| t \right)^{\nu - 1} \left\langle \xi t \right\rangle^{-1-\nu} \right\|_{\mathbf{L}^1} \left\| \phi \right\|_{\mathbf{L}^2}^2 \leq C \left\| \phi \right\|_{\mathbf{L}^2}^2
\end{aligned} \tag{46}$$

if $\nu \in (0, 1)$. Lemma 14 is proved. □

Next we estimate $\mathscr{W}\phi(0)$.

Lemma 7. *Suppose that $|(\xi\partial_\xi)^k q(t,\xi,\eta)| \leq C(|\eta| + |\xi|)^{-1}$ for all $\xi, \eta \in \mathbb{R}$, $k = 0, 1$. Then the estimate $\||\eta|^{1/2}\mathscr{W}1\|_{\mathbf{L}^2} \leq Ct^{1/6}$ is true.*

Proof. As in the proof of Lemma 13 we decompose

$$\begin{aligned}
\mathscr{W}1 &= t^{1/2} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q(t,\xi,\eta)\, d\xi \\
&= t^{1/2} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q(t,\xi,\eta)\, \Theta\left(\xi\eta^{-1}\right) d\xi \\
&+ t^{1/2} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q(t,\xi,\eta) \\
&\cdot \left(\chi_1\left(\xi\eta^{-1}\right) - \Theta\left(\xi\eta^{-1}\right)\right) d\xi \\
&+ t^{1/2} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q(t,\xi,\eta)\, \chi_2\left(\xi\eta^{-1}\right) d\xi = I_1 \\
&+ I_2 + I_3
\end{aligned} \tag{47}$$

for $\eta \neq 0$. In the first summand I_1 we integrate by parts via identity (19), to get

$$\begin{aligned}
I_1 &= Ct^{1/2} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} (\xi - \eta) \\
&\cdot \partial_\xi \left(H_1 q(t,\xi,\eta)\, \Theta\left(\xi\eta^{-1}\right)\right) d\xi.
\end{aligned} \tag{48}$$

Using the estimate $|(\xi - \eta)\partial_\xi(H_1 q(t,\xi,\eta)\Theta(\xi\eta^{-1}))| \leq C|\eta|^{-1}/(1 + t|\eta|(\xi - \eta)^2)$ then changing $\xi = \eta y$, we obtain

$$\begin{aligned}
\left|I_1\right| &\leq Ct^{1/2} |\eta|^{-1} \int_{1/3<\xi/\eta<3} \frac{d\xi}{1 + t|\eta|(\xi - \eta)^2} \\
&\leq Ct^{1/2} \int_{1/3}^{3} \frac{dy}{1 + t|\eta|^3 (y-1)^2} \leq Ct^{1/2} \langle \hat{\eta} \rangle^{-3/2}.
\end{aligned} \tag{49}$$

Thus we get $\||\eta|^{1/2} I_1\|_{\mathbf{L}^2} \leq Ct^{1/2}\||\eta|^{1/2}\langle\hat{\eta}\rangle^{-3/2}\|_{\mathbf{L}^2} \leq Ct^{1/6}$. To estimate the integral I_2 we integrate by parts via identity (24), to get

$$\begin{aligned}
I_2 &= Ct^{1/2} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \xi\partial_\xi \left(\left(\chi_1\left(\xi\eta^{-1}\right) - \Theta\left(\xi\eta^{-1}\right)\right)\right. \\
&\left.\cdot H_2 q(t,\xi,\eta)\right) d\xi.
\end{aligned} \tag{50}$$

We find the estimates $|\xi\partial_\xi((\chi_1(\xi\eta^{-1}) - \Theta(\xi\eta^{-1}))H_2 q(t,\xi,\eta))| \leq C|\eta|^{-1}/(1 + t|\xi|\eta^2)$, then we obtain

$$\begin{aligned}
\left|I_2\right| &\leq Ct^{1/2} |\eta|^{-1} \int_{|\xi|\leq 3|\eta|} \frac{d\xi}{1 + t|\xi|\eta^2} \\
&\leq Ct^{1/2} \int_0^3 \frac{dy}{1 + t|\eta|^3 y} \leq Ct^{1/2} \langle \hat{\eta} \rangle^{-3/2}.
\end{aligned} \tag{51}$$

Thus as above we get $\||\eta|^{1/2} I_2\|_{\mathbf{L}^2} \leq Ct^{1/6}$ for all $t \geq 1$ if $\nu > -1/2$. To estimate I_3 we integrate by parts via identity (24)

$$I_3 = Ct^{1/2} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \xi\partial_\xi \left(H_2 \chi_2\left(\xi\eta^{-1}\right) q(t,\xi,\eta)\right) d\xi. \tag{52}$$

We find the estimates $|\xi\partial_\xi(H_2\chi_2(\xi\eta^{-1})q(t,\xi,\eta))| \leq C/(|\eta| + |\xi|)(1 + t|\xi|^3)$ in the domain $|\xi| \geq (3/2)|\eta|$, and then we obtain

$$\begin{aligned}
\left|I_3\right| &\leq Ct^{1/2} \int_{(3/2)|\eta|}^{\infty} \frac{d\xi}{\left(|\eta| + |\xi|\right)\left(1 + t|\xi|^3\right)} \\
&\leq Ct^{1/2} \int_{3/2}^{(3/2)\langle\eta\rangle|\eta|^{-1}} \frac{dy}{(1 + y)\left(1 + t|\eta|^3 y^3\right)} \\
&+ Ct^{-1/2} \langle \eta \rangle^{-1} \int_{(3/2)\langle\eta\rangle}^{\infty} \xi^{-3} d\xi \\
&\leq Ct^{1/2} |\hat{\eta}|^{-\gamma} \langle \hat{\eta} \rangle^{\gamma - 3} + Ct^{-1/2} \langle \eta \rangle^{-3}
\end{aligned} \tag{53}$$

if $0 < \gamma < 1$. Thus we find

$$\begin{aligned}
\left\| |\eta|^{1/2} I_3 \right\|_{\mathbf{L}^2} &\leq Ct^{1/2} \left\| |\eta|^{1/2} |\hat{\eta}|^{-\gamma} \langle \hat{\eta} \rangle^{\gamma - 3} \right\|_{\mathbf{L}^2} \\
&+ Ct^{-1/2} \left\| \langle \eta \rangle^{-5/2} \right\|_{\mathbf{L}^2} \leq Ct^{1/6}
\end{aligned} \tag{54}$$

for all $t \geq 1$. Lemma 7 is proved. □

In the next lemma we estimate the commutator $[\eta, \mathscr{V}_1]$. Define the norm $\|\phi\|_{\mathbf{Y}} = \|\phi\|_{\mathbf{L}^\infty} + t^{-1/6}\|\partial_\xi\phi\|_{\mathbf{L}^2}$.

Lemma 8. *Let $j \geq 0$, $\nu \in (0, 1)$. Then the estimate $\||\eta|^{3/2-j}\{\hat{\eta}\}^\nu\langle\hat{\eta}\rangle^{-\nu} t[\eta, \mathscr{V}_1]\xi^j\phi\|_{\mathbf{L}^2} \leq Ct^{1/6}\|\phi\|_{\mathbf{Y}}$ is true for all $t \geq 1$.*

Proof. For $\eta \neq 0$ we integrate by parts

$$\begin{aligned}
&|\eta|^{1-j} \{\hat{\eta}\}^\nu \langle \hat{\eta} \rangle^{-\nu} t \left[\eta, \mathscr{V}_1\right] \xi^j \phi = |\eta|^{1-j} \{\hat{\eta}\}^\nu \langle \hat{\eta} \rangle^{-\nu} \\
&\cdot \sqrt{\frac{|t|}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \partial_\xi \left(\frac{\eta - \xi}{i\partial_\xi S(\xi,\eta)} \chi_1\left(\xi\eta^{-1}\right)\right. \\
&\left.\cdot \xi^j \phi(\xi)\right) d\xi = \sqrt{\frac{|t|}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q_1(t,\xi,\eta)
\end{aligned}$$

$$\cdot \partial_\xi \phi(\xi)\, d\xi + \sqrt{\frac{|t|}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q_2(t,\xi,\eta) \cdot \frac{\phi(\xi)-\phi(0)}{\xi} d\xi + \phi(0) \cdot \sqrt{\frac{|t|}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} q_3(t,\xi,\eta)\, d\xi, \tag{55}$$

where $q_1(t,\xi,\eta) = \{\widehat{\eta}\}^\nu \langle\widehat{\eta}\rangle^{-\nu}|\eta|^{1-j}((\eta-\xi)/i\partial_\xi S(\xi,\eta))\xi^j\chi_1(\xi\eta^{-1})$, $q_2(t,\xi,\eta) = \xi\partial_\xi q_1(t,\xi,\eta)$, and $q_3(t,\xi,\eta) = \partial_\xi q_1(t,\xi,\eta)$. Since $\partial_\xi S(\xi,\eta) = \xi|\xi| - \eta|\eta|$ and $(\eta-\xi)/\partial_\xi S(\xi,\eta) = (|\xi|+|\eta|)/(\xi^2+\eta^2+|\xi||\eta|+\eta\xi)$, we have

$$q_1(t,\xi,\eta) = |\eta|^{1-j}\{\widehat{\eta}\}^\nu \langle\widehat{\eta}\rangle^{-\nu} \frac{\eta-\xi}{\xi|\xi|-\eta|\eta|}\xi^j\chi_1(\xi\eta^{-1}) = O\left(\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}\right) \tag{56}$$

and similarly $q_2(t,\xi,\eta) = \xi\partial_\xi q_1(\xi,\eta) = O(\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu})$ for all $\xi,\eta \in \mathbb{R}$ in the domain $|\xi| \le 3|\eta|$. Hence we have $|(\eta\partial_\eta)^k q_l(t,\xi,\eta)| \le C\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}$ for all $\xi,\eta\in\mathbb{R}$, $k=0,1,2$, and $l=1,2$, where $\nu\in(0,1)$, and by Lemma 14

$$\left\| |\eta|^{1/2}\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)} q_1(t,\xi,\eta)\partial_\xi\phi(\xi)\,d\xi \right\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2} \tag{57}$$

and by the Hardy inequality and Lemma 14

$$\left\| |\eta|^{1/2}\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)} q_2(t,\xi,\eta)\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \right\|_{\mathbf{L}^2} \le C\left\|\frac{\phi(\xi)-\phi(0)}{\xi}\right\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}. \tag{58}$$

Also we have

$$q_3(t,\xi,\eta) = |\eta|^{1-j}\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}\partial_\xi\left(\frac{\eta-\xi}{(\xi|\xi|-\eta|\eta|)}\xi^j\chi_1(\xi\eta^{-1})\right) = O\left((|\eta|+|\xi|)^{-1}\langle\eta\rangle^{\beta+j-1}\right) \tag{59}$$

for all $\xi,\eta\in\mathbb{R}$, in the domain $|\xi|\le 3|\eta|$. Hence we get $|(\xi\partial_\xi)^k q_3(t,\xi,\eta)| \le C(|\eta|+|\xi|)^{-1}$ for all $\xi,\eta\in\mathbb{R}$, $k=0,1$. Therefore applying Lemma 7 we obtain $\||\eta|^{1/2}\sqrt{|t|/2\pi}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_3(t,\xi,\eta)d\xi\|_{\mathbf{L}^2} \le C^{1/6}$. Lemma 8 is proved. □

In the next lemma we estimate the operator $\mathscr{V}_2$.

Lemma 9. *Let $j=0,1,2$, $\nu\in(0,1)$. Then the estimate $\||\eta|^{5/2-j}\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}t\mathscr{V}_2\xi^j\phi\|_{\mathbf{L}^2} \le Ct^{1/6}\|\phi\|_{\mathbf{Y}}$ is true for all $t\ge1$.*

Proof. We integrate by parts

$$\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}|\eta|^{2-j}t\mathscr{V}_2\xi^j\phi = \{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}|\eta|^{2-j} \cdot \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\partial_\xi\left(\frac{\xi^j\chi_2(\xi\eta^{-1})}{i\partial_\xi S(\xi,\eta)}\phi(\xi)\right)d\xi$$
$$= \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_1(t,\xi,\eta)\partial_\xi\phi(\xi)\,d\xi + \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_2(t,\xi,\eta)\frac{\phi(\xi)-\phi(0)}{\xi}d\xi + \phi(0)\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_3(t,\xi,\eta)\,d\xi, \tag{60}$$

where $q_1(t,\xi,\eta) = (\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}|\eta|^{2-j}\xi^j/i\partial_\xi S(\xi,\eta))\chi_2(\xi\eta^{-1})$, $q_2(t,\xi,\eta) = \xi\partial_\xi q_1(t,\xi,\eta)$, and $q_3(t,\xi,\eta) = \partial_\xi q_1(t,\xi,\eta)$. We find

$$q_1(t,\xi,\eta) = O\left(\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}|\eta|^{2-j}|\xi|^{j-2}\right) = O\left(\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}\right), \tag{61}$$

and $q_2(t,\xi,\eta) = O(\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu})$ for all $\xi,\eta\in\mathbb{R}$, in the domain $|\xi|\ge(3/2)|\eta|$ if $j=0,1,2$. Then $|(\eta\partial_\eta)^k q_l(t,\xi,\eta)| \le C\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}$ for all $\xi,\eta\in\mathbb{R}$, $k=0,1,2$, and $l=1,2$. Hence by Lemma 14 we find

$$\left\| |\eta|^{1/2}\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_1(t,\xi,\eta)\partial_\xi\phi(\xi)\,d\xi \right\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}, \tag{62}$$

and by the Hardy inequality

$$\left\| |\eta|^{1/2}\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_2(t,\xi,\eta)\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \right\|_{\mathbf{L}^2} \le C\left\|\frac{\phi(\xi)-\phi(0)}{\xi}\right\|_{\mathbf{L}^2} \le C\left\|\partial_\xi\phi\right\|_{\mathbf{L}^2}. \tag{63}$$

Also we have $q_3(t,\xi,\eta) = O((|\eta|+|\xi|)^{-1})$ for all $\xi,\eta\in\mathbb{R}$, in the domain $|\xi|\ge(3/2)|\eta|$, if $j=0,1,2$. Hence $|(\xi\partial_\xi)^k q_3(t,\xi,\eta)| \le C(|\eta|+|\xi|)^{-1}$ for all $\xi,\eta\in\mathbb{R}$, $k=0,1$, and by Lemma 7

$$\left\| |\eta|^{1/2}\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_3(t,\xi,\eta)\,d\xi \right\|_{\mathbf{L}^2} \le Ct^{1/6}. \tag{64}$$

Lemma 9 is proved. □

In the next lemma, we estimate the derivative $\partial_\eta\mathscr{V}$.

Lemma 10. *Let $j=0,1$, $\nu\in(0,1)$. Then the estimate $\||\eta|^{1/2-j}\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^{-\nu}\partial_\eta\mathscr{V}\xi^j\phi\|_{\mathbf{L}^2} \le Ct^{1/6}\|\phi\|_{\mathbf{Y}}$ is true for all $t\ge1$.*

Proof. Since $\mathscr{A}_1\mathscr{V} = \mathscr{V}i\xi$ with $\mathscr{A}_1 = \mathscr{A}_0 + i\eta$, $\mathscr{A}_0 = (1/2t|\eta|)\partial_\eta$, then we obtain the commutator $\partial_\eta\mathscr{V} = 2t|\eta|[i\eta,\mathscr{V}]$. Also $\mathscr{V} = \mathscr{V}_1 + \mathscr{V}_2$. Hence

$$\begin{aligned}&\left\||\eta|^{1/2-j}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\partial_\eta\mathscr{V}\xi^j\phi\right\|_{\mathbf{L}^2}\\&\quad\le Ct\left\||\eta|^{3/2-j}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}[\eta,\mathscr{V}_1]\xi^j\phi\right\|_{\mathbf{L}^2}\\&\quad+Ct\left\||\eta|^{3/2-j}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\mathscr{V}_2\xi^j\phi\right\|_{\mathbf{L}^2}\\&\quad+Ct\left\||\eta|^{1/2-j}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\mathscr{V}_2\xi^{j+1}\phi\right\|_{\mathbf{L}^2}.\end{aligned}\tag{65}$$

By Lemma 8 we find $\||\eta|^{3/2-j}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}t[\eta,\mathscr{V}_1]\xi^j\phi\|_{\mathbf{L}^2} \le Ct^{1/6}\|\phi\|_{\mathbf{Y}}$ for all $t \ge 1$, if $j \ge 0$, $\nu \in (0,1)$. Also by Lemma 9 we get $\||\eta|^{5/2-j}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}t\mathscr{V}_2\xi^j\phi\|_{\mathbf{L}^2} + \||\eta|^{3/2-j}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}t\mathscr{V}_2\xi^{j+1}\phi\|_{\mathbf{L}^2} \le Ct^{1/6}\|\phi\|_{\mathbf{Y}}$ for all $t \ge 1$, if $j = 0,1$, $\nu \in (0,1)$. Hence the result of the lemma follows. Lemma 10 is proved. □

4. A Priori Estimates

Local existence and uniqueness of solutions to Cauchy problem (1) were shown in [19, 20] when $u_0 \in \mathbf{H}^1$. By using the local existence result, we have the following.

Theorem 11. *Assume that the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{H}^{0,1}$. Then there exists a unique local solution u of Cauchy problem (1) such that $\mathscr{U}(-t)u \in \mathbf{C}([0,T];\mathbf{H}^1 \cap \mathbf{H}^{0,1})$.*

We can take $T > 1$ if the data are small in $\mathbf{H}^1 \cap \mathbf{H}^{0,1}$ and we may assume that $\|u\|_{\mathbf{X}_1} \le \varepsilon$. To get the desired results, we prove a priori estimates of solutions uniformly in time. Define the following norm

$$\begin{aligned}\|u\|_{\mathbf{X}_T} = \sup_{t\in[1,T]}&\left(\|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty} + t^{-1/6}\|\mathscr{J}u(t)\|_{\mathbf{L}^2}\right.\\&\left.+t^{(1/3)(1-1/p)}\|u(t)\|_{\mathbf{L}^p}\right),\end{aligned}\tag{66}$$

where $\mathscr{J} = \mathscr{U}(t)x\mathscr{U}(-t)$, $\widehat{\varphi}(t) = \mathscr{F}\mathscr{U}(-t)u(t)$, and $p > 4$.

First we obtain the large time asymptotic behavior of the nonlinearity $\mathscr{F}\mathscr{U}(-t)\partial_x(|u|^2u)$.

Lemma 12. *The asymptotics $t\mathscr{F}\mathscr{U}(-t)\partial_x(|u|^2u) = i\widehat{\xi}|\widehat{\xi}|^5\langle\widehat{\xi}\rangle^{-6}|\widehat{\varphi}|^2\widehat{\varphi} + O(\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-\nu}\|\widehat{\varphi}\|^3_{\mathbf{Y}})$ is true for all $t \ge 1$ and $\xi \in \mathbb{R}$, where $\widehat{\varphi}(t) = \mathscr{F}\mathscr{U}(-t)u(t)$ and $\nu > 0$ is small.*

Proof. In view of factorization formula (11) we find $t\mathscr{F}\mathscr{U}(-t)\partial_x(|u|^2u) = i\xi\mathscr{V}^*(t)|\psi|^2\psi = \mathscr{V}^*(t)\mathscr{A}_1(t)|\psi|^2\psi$, where $\psi = \mathscr{V}\widehat{\varphi}$. Then by Lemma 5 with $\alpha = 1/2+\nu$, $\beta = 2\nu$, and $\nu > 0$ small, we get

$$\begin{aligned}&t\mathscr{F}\mathscr{U}(-t)\partial_x\left(|u|^2u\right) = i\xi\mathscr{V}^*(t)|\psi|^2\psi = i\xi A^*|\psi|^2\psi\\&\quad+O\left(t^{-1/2+\nu/3}\widehat{\xi}\left(\left\||\eta|^{1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\partial_\eta\left(|\psi|^2\psi\right)\right\|_{\mathbf{L}^2}\right.\right.\\&\quad\left.\left.+\left\||\eta|^{-1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}|\psi|^2\psi\right\|_{\mathbf{L}^2}\right)\right)\end{aligned}\tag{67}$$

in the case of $|\xi| < t^{-1/3}$ and

$$\begin{aligned}&t\mathscr{F}\mathscr{U}(-t)\partial_x\left(|u|^2u\right) = \mathscr{V}^*\mathscr{A}_1|\psi|^2\psi = A^*\mathscr{A}_1|\psi|^2\psi\\&\quad+O\left(t^{\nu/3-1/6}\langle\widehat{\xi}\rangle^{-2\nu}\right.\\&\quad\cdot\left(\left\||\eta|^{1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\partial_\eta\mathscr{A}_1\left(|\psi|^2\psi\right)\right\|_{\mathbf{L}^2}\right.\\&\quad\left.\left.+\left\||\eta|^{-1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\mathscr{A}_1\left(|\psi|^2\psi\right)\right\|_{\mathbf{L}^2}\right)\right)\end{aligned}\tag{68}$$

in the case of $|\xi| > t^{-1/3}$. Via identity $|\eta|^\nu = t^{-\nu/3}\{\widehat{\eta}\}^\nu\langle\widehat{\eta}\rangle^\nu$, we consider the remainder terms

$$\begin{aligned}&\left\||\eta|^{1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\partial_\eta\left(|\psi|^2\psi\right)\right\|_{\mathbf{L}^2}\\&\quad\le C\left\||\eta|^{1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}|\psi|^2\partial_\eta\psi\right\|_{\mathbf{L}^2}\\&\quad\le C\left\||\eta|^{\nu}\{\widehat{\eta}\}^{-\nu}\langle\widehat{\eta}\rangle^{-\nu}|\psi|^2\right\|_{\mathbf{L}^\infty}\\&\quad\cdot\left\||\eta|^{1/2}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\partial_\eta\psi\right\|_{\mathbf{L}^2}\le Ct^{1/6-\nu/3}\|\phi\|_{\mathbf{Y}}\\&\quad\cdot\left\||\psi|^2\right\|_{\mathbf{L}^\infty},\\&\left\||\eta|^{-1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\partial_\eta\mathscr{A}_1\left(|\psi|^2\psi\right)\right\|_{\mathbf{L}^2}\\&\quad\le C\left\||\eta|^{1+\nu}\{\widehat{\eta}\}^{-\nu}\langle\widehat{\eta}\rangle^{-\nu}|\psi|^2\right\|_{\mathbf{L}^\infty}\\&\quad\cdot\left\||\eta|^{-1/2}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\partial_\eta\psi_1\right\|_{\mathbf{L}^2}\\&\quad+C\left\||\eta|^{\nu}\{\widehat{\eta}\}^{-\nu}\langle\widehat{\eta}\rangle^{-\nu}\psi\psi_1\right\|_{\mathbf{L}^\infty}\\&\quad\cdot\left\||\eta|^{1/2}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\partial_\eta\psi\right\|_{\mathbf{L}^2}\le Ct^{1/6-\nu/3}\|\phi\|_{\mathbf{Y}}\\&\quad\cdot\left(\left\|\eta|\psi|^2\right\|_{\mathbf{L}^\infty}+\|\psi\psi_1\|_{\mathbf{L}^\infty}\right),\end{aligned}\tag{69}$$

where $\psi_1 = \mathscr{V}i\xi\widehat{\varphi}$. By Lemma 10 with $j = 0,1$, we have

$$\begin{aligned}&\left\||\eta|^{1/2}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\partial_\eta\psi\right\|_{\mathbf{L}^2}+\left\||\eta|^{-1/2}\{\widehat{\eta}\}^{\nu}\langle\widehat{\eta}\rangle^{-\nu}\partial_\eta\psi_1\right\|_{\mathbf{L}^2}\\&\quad\le Ct^{1/6}\|\widehat{\varphi}\|_{\mathbf{Y}}.\end{aligned}\tag{70}$$

Using Lemma 13 we get $|\psi| \le C(t^{1/2}|\eta|\langle\widehat{\eta}\rangle^{-3/2} + t^{1/6}\langle\widehat{\eta}\rangle^{-3/2})\|\phi\|_{\mathbf{Y}} \le Ct^{1/6}\langle\widehat{\eta}\rangle^{-1/2}\|\widehat{\varphi}\|_{\mathbf{Y}}$ and $|\psi_1| \le C(t^{1/2}|\eta|^2\langle\widehat{\eta}\rangle^{-3/2} + t^{-1/6}\langle\widehat{\eta}\rangle^{-1/2})\|\phi\|_{\mathbf{Y}} \le C(|\eta|^{1/2} + t^{-1/6})\|\widehat{\varphi}\|_{\mathbf{Y}}$. Hence $\|\psi^2\|_{\mathbf{L}^\infty} \le Ct^{1/3}\|\widehat{\varphi}\|^2_{\mathbf{Y}}$ and $\|\eta|\psi|^2\|_{\mathbf{L}^\infty} + \|\psi\psi_1\|_{\mathbf{L}^\infty} \le C\|\widehat{\varphi}\|^2_{\mathbf{Y}}$. Also we find $\||\eta|^{-1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}|\psi|^2\psi\|_{\mathbf{L}^2} \le Ct^{1/2}\|\widehat{\varphi}\|^3_{\mathbf{Y}}\||\eta|^{-1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu-3/2}\|_{\mathbf{L}^2} \le Ct^{1/2-\nu/3}\|\widehat{\varphi}\|^3_{\mathbf{Y}}$ and $\||\eta|^{-1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\mathscr{A}_1(|\psi|^2\psi)\|_{\mathbf{L}^2} \le Ct^{1/6}\|\widehat{\varphi}\|^3_{\mathbf{Y}}\||\eta|^{-1/2+\nu}\langle\widehat{\eta}\rangle^{-2\nu-1/2}\|_{\mathbf{L}^2} \le Ct^{1/6-\nu/3}\|\widehat{\varphi}\|^3_{\mathbf{Y}}$. Therefore we obtain $t\mathscr{F}\mathscr{U}(-t)\partial_x(|u|^2u) = i\xi A^*|\psi|^2\psi + O(\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-2\nu}\|\widehat{\varphi}\|^3_{\mathbf{Y}})$ for $|\xi| < t^{-1/3}$ and $t\mathscr{F}\mathscr{U}(-t)\partial_x(|u|^2u) = A^*\mathscr{A}_1|\psi|^2\psi + O(\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-2\nu}\|\widehat{\varphi}\|^3_{\mathbf{Y}})$ for $|\xi| > t^{-1/3}$. Next by Lemma 13 we have $\psi_j(t,\xi) = (t^{1/2}|\xi|(i\xi)^j/\sqrt{2i\langle t\xi^3\rangle})\widehat{\varphi}(\xi) + O(t^{1/6-j/3}\langle\widehat{\xi}\rangle^{j-3/4}\|\widehat{\varphi}\|_{\mathbf{Y}})$ for $j = 0,1$. Then we get $i\xi A^*|\psi|^2\psi = i\widehat{\xi}|\widehat{\xi}|^5\langle\widehat{\xi}\rangle^{-6}|\widehat{\varphi}|^2\widehat{\varphi} +$

$O(\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-\gamma}\|\widehat{\varphi}\|_{\mathbf{Y}}^3)$ and $A^*\mathscr{A}_1|\psi|^2\psi = i\widehat{\xi}|\widehat{\xi}|^5\langle\widehat{\xi}\rangle^{-6}|\widehat{\varphi}|^2\widehat{\varphi} + O(\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-\gamma}\|\widehat{\varphi}\|_{\mathbf{Y}}^3)$. Lemma 12 is proved. □

Next we estimate the solutions in the norm $\mathbf{X}_T$.

Lemma 13. *Assume that $\|u\|_{\mathbf{X}_1} \le \varepsilon$ holds. Then there exists $\varepsilon > 0$ such that the estimate $\|u\|_{\mathbf{X}_T} < C\varepsilon$ is true for all $T > 1$.*

Proof. By continuity of the norm $\|u\|_{\mathbf{X}_T}$ with respect to T, arguing by the contradiction we can find the first time $T > 0$ such that $\|u\|_{\mathbf{X}_T} = C\varepsilon$. To prove the estimate for the norm $\sup_{t\in[1,T]}\|\widehat{\varphi}\|_{\mathbf{L}^\infty} < C\varepsilon$ we use (11). Then in view of Lemma 12, we get

$$\begin{aligned}\partial_t\widehat{\varphi} &= \lambda\mathscr{F}\mathscr{U}(-t)\,\partial_x\left(|u|^2u\right)\\ &= i\lambda t^{-1}\widehat{\xi}\left|\widehat{\xi}\right|^5\left\langle\widehat{\xi}\right\rangle^{-6}\left|\widehat{\varphi}\right|^2\widehat{\varphi} + O\left(\varepsilon^3t^{-1}\left\{\widehat{\xi}\right\}\left\langle\widehat{\xi}\right\rangle^{-\nu}\right).\end{aligned} \tag{71}$$

For the case of $|\xi| < t^{-1/3}$ we can integrate $|\widehat{\varphi}(t,\xi)| \le |\widehat{\varphi}(1,\xi)| + C|\xi|\|\widehat{\varphi}\|_{\mathbf{Y}}^3\int_1^t\tau^{-2/3}d\tau \le \varepsilon + C\xi t^{1/3}\varepsilon^3 \le \varepsilon + C\varepsilon^3$. For the case of $|\xi| \ge t^{-1/3}$ multiplying by $\overline{\widehat{\varphi}}$ and taking the real part of the result we obtain $\partial_t(|\widehat{\varphi}(t,\xi)|^2) = O(t^{-1}\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-\nu}\|\widehat{\varphi}\|_{\mathbf{Y}}^4)$. Integrating in time we obtain

$$\begin{aligned}\left|\widehat{\varphi}(t,\xi)\right|^2 &\le \left|\widehat{\varphi}\left(\xi^{-3},\xi\right)\right|^2\\ &\quad + C\left\|\widehat{\varphi}\right\|_{\mathbf{Y}}^4\int_{\xi^{-3}}^t\left\{\xi\tau^{1/3}\right\}\left\langle\xi\tau^{1/3}\right\rangle^{-\nu}\frac{d\tau}{\tau}\\ &\le \varepsilon^2 + C\varepsilon^4\int_1^{\xi t^{1/3}}\langle y\rangle^{-1-\nu}dy \le \varepsilon^2 + C\varepsilon^4.\end{aligned} \tag{72}$$

Therefore $\|\mathscr{F}\mathscr{U}(-t)u(t)\|_{\mathbf{L}^\infty} < C\varepsilon$. Applying estimate of Lemma 4 we find $|\partial_x^ju(t,x)| \le C\varepsilon\langle xt^{-1/3}\rangle^{j/2-1/4}t^{-1/3-j/3}$, $|u(t,x)\partial_xu(t,x)| \le C\varepsilon^2t^{-1}$, and

$$\begin{aligned}\|u\|_{\mathbf{L}^p} &\le Ct^{-1/3}\|u\|_{\mathbf{X}_T}\left\|\left\langle xt^{-1/3}\right\rangle^{-1/4}\right\|_{\mathbf{L}^p}\\ &\le C\varepsilon t^{-(1/3)(1-1/p)}\end{aligned} \tag{73}$$

if $p > 4$. Consider a priori estimates for $\|\mathscr{J}u(t)\|_{\mathbf{L}^2}$. Using the identity $\partial_x^{-1}\mathscr{P}u - \mathscr{J}u = 3t\partial_x^{-1}\mathscr{L}u$, we get $\|\mathscr{J}u\|_{\mathbf{L}^2} \le C\|\partial_x^{-1}\mathscr{P}u\|_{\mathbf{L}^2} + Ct\|u\|_{\mathbf{L}^6}^3 \le C\|\partial_x^{-1}\mathscr{P}u\|_{\mathbf{L}^2} + C\varepsilon^3t^{1/6}$. Applying the operator $\partial_x^{-1}\mathscr{P}$ to (1), in view of the commutators $[\mathscr{L},\mathscr{P}] = 3\mathscr{L}$, $[\mathscr{P},\partial_x] = -\partial_x$, we get $\mathscr{L}\partial_x^{-1}\mathscr{P}u = \partial_x^{-1}(\mathscr{P}+3)\mathscr{L}u = \lambda\partial_x^{-1}(\mathscr{P}+3)\partial_x(|u|^2u) = \lambda(\mathscr{P}+2)(|u|^2u)$. Then by the energy method we obtain

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^{-1}\mathscr{P}u\right\|_{\mathbf{L}^2}^2 &\le C\left\|uu_x\right\|_{\mathbf{L}^\infty}\left\|\partial_x^{-1}\mathscr{P}u\right\|_{\mathbf{L}^2}^2\\ &\quad + \|u\|_{\mathbf{L}^6}^3\left\|\partial_x^{-1}\mathscr{P}u\right\|_{\mathbf{L}^2}\\ &\le C\varepsilon^2t^{-1}\left\|\partial_x^{-1}\mathscr{P}u\right\|_{\mathbf{L}^2}^2\\ &\quad + C\varepsilon^3t^{-5/6}\left\|\partial_x^{-1}\mathscr{P}u\right\|_{\mathbf{L}^2}\end{aligned} \tag{74}$$

from which it follows that $\|\partial_x^{-1}\mathscr{P}u\|_{\mathbf{L}^2} \le \varepsilon t^{C\varepsilon^2} + C\varepsilon^3t^{1/6}$. Therefore $\|\mathscr{J}u\|_{\mathbf{L}^2} \le C\|\partial_x^{-1}\mathscr{P}u\|_{\mathbf{L}^2} + Ct\|u\|_{\mathbf{L}^6}^3 \le C\varepsilon^3t^{1/6}$ for all $t \in [1,T]$. Thus we obtain $\|u\|_{\mathbf{X}_T} < C\varepsilon$. Lemma 13 is proved. □

5. Proof of Theorem 1

By Lemma 13 we see that a priori estimate $\|u\|_{\mathbf{X}_T} \le C\varepsilon$ is true for all $T > 0$. Therefore the global existence of solutions of Cauchy problem (1) satisfying the estimate $\|u\|_{\mathbf{X}_\infty} \le C\varepsilon$ follows by a standard continuation argument by local existence Theorem 11.

6. Proof of Theorem 2

In this section we prove the existence of a unique self-similar solution $v_m(t,x) \equiv t^{-1/3}f_m(xt^{-1/3})$ for (1), which is uniquely determined by the total mass condition $m = (1/\sqrt{2\pi})\int_{\mathbf{R}}v_m(t,x)dx \ne 0$. Define the operators

$$\begin{aligned}\widetilde{\mathscr{V}}\phi &= \sqrt{\frac{1}{2\pi}}\int_{\mathbb{R}}e^{-iS(\xi,\eta)}\phi(\xi)\,d\xi,\\ \widetilde{\mathscr{V}}^*\phi &= \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}}e^{iS(\xi,\eta)}\phi(\eta)\,|\eta|\,d\eta.\end{aligned} \tag{75}$$

Then for the self-similar solutions $v_m(t,x) = t^{-1/3}f_m(xt^{-1/3}) = \mathscr{D}_t\mathscr{B}M\mathscr{V}\widehat{\varphi_m}$, where $\widehat{\varphi_m}(t,\xi) \equiv \mathscr{F}\mathscr{U}(-t)v_m(t)$, we find that $\widehat{\varphi_m}$ have a self-similar form, that is, $\widehat{\varphi_m}(t,\xi) \equiv \phi_m(\widehat{\xi})$ with $\widehat{\xi} = \xi t^{1/3}$. Using the relation $\partial_t\widehat{\varphi_m}(t,\xi) = (1/3)t^{-1}\eta\phi_m'(\eta)$ by factorization formula (11) we get $(1/3)\eta\partial_\eta\phi_m(\eta) = i\lambda\eta\widetilde{\mathscr{V}}^*|\widetilde{\mathscr{V}}\phi_m|^2\widetilde{\mathscr{V}}\phi_m$. Therefore $\partial_\eta\phi_m(\eta) = 3i\lambda\widetilde{\mathscr{V}}^*|\widetilde{\mathscr{V}}\phi_m|^2\widetilde{\mathscr{V}}\phi_m \equiv F(\phi_m)$. Note that $F(\phi_m)$ is not in $\mathbf{L}^2$. Therefore we need the approximate equation. Define $\Theta(\eta) = 1$ for $|\eta| \le 1$ and $\Theta(\eta) = 0$ for $|\eta| > 2$, and denote $\Theta_R(\eta) = \Theta(\eta/R)$. Also define the approximate equation

$$\begin{aligned}\partial_\eta\phi_{m,R}(\eta) &= 3i\lambda\widetilde{\mathscr{V}}^*\Theta_R\left|\widetilde{\mathscr{V}}\Theta_R\phi_{m,R}\right|^2\widetilde{\mathscr{V}}\Theta_R\phi_{m,R}\\ &\equiv F_R\left(\phi_{m,R}(\eta)\right).\end{aligned} \tag{76}$$

Let us show a priori estimate $\|\phi_{m,R}\|_{\mathbf{Z}} = \|\phi_{m,R}\|_{\mathbf{L}^\infty} + \|\partial_\eta\phi_{m,R}\|_{\mathbf{L}^2} \le 3|m|$ uniformly in R. Applying Lemma 12 with $t = 1$ we get

$$\begin{aligned}\partial_\eta\phi_{m,R}(\eta) &= 3i\lambda|\eta|^5\langle\eta\rangle^{-6}\left|\Theta_R\phi_{m,R}(\eta)\right|^2\Theta_R\phi_{m,R}(\eta)\\ &\quad + O\left(\langle\eta\rangle^{-1-\nu}\|\phi_{m,R}\|_{\mathbf{Z}}^3\right).\end{aligned} \tag{77}$$

Integrating with respect to η, we obtain $\|\partial_\eta\phi_{m,R}\|_{\mathbf{L}^2} \le C\|\phi_{m,R}\|_{\mathbf{Z}}^3$. Also multiplying by $\overline{\phi_{m,R}}$ and integrating with respect to η we get $|\phi_{m,R}(\eta)| = m + O(\int_0^\eta\langle\eta\rangle^{-1-\gamma}\|\phi_{m,R}\|_{\mathbf{Z}}^3d\eta)$. Hence $\|\phi_{m,R}\|_{\mathbf{L}^\infty} \le |m| + C\|\phi_{m,R}\|_{\mathbf{Z}}^3$. Thus we obtain $\|\phi_{m,R}\|_{\mathbf{Z}} \le 2|m| + C\|\phi_{m,R}\|_{\mathbf{Z}}^3 \le 3|m|$ for some small m. Taking the limit $R \to \infty$, we find that there exists a unique solution

ϕ_m of equation $\partial_\eta \phi_m(\eta) = 3i\lambda \widetilde{\mathcal{V}}^* |\widetilde{\mathcal{V}}\phi_m|^2 \widetilde{\mathcal{V}}\phi_m$ in $\mathbf{Z}$. By the definition of $\phi_m(\eta)$, we obtain $\|\partial_\xi \widehat{\varphi_m}\|_{\mathbf{L}^2} = t^{1/6}\|\partial_\eta \phi_m\|_{\mathbf{L}^2} \leq C|m|t^{1/6}$ and $\|\widehat{\varphi_m}\|_{\mathbf{L}^\infty} \leq 3|m|$. In the same way as in the proof of (73) we have $\mathbf{L}^p$ estimate of v_m for $p > 4$.

7. Proof of Theorem 3

Define the norm $\|u\|_{\mathbf{Y}_T} = \sup_{t\in[1,T]}(t^{1/2-\gamma}\|u\|_{\mathbf{L}^\infty} + t^{-\gamma}\|\mathcal{J}u\|_{\mathbf{L}^2})$ with a small $\gamma > 0$.

Lemma 14. *Suppose that $\|u_j\|_{\mathbf{X}_T} \leq C\varepsilon$, $j = 1, 2$, where ε is sufficiently small. Let $\widehat{\varphi_1}(t, 0) = \widehat{\varphi_2}(t, 0)$ for $j = 1, 2$, $t \geq 1$, where $\widehat{\varphi_j}(t, \xi) = \mathcal{F}\mathcal{U}(-t)u_j(t)$. Let $u_2 = t^{-1/3} f(xt^{-1/3})$ be a self-similar solution. Then the estimate $\|u_1 - u_2\|_{\mathbf{Y}_T} < C\varepsilon$ is true for all $T > 1$.*

Proof. By the continuity of the norm $\|u_1 - u_2\|_{\mathbf{Y}_T}$ with respect to T, arguing by the contradiction we can find for the first time $T > 0$ such that $\|u_1 - u_2\|_{\mathbf{Y}_T} = C\varepsilon$. We denote $\widehat{w} = \widehat{\varphi_1} - \widehat{\varphi_2}$ and $y = u_1 - u_2$. Applying estimate of Lemma 4 we find

$$\begin{aligned} |y(t,x)| &\leq Ct^{-1/3}\left\langle x^{1/2}t^{-1/6}\right\rangle^{-1/2}\left|\widehat{w}\left(t, x^{1/2}t^{-1/2}\right)\right| \\ &\quad + Ct^{-1/2}\left\langle x^{1/2}t^{-1/6}\right\rangle^{-3/4}\left\|\partial_\xi \widehat{w}\right\|_{\mathbf{L}^2} \\ &\leq Ct^{-1/2}\left\|\partial_\xi \widehat{w}\right\|_{\mathbf{L}^2}. \end{aligned} \tag{78}$$

Thus we need to estimate the norm $\|\partial_\xi \widehat{w}\|_{\mathbf{L}^2} = \|\mathcal{J}y\|_{\mathbf{L}^2}$. For the difference y we get from (1) $\mathcal{L}\partial_x^{-1}\mathcal{P}y = \lambda(\mathcal{P} + 2)(|u_1|^2 u_1 - |u_2|^2 u_2)$. Hence by the energy method

$$\begin{aligned} &\frac{d}{dt}\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}^2 \\ &\quad = 4\operatorname{Re}\int_{\mathbf{R}} \overline{\partial_x^{-1}\mathcal{P}y}\left(|u_1|^2\mathcal{P}u_1 - |u_2|^2\mathcal{P}u_2\right)dx \\ &\qquad + 2\operatorname{Re}\int_{\mathbf{R}} \overline{\partial_x^{-1}\mathcal{P}y}\left(u_1^2\overline{\mathcal{P}u_1} - u_2^2\overline{\mathcal{P}u_2}\right)dx \\ &\qquad + 4\operatorname{Re}\int_{\mathbf{R}} \overline{\partial_x^{-1}\mathcal{P}y}\left(|u_1|^2 u_1 - |u_2|^2 u_2\right)dx. \end{aligned} \tag{79}$$

Next we get

$$\begin{aligned} &4\operatorname{Re}\int_{\mathbf{R}} \overline{\partial_x^{-1}\mathcal{P}y}\left(|u_1|^2\mathcal{P}u_1 - |u_2|^2\mathcal{P}u_2\right)dx \\ &\quad = 4\operatorname{Re}\int_{\mathbf{R}} |u_1|^2\,\overline{\partial_x^{-1}\mathcal{P}y}\mathcal{P}y\,dx \\ &\qquad + 4\operatorname{Re}\int_{\mathbf{R}} \left(|u_1|^2 - |u_2|^2\right)\mathcal{P}u_2\partial_x^{-1}\mathcal{P}y\,dx \\ &\quad = 4\operatorname{Re}\int_{\mathbf{R}} \partial_x|u_1|^2\left|\partial_x^{-1}\mathcal{P}y\right|^2 dx \\ &\qquad + 4\operatorname{Re}\int_{\mathbf{R}} \left(|u_1|^2 - |u_2|^2\right)\mathcal{P}u_2\overline{\partial_x^{-1}\mathcal{P}y}\,dx \\ &\quad \leq C\left\||u_1|\,u_{1x}\right\|_{\mathbf{L}^\infty}\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}^2 \\ &\qquad + C\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}\left\|\left(|u_1|^2 - |u_2|^2\right)\mathcal{P}u_2\right\|_{\mathbf{L}^2}. \end{aligned} \tag{80}$$

In the same manner

$$\begin{aligned} &2\operatorname{Re}\int_{\mathbf{R}} \overline{\partial_x^{-1}\mathcal{P}y}\left(u_1^2\overline{\mathcal{P}u_1} - u_2^2\overline{\mathcal{P}u_2}\right)dx \\ &\quad = 2\operatorname{Re}\int_{\mathbf{R}} u_1^2\overline{\partial_x^{-1}\mathcal{P}y\mathcal{P}y}\,dx \\ &\qquad + 2\operatorname{Re}\int_{\mathbf{R}} \left(u_1^2 - u_2^2\right)\overline{\mathcal{P}u_2\partial_x^{-1}\mathcal{P}y}\,dx \\ &\quad = 2\operatorname{Re}\int_{\mathbf{R}} \partial_x u_1^2\overline{\partial_x^{-1}\mathcal{P}y}^2\,dx \\ &\qquad + 2\operatorname{Re}\int_{\mathbf{R}} \left(u_1^2 - u_2^2\right)\overline{\mathcal{P}u_2\partial_x^{-1}\mathcal{P}y}\,dx \\ &\quad \leq C\left\|u_1 u_{1x}\right\|_{\mathbf{L}^\infty}\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}^2 \\ &\qquad + C\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}\left\|\left(u_1^2 - u_2^2\right)\mathcal{P}u_2\right\|_{\mathbf{L}^2}. \end{aligned} \tag{81}$$

Note that $\mathcal{P}u_2 = \partial_x x t^{-1/3} f(xt^{-1/3}) + 3t\partial_t t^{-1/3} f(xt^{-1/3}) = 0$ for the case of self-similar solution $u_2 = t^{-1/3} f(xt^{-1/3})$. Hence

$$\begin{aligned} &\frac{d}{dt}\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}^2 \\ &\quad \leq C\left\||u_1|\,u_{1x}\right\|_{\mathbf{L}^\infty}\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}^2 \\ &\qquad + C\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2}\left\||u_1|^2 u_1 - |u_2|^2 u_2\right\|_{\mathbf{L}^2}. \end{aligned} \tag{82}$$

By (78) we have $\langle x^{1/2}t^{-1/6}\rangle^{1/2}|u(t,x)| \leq C\varepsilon t^{-1/3}$. To estimate $\||u_1|^2u_1 - |u_2|^2u_2\|_{\mathbf{L}^2}$ we use the above estimates to get

$$\begin{aligned} &\left\||u_1|^2 u_1 - |u_2|^2 u_2\right\|_{\mathbf{L}^2} \\ &\quad \leq C\sum_{j=1}^{2}\left\|\left\langle|x|^{1/2}t^{-1/6}\right\rangle^{1/2}u_j\right\|_{\mathbf{L}^\infty}^2\left\|\left\langle|x|^{1/2}t^{-1/6}\right\rangle^{-1}y\right\|_{\mathbf{L}^2} \\ &\quad \leq C\varepsilon^2 t^{-2/3}\left\|\left\langle|x|^{1/2}t^{-1/6}\right\rangle^{-1}y\right\|_{\mathbf{L}^2}. \end{aligned} \tag{83}$$

In view of Lemma 4

$$\begin{aligned} &\left\|\left\langle|x|^{1/2}t^{-1/6}\right\rangle^{-1}y\right\|_{\mathbf{L}^2} \\ &\quad \leq Ct^{-1/3}\left\|\left\langle|x|^{1/2}t^{-1/6}\right\rangle^{-3/2}\widehat{w}\left(t, x^{1/2}t^{-1/2}\right)\right\|_{\mathbf{L}_x^2} \\ &\qquad + Ct^{-1/2}\left\|\left\langle|x|^{1/2}t^{-1/6}\right\rangle^{-7/4}\right\|_{\mathbf{L}^2}\left\|\partial_\xi\widehat{w}\right\|_{\mathbf{L}^2}. \end{aligned} \tag{84}$$

Since $\widehat{w}(0) = 0$, we get by the Hardy inequality $\|\langle|x|^{1/2}t^{-1/6}\rangle^{-3/2}\widehat{w}(t, x^{1/2}t^{-1/2})\|_{\mathbf{L}_x^2} \leq C\|\partial_\xi\widehat{w}\|_{\mathbf{L}^2}$ and by a direct calculation $\|\langle|x|^{1/2}t^{-1/6}\rangle^{-7/4}\|_{\mathbf{L}^2} \leq Ct^{1/6}$. Hence

$\|\langle |x|^{1/2}t^{-1/6}\rangle^{-1}y\|_{\mathbf{L}^2} \le Ct^{-1/3}\|\partial_\xi \widehat{w}\|_{\mathbf{L}^2}$. Therefore by (83) $\||u_1|^2u_1 - |u_2|^2u_2\|_{\mathbf{L}^2} \le C\varepsilon^2 t^{-1}\|\partial_\xi \widehat{w}\|_{\mathbf{L}^2} \le C\varepsilon^3 t^{-1+\gamma}$. Thus we obtain from (82) $(d/dt)\|\partial_x^{-1}\mathscr{P}y\|_{\mathbf{L}^2} \le C\varepsilon^3 t^{-1+\gamma}$ which implies $\|\partial_x^{-1}\mathscr{P}y\|_{\mathbf{L}^2} \le C\varepsilon^3 t^{\gamma}$. Therefore $\|\mathscr{J}y\|_{\mathbf{L}^2} \le \|\partial_x^{-1}\mathscr{P}y\|_{\mathbf{L}^2} + Ct\||u_1|^2u_1 - |u_2|^2u_2\|_{\mathbf{L}^2} \le C\varepsilon^3 t^{\gamma}$. Lemma 14 is proved. □

Now we turn to the proof of asymptotic formula (7) for the solutions u of Cauchy problem (1). Let $v_m(t,x)$ be the self-similar solution with the total mass condition $m = (1/\sqrt{2\pi})\int_{\mathbf{R}} u_0(x)dx = (1/\sqrt{2\pi})\int_{\mathbf{R}} v_m(t,x)dx \neq 0$. Note that $\|v_m\|_{\mathbf{X}_\infty} \le C\varepsilon$ by Theorem 2 and $\|u\|_{\mathbf{X}_\infty} \le C\varepsilon$ by Theorem 1. Also $m = \widehat{\varphi}(t,0) = \widehat{v_m}(t,0)$ for $t \ge 1$. Then by Lemma 14 we find $u(t,x) = v_m(t,x) + O(\varepsilon t^{-1/2+\gamma})$. Thus asymptotics (7) follows. Theorem 3 is proved.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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