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Research Article

Existence of Global Solutions for Nonlinear Magnetohydrodynamics with Finite Larmor Radius Corrections

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We discuss the existence of global solutions to the magnetohydrodynamics (MHD) equations, where the effects of finite Larmor radius corrections are taken into account. Unlike the usual MHD, the pressure is a tensor and it depends on not only the density but also the magnetic field. We show the existence of global solutions by the energy methods. Our techniques of proof are based on the existence of local solution by semigroups theory and a priori estimate.

1. Introduction

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids, such as charged particles in an electromagnetic field, and has an extensive range of applications in mathematics and physics. MHD has been the subject of many studies by physicists and mathematicians because of its environmental importance, rich phenomena, and mathematical challenges. When the magnetic field is strong, the charged particle undergoes gyromotion and it affects the fluid motion. The radius of gyro motion is called the Larmor radius. The fluid description with the finite Larmor radius (FLR) effects is referred to as the gyrofluid or the Landau fluid and such effects are reflected in the pressure and the heat flux. Usually, in the MHD, the pressure is isotropic and scalar, which depends on the density. On the other hand, in the gyrofluid, the pressure is anisotropic and depends on both the density and the magnetic field. Another difference is that the pressure is no longer a scalar. Instead, it is a tensor derived from the moment equations and consists of the gyrotropic and gyroviscous tensors. The gyrotropic part consists of the pressures parallel and perpendicular to the magnetic field, and the gyroviscous tensor is derived from the moment equations for pressure by a Chapman-Enskog expansion.

In this paper, we are interested in the existence of global solutions to the one-dimensional MHD equations with the FLR corrections in the pressure discussed above. We first construct the local solutions by the semigroup theory to linearized equations for a small time. Then, the existence of global solutions is proved by extending the local solutions globally in time based on a priori estimates of solutions.

As far as the derivation of gyrofluid is concerned, Chew et al. [1] proposed the gyrotropic tensor. The gyroviscous tensor was first introduced by Thompson and an improvement was made by Yajima [2] and Khanna and Rajaram [3]. In particular, the general form of the gyroviscous tensor was derived by Hsu et al. [4]. Recently, Ramos [5] discussed the heat flux terms for the gyrofluid, and Passot and Sulem [6] discussed the closure relations for the gyrofluid. For the existence of solutions to MHD, Chen and Wang [7] showed the existence of global solutions to the piston problem. Hu and Wang [8] obtained the existence of global weak solutions in three dimensions for MHD equations, by using the Faedo-Galerkin method. Many authors have investigated the existence of global solutions to viscous systems. Matsumura and Nishida [9] treated the initial value problem for equations of motion of viscous and heat conductive gases in three dimensions, and they obtained the global solution for their equations in H^3 with a method based on iteration and energy method. Many applications of the energy method are discussed in [10]. Slemrod [11] proved the existence of global solutions for nonlinear thermoelasticity with initial data being sufficiently small and smooth in one dimension, applying the contraction mapping theorem for existence of local solutions. In this regard, the existence of a global solution to partial differential equations of the hyperbolic-parabolic type has received much attention in the last decades; see [9, 11].

We organize the rest of the paper as follows: Section 2 presents a derivation of the equations of nonlinear Landau fluid. Section 3 shows the existence of a local solution. In Section 4, we establish a priori estimate. In Section 5, the existence of global solutions is proved by extending the local solutions in time based on a priori estimates of solutions. We also show the asymptotic behavior of solutions.

2. Derivation of Landau Fluid Equations

In this section, we will look at the model for describing Landau fluid and derive the equations from MHD equations in three dimensions.

$$\begin{aligned}
&\rho_t + \nabla\cdot(u\rho) = 0,\\
&(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla\cdot P\\
&\quad = \frac{1}{c} j\times b + \mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u),\\
&b_t - \nabla\times(u\times b) = -\nabla\times(\nu\nabla\times b),\\
&\nabla\cdot b = 0.
\end{aligned} \tag{1}$$

In the above system ρ is a density, $u = \langle u_1, u_2, u_3\rangle$ is a velocity, ∇ is the gradient operator, $j = (c/4\pi)\nabla\times b$ is the current vector, c is the speed of light, μ, λ are viscosity coefficients, $b = \langle b_1, b_2, b_3\rangle$ is the magnetic field, and ν is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. The magnetic field consists of the large ambient field oriented in the x-direction and small perturbation.

In the gyrofluid, we assume that the pressure term P has the following form:

$$P = P^0 + P^1, \tag{2}$$

where P^0 and P^1 are called the gyrotropic tensor and the gyroviscosity tensor, respectively. For P^0, Chew et al. [1] deduced that the lowest order form of the plasma pressure may be expressed in terms of pressures parallel and perpendicular to the field.

$$P^0 = p_\perp\left(I - \hat{b}\otimes\hat{b}\right) + p_\parallel \hat{b}\otimes\hat{b}, \tag{3}$$

where $p_\perp$ is the perpendicular pressure, $p_\parallel$ is the parallel pressure, I is the identity matrix, $\otimes$ denotes the Kronecker tensor product, and $\hat{b}$ is the unit vector in the direction of b. Kulsrud [12] observed that the perpendicular pressure is proportional to B/ν, and parallel pressure is proportional to $1/\nu^3 B^2$; that is,

$$\begin{aligned}
p_\perp &\propto \frac{B}{\nu},\\
p_\parallel &\propto \frac{1}{\nu^3 B^2},
\end{aligned} \tag{4}$$

where $\nu = 1/\rho$, ν is specific volume, and $B = \sqrt{\sum_{i=1}^{3} b_i^2}$ is the magnitude of the magnetic field. Based on his observation, we use in this paper

$$\begin{aligned}
p_\perp &= \frac{B}{\nu},\\
p_\parallel &= \frac{1}{\nu^3 B^2}.
\end{aligned} \tag{5}$$

The gyroviscosity tensor P^1 is derived from the ten-moment equations, where the pressure tensor P satisfies

$$\begin{aligned}
&\frac{d}{d\tau}P + \nabla\cdot uP + (P\cdot\nabla)u + (P\cdot\nabla u)^T\\
&\quad + \frac{e}{m}[b\times P - P\times b] = 0.
\end{aligned} \tag{6}$$

We write the above relation in the following way:

$$\begin{aligned}
&\left[P\times\hat{b} + \mathrm{Tr}\right]\\
&\quad = \frac{1}{\Omega}\left[\frac{d}{d\tau}P + P\nabla\cdot u + P\cdot\nabla u + (P\cdot\nabla u)^T\right],
\end{aligned} \tag{7}$$

where $\mathrm{Tr} = -\hat{b}\times P$ is the transpose of $P\times\hat{b}$ and $d/d\tau = \partial/\partial t + u\cdot\nabla$ denotes the convective derivative, $\Omega = \sigma B_0/mc$ is the gyrofrequency, where σ is the electron charge, m is the mass of particle, c is the speed of light, and B_0 is the magnitude of the constant ambient field assumed to be oriented in the x-direction and approximately equal to B. In the gyrofluid, Ω is large. If we assume $P^0 = O(1)$ and $P^1 = O(1/\Omega)$, then from (7) we obtain

$$\begin{aligned}
&\left[P^1\times\hat{b} + \mathrm{Tr}\right]\\
&\quad \approx \frac{1}{\Omega}\left[\frac{d}{d\tau}P^0 + P^0\nabla\cdot u + P^0\cdot\nabla u + \left(P^0\cdot\nabla u\right)^T\right].
\end{aligned} \tag{8}$$

Solving the above system for P^1, Hsu et al. [4] obtained

$$P^1 = \frac{1}{4\Omega}\left\{\hat{b}\times S\cdot\left(I + 3\hat{b}\hat{b}\right) + \left[\hat{b}\times S\cdot\left(I + 3\hat{b}\hat{b}\right)\right]^T\right\}, \tag{9}$$

where

$$S = \left(\frac{\partial}{\partial t} + u \cdot \nabla\right) P^0 + \left[\left(P^0 \cdot \nabla u\right) + \mathrm{Tr}\right]. \tag{10}$$

When the ambient magnetic field is oriented in the x-direction, P^1 is decomposed as $P^1 = P^{10} + P^{11}$, where P^{10} is the central part of gyroviscous term given by

$$\begin{gathered}
P^{10} = \begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{bmatrix}, \\
p_{xx} = 0, \\
p_{yy} = -p_{zz} = -\frac{p_\perp}{2\Omega}\left(u_{3y} + u_{2z}\right), \\
p_{yz} = p_{zy} = \frac{p_\perp}{2\Omega}\left(u_{2y} - u_{3z}\right), \\
p_{zx} = p_{xz} = \frac{-1}{\Omega}\left[p_\perp\left(u_{2x} - u_{1y}\right) - 2p_\parallel u_{2x}\right], \\
p_{xy} = p_{yx} = \frac{1}{\Omega}\left[\left(p_\perp u_{3x} - u_{1z}\right) - 2p_\parallel u_{3x}\right],
\end{gathered} \tag{11}$$

and P^{11} is a small correction and a tensor whose components are of the order $O(|q||\nabla u|)$, where $q = \langle b_2, b_3\rangle$. The gyroviscous terms are dispersive as discussed in [2, 13]. In the one-dimensional case, the constraint $\nabla \cdot b = 0$ implies that b_1 is a constant, and based on nondimensionalization, we may set $b_1 = 1$ without loss of generality. Then, (1) reduces to

$$\begin{gathered}
\rho_t + \left(u_1 \rho\right)_x = 0, \\
\rho\left(u_t + u_1 u_x\right) + \left(P^0 + P^1\right)_x - \left(\frac{1}{4\pi}\nabla \times b\right) \times b = \left(u_x\right)_x, \\
q_t + \left(u_1 q - w\right)_x = \left(\nu q_x\right)_x,
\end{gathered} \tag{12}$$

where

$$\begin{gathered}
P^0 = \begin{bmatrix} p_\perp\left(1 - \widehat{b_1^2}\right) + p_\parallel \widehat{b_1^2} \\ -p_\perp \widehat{b_1}\widehat{b_2} + p_\parallel \widehat{b_1}\widehat{b_2} \\ -p_\perp \widehat{b_1}\widehat{b_3} + p_\parallel \widehat{b_1}\widehat{b_3} \end{bmatrix}, \\
P^1 = \begin{bmatrix} 0 \\ \dfrac{1}{\Omega}\left(p_\perp - 2p_\parallel\right)\dfrac{u_{3x}}{v} \\ -\dfrac{1}{\Omega}\left(p_\perp - 2p_\parallel\right)\dfrac{u_{2x}}{v} \end{bmatrix}, \\
w = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}.
\end{gathered} \tag{13}$$

We convert the equations to the Lagrangian coordinates so that there are no convective derivatives. Then,

$$v_t - u_{1y} = 0, \tag{14}$$

$$u_{1t} + \left(p_\perp\left(1 - \widehat{b_1^2}\right) + p_\parallel \widehat{b_1^2}\right)_y + \left(\frac{1}{2}\left(b_2^2 + b_3^2\right)\right)_y = \left(\frac{u_{1y}}{v}\right)_y, \tag{15}$$

$$u_{2t} + \left(-p_\perp \widehat{b_1}\widehat{b_2} + p_\parallel \widehat{b_1}\widehat{b_2}\right)_y + \left(-b_2\right)_y = \left(\frac{u_{2y}}{v} + \frac{1}{\Omega}\left(p_\perp - 2p_\parallel\right)\frac{u_{3y}}{v}\right)_y, \tag{16}$$

$$u_{3t} + \left(-p_\perp \widehat{b_1}\widehat{b_3} + p_\parallel \widehat{b_1}\widehat{b_3}\right)_y + \left(-b_3\right)_y = \left(\frac{u_{3y}}{v} - \frac{1}{\Omega}\left(p_\perp - 2p_\parallel\right)\frac{u_{2y}}{v}\right)_y, \tag{17}$$

$$\left(vb_2\right)_t + \left(-u_2\right)_y = \left(\frac{1}{v}b_{2y}\right)_y, \tag{18}$$

$$\left(vb_3\right)_t + \left(-u_3\right)_y = \left(\frac{1}{v}b_{3y}\right)_y. \tag{19}$$

We discuss the Cauchy problem for the above equations with the initial data

$$\left(u, v - \bar{v}, q\right)(y, 0) = \left(u_0, v_0 - \bar{v}, q_0\right). \tag{20}$$

We consider $u = u(y,t)$, $v = v(y,t)$, and $q = q(y,t)$ denoting the unknown functions of $t > 0$ and $y \in R$, the initial data where v approaches to a positive constant $\bar{v} > 0$ as $y \to \pm\infty$, and the magnetic field b approaches to $(1,0,0)$ as $y \to \pm\infty$.

For this purpose, we define the Banach space $X(J)$ as follows:

$$X(J) = \left\{ (u, v, q);\ \begin{array}{l} (u, v - \bar{v}, q) \in C^0\left(J, H^2\right); \\ u_y, q_y \in L^2\left(J; H^2\right); \\ v_y \in L^2\left(J; H^1\right); \\ \sup \left\|(u, v - \bar{v}, q)(t)\right\|_2 \le M,\ \inf v(y,t) \ge m,\ M, m > 0, \end{array} \right. \tag{21}$$

where $J = [0, T]$ is the time interval, $\|\cdot\|_2$ is the H^2 norm, and we denote L^2 norm by $\|\cdot\|$.

3. Existence of Local Solution

The proof of the existence of local solutions is standard. However, for the sake of completion, we briefly explain the proof here. We construct a sequence of functions that converges to a function satisfying the Cauchy problem. This method gives a sequence of approximation on (u, v, q) to the solution where the nth approximation is obtained from one or more previous approximation(s). We linearize (14)–(19) and consider the following iteration:

$$
\begin{aligned}
& v_t^{(n)} - u_{1y}^{(n-1)} = 0, \\
& u_{1t}^{(n)} + \left(p_\perp^{(n-1)} \left(1 - \left(\widehat{b}_1^{(n-1)} \right)^2 \right) + p_\parallel^{(n-1)} \left(\widehat{b}_1^{(n-1)} \right)^2 \right)_y \\
& \quad + \left(\frac{1}{2} \left(b_2^{(n-1)} \right)^2 + \left(b_3^{(n-1)} \right)^2 \right)_y = \left(\frac{u_{1y}^{(n)}}{v^{(n-1)}} \right)_y, \\
& u_{2t}^{(n)} + \left(-p_\perp^{(n-1)} \widehat{b}_1^{(n-1)} \widehat{b}_2^{(n-1)} + p_\parallel^{(n-1)} \widehat{b}_1^{(n-1)} \widehat{b}_2^{(n-1)} \right)_y \\
& \quad + \left(-b_2^{(n-1)} \right)_y \\
& \quad = \left(\frac{u_{2y}^{(n)}}{v^{(n-1)}} + \frac{1}{\Omega} \left(p_\perp^{(n-1)} - 2 p_\parallel^{(n-1)} \right) \frac{u_{3y}^{(n-1)}}{v^{(n-1)}} \right)_y, \\
& u_{3t}^{(n)} + \left(-p_\perp^{(n-1)} \widehat{b}_1^{(n-1)} \widehat{b}_3^{(n-1)} + p_\parallel^{(n-1)} \widehat{b}_1^{(n-1)} \widehat{b}_3^{(n-1)} \right)_y \\
& \quad + \left(-b_3^{(n-1)} \right)_y \\
& \quad = \left(\frac{u_{3y}^{(n)}}{v^{(n-1)}} - \frac{1}{\Omega} \left(p_\perp^{(n-1)} - 2 p_\parallel^{(n-1)} \right) \frac{u_{2y}^{(n-1)}}{v^{(n-1)}} \right)_y, \\
& \left(v^{(n-1)} b_2^{(n)} \right)_t + \left(-u_2^{(n-1)} \right)_y = \left(\frac{1}{v^{(n-1)}} b_{2y}^{(n)} \right)_y, \\
& \left(v^{(n-1)} b_3^{(n)} \right)_t + \left(-u_3^{(n-1)} \right)_y = \left(\frac{1}{v^{(n-1)}} b_{3y}^{(n)} \right)_y.
\end{aligned}
\tag{22}
$$

We find the existence of the local solutions of the nonlinear equations (14)–(19) by iteration. Let $U^{(n)} = (u^{(n)}, v^{(n)} - \overline{v}, q^{(n)})$. We show the existence of $U^{(n)}$ for given $U^{(n-1)}$ by the semigroup theory and the energy estimate for $U^{(n)}$.

Theorem 1 (energy estimate). *For each $m > 0$ and $M > 0$ there exist $T > 0$ and C depending on m and M, such that $U^{(n)} \in C^0(J; H^2)$ and that for each n the initial value problem with the initial data $(u_0, v_0 - \overline{v}, q_0)$ for linearized equations (14)–(19) has a unique solution satisfying the energy estimates:*

$$
\begin{aligned}
& \|(u, q)(t)\|_2^2 \\
& \le \exp^{C(m,M)t} \left(\|(u, q)(0)\|_2^2 + C(m, M) \int_0^t \|g\|_1^2 \, d\tau \right), \\
& \int_0^t \left\| \left(u_y, q_y \right)(\tau) \right\|_2^2 d\tau \\
& \le \exp^{C(m,M)t} \left(\|(u, q)(0)\|_2^2 + C(m, M) \int_0^t \|g\|_1^2 \, d\tau \right), \\
& \|v(t)\|_2^2 \le \exp^{Ct} \left(\|v(0)\|_2^2 + \int_0^t \|h\|_1^2 \, d\tau \right),
\end{aligned}
\tag{23}
$$

where g and h are known functions belonging to $X(J)$.

It is not difficult to show that $\{U^{(n)}\}$ with the initial data $(u^{(n)}, v^{(n)}, q^{(n)})(y, 0) = (u_0, v_0, q_0)$ is the Cauchy sequence in $C^0(J; H^2)$, which proves the following theorem. The proof is omitted.

Theorem 2. *For all $m > 0$ and $M > 0$, there exists $T(m, M)$ such that if $\|u_0, v_0 - \overline{v}, q_0\|_2 \le M$, $\inf v(y, 0) \ge m$, then the initial value problem for (14)–(20) has a local solution in $X(J)$.*

4. A Priori Estimates

To prove the existence of global solutions we need a priori estimates where we estimate u, v, and q, and the first and second derivatives of them. We define the energy and dissipation terms as $E(t)$ and $F(t)$, respectively:

$$
\begin{aligned}
E(t) &= \frac{1}{2} \left(\|u\|_2^2 + \|q\|_2^2 + \|(v - \overline{v})\|_2^2 \right), \\
F(t) &= \frac{1}{2} \int_0^t \left(\left\|u_y\right\|_2^2 + \left\|q_y\right\|_2^2 + \left\|(v - \overline{v})_y\right\|_1^2 \right) d\tau.
\end{aligned}
\tag{24}
$$

First, we multiply (15), (16), and (17) by u_1, u_2, u_3, respectively, and we perform integration by parts with respect to y. Then, by adding the resulting integrals, we obtain

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \frac{1}{2} \left(u_1^2 \right)_t dy + \int_{-\infty}^{\infty} \frac{1}{2} \left(u_2^2 \right)_t dy + \int_{-\infty}^{\infty} \frac{1}{2} \left(u_3^2 \right)_t dy \\
& \quad - \int_{-\infty}^{\infty} u_{1y} \left(p_\perp \left(1 - \widehat{b}_1^2 \right) + p_\parallel \widehat{b}_1^2 - \overline{p}_\parallel \right) dy \\
& \quad - \int_{-\infty}^{\infty} u_{2y} \left(-p_\perp \widehat{b}_1 \widehat{b}_2 + p_\parallel \widehat{b}_1 \widehat{b}_2 \right) dy \\
& \quad - \int_{-\infty}^{\infty} u_{3y} \left(-p_\perp \widehat{b}_1 \widehat{b}_3 + p_\parallel \widehat{b}_1 \widehat{b}_3 \right) dy
\end{aligned}
$$

$$+\int_{-\infty}^{\infty} u_{2y} b_2 dy + \int_{-\infty}^{\infty} u_{3y} b_3 dy$$
$$= \int_{-\infty}^{\infty} \frac{1}{2} u_{1y}\left(b_2^2 + b_3^2\right) dy - \int_{-\infty}^{\infty} u_{1y}\left(\frac{u_{1y}}{v}\right) dy$$
$$- \int_{-\infty}^{\infty} u_{2y}\left(\frac{u_{2y}}{v} + \frac{1}{\Omega}\left(p_\perp - 2p_\parallel\right)\frac{u_{3y}}{v}\right) dy$$
$$- \int_{-\infty}^{\infty} u_{3y}\left(\frac{u_{3y}}{v} - \frac{1}{\Omega}\left(p_\perp - 2p_\parallel\right)\frac{u_{2y}}{v}\right) dy, \tag{25}$$

where $\overline{p}_\parallel = p_\parallel(\overline{v}, \overline{B})$ is a constant of integration, $\overline{B} = 1$. Note that the two terms containing Ω are canceled and they cause no problem in a priori estimates. To carry out the estimates, we use (14), (18), and (19) for u_{1y}, u_{2y}, u_{3y}, and the relations

$$\widehat{b_1} = \frac{1}{B}, \quad \widehat{b_2} = \frac{b_2}{B}, \quad \widehat{b_3} = \frac{b_3}{B}. \tag{26}$$

in (25).

We also combine the fourth, the fifth, and the sixth terms on the left-hand side of (25). Then, the sum of the three terms leads to

$$\int_{-\infty}^{\infty}\left(\frac{1}{2v^2B^2}\right)_t dy + \int_{-\infty}^{\infty} v_t \frac{1}{\overline{v}^3\overline{B}^2} dy + \int_{-\infty}^{\infty} (B)_t dy$$
$$+ \int_{-\infty}^{\infty}\left(\frac{1}{v}b_{2y}\right)_y\left(-\frac{B}{v} + \frac{1}{v^3B^2}\right)\frac{b_2}{B^2} dy \tag{27}$$
$$+ \int_{-\infty}^{\infty}\left(\frac{1}{v}b_{3y}\right)_y\left(-\frac{B}{v} + \frac{1}{v^3B^2}\right)\frac{b_3}{B^2} dy.$$

We now integrate with respect to t and expand the term $1/2v^2B^2$ by the Taylor series. Then, linear terms for v cancel, and we have

$$\int_{-\infty}^{\infty} \frac{1}{2}\sum_1^3 u_i^2(y,t)\,dy + \int_{-\infty}^{\infty}\left(1 - \frac{1}{\overline{v}^2\overline{B}^3}\right)\left(B - \overline{B}\right) dy$$
$$+ \int_{-\infty}^{\infty} \frac{1}{2!}\left(\frac{3}{\overline{v}^4\overline{B}^2}(v - \overline{v})^2 + \frac{2}{\overline{v}^3\overline{B}^3}(v - \overline{v})\left(B - \overline{B}\right)\right.$$
$$\left. + \frac{3}{\overline{v}^2\overline{B}^4}\left(B - \overline{B}\right)^2\right) dy + \int_0^t \int_{-\infty}^{\infty} u_{2y} b_2 dy\,d\tau$$
$$+ \int_0^t \int_{-\infty}^{\infty} u_{3y} b_3 dy\,d\tau + \int_0^t \int_{-\infty}^{\infty} \frac{u_{1y}^2}{v} dy\,d\tau$$
$$+ \int_0^t \int_{-\infty}^{\infty} \frac{u_{2y}^2}{v} dy\,d\tau + \int_0^t \int_{-\infty}^{\infty} \frac{u_{3y}^2}{v} dy\,d\tau$$
$$- \int_0^t \int_{-\infty}^{\infty} \frac{1}{2} u_{1y}\left(b_2^2 + b_3^2\right) dy\,d\tau$$
$$\le \int_{-\infty}^{\infty} \frac{1}{2}\sum_1^3 u_i^2(y,0)\,dy$$
$$+ \int_0^t \int_{-\infty}^{\infty}\left(\frac{1}{v}b_{2y}\right)_y\left(-\frac{B}{v} + \frac{1}{v^3B^2}\right)\frac{b_2}{B^2} dy\,d\tau$$
$$+ \int_0^t \int_{-\infty}^{\infty}\left(\frac{1}{v}b_{3y}\right)_y\left(-\frac{B}{v} + \frac{1}{v^3B^2}\right)\frac{b_3}{B^2} dy\,d\tau. \tag{28}$$

We note that, for the third integral on the left-hand side of (28), the integrand is estimated from below in the following way:

$$\frac{2}{\overline{v}^4\overline{B}^2}(v - \overline{v})^2 + \frac{2}{\overline{v}^2\overline{B}^4}\left(B - \overline{B}\right)^2$$
$$\le \frac{3}{\overline{v}^4\overline{B}^2}(v - \overline{v})^2 + \frac{2}{\overline{v}^3\overline{B}^3}(v - \overline{v})\left(B - \overline{B}\right) \tag{29}$$
$$+ \frac{3}{\overline{v}^2\overline{B}^4}\left(B - \overline{B}\right)^2.$$

Next, we estimate $q = \langle b_2, b_3\rangle$. For this purpose, we multiply (18) and (19) by b_2, b_3, respectively, and perform the integration with respect to y and t. Then we obtain

$$\int_{-\infty}^{\infty} \frac{1}{2} v b_2^2(y,t)\,dy + \int_{-\infty}^{\infty} \frac{1}{2} v b_3^2(y,t)\,dy$$
$$+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{v} b_{2y}^2 dy\,d\tau + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v} b_{3y}^2 dy\,d\tau$$
$$+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{2} u_{1y}\left(b_2^2 + b_3^2\right) dy\,d\tau \tag{30}$$
$$- \int_0^t \int_{-\infty}^{\infty} u_{2y} b_2 dy\,d\tau - \int_0^t \int_{-\infty}^{\infty} u_{3y} b_3 dy\,d\tau$$
$$\le \int_{-\infty}^{\infty} \frac{1}{2} v b_2^2(y,0)\,dy + \int_{-\infty}^{\infty} \frac{1}{2} v b_3^2(y,0)\,dy.$$

To get the estimate for v_y, we multiply (15) by v_y/v and integrate with respect to y, and we obtain

$$\int_{-\infty}^{\infty} \frac{v_y}{v} u_{1t} dy + \int_{-\infty}^{\infty} \frac{v_y}{v}\left(p_\perp\left(1 - \widehat{b_1^2}\right) + p_\parallel \widehat{b_1^2}\right)_y dy$$
$$+ \int_{-\infty}^{\infty} \frac{v_y}{v}\left(\frac{1}{2}\left(b_2^2 + b_3^2\right)\right)_y dy \tag{31}$$
$$= \int_{-\infty}^{\infty} \frac{v_y}{v}\left(\frac{u_{1y}}{v}\right)_y dy.$$

By using (14), the term on the right-hand side becomes

$$\frac{v_y}{v}\left(\frac{u_{1y}}{v}\right)_y = \frac{v_y}{v}\left(\frac{v_t}{v}\right)_y = \frac{v_y}{v}(\log v)_{ty} = \frac{v_y}{v}\left(\frac{v_y}{v}\right)_t. \tag{32}$$

So,

$$\int_{-\infty}^{\infty} \frac{v_y}{v}\left(\frac{v_t}{v}\right)_y dy - \int_{-\infty}^{\infty} \frac{v_y}{v} u_{1t} dy$$
$$= \int_{-\infty}^{\infty} \frac{v_y}{v}\left(p_\perp\left(1 - \widehat{b_1^2}\right) + p_\parallel \widehat{b_1^2}\right)_y dy \tag{33}$$
$$+ \int_{-\infty}^{\infty} \frac{v_y}{v}\left(\frac{1}{2}\left(b_2^2 + b_3^2\right)\right)_y dy.$$

We use the product rule for the second term on the left-hand side of (33):

$$\frac{d}{dt}\left(u_1\frac{v_y}{v}\right)=u_1\left(\frac{v_y}{v}\right)_t+u_{1t}\frac{v_y}{v}. \tag{34}$$

And we use

$$B_y=\frac{b_2b_{2y}+b_3b_{3y}}{B} \tag{35}$$

for the derivatives of $\widehat{b_2}$ and $\widehat{b_3}$. Then integrating with respect to t, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_y}{v}\right)^2(y,t)\,dy-\int_{-\infty}^{\infty}u_1\frac{v_y}{v}dy \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{u_{1y}^2}{v}dy\,d\tau \\
&\le\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_y}{v}\right)^2(y,0)\,dy \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2B}b_2b_{2y}v_ydy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2B}b_3b_{3y}v_ydy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^3B}b_2^2v_y^2dy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^3B}b_3^2v_y^2dy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{3}{v^5B^4}v_y^2dy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2B^3}b_2^2\left(b_2b_{2y}+b_3b_{3y}\right)v_ydy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2B^3}b_3^2\left(b_2b_{2y}+b_3b_{3y}\right)v_ydy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{4}{v^3B^6}\left(b_2b_{2y}+b_3b_{3y}\right)v_ydy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{1}{2}\frac{v_y}{v}\left(b_2^2+b_3^2\right)_ydy\,d\tau.
\end{aligned} \tag{36}$$

Now, by adding (28), (30), and (36) together, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}\sum_1^3u_i^2(y,t)\,dy+\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_y}{v}\right)^2(y,t)\,dy \\
&\quad+\int_{-\infty}^{\infty}\frac{1}{2}vb_2^2(y,t)\,dy+\int_{-\infty}^{\infty}\frac{1}{2}vb_3^2(y,t)\,dy \\
&\quad+\int_{-\infty}^{\infty}\left(1-\frac{1}{\overline{v}^2\overline{B}^3}\right)\left(B-\overline{B}\right)dy \\
&\quad+\int_{-\infty}^{\infty}\frac{2}{\overline{v}^4\overline{B}^2}(v-\overline{v})^2\,dy \\
&\quad+\int_{-\infty}^{\infty}\frac{2}{\overline{v}^2\overline{B}^4}\left(B-\overline{B}\right)^2dy+\int_0^t\int_{-\infty}^{\infty}\zeta v_y^2dy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{u_{2y}^2}{v}dy\,d\tau+\int_0^t\int_{-\infty}^{\infty}\frac{u_{3y}^2}{v}dy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v}b_{2y}^2dy\,d\tau+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v}b_{3y}^2dy\,d\tau \\
&\le\int_{-\infty}^{\infty}\frac{1}{2}\sum_1^3u_i^2(y,0)\,dy+\int_{-\infty}^{\infty}\frac{1}{2}vb_2^2(y,0)\,dy \\
&\quad+\int_{-\infty}^{\infty}\frac{1}{2}vb_3^2(y,0)\,dy+\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_y}{v}\right)^2(y,0)\,dy \\
&\quad+\int_{-\infty}^{\infty}u_1\frac{v_y}{v}dy+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2B}b_2b_{2y}v_ydy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2B}b_3b_{3y}v_ydy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2B^3}b_2^2\left(b_2b_{2y}+b_3b_{3y}\right)v_ydy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2B^3}b_3^2\left(b_2b_{2y}+b_3b_{3y}\right)v_ydy\,d\tau \\
&\quad-\int_0^t\int_{-\infty}^{\infty}\frac{4}{v^3B^6}\left(b_2b_{2y}+b_3b_{3y}\right)v_ydy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{1}{2}\frac{v_y^2b_3^2}{v^2}dy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\frac{1}{2}\frac{v_y}{v}\left(b_2^2+b_3^2\right)_ydy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\left(\frac{1}{v}b_{2y}\right)_y\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_2}{B^2}dy\,d\tau \\
&\quad+\int_0^t\int_{-\infty}^{\infty}\left(\frac{1}{v}b_{3y}\right)_y\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_3}{B^2}dy\,d\tau,
\end{aligned} \tag{37}$$

where

$$\zeta=\frac{b_2^2}{v^3B}+\frac{b_3^2}{v^3B}+\frac{3}{v^5B^4}. \tag{38}$$

Using Cauchy Schwarz inequality, we estimate the rest of the terms on the right-hand side. For example, consider

$$\int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_2 b_{2y} v_y dy\, d\tau$$

$$\left| \int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_2 b_{2y} v_y dy\, d\tau \right| \leq C \int_0^t \sup |b_2| \int_{-\infty}^{\infty} \left| v_y b_{2y} \right| dy\, d\tau, \tag{39}$$

by Sobolev embedding theorem, and we have

$$\sup_y |b_2| \leq \left(\int_{-\infty}^{\infty} \left(b_2^2 + b_{2y}^2 \right) dy \right)^{1/2}. \tag{40}$$

Thus,

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_2 b_{2y} v_y dy\, d\tau \\ &\quad \leq C \sup_t \left(\int_{-\infty}^{\infty} \left(b_2^2 + b_{2y}^2 \right) dy \right)^{1/2} \\ &\quad \cdot \int_0^t \int_{-\infty}^{\infty} \left(v_y^2 + b_{2y}^2 \right) dy\, d\tau \leq C E(t)^{1/2} F(t). \end{aligned} \tag{41}$$

Therefore,

$$E_1(t) + F_1(t) \leq E_1(0) + C \left(E(t)^{1/2} + E(t) \right) F(t), \tag{42}$$

where

$$\begin{aligned} E_1(t) &= \frac{1}{2} \left\{ \|u\|^2 + \|(v - \overline{v})\|^2 + \|q\|^2 + \left\| \frac{v_y}{v} \right\|^2 \right\}, \\ F_1(t) &= \int_0^t \left\{ \left\| u_y \right\|^2 + \left\| q_y \right\|^2 + \left\| v_y \right\|^2 \right\} d\tau. \end{aligned} \tag{43}$$

Second, we need to estimate the first derivatives of u and q. For that purpose, we multiply (15), (16), and (17) by $-u_{1yy}$, $-u_{2yy}$, and $-u_{3yy}$, respectively, and integrate with respect to y and t. Similarly, we differentiate (18) and (19) in y, multiply them by b_{2y} and b_{3y}, respectively, and integrate with respect to y and t. Then we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1}{2} u_{1y}^2 (y,t)\, dy + \int_{-\infty}^{\infty} \frac{1}{2} u_{2y}^2 (y,t)\, dy \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2} u_{3y}^2 (y,t)\, dy + \int_{-\infty}^{\infty} \frac{1}{2} v b_{2y}^2 (y,t)\, dy \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2} v b_{3y}^2 (y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v} u_{1yy}^2 dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v} u_{2yy}^2 dy\, d\tau + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v} u_{3yy}^2 dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v} b_{2yy}^2 dy\, d\tau + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v} b_{3yy}^2 dy\, d\tau \\ &\leq \int_{-\infty}^{\infty} \frac{1}{2} u_{1y}^2 (y,0)\, dy + \int_{-\infty}^{\infty} \frac{1}{2} u_{2y}^2 (y,0)\, dy \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2} u_{3y}^2 (y,0)\, dy \int_{-\infty}^{\infty} \frac{1}{2} v b_{2y}^2 (y,0)\, dy \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2} v b_{3y}^2 (y,0)\, dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{v_y}{v^2} u_{1y} u_{1yy} dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{v_y}{v^2} u_{2y} u_{2yy} dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{v_y}{v^2} u_{3y} u_{3yy} dy\, d\tau \\ &\quad - \int_0^t \int_{-\infty}^{\infty} u_{1y} b_{2y} b_{2y} dy\, d\tau \\ &\quad - \int_0^t \int_{-\infty}^{\infty} u_{1y} b_{3y} b_{3y} dy\, d\tau \\ &\quad - \int_0^t \int_{-\infty}^{\infty} v_y b_{2y} b_{2t} dy\, d\tau \\ &\quad - \int_0^t \int_{-\infty}^{\infty} v_y b_{3y} b_{3t} dy\, d\tau \\ &\quad - \int_0^t \int_{-\infty}^{\infty} u_{1yy} b_2 b_{2y} dy\, d\tau \\ &\quad - \int_0^t \int_{-\infty}^{\infty} u_{1yy} b_3 b_{3y} dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v^2} b_{2y} v_y dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{1}{v^2} b_{3y} v_y dy\, d\tau + 2\varepsilon \int_0^t \int_{-\infty}^{\infty} u_{1yy}^2 dy\, d\tau \\ &\quad + 2\varepsilon \int_0^t \int_{-\infty}^{\infty} u_{2yy}^2 dy\, d\tau + 2\varepsilon \int_0^t \int_{-\infty}^{\infty} u_{3yy}^2 dy\, d\tau \\ &\quad + \frac{3C}{\varepsilon} \int_0^t \int_{-\infty}^{\infty} v_y^2 dy\, d\tau + \frac{3C}{\varepsilon} \int_0^t \int_{-\infty}^{\infty} B_y^2 dy\, d\tau. \end{aligned} \tag{44}$$

For the second derivative of v, take the derivative with respect to y of (15) and multiply by v_{yy}/v. Then we perform the integration with respect to y and t.

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} u_{1ty} dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(p_\perp \left(1 - \widehat{b_1^2} \right) + p_\parallel \widehat{b_1^2} \right)_{yy} dy\, d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(\frac{1}{2} \left(b_2^2 + b_3^2 \right) \right)_{yy} dy\, d\tau \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(\frac{u_{1y}}{v} \right)_{yy} dy\, d\tau, \end{aligned} \tag{45}$$

and by using (14), the term on the right-hand side of (45) becomes

$$\begin{aligned}\frac{v_{yy}}{v}\left(\frac{u_{1y}}{v}\right)_{yy} &= \frac{v_{yy}}{v}\left(\frac{v_t}{v}\right)_{yy} = \frac{v_{yy}}{v}(\log v)_{tyy} \\ &= \frac{v_{yy}}{v}\left(\frac{v_y}{v}\right)_{ty} = \frac{v_{yy}}{v}\left(\frac{v_{yy}}{v} - \frac{v_y^2}{v^2}\right)_t.\end{aligned} \tag{46}$$

So,

$$\begin{aligned}&\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(\frac{v_{yy}}{v}\right)_t dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(\frac{v_y^2}{v^2}\right)_t dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}v_{tt}}{v}dy\,d\tau \\ &= \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(p_\perp\left(1-\widehat{b_1^2}\right)+p_\parallel\widehat{b_1^2}\right)_{yy}dy\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(\frac{1}{2}\left(b_2^2+b_3^2\right)\right)_{yy}dy\,d\tau.\end{aligned} \tag{47}$$

For the estimate of v_{yy}, we perform a few estimates. The third term on the left-hand side of the above equation becomes

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}v_{tt}}{v}dy\,d\tau &= \int_{-\infty}^{\infty}\frac{v_{yy}}{v}v_t dy \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\left(\frac{v_{yy}}{v}\right)_t v_t dy\,d\tau \\ &= \int_{-\infty}^{\infty}\frac{v_{yy}}{v}v_t dy \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{v_{yyt}}{v}v_t dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}v_t^2}{v^2}dy\,d\tau.\end{aligned} \tag{48}$$

We estimate the third term on the right-hand side from the above equation, and, by using equation (14), we have

$$\begin{aligned}&\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}v_t^2}{v^2}dy\,d\tau \\ &= \int_0^t\int_{-\infty}^{\infty}\left(\frac{2v_y u_{1y}u_{1yy}}{v^2} - \frac{2v_y^2 u_{1y}^2}{v^3}\right)dy\,d\tau.\end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,t)\,dy - \int_{-\infty}^{\infty}\frac{v_{yy}}{v}u_{1y}dy \\ &\le \int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,0)\,dy \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(\frac{v_y^2}{v^2}\right)_t dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{v_{yyt}}{v}u_{1y}dy\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{2v_y u_{1y}u_{1yy}}{v^2}dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{2v_y^2 u_{1y}^2}{v^3}dy\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(\frac{B}{v}\left(1-\frac{1}{B^2}\right)+\frac{1}{v^3B^4}\right)_{yy}dy\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(\frac{1}{2}\left(b_2^2+b_3^2\right)\right)_{yy}dy\,d\tau.\end{aligned} \tag{50}$$

Take the second derivative with respect to y of the last two terms on the right-hand side. Then, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,t)\,dy - \int_{-\infty}^{\infty}\frac{v_{yy}}{v}u_{1y}dy \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{u_{1yy}^2}{v}dy\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}v_{yy}^2\left[\frac{3}{v^5B^4}+\frac{B}{v^3}-\frac{1}{v^3B}\right]dy\,d\tau \\ &\le \int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,0)\,dy \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{2v_y u_{1yy}v_{yy}}{v^3}dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{2u_{1y}v_y^2 v_{yy}}{v^4}dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{u_{iy}u_{1yy}v_y}{v^2}dy\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{2v_y u_{1y}u_{1yy}}{v^2}dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{2v_y^2 u_{1y}^2}{v^3}dy\,d\tau \\ &\quad + \int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2}v_{yy}B_{yy}dy\,d\tau \\ &\quad - \int_0^t\int_{-\infty}^{\infty}\frac{2}{v^3}v_y v_{yy}B_y dy\,d\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^4}v_y^2v_{yy}B\,dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2B^2}v_{yy}B_{yy}dy\,d\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2B^3}v_{yy}{B_y}^2dy\,d\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^3B^2}v_yv_{yy}B_ydy\,d\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^4B}{v_y}^2v_{yy}dy\,d\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{4}{v^4B^5}v_{yy}B_{yy}dy\,d\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{20}{v^4B^6}v_{yy}{B_y}^2dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{24}{v^5B^5}v_yv_{yy}B_ydy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{12}{v^6B^4}{v_y}^2v_{yy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}b_2b_{2yy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}b_3b_{3yy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}b_{2y}b_{2y}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}b_{3y}b_{3y}dy\,d\tau.
\end{aligned}
\tag{51}
$$

We combine (44) and (51) and we estimate the terms on the right-hand side as $(E(t)^{1/2}+E(t))F(t)$. Therefore,

$$
E_2(t)+F_2(t)\le E_2(0)+C\left(E(t)^{1/2}+E(t)\right)F(t), \tag{52}
$$

where

$$
\begin{aligned}
E_2(t)&=\frac{1}{2}\left\{\left\|u_y\right\|^2+\left\|q_y\right\|^2+\left\|\frac{v_{yy}}{v}\right\|^2\right\},\\
F_2(t)&=\int_0^t\left\{\left\|u_{yy}\right\|^2+\left\|q_{yy}\right\|^2+\left\|v_{yy}\right\|^2\right\}d\tau.
\end{aligned}
\tag{53}
$$

Third, we differentiate (15), (16), and (17) in y, multiply them by $-u_{1yyy}, -u_{2yyy}$, and $-u_{3yyy}$, and integrate with respect to y and t. Similarly, we multiply (18) and (19) by $-b_{2yyy}$ and $-b_{3yyy}$, respectively, and integrate with respect to y and t. Then, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}\sum_1^3u_{iyy}^2(y,t)\,dy+\int_{-\infty}^{\infty}\frac{1}{2}vb_{2yy}^2(y,t)\,dy\\
&+\int_{-\infty}^{\infty}\frac{1}{2}vb_{3yy}^2(y,t)\,dy+\int_0^t\int_{-\infty}^{\infty}\sum_1^3\frac{u_{iyyy}^2}{v}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{b_{2yyy}^2}{v}dy\,d\tau+\int_0^t\int_{-\infty}^{\infty}\frac{b_{3yyy}^2}{v}dy\,d\tau\\
&\le\int_{-\infty}^{\infty}\frac{1}{2}\sum_1^3u_{iyy}^2(y,0)\,dy+\int_{-\infty}^{\infty}\frac{1}{2}vb_{2yy}^2(y,0)\,dy\\
&+\int_{-\infty}^{\infty}\frac{1}{2}vb_{3yy}^2(y,0)\,dy\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_1^3\frac{2}{v^2}u_{iyy}u_{iyyy}v_ydy\,d\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\sum_1^3\frac{v_{yy}}{v^2}u_{iy}u_{iyyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_1^3\frac{2v_y^2}{v^3}u_{iy}u_{iyyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2}v_yb_{2yy}b_{2yyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2}v_yb_{3yy}b_{3yyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2}v_yb_{2y}b_{2yyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2}v_yb_{3y}b_{3yyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^3}v_y^2b_{2y}b_{2yyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^3}v_y^2b_{3y}b_{3yyy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}v_yb_{2yyy}b_{2t}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}v_yb_{3yyy}b_{3t}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}u_{1y}b_{2yyy}b_{2y}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}u_{1y}b_{3yyy}b_{3y}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty}u_{1yy}b_2b_{2yyy}dy\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_{-\infty}^{\infty} u_{1yy}b_3b_{3yyy}\,dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty} u_{1yyy}\left(-\frac{B}{v}\left(1-\frac{1}{B^2}\right)+\frac{1}{v^3B^2}\right.\\
&\left.\cdot\frac{1}{B^2}\right)_{yy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty} u_{2yyy}\left(\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_2}{B^2}\right)_{yy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty} u_{3yyy}\left(\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_3}{B^2}\right)_{yy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty} u_{1yyy}\left(\frac{1}{2}\left(b_2^2+b_3^2\right)\right)_{yy}dy\,d\tau\\
&-\int_0^t\int_{-\infty}^{\infty} u_{2yyy}\left(\frac{1}{\Omega}\left(\frac{B}{v}-\frac{2}{v^3B^2}\right)\frac{u_{3y}}{v}\right)_{yy}dy\,d\tau\\
&+\int_0^t\int_{-\infty}^{\infty} u_{3yyy}\left(\frac{1}{\Omega}\left(\frac{B}{v}-\frac{2}{v^3B^2}\right)\frac{u_{2y}}{v}\right)_{yy}dy\,d\tau.
\end{aligned}
\tag{54}
$$

We estimate the terms on the right-hand side as $(E(t)^{1/2}+E(t))F(t)$. Therefore,

$$E_3(t)+F_3(t)\le E_3(0)+C\left(E(t)^{1/2}+E(t)\right)F(t), \tag{55}$$

where

$$
\begin{aligned}
E_3(t)&=\frac{1}{2}\left\{\left\|u_{yy}\right\|^2+\left\|q_{yy}\right\|^2\right\},\\
F_3(t)&=\int_0^t\left\{\left\|u_{yyy}\right\|^2+\left\|q_{yyy}\right\|^2\right\}d\tau.
\end{aligned}
\tag{56}
$$

Finally, by adding (42), (52), and (55), we obtain the following a priori estimate:

$$
\begin{aligned}
&\left\|(u,v-\overline{v},q)(t)\right\|_2^2\\
&\quad+\int_0^t\left[\sum_{k=1}^{2}\left\|D^k(u,v,q)(t)\right\|^2+\left\|D^3(u,q)(t)\right\|^2\right]d\tau\\
&\leqslant C\left\|(u,v-\overline{v},q)(0)\right\|_2^2+C\left(E(t)^{1/2}+E(t)\right)F(t),
\end{aligned}
\tag{57}
$$

where C is a constant independent of t.

5. Global Existence and Asymptotic Behavior

In this section, we state and prove our global existence result of the system (22) by combining local solution and a priori estimate and also show the asymptotic behavior of the solution. Our main theorem reads as follows.

Theorem 3. *Suppose the initial data $(u,v-\overline{v},q)(0)\in H^2$. Then, the initial value problem for (14)–(19) has a unique solution $(u,v-\overline{v},q)(t)$ globally in time such that $(u,v-\overline{v},q)(t)\in C(0,\infty;H^2)$, $D(u,q)\in L^2(0,\infty;H^2)$, $D(v)\in L^2(0,\infty;H^1)$ for $t>0$ and has estimate for any $t\ge 0$.*

$$
\begin{aligned}
&\left\|(u,v-\overline{v},q)(t)\right\|_2^2\\
&\quad+\int_0^t\left[\sum_{k=1}^{2}\left\|D^k(u,v,q)(\tau)\right\|^2+\left\|D^3(u,q)(\tau)\right\|^2\right]d\tau\\
&\leqslant C\left\|(u,v-\overline{v},q)(0)\right\|_2^2.
\end{aligned}
\tag{58}
$$

Furthermore, the solution has the following decay property:

$$
\begin{aligned}
\left\|\left(u_y,v_y,q_y\right)(t)\right\|&\longrightarrow 0,\\
\left\|(u,v-\overline{v},q)(t)\right\|_{L^\infty}&\longrightarrow 0\\
&\text{as } t\longrightarrow\infty.
\end{aligned}
\tag{59}
$$

Proof. To complete the proof of the existence of global solutions, we use induction on the local solution and extend the time. We also prove that a solution exists for all time and satisfies (58). If M is small and the initial data $t=0$ satisfies (58), then there exists a local solution and therefore there exists $T_1>0$ such that the solution exists and satisfies

$$\sup_{0\le t\le T_1}\left\|(u,v-\overline{v},q)(t)\right\|_2<M. \tag{60}$$

Then we see that the local solution exists for $t\le T_1$ and satisfies a priori estimate (58). Therefore, the Cauchy problem has a solution

$$(u,v,q)\in\left(T_1,2T_1;H^2\right), \tag{61}$$

satisfying the estimate (60). Then, the energy estimate for the local solution and the a priori estimate imply that, for $T_1\le t\le 2T_1$, with the initial data $(u,v-\overline{v},q)(T_1)$, the Cauchy problem (14)–(20) has a local solution satisfying

$$
\begin{aligned}
\sup_{T_1\le t\le 2T_1}&\left\|(u,v-\overline{v},q)(t)\right\|_2\le\left\|(u,v-\overline{v},q)(T_1)\right\|_2\\
&<M.
\end{aligned}
\tag{62}
$$

Then, a priori estimates hold for all t. We continue with the same process for $0\leqslant t\leqslant nT_1$, $n=3,4,\ldots$. Thus, we have a global solution $(u,v,q)(t)\in C(0,\infty;H^2)$ which satisfies the estimate for all $t\geqslant 0$.

To prove the assertion (59) concerning the large-time behavior, we show the decay estimates in detail for u.

Set

$$f(t)=\int_{-\infty}^{\infty}u_y^2(y,t)\,dy. \tag{63}$$

Then, since f is a function of bounded variation, by the Cauchy Schwarz inequality, we have

$$
\begin{aligned}
\underset{0<s<t}{TV}f(t)&\le\left|\int_0^t f'(s)\,ds\right|\le 2\int_0^t\int_{-\infty}^{\infty}\left|u_yu_{ys}\right|dy\,ds\\
&\le 2\int_0^t\left\|u_y\right\|\left\|u_{ys}\right\|ds\\
&\le\sup_{0\le t\le T}\int_0^t\int_{-\infty}^{\infty}\left(u_y^2+u_{ys}^2\right)dy\,ds.
\end{aligned}
\tag{64}
$$

This implies that $f(t)$ has a limit as $t \to \infty$ and it must be zero. The similar conclusion holds for $v - \overline{v}$ and q. Therefore,

$$\left\|\left(u_y, v_y, q_y\right)(t)\right\| \longrightarrow 0. \tag{65}$$

For the decay estimate of L^{∞} norms, using the following inequality,

$$\|u\|_{L^{\infty}}^2 = \sup_y u^2 \leq 2\int_{-\infty}^{\infty} \left|uu_y\right| dy \leq 2\|u\| \left\|u_y\right\|, \tag{66}$$

we have

$$\left\|(u, v - \overline{v}, q)(t)\right\|_{L^{\infty}} \longrightarrow 0. \tag{67}$$

□

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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