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Research Article

Linearization of Fifth-Order Ordinary Differential Equations by Generalized Sundman Transformations

Supaporn Suksern [1,2] and Kwanpaka Naboonmee [1]

[1] Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
[2] Research Center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Supaporn Suksern; supapornsu@nu.ac.th

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In this article, the linearization problem of fifth-order ordinary differential equation is presented by using the generalized Sundman transformation. The necessary and sufficient conditions which allow the nonlinear fifth-order ordinary differential equation to be transformed to the simplest linear equation are found. There is only one case in the part of sufficient conditions which is surprisingly less than the number of cases in the same part for order 2, 3, and 4. Moreover, the derivations of the explicit forms for the linearizing transformation are exhibited. Examples for the main results are included.

1. Introduction

Nonlinear problems are of interest to engineers, physicists, mathematicians and many other scientists since most equations are inherently nonlinear in nature. Although linear ordinary differential equations can be solved by a large number of methods but this situation does not hold for nonlinear equations. One common method to solve nonlinear ordinary differential equations is to change their unknowns by suitable variables so as to get linear ordinary differential equations.

The main tools used to solve the linearization problem are transformations such as point, contact, tangent, and generalized Sundman transformations.

It was recognized that Lie [1] is the first person who solved linearization problem for ordinary differential equations in 1883. He discovered the linearization of second-order ordinary differential equations by point transformations. Later, Liouville [2] and Tresse [3] attacked the equivalence problems for second-order ordinary differential equations via group of point transformations. Moreover, Cartan [4] approached the second-order ordinary differential equations by geometric structure of a certain form.

Mahomed and Leach [5] indicated that the nth-order ($n > 3$) linear ordinary differential equation has exactly one of $n+1$, $n+2$, or $n+4$ point symmetries. They suggested that the necessary and sufficient conditions for the nth-order ($n \geq 3$) to be linearizable by a point transformation must admit the n dimensional Abelian algebra.

The linearization problem of third-order ordinary differential equations under point transformations was solved by Bocharov et al. [6], Grebot [7], and Ibragimov and Meleshko [8]. Fourth-order ordinary differential equation was studied by Ibragimov et al. [9]. They found the necessary and sufficient conditions for a complete linearization problem. The linearization problem of a fifth-order ordinary differential equation with respect to fiber preserving transformations was considered by Suksern and Pinyo [10].

In the series of articles [8, 11–14] the linearization problem of a third-order ordinary differential equation via the contact transformations was solved. For a fourth-order ordinary differential equation, this problem was studied in [15, 16]. The criteria of the linearization problem of fifth-order ordinary differential equations were discovered by Suksern [17].

The linearization problems of third-order and fourth-order ordinary differential equations by the tangent transformations are examined in [18, 19]. These are the first application of tangent (essentially) transformations to the linearization problems of third-order and fourth-order ordinary differential equations. Necessary and sufficient conditions for third-order and fourth-order ordinary differential equations to be linearizable are obtained there.

Sundman introduced the generalized Sundman transformations in 1992. Later on Duarte et al. [20] applied this method to transform second-order ordinary differential equations into free particle equations. In addition, Muriel and Romero [21] characterized the equations that can be linearized by means of generalized Sundman transformations in terms of first integral. A new characterization of linearizable equations in terms of the coefficients of ordinary differential equation and one auxiliary function was given by Mustafa et al. [22]. Moreover, Nakpim and Meleshko [23] pointed out that the solution given by Duarte et al. using the Laguerre form is not complete.

For the third-order ordinary differential equations, the linearization by the generalized Sundman transformation was investigated by [24] for the form $X'''(T) = 0$ and [25] for the Laguerre form. Some applications of the generalized Sundman transformation to ordinary differential equations can be found in [26]. More information of the generalized Sundman transformation are collected in the book [27].

The linearization problem of a fourth-order ordinary differential equation with respect to generalized Sundman transformations was studied in [28]. They found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation to be transformed to the simplest linear equation.

In this article, we intend to use the generalized Sundman transformations to linearize the fifth-order ordinary differential equations in some particular cases. We use computer algebra system Reduce to compute the necessary and sufficient conditions of the linearization. We provide some examples to illustrate the conditions that we have found and also obtain the linearizing transformations.

2. Necessary Conditions

We now concentrate on finding the fifth-order ordinary differential equations

$$x^{(5)} = f\left(t, x, x', x'', x''', x^{(4)}\right), \tag{1}$$

which can be transformed to the linear equation

$$X^{(5)}(T) = 0, \tag{2}$$

under the generalized Sundman transformation

$$\begin{aligned} X &= F(t,x), \\ dT &= G(t,x)\,dt. \end{aligned} \tag{3}$$

It turns out that those equations must be in the form of the following theorem.

Theorem 1. *Any linearizable fifth-order ordinary differential equations that can be transformed by a generalized Sundman transformation has to be in the form*

$$\begin{aligned} x^{(5)} &+ \left(A_1 x' + A_0\right) x^{(4)} \\ &+ \left(B_3 x'' + B_2 x'^2 + B_1 x' + B_0\right) x''' \\ &+ \left(C_1 x' + C_0\right) x''^2 \\ &+ \left(D_3 x'^3 + D_2 x'^2 + D_1 x' + D_0\right) x'' + H_5 x'^5 \\ &+ H_4 x'^4 + H_3 x'^3 + H_2 x'^2 + H_1 x' + H_0 = 0. \end{aligned} \tag{4}$$

Here $A_i = A_i(t,x)$, $B_i = B_i(t,x)$, $C_i = C_i(t,x)$, $D_i = D_i(t,x)$, and $H_i = H_i(t,x)$ are some functions of t and x. Expressions of these coefficients are presented in the appendix.

Proof. By a generalized Sundman transformation (3), we have

$$X'(T) = \frac{D_t F(t,x)}{D_t \int G(t,x)\,dt} = \frac{F_t + x' F_x}{G} = P\left(t, x, x'\right),$$

$$\begin{aligned} X''(T) &= \frac{D_t P\left(t,x,x'\right)}{D_t \int G(t,x)\,dt} = \frac{P_t + x' P_x + x'' P_{x'}}{G} \\ &= \frac{1}{G^3}\left[2F_{tx} G x' + F_{tt} G - F_t G_t - F_t G_x x' + F_{xx} G x'^2 \right. \\ &\quad \left. - F_x G_t x' - F_x G_x x'^2 + F_x G x''\right] = Q\left(t, x, x', x''\right), \end{aligned}$$

$$\begin{aligned} X'''(T) &= \frac{D_t Q}{D_t \int G(t,x)\,dt} \\ &= \frac{Q_t + x' Q_x + x'' Q_{x'} + x''' Q_{x''}}{G} = \frac{1}{G^5}\left[\left(F_x G^2\right) x''' \right. \\ &\quad + G\left(3F_{xx} G - 4F_x G_x\right) x' x'' \\ &\quad \left. + G\left(3F_{tx} G - F_t G_x - 3F_x G_t\right) x'' + \cdots\right] \\ &= R\left(t, x, x', x'', x'''\right), \end{aligned}$$

$$\begin{aligned} X^{(4)}(T) &= \frac{D_t R}{D_t \int G(t,x)\,dt} \\ &= \frac{R_t + x' R_x + x'' R_{x'} + x''' R_{x''} + x^{(4)} R_{x'''}}{G} \\ &= \frac{1}{G^7}\left[\left(F_x G^3\right) x^{(4)} + G^2\left(4F_{xx} G - 7F_x G_x\right) x' x''' \right. \\ &\quad \left. + G^2\left(4F_{tx} G - F_t G_x - 6F_x G_t\right) x''' + \cdots\right] \\ &= S\left(t, x, x', x'', x''', x^{(4)}\right), \end{aligned}$$

$$\begin{aligned} X^{(5)}(T) &= \frac{D_t S}{D_t \int G(t,x)\,dt} \\ &= \frac{S_t + x' S_x + x'' S_{x'} + x''' S_{x''} + x^{(4)} S_{x'''} G + x^{(5)} S_{x^{(4)}}}{G} \\ &= \frac{1}{G^9}\left[\left(F_x G^4\right) x^{(5)} + G^3\left(5F_{xx} G - 11F_x G_x\right) x' x^{(4)}\right. \end{aligned}$$

$$+G^3\left(5F_{tx}G-F_tF_x-10F_xG_t\right)x^{(4)}+\cdots\Big]$$
$$=V\left(t,x,x',x'',x''',x^{(4)},x^{(5)}\right), \tag{5}$$

where $D_t = \partial/\partial t + x'(\partial/\partial x) + x''(\partial/\partial x') + x'''(\partial/\partial x'') + x^{(4)}(\partial/\partial x''')+x^{(5)}(\partial/\partial x^{(4)})+\cdots$ is a total derivative. Replacing $X^{(5)}(T)$ in (2), we get that

$$\begin{aligned}
&x^{(5)}+\left(\left(\frac{5F_{xx}G-11F_xG_x}{F_xG}\right)x'+\left(\frac{5F_{tx}G-F_tG_x-10F_xG_t}{F_xG}\right)\right)x^{(4)}+\left(\left(\frac{5\left(2F_{xx}G-3F_xG_x\right)}{F_xG}\right)x''\right.\\
&+\left(\frac{10F_{xxx}G^2-45F_{xx}G_xG-14F_xG_{xx}G+\cdots}{F_xG^2}\right)x'^2\\
&+\left.\left(\frac{20F_{txx}G^2-50F_{tx}G_xG-4F_tG_{xx}G+15F_tG_x^2-40F_{xx}G_tG+\cdots}{F_xG^2}\right)x'+\cdots\right)x'''\\
&+\left(\left(\frac{15F_{xxx}G^2-60F_{xx}G_xG-18F_xG_{xx}G+70F_xG_x^2}{F_xG^2}\right)x'\right.\\
&+\left.\left(\frac{15F_{txx}G^2-30F_{tx}G_xG-3F_tG_{xx}G+10F_tG_x^2-30F_{xx}G_tG+\cdots}{F_xG^2}\right)\right)x''^2\\
&+\left(\left(\frac{10F_{xxxx}G^3-70F_{xxx}G_xG^2-45F_{xx}G_{xx}G^2+195F_{xx}G_x^2G+\cdots}{F_xG^3}\right)x'^3+\cdots\right)x''\\
&+\left(\frac{F_{xxxxx}G^4-10F_{xxxx}G_xG^3-10F_{xxx}G_{xx}G^3+45F_{xxx}G_x^2G^2+\cdots}{F_xG^4}\right)x'^5+\cdots=0.
\end{aligned} \tag{6}$$

Denoting A_i, B_i, C_i, D_i, and H_i as (A.1)–(A.18), we obtain the necessary form (4). This proves the theorem. □

3. Sufficient Conditions and Linearizing Transformation

To get the sufficient conditions, we consider (A.1)–(A.18) appearing in the previous section. After using the compatibility theory to those equations, we derive the following results.

Theorem 2. *Equation (4) can be linearizable by the generalized Sundman transformation if its coefficients satisfy the following equations:*

$$S_{4x}=\frac{\left(7A_1S_4+49S_1+23S_4^2\right)}{280}, \tag{7}$$

$$S_{8t}=\left(27720S_2S_8-8520S_5S_8+62S_6S_8+4928S_7-115S_8^2\right)\left(277200S_4\right)^{-1}, \tag{8}$$

$$S_{8x}=\left(21A_1S_4S_8+147S_1S_8+73920S_2S_4^2-22720S_4^2S_5-40S_4^2S_6+69S_4^2S_8\right)\left(840S_4\right)^{-1}, \tag{9}$$

$$\begin{aligned}
B_{2xx}=&\left(-2195200B_{2x}A_1+823200B_{2x}S_4-548800S_{1xx}-644840S_{1x}A_1+82320S_{1x}S_4+10633A_1^3S_4-10633A_1^2S_1\right.\\
&+1519A_1^2S_4^2+89180A_1B_2S_4+72716A_1S_1S_4+2604A_1S_4^3-548800B_2^2-260680B_2S_1+11760B_2S_4^2+54880000H_5\\
&\left.+43218S_1^2+1176S_1S_4^2+558S_4^4\right)\left(5488000\right)^{-1},
\end{aligned} \tag{10}$$

$$\begin{aligned}
S_{1x}=&\left(-78400B_{2x}-637A_1^2S_4-15680A_1B_2-5635A_1S_1+105A_1S_4^2+2940B_2S_4+78400D_3+294S_1S_4+30S_4^3\right)\\
&\cdot\left(5880\right)^{-1},
\end{aligned} \tag{11}$$

$$C_{0x} = \left(-5488560A_{1t}S_4^2 - 38419920S_{1t}S_4 + 25347840S_{5x}S_4 + 75600S_{6x}S_4 - 18627840A_1C_0S_4 - 6137208A_1S_2S_4 + 6337320A_1S_4S_5 + 16929A_1S_4S_6 + 6573A_1S_4S_8 + 23950080C_0S_4^2 + 46569600D_2S_4 + 2794176S_1S_2 - 5188848S_1S_5 - 12096S_1S_6 - 6237S_1S_8 - 5207664S_2S_4^2 + 2080288S_4^2S_5 + 3682S_4^2S_6 + 5925S_4^2S_8\right)\left(93139200S_4\right)^{-1}, \quad (12)$$

$$A_{1t} = \left(-133056A_1S_2S_4 + 168240A_1S_4S_5 + 576A_1S_4S_6 - 231A_1S_4S_8 + 443520C_0S_4^2 + 25872S_1S_5 + 231S_1S_8 - 133056S_2S_4^2 + 21424S_4^2S_5 + 144S_4^2S_6 - 215S_4^2S_8\right)\left(1330560S_4^2\right)^{-1}, \quad (13)$$

$$S_{1t} = \left(135878840S_{5x}S_4 + 40320S_{6x}S_4 - 1862784A_1S_2S_4 + 2662464A_1S_4S_5 + 7056A_1S_4S_6 + 2541A_1S_4S_8 + 6209280C_0S_4^2 + 1862784S_1S_2 - 3235008S_1S_5 - 8064S_1S_6 - 2541S_1S_8 - 1862784S_2S_4^2 + 483392S_4^2S_5 + 504S_4^2S_6 + 2165S_4^2S_8\right) \cdot \left(18627840S_4\right)^{-1}, \quad (14)$$

$$S_{3x} = \left(-\left(\left(246400\left(144\left(3\left(308B_0 + 491S_3\right)S_4 + 5S_{6t}\right)S_4^2 - \left(337S_{5x} + S_{6x}\right)\left(112S_5 + S_8\right)\right) - \left(7020925440S_5^2 + 33795200S_5S_6 - 2623280S_5S_8 + 61440S_6^2 - 41624S_6S_8 + 2776064S_7 + 52785S_8^2\right)S_4\right)S_4 - 77\left(149022720S_3S_4^2 - 23206400S_5^2 - 62720S_5S_6 - 225680S_5S_8 - 224S_6S_8 + 21504S_7 - 765S_8^2\right)S_1 + 443520\left(32S_6 + 23S_8 + 27344S_5\right)S_2S_4^2 + 77\left(949760S_5^2 + 8960S_5S_6 - 10000S_5S_8 + 416S_6S_8 + 21504S_7 - 765S_8^2 + 149022720S_3S_4^2\right) \cdot A_1S_4\right)\right)\left(65569996800S_4^3\right)^{-1}, \quad (15)$$

$$S_{5x} = \left(-155232A_1S_4S_5 - 1617A_1S_4S_8 + 362208S_1S_5 + 1617S_1S_8 - 680064S_2S_4^2 - 69856S_4^2S_5 - 136S_4^2S_6 - 1449S_4^2S_8\right) \cdot \left(1034880S_4\right)^{-1}, \quad (16)$$

$$S_{7x} = \left(77616000S_{6t}S_4^3 - 64680S_{6x}S_4S_8 + 1617A_1S_4S_6S_8 + 206976A_1S_4S_7 + 20490624000B_0S_4^4 + 11319S_1S_6S_8 + 1448832S_1S_7 + 4719052800S_2S_4^2S_5 + 2069760S_2S_4^2S_6 + 17001600S_2S_4^2S_8 + 18255283200S_3S_4^4 - 3786137600S_4^2S_5^2 - 18085760S_4^2S_5S_6 - 5225600S_4^2S_5S_8 - 30520S_4^2S_6^2 - 3887S_4^2S_6S_8 - 1022336S_4^2S_7\right)\left(4139520S_4\right)^{-1}, \quad (17)$$

$$C_1 = \frac{\left(9A_1S_4 + 60B_2 + 3S_1 + S_4^2\right)}{40}, \quad (18)$$

and (A.19), (A.20), (A.21), (A.22), (A.23), (A.24), (A.25) (moved to the appendix in order to avoid the huge expressions), where

$$\begin{aligned}
S_1 &= -10A_{1x} - 2A_1^2 + 5B_2, \\
S_2 &= -20A_{0x} - 4A_0A_1 + 5B_1, \\
S_3 &= 10A_{0t} + 2A_0^2 - 5B_0, \\
S_4 &= -2A_1 + B_3, \\
S_5 &= -80S_{4t} + 2A_0S_4 + 7S_2, \\
S_6 &= -462A_0S_4 + 231S_2 - 337S_5, \\
S_7 &= -13860S_{5t}S_4 + 1386S_2S_5 + 8316S_3S_4^2 - 202S_5^2 + S_5S_6, \\
S_8 &= -41580B_1 + 55440C_0 - 8316S_2 + 9176S_5 + 27S_6.
\end{aligned} \quad (19)$$

Proof. We start with the coefficients A_i, B_i, C_i, D_i, and H_i in Theorem 1 through the unknown functions F and G. From (A.1) and (A.2), we have the derivatives

$$F_{xx} = \frac{\left(F_x\left(11G_x + A_1G\right)\right)}{(5G)}, \quad (20)$$

$$F_{tx} = \frac{\left(F_tG_x + 10F_xG_t + F_xA_0G\right)}{(5G)}. \quad (21)$$

From (A.4), one obtains the derivative

$$G_{xx} = \frac{\left(63G_x^2 + G_xA_1G + G^2S_1\right)}{(40G)}, \quad (22)$$

where

$$S_1 = -10A_{1x} - 2A_1^2 + 5B_2. \tag{23}$$

From (A.5), one gets the derivative

$$G_{tx} = \frac{\left(-9F_tG_x^2 + 135F_xG_tG_x + F_xG\left(2G_xA_0 + GS_2\right)\right)}{\left(80F_xG\right)}, \tag{24}$$

where

$$S_2 = -20A_{0x} - 4A_0A_1 + 5B_1. \tag{25}$$

From (A.6), one finds the derivative

$$F_{tt} = \frac{\left(9F_t^2G_x^2 + 225F_tF_xG_tG_x + F_tF_xG\left(14G_xA_0 - GS_2\right) + 400F_x^2G_{tt}G - 600F_x^2G_t^2 + 8F_x^2G^2S_3\right)}{\left(120F_xG_xG\right)}, \tag{26}$$

where

$$S_3 = 10A_{0t} + 2A_0^2 - 5B_0. \tag{27}$$

From (A.3), one obtains the derivative

$$G_x = \frac{\left(GS_4\right)}{7}, \tag{28}$$

where

$$S_4 = -2A_1 + B_3. \tag{29}$$

We note that, for the case $G_x = 0$, the generalized Sundman transformations are indeed the point transformations. We then suppose $G_x \neq 0$, which also implies $S_4 \neq 0$.

The relations $(G_x)_x = G_{xx}$ and $(G_x)_t = G_{tx}$ provide condition (7) and the derivative

$$F_t = \frac{\left(385F_xG_tS_4 + 7F_xGS_5\right)}{\left(9GS_4^2\right)}, \tag{30}$$

where

$$S_5 = -80S_{4t} + 2A_0S_4 + 7S_2. \tag{31}$$

The relation $(F_t)_t = F_{tt}$ gives the derivative

$$G_{tt} = \frac{\left(-2156000G_t^2S_4^2 + 385G_tGS_4S_6 + 4G^2S_7\right)}{\left(1386000GS_4^2\right)}, \tag{32}$$

where

$$\begin{aligned} S_6 &= -462A_0S_4 + 231S_2 - 337S_5, \\ S_7 &= -13860S_{5t}S_4 + 1386S_2S_5 + 8316S_3S_4^2 - 202S_5^2 \\ &\quad + S_5S_6. \end{aligned} \tag{33}$$

Substituting A_0 into A_{0x} and A_{0t}, one obtains the conditions

$$\begin{aligned} S_{2x} &= \left(94360S_{5x}S_4 + 280S_{6x}S_4 - 11319A_1S_2S_4\right. \\ &\quad + 16513A_1S_4S_5 + 49A_1S_4S_6 + 32340B_1S_4^2 \\ &\quad + 11319S_1S_2 - 16513S_1S_5 - 49S_1S_6 - 1155S_2S_4^2 \\ &\quad \left.- 7751S_4^2S_5 - 23S_4^2S_6\right)\left(64680S_4\right)^{-1}, \\ S_{2t} &= \left(889680S_{5t}S_4 + 2640S_{6t}S_4 + 609840B_0S_4^2\right. \\ &\quad + 70224S_2S_5 + 231S_2S_6 + 121968S_3S_4^2 - 102448S_5^2 \\ &\quad \left.- 641S_5S_6 - S_6^2\right)\left(609840S_4\right)^{-1}. \end{aligned} \tag{34}$$

From (A.8), we have

$$G_t = \frac{\left(GS_8\right)}{\left(6160S_4\right)}, \tag{35}$$

where

$$\begin{aligned} S_8 &= -41580B_1 + 55440C_0 - 8316S_2 + 9176S_5 \\ &\quad + 27S_6. \end{aligned} \tag{36}$$

The relations $(G_t)_t = G_{tt}$ and $(G_t)_x = G_{tx}$ provide conditions (8) and (9). From (A.18), (A.15), (A.17), (A.13), and (A.11), we obtain conditions (A.19)–(A.21), (10), (A.22). Substituting the relation C_{0t} into C_{0tt}, one obtains condition (A.23). Equations (A.9), (A.10), and (A.12) provide conditions (11), (12), (A.24). Comparing the mixed derivatives $(F_{xx})_t = (F_{tx})_x$, $(G_{xx})_t = (G_{tx})_x$, $(F_{tt})_x = (F_{tx})_t$, $(F_t)_x = F_{tx}$, we obtain conditions (13)–(16). Substituting the relation S_{5x} into S_{5xx}, one obtains condition (A.25). Comparing the mixed derivative $(G_{tt})_x = (G_{tx})_t$, one arrives at condition (17). From (A.7), one obtains condition (18). This proves the theorem. □

Corollary 3. *Under the sufficient conditions in Theorem 2, the transformation (3) mapping equation (4) to a linear equation (2) can be solved by the compatible system of (20), (28), (30), and (35).*

Remark 4. In the part of sufficient conditions for second-order, there are 2 cases in [20] and 3 cases in [23]. For the third-order, there are 3 cases in [24] and 4 cases in [25]. For the fourth-order, there are 2 cases in [28]. But for the fifth-order there is only one case.

4. Examples

Example 1. For the fifth-order ordinary differential equation

$$\begin{aligned} x^{(5)}x^4 - 11x'x^{(4)}x^3 - 15x''x'''x^3 + 60x'^2x'''x^2 \\ + 70x'x''^2x^2 - 210x'^3x''x + 105x'^5 = 0, \end{aligned} \tag{37}$$

we can verify that this equation cannot be linearized by a point transformation [10] or contact transformation [17]. However, (37) is in fact the form (4) in Theorem 1 with the coefficients

$$\begin{aligned} A_1 &= \frac{-11}{x}, \\ A_0 &= 0, \\ B_3 &= \frac{-15}{x}, \\ B_2 &= \frac{60}{x^2}, \\ B_1 &= 0, \\ B_0 &= 0, \\ C_1 &= \frac{70}{x^2}, \\ C_0 &= 0, \\ D_3 &= \frac{-210}{x^3}, \\ D_2 &= 0, \\ D_1 &= 0, \\ D_0 &= 0, \\ H_5 &= \frac{105}{x^4}, \\ H_4 &= 0, \\ H_3 &= 0, \\ H_2 &= 0, \\ H_1 &= 0, \\ H_0 &= 0, \\ S_1 &= \frac{-52}{x^2}, \\ S_2 &= 0, \\ S_3 &= 0, \\ S_4 &= \frac{7}{x}, \\ S_5 &= 0, \\ S_6 &= 0, \\ S_7 &= 0, \\ S_8 &= 0. \end{aligned} \tag{38}$$

Moreover, these coefficients also satisfy the conditions in Theorem 2. We now conclude that (37) is linearizable by a generalized Sundman transformation. Corollary 3 yields the linearizing transformation by solving the following equations:

$$F_{xx} = 0, \tag{39}$$

$$G_x = \frac{G}{x}, \tag{40}$$

$$G_t = 0, \tag{41}$$

$$F_t = 0. \tag{42}$$

Considering (40), one arrives at

$$G = xK_1(t). \tag{43}$$

Considering (41), one obtains

$$G = K_2(x). \tag{44}$$

From (43) and (44), one can choose $K_1(t) = 1$ and $K_2(x) = x$; then we have

$$G = x. \tag{45}$$

Considering (39), one gets

$$F = K_3(t)\, x + K_4(t). \tag{46}$$

Considering (42), one arrives at

$$F = K_5(x). \tag{47}$$

From (46) and (47), one can choose $K_3(t) = 1$, $K_4(t) = 0$, and $K_5(x) = x$; then we obtain

$$F = x. \tag{48}$$

So the linearizing transformation is

$$\begin{aligned} X &= x, \\ dT &= x\,dt. \end{aligned} \tag{49}$$

Hence, by (49), (37) becomes

$$X^{(5)} = 0. \tag{50}$$

The general solution of (50) is

$$X = \frac{c_1}{24}T^4 + \frac{c_2}{6}T^3 + \frac{c_3}{2}T^2 + c_4T + c_5, \tag{51}$$

where c_1, c_2, c_3, c_4, and c_5 are arbitrary constants. Substituting (49) into (51), the general solution of (37) is

$$x(t) = \frac{c_1}{24}\phi(t)^4 + \frac{c_2}{6}\phi(t)^3 + \frac{c_3}{2}\phi(t)^2 + c_4\phi(t) + c_5, \quad (52)$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \frac{c_1}{24}T^4 + \frac{c_2}{6}T^3 + \frac{c_3}{2}T^2 + c_4T + c_5. \quad (53)$$

Example 2. For the fifth-order ordinary differential equation

$$\begin{aligned} & x^{(5)}tx^4 - 22x'x^{(4)}tx^3 + 3x^{(4)}x^4 - 30x''x'''tx^3 \\ & \quad + 212x'^2x'''tx^2 - 48x'x'''x^3 + 244x'x''^2tx^2 \\ & \quad - 26x''^2x^3 - 1180x'^3x''tx + 320x'^2x''x^2 \\ & \quad + 880x'^5t - 320x'^4x = 0 \end{aligned} \quad (54)$$

we can verify that this equation cannot be linearized by a point transformation [10] or contact transformation [17]. However, (54) is in fact the form (4) in Theorem 1 with the coefficients

$$\begin{aligned} A_1 &= \frac{-22}{x}, \\ A_0 &= \frac{3}{t}, \\ B_3 &= \frac{-30}{x}, \\ B_2 &= \frac{212}{x^2}, \\ B_1 &= \frac{-48}{tx}, \\ B_0 &= 0, \\ C_1 &= \frac{244}{x^2}, \\ C_0 &= \frac{-26}{tx}, \\ D_3 &= \frac{-1180}{x^3}, \\ D_2 &= \frac{320}{tx^2}, \\ D_1 &= 0, \\ D_0 &= 0, \\ H_5 &= \frac{880}{x^4}, \\ H_4 &= \frac{-320}{tx^3}, \\ H_3 &= 0, \\ H_2 &= 0, \\ H_1 &= 0, \\ H_0 &= 0, \\ S_1 &= \frac{-128}{x^2}, \\ S_2 &= \frac{24}{tx}, \\ S_3 &= \frac{-12}{t^2}, \\ S_4 &= \frac{14}{x}, \\ S_5 &= \frac{252}{tx}, \\ S_6 &= \frac{-98784}{tx}, \\ S_7 &= 0, \\ S_8 &= 0. \end{aligned} \quad (55)$$

Moreover, these coefficients also satisfy the conditions in Theorem 2. We now conclude that (54) is linearizable by a generalized Sundman transformation. Corollary 3 yields the linearizing transformation by solving the following equations:

$$F_{xx} = 0, \quad (56)$$

$$G_x = \frac{(2G)}{x}, \quad (57)$$

$$G_t = 0, \quad (58)$$

$$F_t = \frac{(F_x x)}{t}. \quad (59)$$

Considering (57), one arrives at

$$G = K_1(t)\,x^2. \quad (60)$$

Considering (58), one obtains

$$G = K_2(x). \quad (61)$$

From (60) and (61), one can choose $K_1(t) = 1$ and $K_2(x) = x^2$; then we obtain

$$G = x^2. \quad (62)$$

Equation (59) becomes

$$tF_t - xF_x = 0, \quad (63)$$

and by Cauchy method, one arrives at

$$F = tx. \quad (64)$$

This solution satisfies (56), so the linearizing transformation is

$$X = tx, \quad dT = x^2 dt. \tag{65}$$

Hence, by (65), (54) becomes

$$X^{(5)} = 0. \tag{66}$$

The general solution of (66) is

$$X = \frac{c_1}{24}T^4 + \frac{c_2}{6}T^3 + \frac{c_3}{2}T^2 + c_4 T + c_5, \tag{67}$$

where c_1, c_2, c_3, c_4, and c_5 are arbitrary constants. Substituting (65) into (67), the general solution of (54) is

$$x(t) = \frac{\left((c_1/24)\,\phi(t)^4 + (c_2/6)\,\phi(t)^3 + (c_3/2)\,\phi(t)^2 + c_4\phi(t) + c_5\right)}{t}, \tag{68}$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \left(\frac{(c_1/24)\,T^4 + (c_2/6)\,T^3 + (c_3/2)\,T^2 + c_4 T + c_5}{t}\right)^2. \tag{69}$$

Appendix

Equations for Theorem 1 in Section 2

$$A_1 = \frac{(5F_{xx}G - 11F_xG_x)}{(F_xG)}, \tag{A.1}$$

$$A_0 = \frac{(5F_{tx}G - F_tG_x - 10F_xG_t)}{(F_xG)}, \tag{A.2}$$

$$B_3 = \frac{5\,(2F_{xx}G - 3F_xG_x)}{(F_xG)}, \tag{A.3}$$

$$B_2 = \frac{(10F_{xxx}G^2 - 45F_{xx}G_xG - 14F_xG_{xx}G + 60F_xG_x^2)}{(F_xG^2)}, \tag{A.4}$$

$$B_1 = \left(20F_{txx}G^2 - 50F_{tx}G_xG - 4F_tG_{xx}G + 15F_tG_x^2 - 40F_{xx}G_tG - 24F_xG_{tx}G + 105F_xG_tG_x\right)\left(F_xG^2\right)^{-1}, \tag{A.5}$$

$$B_0 = \left(-40F_{tx}G_tG + 10F_{ttx}G^2 - 5F_{tt}G_xG - 4F_tG_{tx}G + 15F_tG_tG_x - 10F_xG_{tt}G + 45F_xG_t^2\right)\left(F_xG^2\right)^{-1}, \tag{A.6}$$

$$C_1 = \frac{(15F_{xxx}G^2 - 60F_{xx}G_xG - 18F_xG_{xx}G + 70F_xG_x^2)}{(F_xG^2)}, \tag{A.7}$$

$$C_0 = \left(15F_{txx}G^2 - 30F_{tx}G_xG - 3F_tG_{xx}G + 10F_tG_x^2 - 30F_{xx}G_tG - 15F_xG_{tx}G + 60F_xG_tG_x\right)\left(F_xG^2\right)^{-1}, \tag{A.8}$$

$$D_3 = \left(10F_{xxxx}G^3 - 70F_{xxx}G_xG^2 - 45F_{xx}G_{xx}G^2 + 195F_{xx}G_x^2G - 11F_xG_{xxx}G^2 + 125F_xG_{xx}G_xG - 210F_xG_x^3\right)\left(F_xG^3\right)^{-1}, \tag{A.9}$$

$$D_2 = \left(30F_{txxx}G^3 - 150F_{txx}G_xG^2 - 60F_{tx}G_{xx}G^2 + 255F_{tx}G_x^2G - 6F_tG_{xxx}G^2 + 65F_tG_{xx}G_xG - 105F_tG_x^3 - 60F_{xxx}G_tG^2 - 75F_{xx}G_{tx}G^2 + 330F_{xx}G_tG_xG - 27F_xG_{txx}G^2 + 205F_xG_{tx}G_xG + 105F_xG_tG_{xx}G - 525F_xG_tG_x^2\right)\left(F_xG^3\right)^{-1}, \tag{A.10}$$

$$D_1 = \left(-120F_{txx}G_tG^2 - 90F_{tx}G_{tx}G^2 + 390F_{tx}G_tG_xG + 30F_{ttxx}G^3 - 90F_{ttx}G_xG^2 - 15F_{tt}G_{xx}G^2 + 60F_{tt}G_x^2G - 12F_tG_{txx}G^2 + 85F_tG_{tx}G_xG + 45F_tG_tG_{xx}G - 210F_tG_tG_x^2 - 30F_{xx}G_{tt}G^2 + 135F_{xx}G_t^2G + 165F_xG_{tx}G_tG - 21F_xG_{ttx}G^2 + 80F_xG_{tt}G_xG - 420F_xG_t^2G_x\right)\left(F_xG^3\right)^{-1}, \tag{A.11}$$

$$D_0 = \left(-30F_{tx}G_{tt}G^2 + 135F_{tx}G_t^2G + 10F_{tttx}G^3 - 10F_{ttt}G_xG^2 - 60F_{ttx}G_tG^2 - 15F_{tt}G_{tx}G^2 + 60F_{tt}G_tG_xG + 45F_tG_{tx}G_tG - 6F_tG_{ttx}G^2 + 20F_tG_{tt}G_xG - 105F_tG_t^2G_x - 5F_xG_{ttt}G^2 + 60F_xG_{tt}G_tG - 105F_xG_t^3\right)\left(F_xG^3\right)^{-1}, \tag{A.12}$$

$$H_5 = \left(F_{xxxxx}G^4 - 10F_{xxxx}G_xG^3 - 10F_{xxx}G_{xx}G^3 + 45F_{xxx}G_x^2G^2 - 5F_{xx}G_{xxx}G^3 + 60F_{xx}G_{xx}G_xG^2 - 105F_{xx}G_x^3G - F_xG_{xxxx}G^3 + 15F_xG_{xxx}G_xG^2 + 10F_xG_{xx}^2G^2 - 105F_xG_{xx}G_x^2G + 105F_xG_x^4\right) \cdot \left(F_xG^4\right)^{-1}, \tag{A.13}$$

$$H_4 = \left(5F_{txxxx}G^4 - 40F_{txxx}G_xG^3 - 30F_{txx}G_{xx}G^3 + 135F_{txx}G_x^2G^2 - 10F_{tx}G_{xxx}G^3 + 120F_{tx}G_{xx}G_xG^2 - 210F_{tx}G_x^3G - F_tG_{xxxx}G^3 + 15F_tG_{xxx}G_xG^2\right.$$

$$
\begin{aligned}
&+10F_tG_{xx}^2G^2-105F_tG_{xx}G_x^2G+105F_tG_x^4\\
&-10F_{xxxx}G_tG^3-20F_{xxx}G_{tx}G^3+90F_{xxx}G_tG_xG^2\\
&-15F_{xx}G_{txx}G^3+120F_{xx}G_{tx}G_xG^2+60F_{xx}G_tG_{xx}G^2\\
&-315F_{xx}G_tG_x^2G-4F_xG_{txxx}G^3+45F_xG_{txx}G_xG^2\\
&+40F_xG_{tx}G_{xx}G^2-210F_xG_{tx}G_x^2G+15F_xG_tG_{xxx}G^2\\
&-210F_xG_tG_{xx}G_xG+420F_xG_tG_x^3\Big)\left(F_xG^4\right)^{-1},
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
H_3=\Big(&-40F_{txxx}G_tG^3-60F_{txx}G_{tx}G^3+270F_{txx}G_tG_xG^2\\
&-30F_{tx}G_{txx}G^3+240F_{tx}G_{tx}G_xG^2+120F_{tx}G_tG_{xx}G^2\\
&-630F_{tx}G_tG_x^2G+10F_{ttxxx}G^4-60F_{ttxx}G_xG^3\\
&-30F_{ttx}G_{xx}G^3+135F_{ttx}G_x^2G^2-5F_{tt}G_{xxx}G^3\\
&+60F_{tt}G_{xx}G_xG^2-105F_{tt}G_x^3G-4F_tG_{txxx}G^3\\
&+45F_tG_{txx}G_xG^2+40F_tG_{tx}G_{xx}G^2-210F_tG_{tx}G_x^2G\\
&+15F_tG_tG_{xxx}G^2-210F_tG_tG_{xx}G_xG+420F_tG_tG_x^3\\
&-10F_{xxx}G_{tt}G^3+45F_{xxx}G_t^2G^2+120F_{xx}G_{tx}G_tG^2\\
&-15F_{xx}G_{ttx}G^3+60F_{xx}G_{tt}G_xG^2-315F_{xx}G_t^2G_xG\\
&+45F_xG_{txx}G_tG^2+40F_xG_{tx}^2G^2-420F_xG_{tx}G_tG_xG\\
&-6F_xG_{ttxx}G^3+45F_xG_{ttx}G_xG^2+20F_xG_{tt}G_{xx}G^2\\
&-105F_xG_{tt}G_x^2G-105F_xG_t^2G_{xx}G+630F_xG_t^2G_x^2\Big)\\
&\cdot\left(F_xG^4\right)^{-1},
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
H_2=\Big(&-30F_{txx}G_{tt}G^3+135F_{txx}G_t^2G^2+240F_{tx}G_{tx}G_tG^2\\
&-30F_{tx}G_{ttx}G^3+120F_{tx}G_{tt}G_xG^2-630F_{tx}G_t^2G_xG\\
&+10F_{tttxx}G^4-40F_{tttx}G_xG^3-10F_{ttt}G_{xx}G^3\\
&+45F_{ttt}G_x^2G^2-60F_{ttxx}G_tG^3-60F_{ttx}G_{tx}G^3\\
&+270F_{ttx}G_tG_xG^2-15F_{tt}G_{txx}G^3+120F_{tt}G_{tx}G_xG^2\\
&+60F_{tt}G_tG_{xx}G^2-315F_{tt}G_tG_x^2G+45F_tG_{txx}G_tG^2\\
&+40F_tG_{tx}^2G^2-420F_tG_{tx}G_tG_xG-6F_tG_{ttxx}G^3\\
&+45F_tG_{ttx}G_xG^2+20F_tG_{tt}G_{xx}G^2-105F_tG_{tt}G_x^2G\\
&-105F_tG_t^2G_{xx}G+630F_tG_t^2G_x^2-5F_{xx}G_{ttt}G^3\\
&+60F_{xx}G_{tt}G_tG^2-105F_{xx}G_t^3G+40F_xG_{tx}G_{tt}G^2\\
&-210F_xG_{tx}G_t^2G-4F_xG_{tttx}G^3+15F_xG_{ttt}G_xG^2\\
&+45F_xG_{ttx}G_tG^2-210F_xG_{tt}G_tG_xG+420F_xG_t^3G_x\Big)\\
&\cdot\left(F_xG^4\right)^{-1},
\end{aligned}
\tag{A.16}
$$

$$
\begin{aligned}
H_1=\Big(&-10F_{tx}G_{ttt}G^3+120F_{tx}G_{tt}G_tG^2-210F_{tx}G_t^3G\\
&+5F_{ttttx}G^4-10F_{tttt}G_xG^3-40F_{tttx}G_tG^3\\
&-20F_{ttt}G_{tx}G^3+90F_{ttt}G_tG_xG^2-30F_{ttx}G_{tt}G^3\\
&+135F_{ttx}G_t^2G^2+120F_{tt}G_{tx}G_tG^2-15F_{tt}G_{ttx}G^3\\
&+60F_{tt}G_{tt}G_xG^2-315F_{tt}G_t^2G_xG+40F_tG_{tx}G_{tt}G^2\\
&-210F_tG_{tx}G_t^2G-4F_tG_{tttx}G^3+15F_tG_{ttt}G_xG^2\\
&+45F_tG_{ttx}G_tG^2-210F_tG_{tt}G_tG_xG+420F_tG_t^3G_x\\
&-F_xG_{tttt}G^3+15F_xG_{ttt}G_tG^2+10F_xG_{tt}^2G^2\\
&-105F_xG_{tt}G_t^2G+105F_xG_t^4\Big)\left(F_xG^4\right)^{-1},
\end{aligned}
\tag{A.17}
$$

$$
\begin{aligned}
H_0=\Big(&F_{ttttt}G^4-10F_{tttt}G_tG^3-10F_{ttt}G_{tt}G^3\\
&+45F_{ttt}G_t^2G^2-5F_{tt}G_{ttt}G^3+60F_{tt}G_{tt}G_tG^2\\
&-105F_{tt}G_t^3G-F_tG_{tttt}G^3+15F_tG_{ttt}G_tG^2\\
&+10F_tG_{tt}^2G^2-105F_tG_{tt}G_t^2G+105F_tG_t^4\Big)\left(F_xG^4\right)^{-1}.
\end{aligned}
\tag{A.18}
$$

Equations for Theorem 2 in Section 3

$$
\begin{aligned}
S_{3ttt}=\Big(&238560040857600000000S_{6ttt}S_4^3S_5\\
&+852000145920000000S_{6ttt}S_4^3S_8\\
&-451818259200000000S_{6t}^2S_4^2S_5\\
&-1226363846400000S_{6t}^2S_4^2S_8\\
&-817920140083200000S_{7ttt}S_4^3\\
&+12146114080235520000000H_0S_4^6\\
&+1288224220631040000S_2^2S_3S_4^2S_6\\
&+333433269227520000S_2^2S_5^2S_6\\
&+2022424588800000S_2^2S_5S_6^2\\
&+1247633114112000S_2^2S_5S_6S_8\\
&-398786229338112000S_2^2S_5S_7\\
&+2142294739200S_2^2S_6^2S_8\\
&-169957431705600S_2^2S_6S_7\\
&-1775000304000S_2^2S_6S_8^2\\
&-147532492800000S_2^2S_7S_8\\
&+487653901701120000S_2S_3S_4^2S_5S_6\\
&+2581818624000000S_2S_3S_4^2S_6^2\\
&+3947600676096000S_2S_3S_4^2S_6S_8
\end{aligned}
$$

$$
\begin{aligned}
&- 1004306790187008000 S_2 S_3 S_4^2 S_7 \\
&- 332554631270400000 S_2 S_5^3 S_6 \\
&- 1429140303360000 S_2 S_5^2 S_6^2 \\
&- 1233512146944000 S_2 S_5^2 S_6 S_8 \\
&+ 353437928718336000 S_2 S_5^2 S_7 \\
&- 2831431680000 S_2 S_5 S_6^3 \\
&- 6606314668800 S_2 S_5 S_6^2 S_8 \\
&+ 1176324891033600 S_2 S_5 S_6 S_7 \\
&+ 2489610816000 S_2 S_5 S_6 S_8^2 \\
&+ 149604618240000 S_2 S_5 S_7 S_8 \\
&- 10866683520 S_2 S_6^3 S_8 \\
&+ 26073973094400 S_2 S_6^2 S_7 \\
&+ 4354257600 S_2 S_6^2 S_8^2 \\
&+ 130235212800 S_2 S_6 S_7 S_8 \\
&- 3590433000 S_2 S_6 S_8^3 \\
&+ 176538275020800 S_2 S_7^2 \\
&- 512265600000 S_2 S_7 S_8^2 \\
&+ 15982865547264000000 S_3 S_4^2 S_5^3 \\
&+ 2184672479109120000 S_3 S_4^2 S_5^2 S_6 \\
&+ 672295138099200000 S_3 S_4^2 S_5^2 S_8 \\
&+ 10708586380800000 S_3 S_4^2 S_5 S_6^2 \\
&+ 1042969036032000 S_3 S_4^2 S_5 S_6 S_8 \\
&+ 337584386506752000 S_3 S_4^2 S_5 S_7 \\
&+ 5249697868800000 S_3 S_4^2 S_5 S_8^2 \\
&+ 14883644160000 S_3 S_4^2 S_6^3 \\
&+ 4409023449600 S_3 S_4^2 S_6^2 S_8 \\
&- 74094096384000 S_3 S_4^2 S_6 S_7 \\
&+ 4756386096000 S_3 S_4^2 S_6 S_8^2 \\
&+ 291637463040000 S_3 S_4^2 S_7 S_8 \\
&+ 21679812000000 S_3 S_4^2 S_8^3 \\
&- 5524941176832000000 S_5^5 \\
&+ 682894773657600000 S_5^4 S_6 \\
&- 70471188480000000 S_5^4 S_8 \\
&+ 291110400000 S_5^2 S_6^3 \\
&+ 2269611955200 S_5^2 S_6^2 S_8 \\
&- 402813609984000 S_5^2 S_6 S_7 \\
&- 517887216000 S_5^2 S_6 S_8^2 \\
&- 57156096000000 S_5^2 S_7 S_8 \\
&- 2773848000000 S_5^2 S_8^3 \\
&- 1049440000 S_5 S_6^4 \\
&+ 6279324480 S_5 S_6^3 S_8 \\
&- 650224460800 S_5 S_6^2 S_7 \\
&- 9707266800 S_5 S_6^2 S_8^2 \\
&- 637826227200 S_5 S_6 S_7 S_8 \\
&+ 11823801000 S_5 S_6 S_8^3 \\
&- 17672824422400 S_5 S_7^2 \\
&+ 505169280000 S_5 S_7 S_8^2 \\
&- 21954900000 S_5 S_8^4 \\
&+ 4937432 S_6^4 S_8 + 896097408 S_6^3 S_7 \\
&- 14788620 S_6^3 S_8^2 \\
&- 1103392000 S_6^2 S_7 S_8 - 1631850 S_6^2 S_8^3 \\
&+ 151872430080 S_6 S_7^2 \\
&- 536659200 S_6 S_7 S_8^2 \\
&+ 32170875 S_6 S_8^4 - 12909568000 S_7^2 S_8 \\
&+ 848760000 S_7 S_8^3 - 62184375 S_8^5 \\
&+ 7683984000 \big(155232000 S_2 S_5 S_6 \\
&+ 554400 S_2 S_6 S_8 - 170311680 S_2 S_7 \\
&- 838252800 S_3 S_4^2 S_6 \\
&- 35795200 S_5^2 S_6 + 112000 S_5 S_6^2 \\
&- 164800 S_5 S_6 S_8 + 48762880 S_5 S_7 \\
&- 272 S_6^2 S_8 - 96000 S_6 S_7 \\
&+ 1155 S_6 S_8^2 + 64000 S_7 S_8 \big) B_0 S_4^2 \\
&+ 17740800 \big(276623424000 B_0 S_4^2 \\
&- 82987027200 S_2^2 + 69144768000 S_2 S_5
\end{aligned}
$$

$$- 59209920 S_2 S_6 + 55440000 S_2 S_8$$
$$+ 264608467200 S_3 S_4^2$$
$$- 49174841600 S_5^2 - 232872800 S_5 S_6$$
$$- 1724000 S_5 S_8$$
$$- 514314 S_6^2 - 1160 S_6 S_8 - 49754880 S_7$$
$$+ 144375 S_8^2\right) S_{7t} S_4$$
$$+ 25818186240000 \left(4158000 S_{6t} S_4\right.$$
$$- 415800 S_2 S_6 + 1096381440 S_3 S_4^2$$
$$- 129382400 S_5^2$$
$$- 741880 S_5 S_6 - 268800 S_5 S_8$$
$$- 1280 S_6^2 - 684 S_6 S_8$$
$$- 4992 S_7 - 825 S_8^2\right) S_{3t} S_4^3$$
$$+ 1022400175104000000 (77 S_6 + 20 S_8$$
$$+ 23968 S_5) S_{3tt} S_4^4$$
$$- 619636469760000 (24213 S_6$$
$$+ 4880 S_8 + 4866176 S_5) S_3^2 S_4^4$$
$$+ 768000 \left((417386935 S_6\right.$$
$$+ 422682482 S_8) S_6$$
$$- 88 \left(12528444464 S_7 + 2443875 S_8^2\right)\right) S_5^3$$
$$- 511200087552000 (S_6 S_8$$
$$- 384 S_7 + 280 S_5 S_6) S_2^3$$
$$- 8520001459200000 (S_6 S_8$$
$$- 192 S_7 + 280 S_5 S_6) B_{0t} S_4^3$$
$$+ 19670999040000 (327 S_6 - 125 S_8$$
$$+ 26080 S_5 + 249480 S_2) S_{7tt} S_4^2$$
$$- 76839840000 \left(332640 \left((280 S_5\right.\right.$$
$$+ S_8) S_2 - 2520 S_3 S_4^2\right)$$
$$+ 54790400 S_5^2 + 380800 S_5 S_6$$
$$+ 158720 S_5 S_8 + 688 S_6 S_8$$
$$- 112896 S_7 + 1155 S_8^2\right) S_{6tt} S_4^2$$
$$- 277200 \left(6147187200 \left(3 \left(5 B_0 S_4^2\right.\right.\right.$$
$$- S_2^2\right) (280 S_5 + S_8)$$
$$- 16 S_{7t} S_4\right) - \left(5438522880000 S_5^3\right.$$
$$+ 30283456000 S_5^2 S_6$$
$$+ 18086822400 S_5^2 S_8 + 96768000 S_5 S_6^2$$
$$+ 99126880 S_5 S_6 S_8$$
$$+ 6218168320 S_5 S_7 + 3603600 S_5 S_8^2$$
$$+ 211536 S_6^2 S_8$$
$$- 29151360 S_6 S_7 + 120120 S_6 S_8^2$$
$$+ 10810240 S_7 S_8 + 129525 S_8^3\right)$$
$$+ 93139200 (3160 S_6 + 1397 S_8$$
$$+ 879840 S_5) S_3 S_4^2$$
$$+ 55440 \left(81088000 S_5^2 - 565600 S_5 S_6\right.$$
$$+ 326560 S_5 S_8 - 844 S_6 S_8 + 467712 S_7$$
$$- 1155 S_8^2 + 838252800 S_3 S_4^2\right) S_2\right) S_{6t} S_4\right)$$
$$\cdot \left(566818657077657600000000 S_4^5\right)^{-1}, \tag{A.19}$$

$$S_{6tx} = \left(\left(2 \left(40320 \left(2 \left(99 \left(259952 S_2^2\right.\right.\right.\right.\right.\right.$$
$$- 436299 S_3 S_4^2 - 19404000 H_3$$
$$+ 1724800 C_0^2\right) S_4$$
$$+ 24736 S_{7x} - 285959520 S_{3x} S_4^2$$
$$+ 672348600 S_{1tt} S_4$$
$$+ 1280664000 C_{0tx} S_4$$
$$- 384199200 B_{0x} S_4^2$$
$$+ 96049800 A_{1tt} S_4^2\right) S_4$$
$$- 1334025 (112 S_5 + S_8) S_{1tx}\right)$$
$$- \left(48428561920 S_5^2\right.$$
$$+ 654609600 S_5 S_6$$
$$+ 600371360 S_5 S_8$$
$$+ 1929680 S_6^2 + 9171338 S_6 S_8$$
$$+ 552802176 S_7$$
$$- 10786905 S_8^2\right) S_4^2$$

$$
\begin{aligned}
&- 1890\left(819624960S_2^2\right.\\
&- 2209320960S_2S_5\\
&- 7096320S_2S_6\\
&+ 1626240S_2S_8\\
&+ 204906240S_3S_4^2\\
&+ 1478702080S_5^2\\
&+ 9564160S_5S_6\\
&- 2372480S_5S_8\\
&+ 15360S_6^2 - 10274S_6S_8\\
&\left.-\ 206976S_7 + 5775S_8^2\right) B_2\Big)\\
&+ 189\left(13113999360S_2^2\right.\\
&- 35713413120S_2S_5\\
&- 113541120S_2S_6\\
&+ 22767360S_2S_8\\
&+ 1434343680S_3S_4^2\\
&+ 24190668800S_5^2\\
&+ 154603520S_5S_6\\
&- 33214720S_5S_8\\
&+ 245760S_6^2 - 95326S_6S_8\\
&\left.+\ 206976S_7 - 5775S_8^2\right) A_1^2\\
&- 42\left(29506498560S_2^2\right.\\
&- 79535554560S_2S_5\\
&- 255467520S_2S_6\\
&+ 105114240S_2S_8\\
&- 101056032000S_3S_4^2\\
&+ 31943598080S_5^2\\
&+ 253961600S_5S_6\\
&- 60480320S_5S_8 + 552960S_6^2\\
&- 700362S_6S_8\\
&\left.-\ 15044736S_7 + 976015S_8^2\right) S_1\\
&- 663896217600\left(21S_1\right.\\
&\left.-\ 11S_4^2\right) B_0S_4^2\\
&+ 41309097984000\left(A_1\right.\\
&\left.-\ S_4\right) C_{0t}S_4^2\\
&+ 838252800\left(67A_1 + 6S_4\right) S_{6t}S_4^2\\
&- 34927200\left(8624A_1S_5 + 77A_1S_8\right.\\
&- 487872S_2S_4 + 265760S_4S_5\\
&\left.+\ 2112S_4S_6 - 3298S_4S_8\right) S_{1t}\\
&- 73920\left(176576S_6 + 4499S_8\right.\\
&\left.+\ 51980272S_5\right) S_2S_4^2\Big) S_4\\
&- 582120\left(18627840S_3S_4^2\right.\\
&+ 636160S_5^2 + 1120S_5S_6\\
&+ 5680S_5S_8 + 52S_6S_8\\
&+ 2688S_7 - 75S_8^2\\
&\left.-\ 18480\left(112S_5 + S_8\right) S_2\right) S_{1x}\\
&- 3725568000\left(48S_6 - 11S_8\right.\\
&\left.+\ 14944S_5 - 11088S_2\right) C_{0x}S_4^2\\
&+ 698544000\left(192S_6 - 11S_8\right.\\
&\left.+\ 63472S_5 - 44352S_2\right) B_{2t}S_4^2\\
&+ 4300800\left(17555S_6 - 2693S_8\right.\\
&\left.+\ 4417608S_5 - 5580036S_2\right) S_{5x}S_4^2\\
&+ 13440\left(18360S_6 - 2657S_8\right.\\
&+ 3797800S_5\\
&\left.-\ 4241160S_2\right) S_{6x}S_4^2\\
&+ 8870400\left(1617\left(\left(112S_5\right.\right.\right.\\
&\left.\left.+\ S_8\right) S_1 - 1728S_2S_4^2\right)\\
&+ \left(8064S_6 - 4841S_8\right.\\
&\left.+\ 1852816S_5\right) S_4^2\Big) C_0S_4\\
&- 3\left(4851\left(18627840S_3S_4^2\right.\right.\\
&- 47676160S_5^2\\
&- 142240S_5S_6 - 425680S_5S_8\\
&- 1228S_6S_8 + 2688S_7 - 75S_8^2\\
&\left.+\ 277200\left(112S_5 + S_8\right) S_2\right) S_1\\
&- \left(3098182348800B_0S_4^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ 1910545781760 S_2^2 \\
&- 2436527681280 S_2 S_5 \\
&- 11120820480 S_2 S_6 \\
&+ 2970419760 S_2 S_8 \\
&+ 4109618177280 S_3 S_4^2 \\
&+ 394650233600 S_5^2 \\
&+ 6026409760 S_5 S_6 \\
&- 4658100400 S_5 S_8 \\
&+ 12337920 S_6^2 \\
&- 16712320 S_6 S_8 \\
&- 588303744 S_7 \\
&+ 4150685 S_8^2 \big) S_4^2 \\
&- 186278400 \big(64 S_6 - 11 S_8 \\
&+ 20336 S_5 - 14784 S_2\big) C_0 S_4^2 \big) A_1 \\
&- 1663200 \Big(1617 \Big(192 \big(80 C_0 - 57 S_2\big) S_4^2 \\
&+ \big(112 S_5 + S_8\big) S_1\Big) \\
&+ \big(49392 S_6 - 9451 S_8 \\
&+ 15674288 S_5\big) S_4^2 \\
&+ 42 \big(1536 S_6 - 77 S_8 + 509008 S_5 \\
&- 354816 S_2\big) A_1 S_4 \Big) A_{1t} S_4 \Big) \\
&\cdot \Big(167650560000 S_4^3\Big)^{-1},
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
S_{7tt} = \Big(&852000145920000000 B_{0tt} S_4^4 \\
&- 852000145920000000 S_{3tt} S_4^4 \\
&+ 215151552000000 S_{6tt} S_4^2 S_5 \\
&+ 307359360000 S_{6tt} S_4^2 S_8 \\
&+ 852000145920000000 B_0^2 S_4^4 \\
&- 1704000291840000000 H_1 S_4^4 \\
&+ 5553778728960000 S_2 S_3 S_4^2 S_5 \\
&- 18646467840000 S_2 S_3 S_4^2 S_6 \\
&+ 3688312320000 S_2 S_3 S_4^2 S_8 \\
&- 17617797120000 S_2 S_5^3 \\
&- 3671754240000 S_2 S_5^2 S_6 \\
&- 471905280000 S_2 S_5^2 S_8 \\
&- 172480000 S_2 S_5 S_6^2 \\
&- 11718537600 S_2 S_5 S_6 S_8 \\
&+ 4367800729600 S_2 S_5 S_7 \\
&- 2550240000 S_2 S_5 S_8^2 \\
&- 18098080 S_2 S_6^2 S_8 \\
&+ 256650240 S_2 S_6 S_7 \\
&+ 46985400 S_2 S_6 S_8^2 \\
&+ 2848384000 S_2 S_7 S_8 \\
&- 78540000 S_2 S_8^3 \\
&+ 39966552299520000 S_3^2 S_4^4 \\
&- 10295681679360000 S_3 S_4^2 S_5^2 \\
&- 71642672640000 S_3 S_4^2 S_5 S_6 \\
&- 19314408960000 S_3 S_4^2 S_5 S_8 \\
&- 109438560000 S_3 S_4^2 S_6^2 \\
&- 54992044800 S_3 S_4^2 S_6 S_8 \\
&+ 11248234905600 S_3 S_4^2 S_7 \\
&- 48898080000 S_3 S_4^2 S_8^2 \\
&+ 24367472640000 S_5^4 \\
&+ 3398088960000 S_5^3 S_6 \\
&+ 717050880000 S_5^3 S_8 + 86176 S_6^3 S_8 \\
&- 29701056 S_6^2 S_7 - 56120 S_6^2 S_8^2 \\
&+ 4396160 S_6 S_7 S_8 - 267075 S_6 S_8^3 \\
&- 1886760960 S_7^2 - 15064000 S_7 S_8^2 \\
&+ 628125 S_8^4 \\
&- 277200 \Big(86240000 S_2 S_5 \\
&+ 252560 S_2 S_8 - 1397088000 S_3 S_4^2 \\
&+ 72800000 S_5^2 + 672000 S_5 S_6 \\
&- 56160 S_5 S_8 \\
&+ 876 S_6 S_8 - 276864 S_7 - 455 S_8^2\Big) S_{6t} S_4
\end{aligned}
$$

$$+ 6400 \left(\left(2909900 S_6 + 1186419 S_8\right) S_6 - \left(440601056 S_7 - 1100025 S_8^2\right)\right) S_5^2 + 170755200 \cdot \left(41 S_6 S_8 - 22848 S_7 + S_6 + 14000 S_5 S_6\right) S_2^2 - 737662464000000 \left(337 S_5 - 231 S_2\right) B_{0t} S_4^3 + 18441561600000 \cdot \left(51 S_6 + 4 S_8 + 16312 S_5 - 8008 S_2\right) S_{3t} S_4^3 + 17740800 \left(2373 S_6 - 500 S_8 + 152320 S_5 + 1681680 S_2\right) S_{7t} S_4 + 40 \left(2 \left(2 \left(224875 S_6 + 349638 S_8\right) S_6^2 - 75 \left(520192 S_7 - 29095 S_8^2\right) S_8\right) - \left(304667776 S_7 + 2689035 S_8^2\right) S_6\right) S_5 - 30735936000 \left(S_6 S_8 - 672 S_7 + 700 S_5 S_6 + 2217600 S_3 S_4^2\right) B_0 S_4^2\Big) \cdot \left(103272744960000 S_4^2\right)^{-1}, \quad (A.21)$$

$$C_{0t} = \Big(-\Big(\Big(332640 \Big(4 \Big(45 S_{6t} S_4 - 27720 B_0 S_4^2 - 92400 D_1 S_4 - 5544 S_2^2 + 20110 S_3 S_4^2\Big) S_4 - 385 \left(112 S_5 + S_8\right) S_{1t}\Big) - \Big(7120943360 S_5^2 + 38913440 S_5 S_6 - 10308880 S_5 S_8 + 60480 S_6^2 - 64408 S_6 S_8 - 769152 S_7 + 57855 S_8^2\Big) S_4 - 9 \Big(819624960 S_2^2 - 2209320960 S_2 S_5 - 7096320 S_2 S_6 + 1626240 S_2 S_8 + 204906240 S_3 S_4^2 + 1478702080 S_5^2 + 9564160 S_5 S_6 - 2372480 S_5 S_8 + 15360 S_6^2 - 10274 S_6 S_8 - 206976 S_7 + 5775 S_8^2\Big) A_1 + 55440 \left(828 S_6 - 65 S_8 + 330832 S_5\right) S_2 S_4\Big) S_4 - 693 \Big(18627840 S_3 S_4^2 + 636160 S_5^2 + 1120 S_5 S_6 + 5680 S_5 S_8 + 52 S_6 S_8 + 2688 S_7 - 75 S_8^2 - 18480 \left(112 S_5 + S_8\right) S_2\Big) S_1 - 4435200 \left(48 S_6 - 11 S_8 + 14944 S_5 - 11088 S_2\right) C_0 S_4^2 + 1663200 \left(192 S_6 - 11 S_8 + 63472 S_5 - 44352 S_2\right) A_{1t} S_4^2\Big)\Big) \cdot \left(245887488000 S_4^3\right)^{-1}, \quad (A.22)$$

$$S_{6tt} = \Big(-\Big(\Big(36960 \Big(15 \Big(3049200 \Big(672 \Big(10 \left(D_{1t} - 3 H_2\right) - 3 B_{0t} S_4\Big) S_4^2 - \left(A_{1tt} S_4 + 7 S_{1tt}\right) \left(112 S_5 + S_8\right)\Big) S_4 + \Big(86240 A_1 S_4 S_5 + 2387 A_1 S_4 S_8 + 1509200 S_1 S_5 + 5390 S_1 S_8 + 3326400 S_2 S_4^2 - 3437680 S_4^2 S_5 - 14400 S_4^2 S_6 + 7604 S_4^2 S_8\Big) S_{6t} + 1848 \left(72 S_6 + 91 S_8 + 104944 S_5\right) S_2^2 S_4 - 24 \left(330129 S_6 - 20551 S_8 + 94386224 S_5\right) S_3 S_4^3 + 231 \Big(2383360 S_5^2 + 7840 S_5 S_6 + 21280 S_5 S_8 + 28 S_6 S_8 - 2688 S_7 + 75 S_8^2 - 18627840 S_3 S_4^2\Big) S_{1t}\Big)$$

$$
\begin{aligned}
& - \left(3544441600 S_5^2 + 15097240 S_5 S_6 \right.\\
& + 703120 S_5 S_8 \\
& + 17280 S_6^2 + 3970 S_6 S_8 \\
& + 3094464 S_7 + 13785 S_8^2 \\
& \left. - 34379009280 S_3 S_4^2 \right) S_2 S_4 \\
& - 19958400 \left(2695 S_1 - 7099 S_4^2 \right.\\
& \left. + 385 A_1 S_4 \right) S_{3t} S_4^2 \\
& - 2880 \left(2695 S_1 - 3183 S_4^2 - 539 A_1 S_4 \right) S_{7t} \\
& + 5940 \left(896 S_7 - 25 S_8^2 \right.\\
& \left. + 14 S_6 S_8 + 2069760 S_3 S_4^2 \right) C_0 S_4 \\
& - 5544000 \left(24 S_6 - 11 S_8 \right.\\
& \left. + 6856 S_5 - 5544 S_2 \right) D_1 S_4^2 \Big) \\
& + \Big(160 \left(\left(4626885 S_6 - 4733737 S_8 \right) S_6 \right.\\
& \left. + 293259136 S_7 + 5053935 S_8^2 \right) S_5 \\
& + 13742136217600 S_5^3 \\
& + 150598419200 S_5^2 S_6 \\
& - 63989446400 S_5^2 S_8 \\
& + 1382400 S_6^3 - 828024 S_6^2 S_8 \\
& + 190854144 S_6 S_7 - 2697315 S_6 S_8^2 \\
& \left. - 214158080 S_7 S_8 + 5466300 S_8^3 \right) S_4 \Big) S_4 \\
& - 385 \left(776160 \left(S_6 S_8 - 192 S_7 \right.\right.\\
& \left.\left. + 280 S_5 S_6 \right) S_2 \right.\\
& - \left(1341502525440 S_3 S_4^2 S_5 \right.\\
& + 3259872000 S_3 S_4^2 S_6 \\
& + 3520661760 S_3 S_4^2 S_8 \\
& - 40040448000 S_5^3 - 183456000 S_5^2 S_6 \\
& - 371804160 S_5^2 S_8 - 431200 S_5 S_6^2 \\
& - 274176 S_5 S_6 S_8 + 19726336 S_5 S_7 \\
& - 3265920 S_5 S_8^2 - 364 S_6^2 S_8 \\
& + 182784 S_6 S_7 - 441 S_6 S_8^2 \\
& \left.\left. + 103936 S_7 S_8 - 11220 S_8^3 \right) \right) S_1 \\
& + 15367968000 \left(77 \left(192 \left(10 C_0 - 3 S_2 \right) S_4^2 \right.\right.\\
& \left.\left. + \left(112 S_5 + S_8 \right) S_1 \right) \right.\\
& \left. + \left(120 S_6 + 17 S_8 + 60272 S_5 \right) S_4^2 \right) B_0 S_4^2 \\
& - 91476000 \left(3696 \left(5040 \left(4 B_0 + S_3 \right) S_4^2 \right.\right.\\
& \left.\left. + \left(112 S_5 + S_8 \right) S_2 \right) \right.\\
& - \left(603904 S_5^2 + 1792 S_5 S_6 \right.\\
& + 5392 S_5 S_8 - 26 S_6 S_8 \\
& \left.\left. - 2688 S_7 + 75 S_8^2 \right) \right) A_{1t} S_4^2 \\
& + 11 \left(11176704000 \left(24 S_6 \right.\right.\\
& - 11 S_8 + 6856 S_5 \\
& \left. - 5544 S_2 \right) B_0 S_4^2 \\
& + 21451820544000 S_3 S_4^2 S_5 \\
& + 71624044800 S_3 S_4^2 S_6 \\
& - 20676902400 S_3 S_4^2 S_8 \\
& - 4004044800 S_5^3 \\
& + 133593600 S_5^2 S_6 \\
& - 107251200 S_5^2 S_8 \\
& - 39200 S_5 S_6^2 + 49007280 S_5 S_6 S_8 \\
& + 3140157440 S_5 S_7 \\
& - 81867600 S_5 S_8^2 + 188188 S_6^2 S_8 \\
& + 12348672 S_6 S_7 - 476955 S_6 S_8^2 \\
& - 10622080 S_7 S_8 \\
& + 249900 S_8^3 - 55440 \left(847 S_6 S_8 \right.\\
& + 59136 S_7 - 1125 S_8^2 + 7840 S_5 S_6 \\
& \left.\left. + 279417600 S_3 S_4^2 \right) S_2 \right) A_1 S_4 \Big) \Big) \\
& \cdot \left(55324684800000 S_4^4 \right)^{-1},
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
B_{0t} = & \left(3688312320000 S_{3t} S_4^3 \right.\\
& - 620928000 S_{6t} S_4 S_5 \\
& + 277200 S_{6t} S_4 S_8 \\
& + 372556800 S_{7t} S_4
\end{aligned}
$$

$$
\begin{aligned}
&- 1106493696000 B_0 S_2 S_4^2 \\
&+ 1614235392000 B_0 S_4^2 S_5 \\
&+ 4790016000 B_0 S_4^2 S_6 \\
&+ 11064936960000 D_0 S_4^3 \\
&+ 442597478400 S_2 S_3 S_4^2 \\
&+ 62092800 S_2 S_5 S_6 \\
&- 27720 S_2 S_6 S_8 - 74511360 S_2 S_7 \\
&- 826011648000 S_3 S_4^2 S_5 \\
&- 2567980800 S_3 S_4^2 S_6 \\
&- 119750400 S_3 S_4^2 S_8 \\
&+ 572006400 S_5^3 \\
&- 19084800 S_5^2 S_6 \\
&+ 15321600 S_5^2 S_8 + 5600 S_5 S_6^2 \\
&+ 38760 S_5 S_6 S_8 \\
&+ 36305920 S_5 S_7 + 82800 S_5 S_8^2 \\
&+ 512 S_6^2 S_8 + 49728 S_6 S_7 \\
&- 1935 S_6 S_8^2 - 34880 S_7 S_8 \\
&+ 2550 S_8^3 \Big) \\
&\cdot \left(11064936960000 S_4^3\right)^{-1},
\end{aligned}
\tag{A.24}
$$

$$
\begin{aligned}
S_{6xx} = {} & \Big(7\Big(33\Big(240\left(108662400 H_4\right. \\
&- 778447 S_2 S_4^2 \\
&+ 2199120 D_2 S_4 \\
&- 18110400 D_{2x} \\
&- \left.1164240 B_{2t} S_4\right) S_4^2 \\
&- 136171\left(112 S_5 + S_8\right) S_1^2 \\
&- 53760\Big(4851 S_1 \\
&- 2188 S_4^2\Big) C_0 S_4^2\Big) \\
&+ 2\Big(4303031040 B_2 S_2 \\
&- 2283635200 B_2 S_5 \\
&- 18627840 B_2 S_6 \\
&+ 20913200 B_2 S_8 \\
&- 1434343680 C_0 S_4^2 \\
&- 14343436800 D_2 S_4 \\
&+ 1290909312 S_1 S_2 \\
&- 1052792048 S_1 S_5 \\
&- 5588352 S_1 S_6 \\
&+ 6852307 S_1 S_8 \\
&+ 2189453112 S_2 S_4^2 \\
&- 3051480 S_4^2 S_5 \\
&+ 2469168 S_4^2 S_6 \\
&- 4097730 S_4^2 S_8\Big) A_1 S_4 \\
&- 388080\left(48 A_1 - 79 S_4\right) S_{6x} S_4^2\Big) \\
&- \Big(16666080 S_6 + 97969487 S_8 \\
&+ 9031282208 S_5\Big) S_4^4 \\
&+ 1764\Big(7528052 S_5 \\
&- 27160 S_6 + 99515 S_8 \\
&+ 15878940 S_2\Big) S_1 S_4^2 \\
&+ 258720\Big(297 S_6 + 14 S_8 \\
&+ 77780 S_5 - 141372 S_2\Big) B_2 S_4^2 \\
&- 3018400\Big(432 S_6 \\
&- 485 S_8 + 52960 S_5 \\
&- 99792 S_2\Big)\left(B_{2x} - D_3\right) S_4 \\
&- 3773\Big(6912 S_6 - 13183 S_8 \\
&+ 819440 S_5 - 1596672 S_2\Big) A_1^2 S_4^2\Big) \\
&\cdot \left(162993600 S_4^2\right)^{-1}.
\end{aligned}
\tag{A.25}
$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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