

Accurate inference for repeated measures in high dimensions*

Xiaoli Kong and Solomon W. Harrar

Department of Mathematics and Statistics, Loyola University Chicago
e-mail: xkong1@luc.edu

Department of Statistics, University of Kentucky
e-mail: solomon.harrar@uky.edu

Abstract: This paper proposes inferential methods for high-dimensional repeated measures in factorial designs. High-dimensional refers to the situation where the dimension is growing with sample size such that either one could be larger than the other. The most important contribution relates to high-accuracy of the methods in the sense that p-values, for example, are accurate up to the second-order. Second-order accuracy in sample size as well as dimension is achieved by obtaining asymptotic expansion of the distribution of the test statistics, and estimation of the parameters of the approximate distribution with second-order consistency. The methods are presented in a unified and succinct manner that it covers general factorial designs as well as any comparisons among the cell means. Expression for asymptotic powers are derived under two reasonable local alternatives. A simulation study provides evidence for a gain in accuracy and power compared to limiting distribution approximations and other competing methods for high-dimensional repeated measures analysis. The application of the methods are illustrated with a real-data from Electroencephalogram (EEG) study of alcoholic and control subjects.

MSC 2010 subject classifications: Primary 62H10; secondary 62H15.

Keywords and phrases: Asymptotic expansion, characteristic function, covariance estimation, quadratic form, multivariate test, profile analysis.

Received April 2019.

Contents

1	Introduction	4917
2	Model and hypotheses	4919
3	Higher-order asymptotic tests	4920
4	Power under local alternatives	4925
5	Simulation study	4926
6	Real data analysis	4934
7	Discussion and conclusion	4937
A	Proofs	4938
	Acknowledgements	4942
	References	4942

*This is an original methodology paper.

1. Introduction

Repeated measures data arise in various disciplines of the sciences, social sciences, engineering and humanities. Study designs such as time course studies, cross-over designs and split-plot designs naturally lead to repeated measures data. The distinctive feature of repeated measures data is that the observations from the same study unit (observational or experimental) are commensurate and exhibit correlations. Analysis of continuous repeated measures data to make inference on the effects of one or many between- or within-subject crossed or nested factor effects fall into three broad categories: multivariate analysis, univariate analysis and mixed model analyses. Mixed model analyses involve some assumption concerning the correlations of the repeated measures. Despite its generality in modeling the correlation and leading to exact inference, classic multivariate methods are not applicable when the number of repeated measures is larger than the error degrees of freedom.

Univariate methods on the other hand focus on adjusting the analysis of variance (ANOVA) for the within-unit correlation. It is well known that when all observations are independent, ANOVA test statistics have exact F distribution. In the presence of the within-unit correlation, the ANOVA tests are valid only if these correlations satisfy a condition known as *sphericity* (Bock, 1963; Huynh and Feldt, 1970). Box (1954) suggested a correction which involves adjustment of the numerator and denominator degrees of freedom of the F-distribution by a constant multiplying factor, commonly referred to as Box's ϵ . Since the constant factor ϵ depends on the unknown within-unit covariance matrix, solution such as using a lower bound for ϵ (Geisser and Greenhouse, 1958) or estimates of the within-unit covariance matrix in the calculation of ϵ (Huynh and Feldt, 1976; Huynh, 1978; Lecoutre, 1991) have been implemented in practical applications. These solutions have been shown to work satisfactorily in terms of controlling type-I error rate when the number of repeated measures is low compared to the degrees of freedom for estimating the covariance matrix. However, the univariate approach is obviously approximate and Rencher and Christensen (2012, p. 219) argue that it has no power advantage over the exact multivariate test. They continue to say "... The only case in which we need to fall back on univariate test is when there are insufficient degrees of freedom to perform multivariate test...", i.e., when the number of repeated measures is larger than the error degrees of freedom to estimate the within-unit covariance.

Well before most researchers embarked on the high-dimension-low-sample-size (HDLSS) problem, Collier et al. (1967), Stoloff (1970) and Maxwell and Arvey (1982) have numerically demonstrated that the univariate approaches for repeated measures tend to be very conservative. In an attempt to improve the estimation of ϵ in the high dimensional situation, Chi et al. (2012) used "dual" forms of the within sum-of-squares and cross-products matrices. They claim that, besides giving stable estimates of ϵ , the use of the "dual" version has computational advantage. The approaches of Brunner et al. (2012) and Happ et al. (2015), on the other hand, overcome the high dimensional problem by using the so-called ANOVA-type statistic (Brunner et al., 1997, 1999) and then use

F-approximation to their null distribution by matching first two moments of the numerator and denominator quadratic forms with that of a scaled-gamma distribution, an approach shown to be successful in a related problem for independent observations by Brunner et al. (1997). Again, these approaches although were shown to be numerically satisfactory, they are only approximate solutions. On the other hand, by deriving asymptotic distributions of some suitable statistics in the high-dimensional asymptotic framework, Pauly et al. (2015), Takahashi and Shutoh (2016) and Harrar and Kong (2016) devised asymptotically-valid tests. Pauly et al. (2015) consider high-dimensional repeated measures analysis for one sample situation but with the possibility of several within subject factors. The two-sample situation was considered by Takahashi and Shutoh (2016) assuming equal covariance matrices for the two populations. More generally, Harrar and Kong (2016) addressed the multi-group as well as the unequal covariance cases. Other works such as Wang and Akritas (2010a,b) and Wang et al. (2010) are also high-dimensional asymptotic results applicable for repeated measures but additionally assume that the repeated measurements are inherently ordered and the dependence between the measurements decays as the separation between them increases. High-dimensional asymptotic mean vector comparisons have recently received attention in the statistics literature (see Bai and Saranadasa, 1996; Chen and Qin, 2010; Katayama et al., 2013; Cai et al., 2014, and the references there in) under assumptions different from that of Pauly et al. (2015), Takahashi and Shutoh (2016), and Harrar and Kong (2016). These recent results are based on limiting distributions (i.e. first-order asymptotics) and are not applicable in the multi-factor repeated measures settings.

More specifically, Harrar and Kong (2016) have proven asymptotic normality for their test statistics under certain assumptions on the covariances. In their simulation study, Harrar and Kong (2016) noticed that the error of approximation from these asymptotic distributions could be considerable unless both the number of repeated measurements and replication sizes are large. The present paper aims to derive second order asymptotics for the tests considered in Harrar and Kong (2016). In addition, the results in the current manuscript are more general in the sense that they are applicable in situations where there are multiple within and/or between unit crossed and/or nested factors. However, to get second-order asymptotic results that are valid for general design, we need to assume equality of covariance matrices in this paper.

This paper is organized as follows. Section 2 introduces the statistical model, hypotheses and notations used in the remainder of the paper. Test statistics for the various effects are presented in Section 3 together with asymptotic expansions for their null distributions. The asymptotic power are derived in Section 4. Numerical studies are carried out in Section 5. First, Monte Carlo simulations are used to show the gain in accuracy from the asymptotic expansions for a selection of covariance matrices and wide choices of values for the number of repeated measures and replication sizes. Data from a large Electroencephalogram (EEG) study of alcoholic and control subjects is used to illustrate the application of the results in Section 6. Also, simulation results by generating data with similar design parameters as the real data is considered later in the section. Sec-

tion 7 contains discussions and conclusions. All proofs and preliminary results are placed in the Appendix.

2. Model and hypotheses

Suppose n_i independent b -dimensional observations; for $i = 1, \dots, a$; are available from multivariate normal populations $\mathcal{N}_b(\boldsymbol{\mu}_i, \boldsymbol{\Sigma})$ denoted by $\mathbf{X}_{i1}, \dots, \mathbf{X}_{in_i}$ and assume that the a samples are mutually independent. The number of samples a is assumed fixed. The aim of this paper is to derive second order asymptotic result for testing hypotheses in repeated measures analysis when both the total sample size $\sum_{i=1}^a n_i$ and the number of repeated measurements b tend to infinity.

Let $\mathbf{X} = (\mathbf{X}'_{11}, \dots, \mathbf{X}'_{1n_1}, \mathbf{X}'_{21}, \dots, \mathbf{X}'_{2n_2}, \dots, \mathbf{X}'_{a1}, \dots, \mathbf{X}'_{an_a})'$, where $\mathbf{X}_{ik} = (X_{i1k}, \dots, X_{ibk})'$. Further, let $\bar{\mathbf{X}} = (\bar{\mathbf{X}}'_1, \dots, \bar{\mathbf{X}}'_a)'$, where $\bar{\mathbf{X}}_i = n_i^{-1} \sum_{k=1}^{n_i} \mathbf{X}_{ik}$. The usual setting gives the interpretation that X_{ijk} is the response from the k th subject treated with the i th level of factor A (e.g., treatment) and the j th level of factor B (e.g. time). In this model X_{ijk} and $X_{i'j'k'}$ are assumed to be independent only if $i \neq i'$ or $k \neq k'$. Otherwise the dependence is completely unspecified.

Throughout this paper, $\mathbf{0}$ will denote a matrix of all zeros where the dimension will be clear from the context, and $\mathbf{1}_k$ denotes the k -dimensional vector consisting of all ones. The matrix \mathbf{I}_k is the identity matrix, whereas \mathbf{J}_k and \mathbf{P}_k are defined as $\mathbf{J}_k = \mathbf{1}_k \mathbf{1}'_k$ and $\mathbf{P}_k = \mathbf{I}_k - k^{-1} \mathbf{J}_k$, respectively. We will use extensively the Kronecker (or direct) product $\mathbf{A} \otimes \mathbf{B}$ and the direct sum $\mathbf{A} \oplus \mathbf{B}$ of matrices \mathbf{A} and \mathbf{B} . The symbol \xrightarrow{D} stands as an abbreviation for ‘‘converges in distribution to’’ and \xrightarrow{P} for ‘‘converges in probability to’’. In estimating a sequence of parameters $\theta_b = O(1)$ by a sequence of estimators $T_{n,b}$, consistency is meant in the sense of $E(T_{n,b} - \theta_b)^2 \rightarrow 0$ as (n, b) go to infinity.

Note that from the distributional assumption made above $E[\mathbf{X}_{ik}] = \boldsymbol{\mu}_i = (\mu_{i1}, \dots, \mu_{ib})'$ and $\text{Var}(\mathbf{X}_{ik}) = \boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}$ is a $b \times b$ positive definite matrix. Let $\boldsymbol{\mu} = (\mu_{11}, \dots, \mu_{1b}, \dots, \mu_{a1}, \dots, \mu_{ab})'$ and $\tilde{\boldsymbol{\Sigma}} = \bigoplus_{i=1}^a \boldsymbol{\Sigma}/n_i = \mathbf{D} \otimes \boldsymbol{\Sigma}$ where $\mathbf{D} = \text{diag}(1/n_1, \dots, 1/n_a)$. Then we have $E[\bar{\mathbf{X}}] = \boldsymbol{\mu}$ and $\text{Var}(\bar{\mathbf{X}}) = \tilde{\boldsymbol{\Sigma}}$.

The hypotheses of interest can be expressed as

$$\mathcal{H}_0 : \mathbf{K}\boldsymbol{\mu} = \mathbf{0} \quad \text{VS} \quad \mathcal{H}_1 : \mathbf{K}\boldsymbol{\mu} \neq \mathbf{0} \quad (1)$$

with $\mathbf{K} = \mathbf{T}_1 \otimes \mathbf{T}_2$, where \mathbf{T}_1 and \mathbf{T}_2 are $a \times a$ and $b \times b$ matrices respectively. We require that the two matrices \mathbf{T}_1 and \mathbf{T}_2 are symmetric and there exist positive definite matrices \mathbf{G}_i , such that $\mathbf{T}_i \mathbf{G}_i \mathbf{T}_i = \mathbf{T}_i$ (i.e. $\mathbf{G}_i = \mathbf{I}$ if \mathbf{T}_i is idempotent). Actually, we can apply the linear transformation $\mathbf{G}_1^{-1/2} \otimes \mathbf{G}_2^{-1/2}$ on the data $\bar{\mathbf{X}}$ and use the symmetric and idempotent matrix $\mathbf{G}_i^{1/2} \mathbf{T}_i \mathbf{G}_i^{1/2}$ instead of \mathbf{T}_i . With this manipulation, the new \mathbf{K} still defines the same hypotheses as in (1). Without loss of generality, we can assume that \mathbf{T}_1 , \mathbf{T}_2 and, therefore, \mathbf{K} are symmetric and idempotent matrices. For such \mathbf{T}_i , \mathbf{K} is positive semidefinite

matrix. In the following Theorem we establish an equivalent quadratic form expressions for the hypotheses (1).

Theorem 2.1. *The null hypotheses (1) are equivalent to*

$$\mathcal{H}'_0 : \boldsymbol{\mu}'\mathbf{K}\boldsymbol{\mu} = 0 \quad \text{vs.} \quad \mathcal{H}'_1 : \boldsymbol{\mu}'\mathbf{K}\boldsymbol{\mu} > 0.$$

The above setup may give the impression that the paper is dealing with one between-subject and one within-subject factor with levels a and b , respectively. However, the indices $i = 1, \dots, a$ and $j = 1, \dots, b$ are to be viewed as lexicographic order of the between-subject factor level combinations and within-subject factor level combinations, respectively. Therefore, the setup covers repeated measures in factorial designs with crossed and nested factors. The factors \mathbf{T}_1 and \mathbf{T}_2 of matrix \mathbf{K} can be viewed as parts of the contrast matrix concerning the between-subject factors and the within-subject factors, respectively. More specifically, suitable choices of \mathbf{T}_1 and \mathbf{T}_2 allow between-subject and within-subject mean comparisons. For a concrete example, consider a factorial design in which there are two between-subject crossed factors, say A and C with a and c levels, respectively, and two within-subject factors, say B and D , where the levels of D are nested within that of B (see also other specific designs considered in Section 5). Suppose B has b levels and the j th level of B has d_j levels of D nested within it. The mean vector in this set up would be $\boldsymbol{\mu} = (\boldsymbol{\mu}'_{11}, \dots, \boldsymbol{\mu}'_{1c}, \dots, \boldsymbol{\mu}'_{a1}, \dots, \boldsymbol{\mu}'_{ac})'$ where $\boldsymbol{\mu}_{ik} = (\boldsymbol{\mu}'_{ik11}, \dots, \boldsymbol{\mu}'_{ik1d_1}, \dots, \boldsymbol{\mu}'_{ikb1}, \dots, \boldsymbol{\mu}'_{ikbd_b})'$. To test the interaction effect of A and B , for instance, we would use $\mathbf{T}_1 = \mathbf{P}_a \otimes \frac{1}{c}\mathbf{J}_c$ and $\mathbf{T}_2 = \mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{-}\mathbf{Q}'$ where $\mathbf{Q} = \mathbf{P}_b(\bigoplus_{j=1}^b d_j^{-1}\mathbf{1}_{d_j})$ and $(\mathbf{Q}'\mathbf{Q})^{-}$ is the generalized inverse of $\mathbf{Q}'\mathbf{Q}$. Further, the set up above can be reset accordingly. For example a would be replaced by ac , and $\mathbf{D} = \text{diag}\{n_{11}, \dots, n_{1c}, \dots, n_{a1}, \dots, n_{ac}\}^{-1}$. See also Brunner et al. (2017) and Konietzschke et al. (2015) for other covered designs. Another example is found in the data analysis section (Section 6).

3. Higher-order asymptotic tests

We have seen in Theorem 2.1 that $\mathbf{K}\boldsymbol{\mu} = 0$ if and only if $\boldsymbol{\mu}'\mathbf{K}\boldsymbol{\mu} = 0$. A reasonable estimator of $\boldsymbol{\mu}'\mathbf{K}\boldsymbol{\mu}$ is given by $H = \overline{\mathbf{X}}'\mathbf{K}\overline{\mathbf{X}}$. The mean and variance the quadratic form H are

$$\begin{aligned} \text{E}(H) &= \text{tr}(\mathbf{T}_1\mathbf{D})c_1 + \boldsymbol{\mu}'\mathbf{K}\boldsymbol{\mu} \quad \text{and} \\ \text{Var}(H) &= 2\text{tr}(\mathbf{T}_1\mathbf{D})^2c_2 + 4\boldsymbol{\mu}'\mathbf{K}\tilde{\boldsymbol{\Sigma}}\mathbf{K}\boldsymbol{\mu} \end{aligned}$$

where

$$c_k := \text{tr}(\mathbf{T}_2\boldsymbol{\Sigma})^k = \text{tr}(\boldsymbol{\Sigma}^{1/2}\mathbf{T}_2\boldsymbol{\Sigma}^{1/2})^k,$$

for $k = 1, \dots, 8$. Let $n = \sum_{i=1}^a n_i - a$.

In this section, we will devise a test statistic for testing (1) under the following high-dimensional asymptotic framework,

- A_1 : $c_8/c_2^4 = O(b^{-3})$ as $b \rightarrow \infty$.
- A_2 : $n \rightarrow \infty$ and $b \rightarrow \infty$ such that $b/n = \gamma \rightarrow \gamma_0 \in (0, \infty)$.
- A_3 : $\boldsymbol{\mu}'\mathbf{K}\boldsymbol{\mu}/\sqrt{\text{tr}(\mathbf{T}_1\mathbf{D})^2c_2} = O(b^{-3/2})$ as $b \rightarrow \infty$.

It is apparent that assumption A_1 is a sparsity condition on the covariance matrix. Using the well known trace inequality (e.g. Yang et al., 2001)

$$\text{tr}(AB)^m \leq \{\text{tr}(A^{2m})\text{tr}(B^{2m})\}^{1/2},$$

for any positive semidefinite matrices A and B , we have

$$\frac{c_k}{c_2^{k/2}} = O(b^{-k/2+1}) \text{ as } b \rightarrow \infty \text{ for any } 1 \leq k \leq 7. \tag{2}$$

To see this, for example, when $k = 4$ the inequality yields

$$c_4 = \text{tr}(\mathbf{T}_2\boldsymbol{\Sigma})^4 = \text{tr}(\mathbf{T}_2\boldsymbol{\Sigma}\mathbf{T}_2\mathbf{I})^4 \leq \left\{ \text{tr}(\mathbf{T}_2\boldsymbol{\Sigma}\mathbf{T}_2)^8 \text{tr}(\mathbf{I}^8) \right\}^{1/2} \leq (c_8 \cdot b)^{1/2}.$$

Now, using condition A_1 ,

$$c_4/c_2^2 \leq (c_8 \cdot b)^{1/2}/c_2^2 = O(b^{-1}).$$

Similarly, when $k = 6$, $c_6 \leq (c_8c_4)^{1/2}$. Thus,

$$c_6/c_2^3 \leq (c_8c_4/c_2^6)^{1/2} = (c_8/c_2^4)^{1/2}(c_4/c_2^2)^{1/2} = O(b^{-2}).$$

The conditions used in Pauly et al. (2015), Takahashi and Shutoh (2016), and Harrar and Kong (2016) are $c_k = O(b)$ for $k = 1, 2, 3, 4$. These conditions are stronger compared to (2) for $k = 1, 2, 3, 4$. To get higher order accuracy, however, we would in addition need $c_k = O(b)$ for $k = 5, 6, 7, 8$, and condition A_1 is satisfied under these assumptions.

Assumption A_2 states that the sample size and dimension grow at the same rate but otherwise either one can be larger than the other. It is weaker than the usual requirement that each of sample sizes diverge and have the same relation with b . Condition A_3 is local alternative assumption which, according to Theorem 4.1, implies that the power of the test can be accurately calculated for alternatives approach to the null at a rate slower than $b^{-3/2}$.

Assume $\boldsymbol{\Sigma}$ is known. A centered and suitably-scaled version of $H = \overline{\mathbf{X}}'\mathbf{K}\overline{\mathbf{X}}$ given by:

$$T = \frac{H - \text{tr}(\mathbf{T}_1\mathbf{D})c_1 - \boldsymbol{\mu}'\mathbf{K}\boldsymbol{\mu}}{\sqrt{2\text{tr}(\mathbf{T}_1\mathbf{D})^2c_2}}$$

yields a reasonable test statistic for testing \mathcal{H}_0 . In order to generalize the test statistic T for the unknown covariance case, we need to estimate c_1 and c_2 to the appropriate order. The estimators \hat{c}_1 and \hat{c}_2 are defined by

$$\hat{c}_1 = \text{tr}(\mathbf{T}_2\mathbf{S}) \quad \text{and} \quad \hat{c}_2 = \frac{n^2}{(n-1)(n+2)} \left\{ \text{tr}(\mathbf{T}_2\mathbf{S})^2 - \frac{1}{n} \{ \text{tr}(\mathbf{T}_2\mathbf{S}) \}^2 \right\}, \tag{3}$$

where

$$\mathbf{S} = \frac{1}{n} \sum_{k=1}^a (n_i - 1) \mathbf{S}_i \quad \text{and} \quad \mathbf{S}_i = \frac{1}{n_i - 1} \sum_{k=1}^{n_i} (\mathbf{X}_{ik} - \bar{\mathbf{X}}_i)(\mathbf{X}_{ik} - \bar{\mathbf{X}}_i)',$$

have desirable asymptotic properties given in Theorem 3.1. Here, it should be noted that \hat{c}_1 and \hat{c}_2 are unbiased estimators of c_1 and c_2 , respectively (Srivastava, 2005; Harrar and Kong, 2016).

Theorem 3.1. *Under the high-dimensional asymptotic frameworks A_1 and A_2 , the estimators \hat{c}_1 and \hat{c}_2 have the following asymptotic properties:*

- (i) *Asymptotic equivalence:* $(\hat{c}_1 - c_1)/\sqrt{c_2} = O_p(b^{-1/2})$ and $(\hat{c}_2 - c_2)/c_2 = O_p(b^{-1})$.
- (ii) *Ratio consistency:* $\hat{c}_2/c_2 \xrightarrow{p} 1$.

Let $\delta_k = \text{tr}(\mathbf{T}_1 \mathbf{D})^k / \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^k$ for $k = 1, 2, 3, 4$ ($\delta_1 = 1$). Since \mathbf{T}_1 is a symmetric and idempotent matrix, one can see that $0 < \delta_k \leq 1$ and, hence, $\delta_k = O(1)$ as $n \rightarrow \infty$. Next we study the asymptotic sampling distribution of the test-statistic,

$$\hat{T} = \frac{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} H - \hat{c}_1}{\sqrt{2\delta_2 \hat{c}_2}}$$

which is obtained from T by replacing c_1 and c_2 by their empirical counterparts under \mathcal{H}_0 . It is shown in the Appendix that \hat{T} can be expanded as

$$\hat{T} = T - \frac{V}{\sqrt{b}} - \frac{1}{b} \frac{TW}{2} + O_p(b^{-3/2}),$$

where

$$V = \frac{\sqrt{b}(\hat{c}_1 - c_1)}{\sqrt{2\delta_2 c_2}} \quad \text{and} \quad W = \frac{b(\hat{c}_2 - c_2)}{c_2},$$

are $O_p(1)$ by Theorem 3.1. The characteristic function of \hat{T} can be expanded as given in the following Theorem.

Theorem 3.2. *If the null hypothesis \mathcal{H}_0 holds, then under the high-dimensional asymptotic frameworks A_1 and A_2 , the characteristic function of \hat{T} can be expanded as*

$$\phi_{\hat{T}}(t) = e^{\frac{1}{2}t^2} \left\{ 1 + \frac{1}{\sqrt{b}} i^3 t^3 \eta_3 + \frac{1}{b} (i^2 t^2 \frac{\gamma}{2\delta_2} + i^4 t^4 \eta_4 + i^6 t^6 \frac{\eta_3^2}{2}) + O(b^{-3/2}) \right\},$$

$$\text{where } \eta_3 = \frac{4b^{1/2}\delta_3 c_3}{3(2\delta_2 c_2)^{3/2}} \text{ and } \eta_4 = \frac{2b\delta_4 c_4}{(2\delta_2 c_2)^2}.$$

Note that, by Assumption A_1 , η_3 and η_4 are $O(1)$. Inverting the characteristic function term by term, we get an asymptotic expansion for the distribution function of \hat{T} as follows.

Theorem 3.3. *If the null hypothesis \mathcal{H}_0 holds, then under the high-dimensional asymptotic frameworks A_1 and A_2 , the distribution function of \widehat{T} can be expanded as*

$$F_{\widehat{T}}(x) = G_{\widehat{T}}(x) + O(b^{-3/2}),$$

uniformly in x where

$$G_{\widehat{T}}(x) = \Phi(x) - \frac{1}{\sqrt{b}}\eta_3\Phi^{(3)}(x) + \frac{1}{b}\left\{\frac{\gamma}{2\delta_2}\Phi^{(2)}(x) + \eta_4\Phi^{(4)}(x) + \frac{\eta_3^2}{2}\Phi^{(6)}(x)\right\}$$

and $\Phi^{(j)}(x)$ is the j th derivative of the standard normal cumulative distribution function $\Phi(x)$.

More specifically, Theorem 3.3 states that $\sup_{x \in \mathbb{R}} |F_{\widehat{T}}(x) - G_{\widehat{T}}(x)| = O(b^{-3/2})$. The function $G_{\widehat{T}}(x)$ can alternatively be expressed as

$$G_{\widehat{T}}(x) = \Phi(x) - \phi(x) \left[\frac{1}{\sqrt{b}}\eta_3h_2(x) + \frac{1}{b}\left\{\frac{\gamma}{2\delta_2}h_1(x) + \eta_4h_3(x) + \frac{\eta_3^2}{2}h_5(x)\right\} \right]$$

where $\phi(x)$ is the standard normal density functions and $h_i(x)$ is the i th Hermite polynomial. The first five Hermite polynomials are:

$$h_1(x) = x, \quad h_2(x) = x^2 - 1, \quad h_3(x) = x^3 - 3x, \\ h_4(x) = x^4 - 6x^2 + 3 \quad \text{and} \quad h_5(x) = x^5 - 10x^3 + 15x.$$

It should be emphasized that when the terms of orders $b^{-1/2}$ and b^{-1} are ignored, assumptions A_1 and A_2 can be relaxed as: (i) the assumption of proportional divergence of n and b in A_2 is not needed (Harrar and Kong, 2016) and (ii) the sparsity condition on the covariance matrix (assumption A_1) is needed only for $c_4/c_2^2 = o(1)$, in which case, the assumption reduces to that of Chen and Qin (2010).

Let $u(z)$ be defined by $P(\widehat{T} \leq u(z)) = P(Z \leq z)$ where Z is a standard normal random variable. In what follows, asymptotic expansion of $u(z)$ in terms of z known as Cornish-Fisher expansion (Hill and Davis, 1968) is given in Corollary 3.4.

Corollary 3.4. *If the null hypothesis \mathcal{H}_0 holds, then under the high-dimensional asymptotic frameworks A_1 and A_2 , $u(z) = u_A(z) + O(b^{-3/2})$ where*

$$u_A(z) = z + \frac{1}{\sqrt{b}}\eta_3h_2(z) + \frac{1}{b}\left\{\frac{\gamma}{2\delta_2}h_1(z) + \eta_4h_3(z) + \frac{\eta_3^2}{2}h_5(z) - z\eta_3^2h_2(z)\left(\frac{1}{2}h_2(z) - 2\right)\right\}.$$

The expansions $G_{\widehat{T}}(x)$ and $u_A(z)$ are approximations for the CDF and quantile, respectively, of \widehat{T} under the null hypothesis. In these approximations, η_3 and η_4 depend on c_2 , c_3 , and c_4 which are unknown quantities. Therefore, for practical applications, we need an estimated version of the expansions which is uniformly correct up to $O_p(b^{-3/2})$ in the sense that $\sup_{x \in \mathbb{R}} |F_{\widehat{T}}(x) - \widehat{F}_{\widehat{T}}(x)| =$

$O_p(b^{-3/2})$ where $\widehat{F}_{\widehat{T}}(x)$ is the estimated version of $F_{\widehat{T}}(x)$. To that end, let \widehat{c}_1 and \widehat{c}_2 be as defined in (3) and define \widehat{c}_3 and \widehat{c}_4 as

$$\begin{aligned} \widehat{c}_3 &= \frac{n^4}{m_1} \left[\text{tr}(\mathbf{T}_2 \mathbf{S})^3 - \frac{3}{n} \text{tr}(\mathbf{T}_2 \mathbf{S})^2 \text{tr}(\mathbf{T}_2 \mathbf{S}) + \frac{2}{n^2} \{\text{tr}(\mathbf{T}_2 \mathbf{S})\}^3 \right] \quad \text{and} \\ \widehat{c}_4 &= \frac{n^5(n^2 + n + 2)}{m_2} \left[\text{tr}(\mathbf{T}_2 \mathbf{S})^4 - \frac{4}{n} \text{tr}(\mathbf{T}_2 \mathbf{S})^3 \text{tr}(\mathbf{T}_2 \mathbf{S}) \right. \\ &\quad - \frac{2n^2 + 3n - 6}{n(n^2 + n + 2)} \{\text{tr}(\mathbf{T}_2 \mathbf{S})^2\}^2 + \frac{2(5n + 6)}{n(n^2 + n + 2)} \text{tr}(\mathbf{T}_2 \mathbf{S})^2 \{\text{tr}(\mathbf{T}_2 \mathbf{S})\}^2 \\ &\quad \left. - \frac{5n + 6}{n^2(n^2 + n + 2)} \{\text{tr}(\mathbf{T}_2 \mathbf{S})\}^4 \right], \end{aligned}$$

where $m_1 = (n - 2)(n - 1)(n + 2)(n + 4)$ and $m_2 = m_1(n + 1)(n - 3)(n + 6)$. These estimators are unbiased and enjoy some higher order asymptotic properties that makes them suitable for use in asymptotic expansions.

Theorem 3.5. *Under the high-dimensional asymptotic frameworks A_1 and A_2 , the estimators $\widehat{c}_2, \widehat{c}_3$ and \widehat{c}_4 have the following properties:*

- (i) *Unbiasedness:* $E[\widehat{c}_k] = c_k$ for $k = 2, 3, 4$.
- (ii) *Asymptotic equivalence:* $\frac{b^{1/2}\widehat{c}_3}{c_2^{3/2}} = \frac{b^{1/2}c_3}{c_2^{3/2}} + O_p(b^{-1})$ and $\frac{b\widehat{c}_4}{c_2^2} = \frac{bc_4}{c_2^2} + O_p(b^{-1})$

Now, by using Theorems 3.1 and 3.5, we know that

$$\begin{aligned} \widehat{\eta}_3 &:= \frac{4b^{1/2}\delta_3\widehat{c}_3}{3(2\delta_2\widehat{c}_2)^{3/2}} = \eta_3 + O_p(b^{-1}), \quad \widehat{\eta}_3^2 = \eta_3^2 + O_p(b^{-1}) \\ \text{and } \widehat{\eta}_4 &:= \frac{2b\delta_4\widehat{c}_4}{(2\delta_2\widehat{c}_2)^2} = \eta_4 + O_p(b^{-1}). \end{aligned}$$

Therefore, we can define the estimated version $\widehat{G}_{\widehat{T}}(x)$ of $G_{\widehat{T}}(x)$ of Theorem 3.3 by replacing η_3 and η_4 with $\widehat{\eta}_3$ and $\widehat{\eta}_4$, respectively.

Before we close this section we shed some light on how the asymptotics $(b, n) \rightarrow \infty$ is working. Let $R = \text{rank}(\mathbf{T}_2)$ be the number of nonzero eigenvalues of $\mathbf{T}_2 \mathbf{\Sigma}$ and denote the eigenvalues by $\lambda_{b,k}$ in decreasing order, i.e. $\lambda_{b,1} \geq \lambda_{b,2} \geq \dots \geq \lambda_{b,R} > 0 = \lambda_{b,R+1} = \dots = \lambda_{b,b}$. Furthermore, assume there are r dominant eigenvalues. That is, there exists $0 < M \leq 1$ such that

$$\liminf_{b \rightarrow \infty} \frac{\lambda_{b,r}}{\lambda_{b,1}} = M \quad \text{and} \quad \limsup_{b \rightarrow \infty} \frac{\lambda_{b,r+1}}{\lambda_{b,1}} = 0.$$

If the rank of \mathbf{T}_2 is finite, then r is bounded and, consequently, condition A_1 is violated. Thus, for condition A_1 to be satisfied, the rank of \mathbf{T}_2 needs to grow with b . Especially, when all eigenvalues are bounded, A_1 holds only if $r/b \rightarrow 0$. If r grows with b at a slower rate, a limiting normal distribution still holds for T

but not the second order accuracy results. This case is interesting and we plan to investigate asymptotic expansion under this framework in future researches.

Next we provide an approximate solution in the situation where r is bounded. In this case, T has the same distribution as a linear combination of centered and independent Chi-square random variables with one degrees of freedom (see also Pauly et al., 2015). More precisely,

$$T \stackrel{D}{=} \sum_{k=1}^r \frac{\lambda_{b,k}}{\sqrt{c_2}} \frac{(Y_k - 1)}{\sqrt{2}},$$

where Y_k are iid χ_1^2 random variables. When $n, p \rightarrow \infty$, $\widehat{T} = T + o_P(1)$, if the limit of $\lambda_{b,k}/\sqrt{c_2}$ exists, for all $k = 1, \dots, r$.

The exact distribution cannot be used directly as the eigenvalues are unknown and their consistent estimators are difficult to come by. We may use the following approximations

$$H \stackrel{\text{approx}}{\sim} \frac{\text{tr}(\mathbf{T}_1 \mathbf{D}) \delta_2 c_2}{c_1} \chi_{c_1^2/(c_2 \delta_2)}^2 \quad \text{and} \quad \widehat{c}_1 \stackrel{\text{approx}}{\sim} \frac{n^{-1} c_2}{c_1} \chi_{nc_1^2/c_2}^2.$$

These approximations are obtained by matching the first two moments with that of a scaled Chi-Square distribution under \mathcal{H}_0 . Further, it is known that H is independent of \widehat{c}_1 . Thus, a reasonable test statistic is,

$$\widetilde{T} = \frac{H}{\text{tr}(\mathbf{T}_1 \mathbf{D}) \widehat{c}_1}.$$

Under \mathcal{H}_0 , the distribution of \widetilde{T} can be approximated by $\mathcal{F}_{c_1^2/(c_2 \delta_2), nc_1^2/c_2}$ where $\mathcal{F}_{\nu_1, \nu_2}$ is the F distribution with degrees of freedom ν_1 and ν_2 . When the rank of \mathbf{T}_2 is 1, it turns out that $\mathbf{T}_2 \boldsymbol{\Sigma}$ has only one nonzero eigenvalue c_1 , and $c_2 = c_1^2$. Then \widetilde{T} has exact $\mathcal{F}_{1/\delta_2, n}$ distribution. For example, for testing the main effect of A or C or their interaction in the four-factor design mentioned in the last paragraph of Section 2, we shall use $\mathbf{Q} = \mathbf{J}_b/b(\bigoplus_{j=1}^b d_j^{-1} \mathbf{1}_{d_j}')$, in which $\text{Rank}(\mathbf{T}_2) = 1$. For another example, let $\mathbf{T}_1 = \text{diag}(n_1, \dots, n_a) - (n_1, \dots, n_a)'(n_1, \dots, n_a)/(n+a)$ and $\mathbf{T}_2 = \mathbf{J}_b/b$. These are matrices of special interest in testing the equality of mean vectors given that they are parallel (see Harrar and Kong, 2016). Under this set up, \widehat{T} has exact $\mathcal{F}_{a-1, n}$ distribution. In the more general case, $\widehat{c}_1^2/\widehat{c}_2$ is a consistent estimate of c_1^2/c_2 . From the proof of Theorem 3.1, we know that $E[\widehat{c}_k] = c_k, k = 1, 2$ and $\text{Var}(\widehat{c}_1) = 2c_2/n$. Thus, $\widehat{c}_1/\sqrt{\widehat{c}_2} \xrightarrow{P} c_1/\sqrt{c_2}$ and $c_2/\widehat{c}_2 \xrightarrow{P} 1$ as $n, b \rightarrow \infty$. Therefore, by Slutsky theorem and continuous mapping theorem, $\widehat{c}_1^2/\widehat{c}_2 \xrightarrow{P} c_1^2/c_2$, as $n, b \rightarrow \infty$.

4. Power under local alternatives

It turns out that the power functions under the local alternatives A_3 depend on the mean vectors through $\Delta = \boldsymbol{\mu}' \mathbf{K} \boldsymbol{\mu}$. Specifically, define the power function of \widetilde{T} by

$$\beta(\Delta) = \text{P}(\widehat{T} > u(z_\alpha)).$$

where z_α is the upper α (significant level) quantile of the standard normal distribution. Then, we can obtain the power functions under the local alternative A_3 as given in Theorem 4.1.

Theorem 4.1. *Under the assumption A_1, A_2 and A_3 , the function $\beta(\Delta)$ can be asymptotically expanded as*

$$\beta(\Delta) = 1 - G_{\hat{T}} \left(u(z_\alpha) - \frac{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \Delta}{\sqrt{2\delta_2 c_2}} \right),$$

where $G_{\hat{T}}(\cdot)$ is as defined in Theorem 3.3.

It follows from Corollary 3.4 that $u(z_\alpha) - \hat{u}_A(z_\alpha) = O_p(b^{-3/2})$. Therefore, Theorem 4.1 could still provide accurate approximation to the power function of the test that rejects \mathcal{H}_0 if $\hat{T} > \hat{u}_A(z_\alpha)$. It can be easily seen that when terms of orders $b^{-1/2}$ and b^{-1} are ignored from the quantiles $u_A(z_\alpha)$ and the power function $\beta(\Delta)$ reduces to

$$\beta(\Delta) = \Phi \left(\frac{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \Delta}{\sqrt{2\delta_2 c_2}} - z_\alpha \right).$$

When $a = 2, \mathbf{T}_1 = \mathbf{P}_a$ and $\mathbf{T}_2 = \mathbf{I}_b$, it has the same form as Bai and Saranadasa (1996) and Chen and Qin (2010). It should be noted, however, that for this later test, Theorem 4.1 gives a second order approximation to the power function which is more accurate than Bai and Saranadasa (1996) and Chen and Qin (2010).

5. Simulation study

To exhibit the improvement resulting from the asymptotic expansion and, hence, facilitate comparison with the limiting distributions in Harrar and Kong (2016), the simulation study will mainly focus on the model where there is one between- and one within-subject factors. We generate 10,000 replications of data from $\mathcal{N}_b(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Although the assumed asymptotic frameworks stipulate n to grow proportionally with b , in reality the actual ratio varies from application to application. To investigate the effect of various proportionality of growth, we look at values of several combinations of a, b and n'_i 's. For practical reasons, we also consider small b and large n_1, \dots, n_a (and vice-versa) combinations with balanced as well as unbalanced designs. For the number of groups (number of levels of factor A), we will consider $a = 2, 3, 4, 6$. and we set significant level α at 0.01 and 0.05.

Tables 2–7 present actual type I error rates (test sizes) for the covariance structures $\boldsymbol{\Sigma} = \rho \mathbf{I}_b + (1 - \rho) \mathbf{J}_b$, $\boldsymbol{\Sigma} = (\rho^{j-j'})$ and $\boldsymbol{\Sigma} = (\sigma_{jj'})$ where $\sigma_{jj} = 1$ and, for $j \neq j'$, $\sigma_{jj'} = \rho/(j - j')^{1/4}$. We consider a range of values for ρ . For the first covariance structure, the assumptions in A_1 are satisfied uniformly in ρ because $c_i = (1 - \rho)^i (b - 1)$. However, for the other two covariance structures these assumptions do not hold except for $\rho = 0$. Especially, when ρ is close

to 1, the quantities $b^2 c_6/c_2^3$ and $b^3 c_8/c_2^4$ could diverge very fast. To illustrate this concretely, Table 1 shows the values of $b^{3/2} c_5/c_2^{5/2}$ and $b^3 c_8/c_2^4$ for different values of b and ρ .

TABLE 1
Covariance Sparsity Assumptions Evaluated for $\Sigma = (\sigma_{jj'})$ where $\sigma_{jj} = 1$ and, for $j \neq j'$, $\sigma_{jj'} = \rho/(j - j')^{1/4}$. Here, $T_2 = P_b$.

	ρ	$b = 12$	$b = 25$	$b = 50$	$b = 100$	$b = 200$	$b = 400$
$b^{3/2} c_5/c_2^{5/2}$	0.5	2.98	6.58	16.65	46.14	133.50	392.58
$b^3 c_8/c_2^4$	0.5	16.69	99.25	694.56	5513.85	46366.55	397877.62
$b^{3/2} c_5/c_2^{5/2}$	0.9	14.84	29.66	61.35	135.15	314.17	763.47
$b^3 c_8/c_2^4$	0.9	302.75	1349.69	6311.86	33055.11	190299.63	1182933.54

It should be noted that the covariance matrix structure $\Sigma = \rho I_b + (1 - \rho) J_b$ will be positive definite if and only if $-1/(b - 1) < \rho < 1$. The empty cells in Tables 2–7 correspond to the cases where b and ρ combinations do not yield positive definitive covariance matrices. In all the three tables we consider the contrast matrices $T_1 = P_a$ or J_a/a and $T_2 = P_b$. Another matrix of particular interest in repeated measures analysis is $T_2 = J_b/b$ which is useful to test the group effect. However, the distribution of \hat{T} in this case does not depend on b . Hence, we do not carry out simulation for this contrast matrix.

First and foremost, comparing Table 2, 4 and 6 (results for $\alpha = 0.05$) with the results in Harrar and Kong (2016), one can clearly see a marked gain in accuracy resulting from the inclusion of higher-order terms in the asymptotic expansion. We can see in Table 2 that for both tests (i.e. $T_1 = P_a$ and J_a/a), a large number of the achieved error rates are within a tenth of the actual values. This phenomena happens more for weaker correlations than for stronger ones. Further, it is clear from the tables that the performance of the tests in controlling type I error rates is excellent when either the sample sizes or the dimension is large. This might be due to the homoscedastic nature. It seems also the case that when n_i 's are small, the tests control type I error rate better as a gets larger. For example, looking at the rows for $a = 6$, performance appear to be satisfactory for small sample sizes but large dimension. Tables 4 and 6 seem to exhibit similar patters and behaviors. Likewise, for $\alpha = 0.01$, the asymptotic expansion provides a gain in accuracy in controlling type-I error rates (see Tables 3, 5 and 7).

To investigate performance in terms of power, we compare the power of the method by Chi et al. (2012) with the methods proposed in this paper taking $T_2 = P_b$ and setting T_1 to either P_a or J_a/a . The power comparison is restricted to Chi et al. (2012) because, to the best of our knowledge, this is the only work for high-dimensional repeated measures data that covers the general factorial design. Their approach uses a property of the Wishart random matrix to avoid the singularity problem that arises due to high dimensionality. This manipulation could, however, potentially compromise the power properties of the test. That is one objective of the power comparison in addition to evaluating the power performance of the proposed method. To keep the comparison

TABLE 2
 Achieved Type I error rates ($\times 100\%$) for the testing procedures when $T_2 = P_b$ and sampling from $N_b(\mu, \Sigma)$ where $\Sigma = (1 - \rho)I_b + \rho J_b$. The nominal size is $\alpha = 0.05$.

a	b	(n_1, \dots, n_a)	$T_1 = P_a$				$T_1 = J_a/a$			
			$\rho =$				$\rho =$			
			-0.01	0	0.2	0.5	-0.01	0	0.2	0.5
2	5	(10,10)	5.60	7.20	5.10	6.70	5.30	5.80	6.60	6.80
2	10	(10,10)	5.70	5.30	4.80	5.70	6.60	5.60	6.00	5.50
2	15	(10,10)	6.30	5.70	6.60	5.00	6.00	5.30	4.70	5.90
2	12	(50,100)	5.19	4.98	4.84	4.99	5.10	5.12	5.35	5.05
2	12	(100,100)	4.96	4.94	5.12	4.91	4.65	5.43	4.78	4.98
2	25	(50,100)	5.12	4.83	5.04	4.90	5.02	4.81	4.84	4.80
2	25	(100,100)	5.20	4.95	5.08	5.14	5.25	5.16	4.93	5.25
2	50	(50,100)	4.73	5.25	5.06	5.30	5.01	4.97	5.41	5.14
2	50	(100,100)	4.82	4.89	5.04	4.96	4.65	5.00	4.97	4.33
2	100	(12,13)	5.11	5.56	5.08	5.22	5.30	4.94	5.31	5.58
2	100	(12,25)	5.02	5.10	5.11	5.42	5.08	5.10	5.02	4.97
2	100	(25,25)	4.96	5.35	4.75	4.83	4.89	4.85	5.06	4.79
2	100	(25,50)	5.28	5.31	5.20	5.21	4.61	5.18	5.28	4.97
2	200	(25,25)		4.87	4.95	5.06		5.36	5.33	5.04
2	200	(25,50)		5.24	5.23	4.69		4.98	4.92	4.95
2	200	(50,50)		5.08	4.99	4.98		4.98	5.45	4.96
2	200	(50,100)		5.20	4.97	4.89		4.96	4.91	5.13
2	400	(50,50)		4.98	4.98	4.71		4.79	5.01	4.81
2	400	(50,100)		4.81	4.96	4.95		4.93	4.74	4.95
2	400	(100,100)		5.04	4.91	5.07		5.36	5.53	5.19
2	400	(100,200)		5.04	4.57	5.35		5.07	5.19	5.31
3	5	(10,10,10)	5.90	6.30	4.60	6.10	5.70	5.60	5.30	5.20
3	10	(10,10,10)	6.70	5.50	5.30	6.10	5.80	5.60	4.80	5.10
3	15	(10,10,10)	5.70	4.60	5.50	6.00	5.50	4.60	4.20	4.30
3	12	(50,100,100)	5.15	5.08	4.82	4.92	4.81	4.61	4.73	5.19
3	12	(100,100,100)	4.80	4.90	5.05	5.15	4.88	4.92	5.02	4.85
3	25	(50,100,100)	5.15	5.05	5.11	5.20	5.14	5.20	4.84	5.16
3	25	(100,100,100)	4.79	4.87	4.94	5.20	4.87	4.87	5.50	4.92
3	50	(50,100,100)	5.06	5.63	5.03	5.18	4.82	4.78	4.90	4.80
3	50	(100,100,100)	5.15	5.18	5.15	4.72	4.66	5.13	5.22	4.97
3	100	(16,17,17)	4.95	5.27	4.87	5.19	4.99	5.32	5.23	5.07
3	100	(16,17,33)	4.86	5.21	4.85	5.46	4.70	5.18	5.17	5.01
3	200	(33,33,34)		5.18	5.20	5.29		5.09	5.09	5.32
3	200	(33,34,67)		4.53	5.10	5.17		4.99	5.10	5.19
3	200	(50,50,50)		4.62	4.88	4.86		5.02	4.92	5.08
3	200	(50,50,100)		4.85	5.03	5.10		4.60	4.68	5.20
4	12	(50,50,100,100)	5.13	5.20	5.32	4.96	4.93	5.05	5.12	5.57
4	12	(100,100,100,100)	5.01	5.04	4.94	5.08	5.03	4.77	5.23	5.06
4	25	(50,50,100,100)	4.81	4.64	4.99	4.65	5.26	5.27	5.01	5.11
4	25	(100,100,100,100)	5.19	5.14	4.89	5.05	4.99	5.14	4.70	5.33
4	50	(50,50,100,100)	5.36	5.10	4.70	5.06	5.22	4.92	5.04	5.08
4	50	(100,100,100,100)	5.11	5.32	5.29	5.08	5.06	4.95	4.94	4.93
4	100	(12,12,13,13)	5.02	5.38	5.63	4.97	5.36	5.16	5.20	5.43
4	100	(12,13,25,25)	5.02	5.29	4.98	5.27	5.11	5.33	5.03	5.12
4	200	(25,25,25,25)		5.19	5.17	4.34		4.82	4.72	4.61
4	200	(25,25,50,50)		4.96	4.89	5.17		4.89	5.04	4.52
4	200	(50,50,50,50)		5.31	4.84	5.66		5.07	5.09	4.91
4	200	(50,50,100,100)		4.97	4.95	4.76		4.71	4.74	5.14
6	100	(8,8,8,8,9,9)	4.98	5.57	5.7	5.34	5.07	5.46	5.22	4.83
6	100	(8,8,9,16,17,17)	5.13	5.08	4.78	4.74	4.88	4.89	5.07	5.18
6	200	(16,16,17,17,17,17)		4.97	5.13	5.15		5.39	4.74	5.25

TABLE 3
 Achieved Type I error rates ($\times 100\%$) for the testing procedures when $T_2 = P_b$ and sampling from $N_b(\mu, \Sigma)$ where $\Sigma = (1 - \rho)I_b + \rho J_b$. The nominal size is $\alpha = 0.01$.

a	b	(n_1, \dots, n_a)	$T_1 = P_a$				$T_1 = J_a/a$			
			-0.01	0	0.2	0.5	-0.01	0	0.2	0.5
2	5	(10,10)	2.00	2.10	1.90	1.80	1.50	1.50	1.50	1.50
2	10	(10,10)	1.10	1.70	0.90	1.60	0.80	1.70	1.60	1.70
2	15	(10,10)	0.80	1.50	1.60	1.90	2.30	1.30	1.70	1.40
2	12	(50,100)	1.00	0.99	1.00	1.13	1.03	1.27	0.81	0.95
2	12	(100,100)	1.16	0.91	1.01	1.07	1.06	0.92	1.00	1.05
2	25	(50,100)	0.99	1.06	1.10	1.10	1.04	1.08	0.99	1.01
2	25	(100,100)	0.95	0.76	0.96	0.96	1.10	1.14	1.12	1.00
2	50	(50,100)	1.27	1.02	1.03	1.04	0.84	1.01	0.94	1.01
2	50	(100,100)	1.04	1.09	0.84	1.05	0.96	0.89	1.04	0.89
2	100	(12,13)	1.17	1.02	1.22	1.07	1.24	1.01	0.99	1.21
2	100	(12,25)	1.06	1.02	0.93	0.94	1.14	0.98	1.00	1.16
2	100	(25,25)	1.07	1.04	0.95	1.01	1.15	1.14	1.07	1.04
2	100	(25,50)	0.91	1.03	0.9	0.98	1.12	0.93	1.01	1.08
2	200	(25,25)		1.17	0.85	1.07		1.00	1.06	1.11
2	200	(25,50)		0.95	0.85	1.00		1.00	0.99	0.94
2	200	(50,50)		1.03	0.92	1.03		0.98	1.05	1.06
2	200	(50,100)		1.02	1.30	1.07		0.95	0.95	0.88
2	400	(50,50)		1.01	1.22	0.94		0.95	0.94	0.88
2	400	(50,100)		1.06	0.95	1.10		1.07	1.04	0.98
2	400	(100,100)		0.92	1.03	0.92		1.05	0.94	0.85
2	400	(100,200)		1.08	1.08	0.82		1.11	1.04	0.92
3	5	(10,10,10)	1.40	1.70	1.90	1.40	1.70	1.40	0.90	1.40
3	10	(10,10,10)	1.70	1.30	1.10	1.60	1.10	0.80	1.20	0.70
3	15	(10,10,10)	0.60	1.10	0.30	0.60	1.70	0.70	1.00	0.60
3	12	(50,100,100)	1.08	1.13	1.19	1.04	1.09	1.17	0.97	0.96
3	12	(100,100,100)	0.94	1.11	1.10	1.03	0.92	1.00	0.99	0.99
3	25	(50,100,100)	1.07	0.94	0.96	1.14	0.93	1.12	1.02	1.05
3	25	(100,100,100)	1.07	1.13	0.79	1.05	1.08	0.81	1.04	0.84
3	50	(50,100,100)	1.00	0.78	0.92	0.97	1.19	1.02	0.82	1.00
3	50	(100,100,100)	1.16	0.94	1.12	1.10	0.94	1.04	0.76	0.93
3	100	(16,17,17)	1.06	0.92	1.13	1.13	1.09	0.94	1.10	0.95
3	100	(16,17,33)	1.17	1.16	0.92	0.91	1.17	1.04	1.03	1.06
3	200	(33,33,34)		0.91	1.17	0.98		0.97	1.00	1.13
3	200	(33,34,67)		0.94	0.91	0.84		1.08	0.99	1.15
3	200	(50,50,50)		0.86	1.07	1.09		0.92	0.85	1.01
3	200	(50,50,100)		0.98	1.01	1.02		1.16	0.95	1.12
4	12	(50,50,100,100)	0.71	0.97	0.92	1.09	0.99	1.00	1.01	0.96
4	12	(100,100,100,100)	1.06	1.04	1.03	1.07	0.98	1.09	1.09	0.90
4	25	(50,50,100,100)	1.00	1.08	1.01	1.21	0.89	1.07	0.90	0.87
4	25	(100,100,100,100)	0.88	0.91	0.72	0.95	0.92	1.09	1.07	0.89
4	50	(50,50,100,100)	1.00	0.95	1.08	1.09	1.11	0.92	1.12	0.96
4	50	(100,100,100,100)	1.04	1.02	1.07	1.24	0.93	1.11	1.00	0.90
4	100	(12,12,13,13)	1.04	0.86	1.14	1.16	1.05	1.03	0.80	0.97
4	100	(12,13,25,25)	1.07	0.99	1.04	1.12	1.02	1.03	1.03	1.03
4	200	(25,25,25,25)		1.04	0.95	1.02		0.84	0.96	1.10
4	200	(25,25,50,50)		1.02	1.06	1.07		0.95	1.19	1.09
4	200	(50,50,50,50)		1.06	0.95	1.02		0.98	1.00	1.25
4	200	(50,50,100,100)		0.95	1.11	1.12		0.96	0.91	0.98
6	100	(8,8,8,8,9,9)	1.03	0.91	1.21	1.23	0.99	1.07	0.99	0.98
6	100	(8,8,9,16,17,17)	0.86	1.12	1.26	1.16	1.01	1.08	0.98	0.95
6	200	(16,16,17,17,17,17)		1.00	1.11	1.07		0.99	0.94	0.98

TABLE 4
 Achieved Type I error rates ($\times 100\%$) for the testing procedures when $T_2 = P_b$ and sampling from $N_b(\mu, \Sigma)$ where $\Sigma = (\rho^{|j-j'|})$. The nominal size is $\alpha = 0.05$.

a	b	(n_1, \dots, n_a)	$T_1 = P_a$				$T_1 = J_a/a$			
			$\rho =$				$\rho =$			
			-0.01	0	0.2	0.5	-0.01	0	0.2	0.5
2	5	(10,10)	5.50	5.60	6.50	7.90	6.90	6.70	6.40	7.00
2	10	(10,10)	5.90	5.70	6.50	4.50	7.20	5.90	6.00	6.80
2	15	(10,10)	4.80	5.80	6.30	5.40	4.50	4.00	6.30	6.20
2	12	(50,100)	4.78	5.10	5.08	4.69	4.89	5.72	5.33	4.69
2	12	(100,100)	4.84	5.04	5.29	5.34	5.00	5.07	4.94	5.01
2	25	(50,100)	4.83	4.92	5.31	4.79	5.23	5.08	5.07	5.31
2	25	(100,100)	5.27	4.89	5.51	5.04	4.89	5.28	5.09	4.87
2	50	(50,100)	4.48	4.82	5.34	5.41	4.98	4.85	4.86	4.85
2	50	(100,100)	4.93	4.48	5.31	5.20	4.81	5.34	4.73	5.12
2	100	(12,13)	5.24	5.04	5.04	5.46	5.61	5.18	5.29	5.30
2	100	(12,25)	4.99	5.47	4.93	5.08	5.11	5.28	4.88	5.19
2	100	(25,25)	5.01	5.13	4.67	5.12	5.56	5.12	4.88	4.78
2	100	(25,50)	5.03	5.11	4.92	4.97	5.23	4.77	4.93	5.19
2	200	(25,25)	5.65	5.33	4.66	4.98	4.96	5.33	5.36	4.58
2	200	(25,50)	4.53	5.00	4.72	4.89	5.01	5.18	5.05	4.94
2	200	(50,50)	5.17	5.09	5.12	4.81	5.22	4.92	5.22	5.12
2	200	(50,100)	5.20	4.67	4.85	5.29	5.16	4.90	5.28	4.82
2	400	(50,50)	4.86	4.74	5.03	5.15	4.92	4.94	4.99	4.99
2	400	(50,100)	4.91	5.03	5.02	4.41	5.10	4.97	5.08	5.07
2	400	(100,100)	4.83	5.10	5.37	4.87	4.76	5.29	4.69	5.19
2	400	(100,200)	5.05	5.08	5.42	4.60	5.18	4.74	4.51	5.19
3	5	(10,10,10)	5.50	6.50	5.60	5.80	5.30	5.60	4.30	5.90
3	10	(10,10,10)	5.60	5.20	6.10	6.10	5.30	4.80	4.70	5.50
3	15	(10,10,10)	5.60	6.20	7.30	5.40	5.70	5.70	4.50	5.70
3	12	(50,100,100)	4.95	5.32	4.63	4.61	4.79	5.14	4.90	4.89
3	12	(100,100,100)	5.10	5.11	5.05	4.75	5.18	4.62	4.79	4.70
3	25	(50,100,100)	4.95	4.78	4.58	5.24	5.13	5.04	5.30	4.73
3	25	(100,100,100)	4.97	4.93	5.17	5.01	5.28	5.03	5.17	5.06
3	50	(50,100,100)	5.03	4.66	4.86	5.09	5.08	5.23	4.80	5.05
3	50	(100,100,100)	5.23	4.79	4.78	5.09	5.32	5.02	4.79	5.24
3	100	(16,17,17)	5.35	5.12	5.04	5.15	4.90	5.02	4.93	5.35
3	100	(16,17,33)	5.39	5.12	5.27	4.83	5.26	5.03	4.98	5.09
3	200	(33,33,34)	5.43	5.23	5.19	5.39	5.16	5.12	5.16	4.80
3	200	(33,34,67)	4.97	5.18	5.32	5.23	5.04	4.95	5.12	4.97
3	200	(50,50,50)	4.74	5.11	4.92	5.35	4.98	5.34	5.08	4.98
3	200	(50,50,100)	5.01	5.18	5.39	5.00	5.38	5.01	5.06	4.98
4	12	(50,50,100,100)	5.02	4.97	5.03	4.95	4.87	4.83	4.68	5.22
4	12	(100,100,100,100)	5.17	5.34	4.49	5.16	4.75	4.83	4.66	4.62
4	25	(50,50,100,100)	5.06	5.10	5.16	4.67	5.47	5.03	5.13	5.13
4	25	(100,100,100,100)	5.38	5.52	4.38	5.13	4.74	4.99	5.22	4.77
4	50	(50,50,100,100)	5.02	5.11	4.76	5.12	5.15	5.15	5.03	5.05
4	50	(100,100,100,100)	4.88	4.83	5.01	4.78	4.76	4.94	4.96	5.07
4	100	(12,12,13,13)	5.31	5.26	4.64	4.99	5.12	5.17	4.91	5.35
4	100	(12,13,25,25)	5.27	5.24	5.45	5.2	5.36	5.38	4.89	5.11
4	200	(25,25,25,25)	5.15	5.26	4.88	5.01	5.18	5.07	5.03	5.15
4	200	(25,25,50,50)	4.98	5.26	4.93	5.00	5.03	4.90	4.93	5.53
4	200	(50,50,50,50)	5.26	5.11	4.70	5.02	5.30	4.92	4.78	4.94
4	200	(50,50,100,100)	5.58	5.10	4.84	5.30	4.81	5.13	4.93	5.22
6	100	(8,8,8,8,9)	5.54	4.96	5.29	5.38	5.37	5.02	5.29	5.38
6	100	(8,8,9,16,17,17)	5.28	4.76	5.01	5.04	4.80	5.11	5.17	5.23
6	200	(16,16,17,17,17,17)	5.12	4.97	5.28	5.08	4.92	5.20	4.75	5.15

TABLE 5
 Achieved Type I error rates ($\times 100\%$) for the testing procedures when $T_2 = P_b$ and sampling from $N_b(\mu, \Sigma)$ where $\Sigma = (\rho^{|j-j'|})$. The nominal size is $\alpha = 0.01$.

a	b	(n_1, \dots, n_a)	$T_1 = P_a$				$T_1 = J_a/a$			
			$\rho =$				$\rho =$			
			-0.01	0	0.2	0.5	-0.01	0	0.2	0.5
2	5	(10,10)	1.30	2.20	1.70	2.10	1.90	1.60	1.80	2.60
2	10	(10,10)	1.70	1.40	1.50	1.40	2.30	1.30	2.00	1.60
2	15	(10,10)	0.70	1.50	1.20	0.80	1.70	1.50	1.20	1.70
2	12	(50,100)	1.12	0.96	1.11	1.07	1.07	1.00	0.95	1.05
2	12	(100,100)	1.23	1.03	0.94	0.89	0.98	1.08	1.04	0.99
2	25	(50,100)	1.01	1.05	1.12	1.22	1.04	1.01	0.99	1.03
2	25	(100,100)	1.00	1.20	0.95	0.86	0.93	1.00	1.01	1.02
2	50	(50,100)	0.79	0.86	0.93	1.00	1.13	1.01	0.96	0.79
2	50	(100,100)	0.98	0.81	1.23	0.96	0.99	1.00	1.04	0.78
2	100	(12,13)	1.17	1.17	1.01	1.26	1.17	1.17	1.14	0.97
2	100	(12,25)	1.18	1.16	0.91	1.36	1.13	1.22	1.02	1.05
2	100	(25,25)	1.07	0.95	1.18	0.98	0.91	0.90	1.15	0.98
2	100	(25,50)	1.00	0.92	0.93	1.19	1.15	0.90	0.81	1.04
2	200	(25,25)	0.95	0.95	1.08	0.87	1.12	1.01	0.95	0.97
2	200	(25,50)	1.11	1.04	0.92	1.07	1.00	0.89	1.14	0.85
2	200	(50,50)	1.04	0.94	1.02	0.92	1.01	1.04	0.98	0.86
2	200	(50,100)	0.85	1.13	1.02	1.06	1.01	0.85	0.93	1.05
2	400	(50,50)	1.09	1.09	1.04	1.08	0.98	1.05	1.12	0.86
2	400	(50,100)	1.07	1.02	1.04	0.83	1.10	1.05	0.87	0.92
2	400	(100,100)	1.01	1.01	1.12	1.07	1.00	1.05	1.03	1.12
2	400	(100,200)	0.97	0.96	0.88	1.06	1.06	0.98	1.02	1.04
3	5	(10,10,10)	1.30	1.50	1.50	2.00	1.20	0.80	1.60	1.70
3	10	(10,10,10)	1.70	1.40	0.70	1.20	1.10	1.70	1.80	1.50
3	15	(10,10,10)	1.00	1.30	1.20	1.60	1.40	1.30	0.70	1.20
3	12	(50,100,100)	0.92	0.98	1.18	1.09	0.93	1.07	1.09	1.05
3	12	(100,100,100)	0.82	1.24	0.99	0.94	1.10	0.89	0.99	0.92
3	25	(50,100,100)	0.90	0.95	1.05	0.99	1.09	0.98	0.84	0.96
3	25	(100,100,100)	1.00	1.25	1.08	1.04	1.00	1.15	0.91	1.06
3	50	(50,100,100)	1.05	1.12	0.92	0.97	1.02	0.82	1.04	1.08
3	50	(100,100,100)	0.85	0.91	1.11	0.90	1.01	0.88	1.06	1.04
3	100	(16,17,17)	1.11	0.92	1.29	1.02	1.07	0.84	0.93	0.98
3	100	(16,17,33)	1.14	1.28	0.98	0.95	1.03	1.01	1.07	1.28
3	200	(33,33,34)	1.26	1.11	0.97	1.12	1.07	0.99	1.04	0.93
3	200	(33,34,67)	1.06	1.04	0.99	1.05	0.99	0.99	1.02	0.99
3	200	(50,50,50)	1.01	0.95	1.04	1.09	1.12	1.05	1.10	1.12
3	200	(50,50,100)	0.99	1.03	0.87	1.26	1.11	1.06	0.98	1.08
4	12	(50,50,100,100)	0.79	1.02	1.10	0.85	1.08	1.03	1.17	1.03
4	25	(50,50,100,100)	1.08	0.90	1.36	0.92	1.12	0.87	1.09	0.95
4	25	(100,100,100,100)	0.89	1.23	1.11	1.08	0.99	1.14	1.12	0.96
4	50	(50,50,100,100)	1.02	0.97	0.87	0.96	1.22	0.98	1.01	0.95
4	50	(100,100,100,100)	0.97	1.03	1.04	0.92	0.98	1.13	1.29	0.92
4	100	(12,12,13,13)	1.06	1.07	1.16	1.03	1.09	1.03	1.08	1.01
4	100	(12,13,25,25)	0.97	1.04	1.10	1.12	1.19	0.93	1.03	0.91
4	200	(25,25,25,25)	1.01	0.99	1.00	0.99	0.90	0.95	0.96	1.00
4	200	(25,25,50,50)	0.91	1.05	0.90	1.04	0.72	1.08	0.94	0.95
4	200	(50,50,100,100)	0.93	0.88	1.04	1.08	0.84	0.87	1.06	0.88
6	100	(8,8,8,8,9,9)	1.11	1.24	1.14	1.33	0.99	0.92	1.14	1.00
6	100	(8,8,9,16,17,17)	1.16	1.05	0.95	1.29	1.04	0.91	1.04	1.19
6	200	(16,16,17,17,17,17)	1.12	1.05	0.90	1.05	0.99	1.05	0.99	0.90

TABLE 6
 Achieved Type I error rates ($\times 100\%$) for the testing procedures when $T_2 = P_b$ and sampling from $N_b(\mu, \Sigma)$ where $\Sigma = (\rho/(j - j')^{1/4})$. The nominal size is $\alpha = 0.05$.

a	b	(n_1, \dots, n_a)	$T_1 = P_a$				$T_1 = J_a/a$			
			-0.01	0	0.2	0.5	-0.01	0	0.2	0.5
2	5	(10,10)	6.00	6.50	5.20	6.80	4.60	6.60	5.70	7.20
2	10	(10,10)	6.10	5.10	6.20	6.40	5.60	5.40	4.30	7.60
2	15	(10,10)	4.50	3.60	5.40	5.80	6.30	5.40	5.90	6.10
2	12	(50,100)	5.04	5.06	5.28	5.41	5.40	4.95	5.63	4.92
2	12	(100,100)	5.12	4.83	5.55	4.88	5.13	4.82	5.29	4.84
2	25	(50,100)	5.24	5.20	4.90	5.02	5.23	5.25	4.89	4.42
2	25	(100,100)	5.15	5.00	4.97	4.63	5.05	4.68	5.26	4.48
2	50	(50,100)	5.24	4.95	5.06	4.97	4.80	4.97	4.83	4.70
2	50	(100,100)	4.85	4.69	5.42	5.15	5.06	5.46	5.13	5.11
2	100	(12,13)	4.88	5.46	5.05	5.58	5.51	5.07	5.25	5.23
2	100	(12,25)	4.94	4.72	5.08	5.15	4.44	5.00	5.24	4.64
2	100	(25,25)	4.94	4.57	5.01	4.45	5.90	5.18	5.18	5.12
2	100	(25,50)	5.32	4.85	5.19	4.98	4.76	5.59	4.87	5.28
2	200	(25,25)	5.12	5.45	5.19	4.88	5.05	4.99	5.18	5.03
2	200	(25,50)	5.07	4.90	4.90	5.21	5.28	5.20	5.12	4.99
2	200	(50,50)	5.24	5.32	5.20	4.60	4.93	5.46	4.42	5.01
2	200	(50,100)	4.53	5.32	5.14	4.97	4.99	5.18	4.91	4.46
2	400	(50,50)		5.45	4.74	4.73		4.70	5.13	4.33
2	400	(50,100)		5.10	4.85	4.65		5.50	5.05	4.84
2	400	(100,100)		4.93	4.88	4.87		4.91	4.87	4.94
2	400	(100,200)		4.77	5.17	5.09		4.61	4.83	4.68
3	5	(10,10,10)	5.70	6.20	7.10	6.50	5.20	5.60	5.80	4.50
3	10	(10,10,10)	4.70	6.30	4.70	5.20	6.20	4.90	6.00	4.90
3	15	(10,10,10)	6.30	3.70	6.00	5.10	4.70	5.00	4.50	6.40
3	12	(50,100,100)	4.97	5.10	4.89	4.71	4.87	5.18	5.21	5.43
3	12	(100,100,100)	4.83	4.61	5.18	5.41	4.84	4.75	4.88	4.84
3	25	(50,100,100)	5.42	4.71	5.12	5.21	4.92	4.64	5.35	4.97
3	25	(100,100,100)	5.13	5.05	5.05	4.69	5.29	4.72	4.83	4.77
3	50	(50,100,100)	5.12	4.76	5.26	4.96	5.02	5.21	4.89	5.12
3	50	(100,100,100)	4.89	5.11	5.06	5.06	5.36	5.12	5.41	4.63
3	100	(16,17,17)	4.70	5.15	5.03	5.33	4.69	5.15	4.97	4.90
3	100	(16,17,33)	5.11	5.18	5.26	5.35	4.99	4.74	5.05	5.07
3	200	(33,33,34)	5.00	4.91	5.12	5.04	4.86	5.15	5.01	4.88
3	200	(33,34,67)	4.80	4.91	5.00	4.41	5.29	5.14	5.03	4.78
3	200	(50,50,50)	5.22	4.65	4.67	4.99	5.23	5.10	5.35	4.75
3	200	(50,50,100)	4.93	4.92	4.75	5.29	5.32	4.98	5.13	4.81
4	12	(50,50,100,100)	4.72	5.04	4.86	5.26	5.14	5.05	4.64	5.05
4	12	(100,100,100,100)	5.41	5.01	5.30	4.90	4.97	4.74	5.15	4.85
4	25	(50,50,100,100)	4.98	4.97	4.81	5.46	5.38	4.93	4.66	4.75
4	25	(100,100,100,100)	5.19	4.57	5.16	4.75	4.79	5.05	5.24	4.81
4	50	(50,50,100,100)	4.87	5.13	4.74	4.94	5.25	5.18	5.38	4.75
4	50	(100,100,100,100)	5.41	5.63	5.32	5.20	5.54	5.03	4.80	4.89
4	100	(12,12,13,13)	5.43	5.39	5.22	4.93	5.13	5.24	5.07	5.07
4	100	(12,13,25,25)	5.21	4.97	5.07	5.16	5.24	5.12	4.97	4.76
4	200	(25,25,25,25)	4.78	5.34	4.86	5.07	5.14	5.08	5.32	5.10
4	200	(25,25,50,50)	4.97	4.83	4.90	4.83	4.85	5.11	4.83	4.72
4	200	(50,50,50,50)	4.96	4.68	5.23	5.00	4.93	5.33	4.97	5.12
4	200	(50,50,100,100)	4.95	4.92	4.88	4.76	4.95	5.15	5.01	4.87
6	100	(8,8,8,8,9,9)	5.27	4.76	5.19	5.57	5.35	5.33	5.12	5.25
6	100	(8,8,9,16,17,17)	5.27	4.91	4.90	5.30	5.02	5.17	5.01	4.92
6	200	(16,16,17,17,17,17)	5.27	5.05	4.93	5.36	4.78	5.20	4.72	4.84

TABLE 7
 Achieved Type I error rates ($\times 100\%$) for the testing procedures when $T_2 = P_b$ and sampling from $N_b(\mu, \Sigma)$ where $\Sigma = (\rho/(j - j')^{1/4})$. The nominal size is $\alpha = 0.01$.

a	b	(n_1, \dots, n_a)	$T_1 = P_a$				$T_1 = J_a/a$			
			$\rho = -0.01$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$	$\rho = -0.01$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$
2	5	(10,10)	1.70	1.90	1.30	1.70	1.80	1.10	2.10	1.90
2	10	(10,10)	1.40	1.70	1.80	1.40	0.70	1.80	1.20	1.50
2	15	(10,10)	1.00	1.80	2.10	1.30	0.90	1.20	1.30	2.60
2	12	(50,100)	1.03	1.10	0.97	1.01	1.17	0.93	1.02	0.90
2	12	(100,100)	0.91	0.86	1.02	0.89	1.26	1.11	0.93	1.12
2	25	(50,100)	1.32	1.13	0.86	0.99	1.14	1.08	1.15	0.92
2	25	(100,100)	0.93	1.04	0.95	1.12	1.04	1.08	1.07	0.97
2	50	(50,100)	0.99	0.91	0.94	1.04	1.06	1.07	0.97	1.09
2	50	(100,100)	1.11	0.97	1.13	1.06	1.13	1.00	1.02	0.78
2	100	(12,13)	1.10	1.09	1.15	1.13	1.04	1.22	1.19	1.24
2	100	(12,25)	1.23	1.04	1.01	1.02	0.93	1.02	1.01	1.26
2	100	(25,25)	1.03	1.05	0.93	1.11	1.08	0.97	1.08	1.00
2	100	(25,50)	0.96	1.08	0.98	1.03	1.00	0.92	1.14	1.21
2	200	(25,25)	0.96	1.09	0.71	0.90	1.02	1.02	1.07	1.12
2	200	(25,50)	0.94	1.03	1.02	0.95	1.13	1.05	1.09	1.08
2	200	(50,50)	1.00	1.02	0.84	1.02	1.03	0.98	1.14	1.06
2	200	(50,100)	0.88	1.20	0.98	0.94	1.21	0.99	0.99	1.20
2	400	(50,50)		1.14	1.02	0.87		1.14	1.06	0.91
2	400	(50,100)		0.98	1.04	1.00		0.85	1.02	0.94
2	400	(100,100)		1.10	0.91	1.07		0.90	1.11	0.86
2	400	(100,200)		0.93	0.97	0.78		0.87	1.04	0.89
3	5	(10,10,10)	1.80	1.80	1.70	1.40	1.30	1.60	1.60	0.50
3	10	(10,10,10)	1.30	1.10	1.30	2.20	0.40	1.50	1.00	1.40
3	15	(10,10,10)	1.00	1.90	0.90	1.20	2.30	1.60	1.10	1.20
3	12	(50,100,100)	1.01	1.22	0.83	1.00	1.05	0.92	1.00	0.86
3	12	(100,100,100)	0.86	1.26	1.09	1.12	0.96	0.84	1.08	0.87
3	25	(50,100,100)	1.03	1.16	1.15	1.06	1.00	0.94	0.94	1.02
3	25	(100,100,100)	1.09	1.02	0.93	0.99	1.12	1.06	0.99	0.87
3	50	(50,100,100)	0.98	1.05	1.11	1.06	0.99	1.06	0.92	0.99
3	50	(100,100,100)	1.13	1.22	1.18	1.20	1.05	1.02	1.10	0.63
3	100	(16,17,17)	1.13	1.21	1.11	1.08	1.04	0.98	0.97	0.96
3	100	(16,17,33)	1.12	1.05	1.14	1.23	1.11	0.91	0.96	0.90
3	200	(33,33,34)	1.15	1.02	1.10	0.90	0.98	0.95	1.22	0.90
3	200	(33,34,67)	0.97	1.04	0.98	1.03	0.98	0.92	1.06	1.06
3	200	(50,50,50)	1.19	0.83	1.11	0.95	0.98	0.98	1.19	0.97
3	200	(50,50,100)	0.84	1.10	1.03	1.09	1.06	1.18	1.19	1.07
4	12	(50,50,100,100)	0.98	1.02	1.10	1.07	1.05	1.05	0.94	0.89
4	12	(100,100,100,100)	0.89	0.95	1.02	1.13	1.06	1.01	1.16	1.07
4	25	(50,50,100,100)	1.05	1.02	0.97	0.99	1.16	0.95	1.00	0.97
4	25	(100,100,100,100)	1.15	1.00	1.09	1.10	1.15	0.93	1.01	0.90
4	50	(50,50,100,100)	0.98	0.97	1.11	1.10	1.00	1.12	0.95	0.87
4	50	(100,100,100,100)	1.04	1.08	0.85	0.94	1.17	0.85	0.97	0.91
4	100	(12,12,13,13)	0.91	1.16	1.02	1.36	1.07	0.96	1.21	1.20
4	100	(12,13,25,25)	0.85	1.15	0.91	1.08	1.10	1.01	1.07	1.09
4	200	(25,25,25,25)	1.12	1.01	1.00	1.08	1.20	1.02	1.12	1.03
4	200	(25,25,50,50)	1.07	0.82	1.21	0.90	1.02	1.04	1.16	0.85
4	200	(50,50,50,50)	0.89	0.77	1.00	0.99	1.00	0.92	0.85	0.83
4	200	(50,50,100,100)	1.05	1.06	1.03	1.07	0.98	1.03	0.89	0.95
6	100	(8,8,8,8,9,9)	0.96	1.08	1.09	1.22	1.02	0.91	1.01	1.18
6	100	(8,8,9,16,17,17)	1.11	0.97	1.15	0.97	0.89	0.98	1.02	1.04
6	200	(16,16,17,17,17,17)	0.98	1.05	1.02	1.33	1.20	1.09	1.04	0.93

manageable, we fix $a = 3$ and $\Sigma = \rho \mathbf{I}_b + (1 - \rho) \mathbf{J}_b$ where $\rho = 0.2$. In regards to sample sizes and dimension, we use the combinations $(b; n_1, n_2, n_3) = (10; 5, 10, 10), (10; 50, 100, 100), (100; 5, 10, 10)$ and $(100; 50, 100, 100)$. For the alternative hypotheses, when $\mathbf{T}_1 = \mathbf{P}_a$, we take $\mu_2 = \mu_3 = \mathbf{0}$ and consider two structures for μ_1 . The first one represents a dense alternative, namely $\mu_{1i} = (1 + \delta)$ for i odd and $\mu_{1i} = (1 - \delta)$ for i even, and the other one represents a sparse alternative, namely $\mu_1 = (1 + \delta, \mathbf{1}'_{b-1})'$. In both cases δ is made to vary from 0 to 1. When $\mathbf{T}_1 = \mathbf{J}_a/a$, we take $\mu_1 = \mathbf{1}_b + \mu_2, \mu_2 = \mu_3$ and consider two structures of μ_2 representing dense and sparse alternatives. For the first one, we take $\mu_{2i} = \delta$ for i odd and $\mu_{2i} = -\delta$ for i even, and for the second one we take $\mu_2 = (\mathbf{0}'_{b-1}, \delta)'$. Here also, δ varies from 0 to 1. The latter structure for both values of \mathbf{T}_1 represent departures that approach to the null hypotheses at the rate $b^{1/2}$. More precisely, the scaled departure from the null $\|\mu_1 - \mathbf{1}_b\|/\text{tr}(\Sigma)^{1/2}$ are δ and $|\delta|/\sqrt{b}$, respectively. Figures 1 and 2 show power results for $\mathbf{T}_1 = \mathbf{P}_a$ and $\mathbf{T}_1 = \mathbf{J}_a/a$, respectively. For dense alternatives (left panels), our methods has a clear advantage in all cases. More pronounced dominance is observed, in particular, when n is small. On the other hand, Chi et al. (2012) has an advantage for sparse alternatives (right panels).

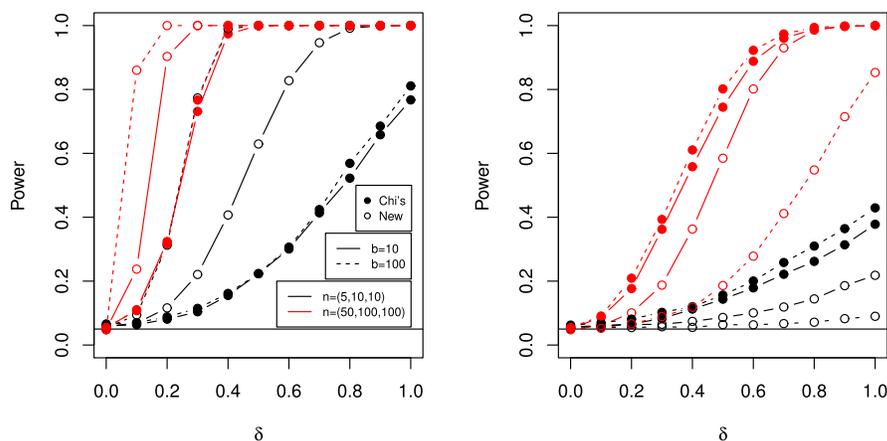


FIG 1. Power comparison of the proposed methods and the test by Chi et al. (2012) for $\mathbf{T}_1 = \mathbf{P}_a$ and $\mathbf{T}_2 = \mathbf{P}_b$. Data is generated from $\mathcal{N}_b(\mu_i, \Sigma)$, where $\Sigma = 0.8\mathbf{I}_b + 0.2\mathbf{J}_b$. In the both panel of the plot, $\mu_2 = \mu_3 = \mathbf{0}$. In the left panel $\mu_{1i} = (1 + \delta)$ for i odd, $\mu_{1i} = (1 - \delta)$ for i even and in the right panel $\mu_1 = (1 + \delta, \mathbf{1}'_{b-1})'$

6. Real data analysis

In this section, we analyze a publicly available data obtained from the University of California-Irvine Machine Learning Repository.¹ The data arose from a

¹web address: <https://archive.ics.uci.edu/ml/datasets/EEG+Database>

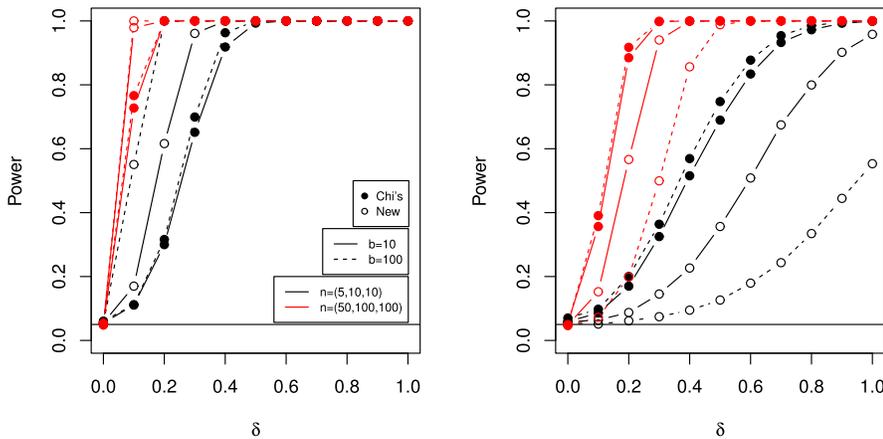


FIG 2. Power comparison of the proposed methods and the test by Chi et al. (2012) for $T_1 = J_a/a$ and $T_2 = P_b$. Data is generated from $N_b(\mu_i, \Sigma)$, where $\Sigma = 0.8I_b + 0.2J_b$. In the both panel of the plot, $\mu_1 = \mathbf{1}_b + \mu_2$ and $\mu_2 = \mu_3$. In the left panel $\mu_{2i} = \delta$ for i odd and $\mu_{2i} = -\delta$ for i even and in the right panel $\mu_2 = (\mathbf{0}_{b-1}^t, \delta)^t$

large study to examine Electroencephalograph (EEG) correlates of genetic predisposition to alcoholism. Measurements from 64 electrodes placed on subject’s scalps were recorded 256 times for 1 second. The study involved two groups of subjects: alcoholic ($n_1 = 77$) and control ($n_2 = 45$). Each subject was exposed to either a single stimulus (S1) or two stimuli (S1 and S2) which were pictures of objects chosen from a picture set. The sixty-four electrodes (channels) are divided into groups based on their location on the scalp (frontal, temporal, parietal and occipital lobes). To illustrate the application of the methods concisely, we focus the analysis on data from the stimulus S1 and the seventeen frontal-lobe channels. The outcome measurements are Event-Related Potentials (ERP) indicating the level of electrical activity (in volts) in the region of the brain where each of the electrodes is placed. This repeated measures data has two within-subject factors (time and channels) and one between-subject factor (alcohol use). The within-subject factors time and channel have 256 and 17 levels, respectively.

To assess the plausibility of the assumptions in the model, we checked channel-by-channel marginal normality after testing for equality of covariance matrices in the alcoholic and control groups. To date, there does not exist a satisfactory test of multivariate normality in the high-dimensional repeated measures. Neither of the high-dimensional tests Li and Chen (2012) nor Zhang et al. (2018) indicated violation of equality of covariance for any of the channels (p -values ≥ 0.512). However, the tests did not give a clean bill of normality. The proportion of fail to rejections (not adjusted for multiplicity) ranged from 24% to 41% for all but five of the frontal periphery channels. For these later

TABLE 8
Analysis for EEG data for $a = 2$, $b = 256$ and $d = 17$.

Hypothesis	T_1	T_2	\hat{T}	p -value
\mathcal{H}_{01}	\mathbf{P}_2	$\mathbf{P}_{256} \otimes \mathbf{P}_{17}$	0.535	0.196
\mathcal{H}_{02}	$\mathbf{J}_2/2$	$\mathbf{P}_{256} \otimes \mathbf{P}_{17}$	26.252	<0.001
\mathcal{H}_{03}	\mathbf{P}_2	$\mathbf{P}_{256} \otimes \mathbf{J}_{17}/17$	0.489	0.205
\mathcal{H}_{04}	$\mathbf{J}_2/2$	$\mathbf{P}_{256} \otimes \mathbf{J}_{17}/17$	42.430	<0.001

channels, the proportions (unadjusted) ranged from 57% to 82%. Considering the multiplicity of the tests, this may only represent a mild violation of normality.

The main research questions of interest are: (\mathcal{H}_{01}) whether the ERP profiles over time differ between channels and groups (three-way interaction: alcohol \times time \times channel); (\mathcal{H}_{02}) whether ERP profiles are similar between the channels when averaged over groups (similar time profiles for all the channels); (\mathcal{H}_{03}) if the time profiles of ERP are similar between the two groups averaged over channels; (\mathcal{H}_{04}) whether the ERP profiles are constant (flat) when averaged over channels and groups. For describing the contrast matrices, we assume the data vectors from each subject are arranged by grouping the 17 channels within each time point, i.e. the data vector from the j th subject in the i th group is $\mathbf{X}_{ij} = (X_{ij11}, \dots, X_{ij1,17}, \dots, X_{ij,256,1}, \dots, X_{ij,256,17})'$. In the notations of the paper, the four hypotheses of interest, \mathcal{H}_{0i} for $i = 1, 2, 3, 4$, can be tested by using the contrast matrices $\mathbf{T}_1 = \mathbf{P}_2$ and $\mathbf{T}_2 = \mathbf{P}_{256} \otimes \mathbf{P}_{17}$; $\mathbf{T}_1 = \mathbf{J}_2/2$ and $\mathbf{T}_2 = \mathbf{P}_{256} \otimes \mathbf{P}_{17}$; $\mathbf{T}_1 = \mathbf{P}_2$ and $\mathbf{T}_2 = \mathbf{P}_{256} \otimes \mathbf{J}_{17}/17$; and $\mathbf{T}_1 = \mathbf{J}_2/2$ and $\mathbf{T}_2 = \mathbf{P}_{256} \otimes \mathbf{J}_{17}/17$, respectively. The results of the analysis are presented in Table 8. Overall time-profile similarity across groups (averaged over channels) cannot be rejected (p -values = 0.205). In fact, channel-by-channel similarity of time profiles of ERP across groups cannot be rejected (p -values = 0.196). However, the flatness over time is rejected overall for all channels as well as channel-by-channel.

As a way of ascertaining the reproducibility and reliability of the results in Table 8, we conducted a simulation study using parameters similar to that of the EEG data. For table 9, we generate 1000 replications of data from $\mathcal{N}_{bd}(\mathbf{0}, \boldsymbol{\Sigma}_i)$. We look at values of $b = 256$, $d = 17$, $a = 2$ and $n_1 = 77$ $n_2 = 45$ and take $\alpha = 0.05$. Table 9 present actual type I error rates (test sizes) for the covariance structures $\boldsymbol{\Sigma}_1 = \rho \mathbf{I}_b + (1 - \rho) \mathbf{J}_b$, for $\rho = 0.2$ and random matrices $\boldsymbol{\Sigma}_i$ for $i = 2, 3, 4$ defined as follows. Let $\boldsymbol{\Sigma}_i = \mathbf{Q}'_i \boldsymbol{\Lambda}_i \mathbf{Q}_i$, where $\boldsymbol{\Lambda}_i$ is a diagonal matrix with diagonal entries taken from $Unif(0, 1)$ and \mathbf{Q}_i is orthogonal matrix. Indeed, \mathbf{Q}_i can be defined from the QR decomposition of a random matrix $\mathbf{Z}_i = (Z_{i,jj'})$ where $Z_{i,jj'}$ are iid random variables. Here, we consider three distributions for $Z_{i,jj'}$, namely $Z_{2,jj'} = 1_{\{j=j'\}}$ with probability 1, $Z_{3,jj'} \sim \mathcal{Exp}(1)$ and $Z_{4,jj'} \sim \mathcal{N}(0, 1)$. Here, it is clear that assumption A_1 is satisfied only for $\boldsymbol{\Sigma}_1$. It is clear from Table 9 that the achieved type I error rates are satisfactorily close to 5% regardless of the covariance matrix assumed.

TABLE 9

Achieved Type I error rates ($\times 100\%$) for the testing procedures with parameters similar to EEG data, i.e. $a = 2, b = 256, d = 17, n_1 = 77, n_2 = 45$.

T_1	T_2	Σ_1	Σ_2	Σ_3	Σ_4
P_2	$P_{256} \otimes P_{17}$	5.0	5.9	4.5	5.8
$J_2/2$	$P_{256} \otimes P_{17}$	4.5	4.0	5.4	5.8
P_2	$P_{256} \otimes J_{17}/17$	4.6	4.6	5.4	5.4
$J_2/2$	$P_{256} \otimes J_{17}/17$	5.6	4.8	5.2	4.6

7. Discussion and conclusion

The paper derives approximations for the null distributions and quantiles of some test statistics in repeated measures. The approximations ensure the errors to be of order $O(b^{-3/2})$ where b is the dimension, i.e. the number of repeated measures. Factorial designs are treated in a unified manner where multiple between- and within-subjects factors which may be crossed or nested are allowed. General covariance structure is allowed where no pre-determined sequence is assumed among the repeated measurements. Therefore, the repeated measurements could be over time or under different treatment conditions.

The asymptotic results require some regularity condition on the covariance matrix. Such assumption appears to be inevitable as long as one prefers to consider unstructured covariance matrix. Our observation from the simulation is that this assumption does not appear to restrict the utility of the results for application in more general situations. Nevertheless, we made somewhat milder requirements compared to similar works (see, for example, Bai and Saranadasa, 1996; Takahashi and Shutoh, 2016). The paper also assumes proportional divergence of the sample size and dimension, i.e. $n/b \rightarrow \gamma_0 \in (0, \infty)$, but either one can be larger than the other. We should point out that this assumption can be relaxed to cover other cases, namely $n = O(b^\epsilon)$ for $\epsilon > 1$ or $\epsilon < 1$. However, the expanded cumulative distribution function may have terms with order different from $b^{-j/2}$ for $j = 1, 2, \dots$ in which case the standard Cornish-Fisher formula for the quantile will not apply. Non standard expansions will need to be derived for the quantiles. Regardless, our impression from the simulation is that the effect of these terms is likely to be insignificant. This is an open problem that needs further investigation.

The development of the paper is under normality. We recommend assessing the plausibility of this assumption before applying the methods. Transformation that improve normality could also be attempted in the event non-normality is detected or suspected. In the proofs, multivariate normality of the repeated measures is mostly needed for its nice property of independence up to correlation and independence of some linear and quadratic forms. Limiting distribution results under statistical models that include independence up to correlation assumption have been derived in Bai and Saranadasa (1996) and Chen and Qin (2010) for two-sample and Katayama et al. (2013) for multiple-sample comparison of mean vectors. In the interest of space, we opted to relegate the investigation of these models for limiting distribution as well as asymptotic expansion in the repeated measures context to a follow-up manuscript.

Appendix A: Proofs

Proof of Theorem 3.1 and Theorem 3.5. We know that $n\mathbf{S} \sim \mathcal{W}_b(\boldsymbol{\Sigma}, n)$, where $\mathcal{W}_b(\boldsymbol{\Sigma}, n)$ stands for b -dimensional Wishart distribution with degrees of freedom n and scale matrix $\boldsymbol{\Sigma}$. Denote $a_k = \text{tr}(\mathbf{T}_2\mathbf{S})^k$ for $k = 1, \dots, 8$ and define

$$b_{\mathbf{T}_2\boldsymbol{\Sigma}} = (c_1^k, c_1^{k-2}c_2, \dots, c_k)' \quad \text{and} \quad b_{\mathbf{T}_2\mathbf{S}} = (a_1^k, a_1^{k-2}a_2, \dots, a_k)'$$

to be the vector of traces of the k th order moments. That means, for each partition of k , for example $k = \nu_1 + \dots + \nu_q$, where $\nu_1 \leq \dots \leq \nu_q$ and $q \leq k$, we include $c_{\nu_1} \dots c_{\nu_q}$ and $a_{\nu_1} \dots a_{\nu_q}$ to the vectors $b_{\mathbf{T}_2\boldsymbol{\Sigma}}$ and $b_{\mathbf{T}_2\mathbf{S}}$, respectively, at same position. It is known (Fujikoshi, 1973) that $E[b_{\mathbf{T}_2\mathbf{S}}] = \mathbf{F}_k b_{\mathbf{T}_2\boldsymbol{\Sigma}}$, where the matrices \mathbf{F}_k have been calculated by Fujikoshi (1973) up to $k = 6$ and by Watamori (1990) for $k = 7, 8$. Using these result, it can be shown that

$$E[\widehat{c}_k] = c_k, \text{ for } k = 1, 2, 3, 4.$$

Further, under the high-dimensional asymptotic frameworks A_1 and A_2 and after lengthy algebraic calculation, it can be seen that

$$\begin{aligned} \text{Var}\left(\frac{\widehat{c}_1}{c_2^{1/2}}\right) &= \frac{2}{n} = O(n^{-1}), \\ \text{Var}\left(\frac{\widehat{c}_2}{c_2}\right) &= \frac{4}{n(n-1)(n+2)c_2^2} [nc_2^2 + (2n^2 + 3n - 6)c_4] = O(n^{-2}), \\ \text{Var}\left(\frac{\sqrt{b}\widehat{c}_3}{c_2^{3/2}}\right) &= \frac{6b}{nm_1c_2^3} [n^2c_2^3 + 3n(n-1)(n+4)c_3^2 + 3n(n^2 + 3n - 12)c_2c_4 + \\ &\quad (3n^4 + 15n^3 - 20n^2 - 120n + 160)c_6] = O(n^{-2}) \quad \text{and} \\ \text{Var}\left(\frac{b\widehat{c}_4}{c_2^2}\right) &= \frac{8b^2}{nm_2c_2^4} [f_1c_2^4 + f_2c_2c_3^2 + f_3c_2^2c_4 + f_4c_4^2 + f_5c_3c_5 + f_6c_2c_6 + f_7c_8] \\ &= O(n^{-2}), \end{aligned}$$

where

$$\begin{aligned} f_1 &= n^2(n^2 + n + 2), \quad f_2 = 8n^2(n+1)(n-3)(n+6), \\ f_3 &= 2n^2(2n^3 + 11n^2 - 47n + 54), \\ f_4 &= n(6n^5 + 40n^4 - 85n^3 - 631n^2 + 726n + 1224), \\ f_5 &= 8n(n+1)(n-3)(n+6)(n^2 + 4n - 16), \\ f_6 &= 4n(n^5 + 10n^4 - 11n^3 - 220n^2 + 276n + 480) \quad \text{and} \\ f_7 &= 2(2n^7 + 23n^6 + 38n^5 - 423n^4 - 992n^3 + 4066n^2 - 420n - 5040). \end{aligned}$$

Obviously, \widehat{c}_2/c_2 is L^2 , so that in probability, convergent to 1. All the asymptotic equivalence results in Theorem 3.1 and Theorem 3.5 can be proved similarly, and

we only include the proof of $(\widehat{c}_1 - c_1)/\sqrt{c_2} = O_p(b^{-1/2})$ here. By Chebyshev's inequality and under the condition A_2 , we have

$$Pr \left(\left| \frac{\sqrt{b}(\widehat{c}_1 - c_1)}{\sqrt{c_2}} \right| > M \right) \leq \frac{b\text{Var}(\widehat{c}_1/c_2^{1/2})}{M^2} = \frac{2b}{nM^2} \rightarrow \frac{2\gamma_0}{M^2},$$

for any $M > 0$. Therefore, for any $\varepsilon > 0$, there exists $M(> \sqrt{2\gamma_0/\varepsilon})$, such that the above probability is bounded by ε . For similar results see Srivastava (2005) and Hyodo et al. (2014). \square

Theorem A.1. *Under the high-dimensional asymptotic frameworks A_1 and A_3 , the characteristic function of T can be expanded as*

$$\phi_T(t) = e^{\frac{1}{2}\iota^2 t^2} \left\{ 1 + \frac{1}{\sqrt{b}} \iota^3 t^3 \eta_3 + \frac{1}{b} (\iota^4 t^4 \eta_4 + \iota^6 t^6 \eta_3^2 / 2) + O(b^{-3/2}) \right\},$$

as $n, b \rightarrow \infty$, where $\iota = \sqrt{-1}$.

Proof of Theorem A.1. Let $\mathbf{Z} = \widetilde{\Sigma}^{-1/2}(\overline{\mathbf{X}} - \boldsymbol{\mu}) \sim \mathcal{N}_{ab}(\mathbf{0}, \mathbf{I}_{ab})$. Then

$$T = \frac{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} (\mathbf{Z}' \widetilde{\Sigma}^{1/2} \mathbf{K} \widetilde{\Sigma}^{1/2} \mathbf{Z} + 2\boldsymbol{\mu}' \mathbf{K} \widetilde{\Sigma}^{1/2} \mathbf{Z}) - c_1}{\sqrt{2\delta_2 c_2}}.$$

Denote $\sigma_1^2 = 2\delta_2 c_2$ with $\sigma_1 > 0$. The characteristic function of T is derived as follows,

$$\begin{aligned} \phi_T(t) &= \mathbb{E} \left[\exp \left\{ \frac{\iota t}{\sigma_1} \left\{ \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} (\mathbf{Z}' \widetilde{\Sigma}^{1/2} \mathbf{K} \widetilde{\Sigma}^{1/2} \mathbf{Z} + 2\boldsymbol{\mu}' \mathbf{K} \widetilde{\Sigma}^{1/2} \mathbf{Z}) - c_1 \right\} \right\} \right] \\ &= \exp \left(\frac{-\iota t c_1}{\sigma_1} \right) \int_{\mathbf{Z}} (2\pi)^{-ab/2} \\ &\quad \times \exp \left\{ -\frac{1}{2} \mathbf{Z}' \mathbf{M} \mathbf{Z} + \frac{2\iota t}{\sigma_1} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \boldsymbol{\mu}' \mathbf{K} \widetilde{\Sigma}^{1/2} \mathbf{Z} \right\} d\mathbf{Z} \\ &= \exp \left(\frac{-\iota t c_1}{\sigma_1} \right) \int_{\mathbf{Z}} (2\pi)^{-ab/2} \\ &\quad \times \exp \left\{ -\frac{1}{2} \left(\mathbf{Z} - \frac{2\iota t}{\sigma_1} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \mathbf{M}^{-1} \widetilde{\Sigma}^{1/2} \mathbf{K} \boldsymbol{\mu} \right)' \right. \\ &\quad \cdot \mathbf{M} \cdot \left. \left(\mathbf{Z} - \frac{2\iota t}{\sigma_1} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \mathbf{M}^{-1} \widetilde{\Sigma}^{1/2} \mathbf{K} \boldsymbol{\mu} \right) \right. \\ &\quad \left. + \frac{2\iota^2 t^2}{\sigma_1^2} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-2} \boldsymbol{\mu}' \mathbf{K} \widetilde{\Sigma}^{1/2} \mathbf{M}^{-1} \widetilde{\Sigma}^{1/2} \mathbf{K} \boldsymbol{\mu} \right\} d\mathbf{Z} \\ &= \exp \left(\frac{-\iota t c_1}{\sigma_1} \right) |\mathbf{M}|^{-1/2} \exp \left\{ \frac{2\iota^2 t^2}{\sigma_1^2} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-2} \boldsymbol{\mu}' \mathbf{K} \widetilde{\Sigma}^{1/2} \mathbf{M}^{-1} \widetilde{\Sigma}^{1/2} \mathbf{K} \boldsymbol{\mu} \right\}, \end{aligned}$$

where $\mathbf{M} = \mathbf{I} - \frac{2\iota t}{\sigma_1} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \widetilde{\Sigma}^{1/2} \mathbf{K} \widetilde{\Sigma}^{1/2}$ and $|\mathbf{M}|$ is the determinant of \mathbf{M} . Let α_i 's be the eigenvalues of $\mathbf{T}_1 \mathbf{D}$, β_j 's be the eigenvalues of $\Sigma^{1/2} \mathbf{T}_2 \Sigma^{1/2}$, then

$$|\mathbf{M}| = \prod_{i=1}^a \prod_{j=1}^b \left(1 - \frac{2\iota t}{\sigma_1} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \alpha_i \beta_j \right).$$

Thus, by Taylor’s series expansion, we have

$$\begin{aligned} \log |\mathbf{M}|^{-1/2} &= -\frac{1}{2} \sum_{i=1}^a \sum_{j=1}^b \log \left(1 - \frac{2it}{\sigma_1} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \alpha_i \beta_j \right) \\ &= \frac{1}{2} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{\infty} \frac{2^k (it)^k}{k \sigma_1^k} \cdot \frac{\alpha_i^k \beta_j^k}{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^k} \\ &= \sum_{k=1}^{\infty} \frac{2^{k/2-1} (it)^k}{k} \cdot \frac{\delta_k c_k}{\delta_2^{k/2} c_2^{k/2}} \\ &= \frac{itc_1}{\sigma_1} + \frac{1}{2} i^2 t^2 + \frac{1}{\sqrt{b}} i^3 t^3 \eta_3 + \frac{1}{b} i^4 t^4 \eta_4 + O(b^{-3/2}), \end{aligned}$$

where the last equality holds by condition A_1 and equation (2).

Since

$$\mathbf{M}^{-1} = \mathbf{I} + \frac{2it}{\sigma_1} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \tilde{\Sigma}^{1/2} \mathbf{K} \tilde{\Sigma}^{1/2} + \frac{4i^2 t^2}{\sigma_1^2} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-2} \tilde{\Sigma}^{1/2} \mathbf{K} \tilde{\Sigma} \mathbf{K} \tilde{\Sigma}^{1/2} + \dots$$

Under assumption A_3 , as $n, b \rightarrow \infty$, we have

$$\boldsymbol{\mu}' \mathbf{K} \tilde{\Sigma} \mathbf{K} \boldsymbol{\mu} \leq \boldsymbol{\mu}' \mathbf{K} \boldsymbol{\mu} \cdot \sqrt{\text{tr}(\mathbf{K} \tilde{\Sigma})^2} = \boldsymbol{\mu}' \mathbf{K} \boldsymbol{\mu} \cdot \sqrt{\text{tr}(\mathbf{T}_1 \mathbf{D})^2 \cdot c_2},$$

and, therefore,

$$\frac{1}{\sigma_1^2} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-2} \boldsymbol{\mu}' \mathbf{K} \tilde{\Sigma} \mathbf{K} \boldsymbol{\mu} \leq \frac{\boldsymbol{\mu}' \mathbf{K} \boldsymbol{\mu}}{2\sqrt{\text{tr}(\mathbf{T}_1 \mathbf{D})^2 \cdot c_2}} = O(b^{-3/2}).$$

Similarly,

$$\begin{aligned} \frac{1}{\sigma_1^3} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-3} \boldsymbol{\mu}' \mathbf{K} \tilde{\Sigma} \mathbf{K} \tilde{\Sigma} \mathbf{K} \boldsymbol{\mu} &\leq \frac{1}{\sigma_1^2} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-2} \boldsymbol{\mu}' \mathbf{K} \tilde{\Sigma} \mathbf{K} \boldsymbol{\mu} = O(b^{-3/2}), \\ \frac{1}{\sigma_1^4} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-4} \boldsymbol{\mu}' \mathbf{K} \tilde{\Sigma} \mathbf{K} \tilde{\Sigma} \mathbf{K} \tilde{\Sigma} \mathbf{K} \boldsymbol{\mu} &\leq \frac{1}{\sigma_1^2} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-2} \boldsymbol{\mu}' \mathbf{K} \tilde{\Sigma} \mathbf{K} \boldsymbol{\mu} = O(b^{-3/2}), \end{aligned}$$

and so on. Thus, we have

$$\frac{1}{\sigma_1^2} \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \boldsymbol{\mu}' \mathbf{K} \tilde{\Sigma}^{1/2} \mathbf{M}^{-1} \tilde{\Sigma}^{1/2} \mathbf{K} \boldsymbol{\mu} \Big\} = O(b^{-3/2}).$$

By Taylor’s series expansion, the characteristic function of T can be expanded as

$$\phi_T(t) = e^{\frac{1}{2} i^2 t^2} \left\{ 1 + \frac{1}{\sqrt{b}} i^3 t^3 \eta_3 + \frac{1}{b} (i^4 t^4 \eta_4 + i^6 t^6 \eta_3^2 / 2) + O(b^{-3/2}) \right\}. \quad \square$$

Proof of Theorem 3.2. Denote $H^* = \{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} H$. Note that

$$\frac{\hat{c}_2}{c_2} = 1 + \frac{1}{b} W \quad \text{and} \quad \frac{\hat{c}_1 - c_1}{\sqrt{2\delta_2 c_2}} = \frac{V}{\sqrt{b}}.$$

By Taylor's expansion, we have

$$\left(\frac{\widehat{c}_2}{c_2}\right)^{-1/2} = 1 - \frac{1}{2b}W + O_p(b^{-2}).$$

Then

$$\begin{aligned} \widehat{T} &= \frac{H^* - \widehat{c}_1}{\sqrt{2\delta_2\widehat{c}_2}} = \frac{(H^* - c_1) - (\widehat{c}_1 - c_1)}{\sqrt{2\delta_2c_2} \sqrt{\widehat{c}_2/c_2}} \\ &= \frac{(H^* - c_1) - (\widehat{c}_1 - c_1)}{\sqrt{2\delta_2c_2}} \left\{ 1 - \frac{1}{2b}W + O_p(b^{-2}) \right\} \\ &= T - \frac{V}{\sqrt{b}} - \frac{TW}{2b} + O_p(b^{-3/2}), \end{aligned}$$

where $T = \frac{(H^* - c_1)}{\sqrt{2\delta_2c_2}}$. So the characteristic function of \widehat{T} is

$$\begin{aligned} \phi_{\widehat{T}}(t) &= E[e^{it\widehat{T}}] = E\left[e^{itT} \cdot e^{it\left(-\frac{V}{\sqrt{b}} - \frac{TW}{2b} + O_p(b^{-3/2})\right)}\right] \\ &= E\left[e^{itT} \cdot \left\{ 1 - \frac{itV}{\sqrt{b}} - \frac{itTW}{2b} + \frac{i^2t^2V^2}{b} + O_p(b^{-3/2}) \right\}\right] \\ &= E[e^{itT}] + \frac{i^2t^2}{2b}E[e^{itT}]E[V^2] + O(b^{-3/2}), \end{aligned}$$

since T is independent with both (V, W) and $E[V] = E[W] = 0$.

Finally, using Theorem A.1 and the fact (see Proof of Theorem 3.1) that $E[V^2] = \frac{b}{2\delta_2c_2}\text{Var}(\widehat{c}_1) = \frac{b}{n\delta_2} = \frac{\gamma}{\delta_2}$, we have the desired result. \square

Proof of Theorem 3.3. In view of the expanded characteristic function in Theorem 3.2, the desired result follows from a term-by-term inversion by using the inversion formula

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^x \int_{-\infty}^{\infty} (-it)^j e^{-it u} e^{-t^2/2} dt du &= \int_{-\infty}^x \frac{d^j}{du^j} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it u} e^{-t^2/2} dt \right\} du \\ &= \int_{-\infty}^x \phi^{(j)}(u) du = \Phi^{(j)}(x) \end{aligned}$$

for any nonnegative integer j . For more details see, for example, Theorem 3 in (Shiryaev and Chibisov, 2016, pp. 340). \square

Proof of Theorem 4.1.

$$\widehat{T} - \frac{\{\text{tr}(\mathbf{T}_1\mathbf{D})\}^{-1}\Delta}{\sqrt{2\delta_2c_2}} = T - \frac{V}{\sqrt{b}} - \frac{WU}{2b} + O_p(b^{-3/2}),$$

where $U = \{\text{tr}(\mathbf{T}_1\mathbf{D})\}^{-1}(H^* - c_1)/\sqrt{2\delta_2c_2}$. Since U and W are independent and $E(U) = E(V) = E(W) = 0$, application of Theorem A.1 leads to,

$$\begin{aligned}
\beta(\Delta) &= P(\widehat{T} > u(z_\alpha)) \\
&= P\left(\widehat{T} - \frac{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \Delta}{\sqrt{2\delta_2 c_2}} > u(z_\alpha) - \frac{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \Delta}{\sqrt{2\delta_2 c_2}}\right) \\
&= 1 - G_{\widehat{T}}\left(u(z_\alpha) - \frac{\{\text{tr}(\mathbf{T}_1 \mathbf{D})\}^{-1} \Delta}{\sqrt{2\delta_2 c_2}}\right),
\end{aligned}$$

as to be proved. □

Acknowledgements

The authors are grateful to the two anonymous referees for critically reading the original and revised versions of the manuscript and their valuable suggestions that led to great improvements. Authors are also thankful to the editor and the associate editor for their constructive comments and orderly handling of the manuscript. The research of Xiaoli Kong was partially done while she was pursuing her Ph.D. She would like to express her gratitude to the Department of Statistics, University of Kentucky.

References

- Bai, Z. and Saranadasa, H. (1996). Effect of high dimension: By an example of a two sample problem. *Statistica Sinica*, 6(2):311–329. [MR1399305](#)
- Bock, R. (1963). *Multivariate analysis of variance of repeated measurements. (Problems in measuring change. Editor C. W. Harris)*. University of Wisconsin Press, Madison, Wisconsin.
- Box, G. E. P. (1954). Some theorems on quadratic forms applied in the study of analysis of variance problems, I. Effect of inequality of variance in the one-way classification. *The Annals of Mathematical Statistics*, 25(2):290–302. [MR0061787](#)
- Brunner, E., Bathke, A. C., and Placzek, M. (2012). Estimation of box’s ε for low- and high-dimensional repeated measures designs with unequal covariance matrices. *Biometrical Journal*, 54(3):301–316. [MR2931076](#)
- Brunner, E., Dette, H., and Munk, A. (1997). Box-type approximations in non-parametric factorial designs. *Journal of the American Statistical Association*, 92(440):1494–1502. [MR1615259](#)
- Brunner, E., Konietzschke, F., Pauly, M., and Puri, M. L. (2017). Rank-based procedures in factorial designs: hypotheses about non-parametric treatment effects. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79(5):1463–1485. [MR3731671](#)
- Brunner, E., Munzel, U., and Puri, M. L. (1999). Rank-score tests in factorial designs with repeated measures. *Journal of Multivariate Analysis*, 70(2):286–317. [MR1711514](#)
- Cai, T. T., Liu, W., and Xia, Y. (2014). Two-sample test of high dimensional means under dependence. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(2):349–372. [MR3164870](#)

- Chen, S. X. and Qin, Y. L. (2010). A two-sample test for high-dimensional data with applications to gene-set testing. *The Annals of Statistics*, 38(2):808–835. [MR2604697](#)
- Chi, Y. Y., Gribbin, M., Lamers, Y., Gregory, J. F., and Muller, K. E. (2012). Global hypothesis testing for high-dimensional repeated measures outcomes. *Statistics in Medicine*, 31(8):724–742. [MR2901796](#)
- Collier, R. O., Baker, F. B., Mandeville, G. K., and Hayes, T. F. (1967). Estimates of test size for several test procedures based on conventional variance ratios in the repeated measures design. *Psychometrika*, 32(3):339–353. [MR0219171](#)
- Fujikoshi, Y. (1973). Asymptotic formulas for the distributions of three statistics for multivariate linear hypothesis. *Annals of the Institute of Statistical Mathematics*, 25(1):423–437. [MR0331639](#)
- Geisser, S. and Greenhouse, S. W. (1958). An extension of box's results on the use of the F distribution in multivariate analysis. *The Annals of Mathematical Statistics*, 29(3):885–891. [MR0096331](#)
- Happ, M., Harrar, S. W., and Bathke, A. C. (2015). Inference for low- and high-dimensional multi-group repeated measures designs with unequal covariance matrices. *Biometrical Journal*, 58(4):810–830. [MR3527417](#)
- Harrar, S. W. and Kong, X. (2016). High-dimensional multivariate repeated measures analysis with unequal covariance matrices. *Journal of Multivariate Analysis*, 145:1–21. [MR3459934](#)
- Hill, G. W. and Davis, A. W. (1968). Generalized asymptotic expansions of cornish-fisher type. *The Annals of Mathematical Statistics*, 39(4):1264–1273. [MR0226762](#)
- Huynh, H. (1978). Some approximate tests for repeated measurement designs. *Psychometrika*, 43(2):161–175. [MR0514724](#)
- Huynh, H. and Feldt, L. S. (1970). Conditions under which mean square ratios in repeated measurements designs have exact F -distributions. *Journal of the American Statistical Association*, 65(332):1582–1589.
- Huynh, H. and Feldt, L. S. (1976). Estimation of the box correction for degrees of freedom from sample data in randomized block and split-plot designs. *Journal of Educational Statistics*, 1(1):69–82.
- Hyodo, M., Takahashi, S., and Nishiyama, T. (2014). Multiple comparisons among mean vectors when the dimension is larger than the total sample size. *Communications in Statistics – Simulation and Computation*, 43(10):2283–2306. [MR3223670](#)
- Katayama, S., Kano, Y., and Srivastava, M. S. (2013). Asymptotic distributions of some test criteria for the mean vector with fewer observations than the dimension. *Journal of Multivariate Analysis*, 116:410–421. [MR3049913](#)
- Konietschke, F., Bathke, A. C., Harrar, S. W., and Pauly, M. (2015). Parametric and nonparametric bootstrap methods for general manova. *Journal of Multivariate Analysis*, 140:291–301. [MR3372569](#)
- Lecoutre, B. (1991). A correction for the formula approximate test in repeated measures designs with two or more independent groups. *Journal of Educational and Behavioral Statistics*, 16(4):371–372.

- Li, J. and Chen, S. X. (2012). Two sample tests for high-dimensional covariance matrices. *The Annals of Statistics*, 40(2):908–940. [MR2985938](#)
- Maxwell, S. E. and Arvey, R. D. (1982). Small sample profile analysis with many variables. *Psychological Bulletin*, 92(3):778–785.
- Pauly, M., Ellenberger, D., and Brunner, E. (2015). Analysis of high-dimensional one group repeated measures designs. *Statistics*, 49(6):1243–1261. [MR3415218](#)
- Rencher, A. C. and Christensen, W. F. (2012). *Methods of multivariate analysis*. John Wiley & Sons, United States, 3rd edition. [MR2962097](#)
- Shiryayev, A. and Chibisov, D. (2016). *Probability-1*. Graduate Texts in Mathematics. Springer New York. [MR3467826](#)
- Srivastava, M. S. (2005). Some tests concerning the covariance matrix in high dimensional data. *Journal of the Japan Statistical Society*, 35(2):251–272. [MR2328427](#)
- Stoloff, P. H. (1970). Correcting for heterogeneity of covariance for repeated measures designs of the analysis of variance. *Educational and Psychological Measurement*, 30(4):909–924.
- Takahashi, S. and Shutoh, N. (2016). Tests for parallelism and flatness hypotheses of two mean vectors in high-dimensional settings. *Journal of Statistical Computation and Simulation*, 86(6):1150–1165. [MR3441561](#)
- Wang, H. and Akritas, M. G. (2010a). Inference from heteroscedastic functional data. *Journal of Nonparametric Statistics*, 22(2):149–168. [MR2598959](#)
- Wang, H. and Akritas, M. G. (2010b). Rank test for heteroscedastic functional data. *Journal of Multivariate Analysis*, 101(8):1791–1805. [MR2651955](#)
- Wang, H., Higgins, J., and Blasi, D. (2010). Distribution-free tests for no effect of treatment in heteroscedastic functional data under both weak and long range dependence. *Statistics & Probability Letters*, 80(5-6):390–402. [MR2593579](#)
- Watamori, Y. (1990). On the moments of traces of wishart and inverted wishart matrices. *South African Statistical Journal*, 24(2):153–176. [MR1128480](#)
- Yang, X. M., Yang, X. Q., and Teo, K. L. (2001). A matrix trace inequality. *Journal of Mathematical Analysis and Applications*, 263(1):327–331. [MR1865285](#)
- Zhang, C., Bai, Z., Hu, J., and Wang, C. (2018). Multi-sample test for high-dimensional covariance matrices. *Communications in Statistics – Theory and Methods*, 48(13):3161–3177. [MR3804216](#)