

Invariant test based on the modified correction to LRT for the equality of two high-dimensional covariance matrices

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Abstract: In this paper, we propose an invariant test based on the modified correction to the likelihood ratio test (LRT) of the equality of two high-dimensional covariance matrices. It is well-known that the classical log-LRT is not well defined when the dimension is larger than or equal to one of the sample sizes. Or even the log-LRT is well-defined, it is usually perceived as a bad statistic in the high-dimensional cases because of their low powers under some alternatives. In this paper, we will justify the usefulness of the modified log-LRT, and an invariant test that works well in cases where the dimension is larger than the sample sizes. Besides, the test is established under the weakest conditions on the dimensions and the moments of the samples. The asymptotic distribution of the proposed test statistic is also obtained under the null hypothesis. What is more, we also propose a lite version of the modified LRT in the paper. A simulation study and a real data analysis show that the performances of the two proposed statistics are invariant under affine transformations.

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1. Introduction

Since the assumption of homogeneity of covariance matrices is needed in many multivariate statistical analyses based on two populations, the equality of two covariance matrices is among the most active hypothesis tests. These tests date back to the work of [19], which was followed by a huge amount of literature. Suppose we have $N := N_1 + N_2$ observations $\{\mathbf{z}_i^{(l)} \sim N(\boldsymbol{\mu}_l, \Sigma_l), i = 1, \dots, N_l, l = 1, 2\}$ and wish to test the hypothesis

$$H_0 : \Sigma_1 = \Sigma_2 \quad \text{v.s.} \quad H_1 : \Sigma_1 \neq \Sigma_2, \quad (1.1)$$

where $\boldsymbol{\mu}_l$ and Σ_l are the population mean vectors and covariance matrices of the p -dimensional vectors $\mathbf{z}_i^{(l)}$, $l = 1, 2$ respectively. It is natural to first consider the

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likelihood ratio test (LRT) if it is “applicable”. But when is the LRT applicable to testing the equality of two covariance matrices? The traditional viewpoint is that the LRT is applicable if the sample sizes are both much larger than the dimensions based on the χ^2 approximation [20]. However, [5] and [12] showed that when the dimensions are large but smaller than the sample sizes, the traditional χ^2 approximation of the LRT fails. This problem encouraged us to investigate the conditions under which the high-dimensional LRT is applicable. Additionally, we consider a lite LRT proposed by [16, 17].

Currently, there are three general types of test procedures for the high-dimensional hypothesis (1.1) that are widely discussed in the literature: (i) Corrected classical LRTs, see [5, 11, 12]; (ii) Nonparametric methods, see [13, 14, 18]; (iii) Maximum element methods, see [7, 8]. The strengths and weaknesses of these three methods are significant. Nonparametric methods and maximum element methods can address cases where the dimensions are much larger than the sample sizes but are strongly restricted by the structure or eigenvalues of the population covariance matrices. By contrast, corrected classical LRT requires the dimensions to be smaller than the sample sizes, but there is no assumption on the population covariance matrices. In addition, if the dimensions are fixed, LRT has been shown to be unbiased and uniformly most powerful among affine-transform-invariant tests. Therefore, we focus on the LRT.

In the following, we denote $\mathbf{X}_{N_1}^{(1)} = (x_{ij}^{(1)})_{p \times N_1}$, where $\{x_{ij}^{(1)}\}$ are independent and identically distributed (i.i.d.) random variables with mean zero and variance one. Similarly $\mathbf{X}_{N_2}^{(2)} = (x_{ij}^{(2)})_{p \times N_2}$, which is independent with $\mathbf{X}_{N_1}^{(1)}$, constitutes another i.i.d. sample with mean zero and variance one. For $l = 1, 2$, we assume that the N_l observations $\mathbf{z}_j^{(l)}$ satisfy the *linear transformation model*

$$\mathbf{z}_j^{(l)} = \Sigma_l^{1/2} \mathbf{x}_j^{(l)} + \boldsymbol{\mu}_l, \tag{1.2}$$

where $\mathbf{x}_j^{(l)}$ is the j th column of $\mathbf{X}_{N_l}^{(l)}$ and $\boldsymbol{\mu}_l$ and Σ_l are the population mean vectors and covariance matrices of $\{\mathbf{z}_j^{(l)}\}$, respectively. Here, $\Sigma_l^{1/2}$ can be chosen as any square root of matrix Σ_l . The linear transformation model covers the case where the samples $\mathbf{z}_i^{(l)}$ are normally distributed, although we are not restricted here. Denote $n_l = N_l - 1$, $\bar{\mathbf{z}}^{(l)} = \frac{1}{N_l} \sum_{i=1}^{N_l} \mathbf{z}_i^{(l)}$, $\hat{\mathbf{z}}_i^{(l)} = \mathbf{z}_i - \bar{\mathbf{z}}^{(l)}$ and $\mathbf{S}_l^z := \mathbf{S}_{n_l}^{(l)} = \frac{1}{n_l} \sum_{i=1}^{N_l} \hat{\mathbf{z}}_i^{(l)} (\hat{\mathbf{z}}_i^{(l)})'$. We recall the famous Bartlett corrected LRT statistic L proposed by [6], which is given by,

$$L = \frac{2}{n_1 + n_2} \log \left(\frac{|\mathbf{S}_1^z|^{\frac{n_1}{2}} \cdot |\mathbf{S}_2^z|^{\frac{n_2}{2}}}{|c_1 \mathbf{S}_1^z + c_2 \mathbf{S}_2^z|^{\frac{n_1+n_2}{2}}} \right).$$

Here, and in the following, we denote $c_1 = n_1/(n_1 + n_2)$ and $c_2 = n_2/(n_1 + n_2)$. Moreover, under the null hypothesis of (1.1) and linear transformation model (1.2), it is not difficult to rewrite this statistic as

$$L = \frac{2}{n_1 + n_2} \log \left(\frac{|\mathbf{S}_1^x|^{\frac{n_1}{2}} \cdot |\mathbf{S}_2^x|^{\frac{n_2}{2}}}{|c_1 \mathbf{S}_1^x + c_2 \mathbf{S}_2^x|^{\frac{n_1+n_2}{2}}} \right),$$

where $\mathbf{S}_l^{\mathbf{x}} = \frac{1}{n_l} \sum_{i=1}^{N_l} (\mathbf{x}_i^{(l)} - \frac{1}{N_l} \sum_{i=1}^{N_l} \mathbf{x}_i^{(l)}) (\mathbf{x}_i^{(l)} - \frac{1}{N_l} \sum_{i=1}^{N_l} \mathbf{x}_i^{(l)})'$, $l = 1, 2$. Thus we know that L is independent with the population means $\boldsymbol{\mu}_l$ and covariance matrices Σ_l under the null hypothesis H_0 . In the following, we drop the superscripts of \mathbf{S} and denote $\mathbf{S}_1 := \mathbf{S}_1^{\mathbf{x}}$ and $\mathbf{S}_2 := \mathbf{S}_2^{\mathbf{x}}$ for simplicity. In addition, by simple calculation, we can rewrite

$$L = c_1 \log(c_1^{-1} |\mathbf{B}_n|) + c_2 \log(c_2^{-1} |\mathbf{I}_p - \mathbf{B}_n|),$$

where

$$\mathbf{B}_n = n_1 \mathbf{S}_1 (n_1 \mathbf{S}_1 + n_2 \mathbf{S}_2)^{-1}.$$

We now analyze L . If $p > n_1$ or $p > n_2$, then L is undefined because: (1) if $p \geq n_1 + n_2$, then matrix $n_1 \mathbf{S}_1 + n_2 \mathbf{S}_2$ is singular, which makes the inverse of matrix $n_1 \mathbf{S}_1 + n_2 \mathbf{S}_2$ undefined; and (2) if $p < n_1 + n_2$, i.e., the inverse of $n_1 \mathbf{S}_1 + n_2 \mathbf{S}_2$ is well-defined almost surely (with the fourth moments finite assumption), but $p > n_1$ or $p > n_2$, then the determinant of \mathbf{B}_n or $\mathbf{I}_p - \mathbf{B}_n$ is zero, which makes the logarithm functions undefined. However, from random matrix theory (RMT), we know that if the fourth moment of $x_{ij}^{(l)}$ exists, matrix \mathbf{B}_n almost certainly has $p - n_1$ zero eigenvalues and $p - n_2$ one eigenvalues according to the condition $p > n_1$ and $p > n_2$, respectively (see [1]). Therefore, we can naturally redefine the LRT L by restricting the non-zero and non-one eigenvalues of \mathbf{B}_n , i.e.,

$$\mathcal{L} = \sum_{\lambda_i^{\mathbf{B}_n} \in (0,1)} [c_1 \log \lambda_i^{\mathbf{B}_n} + c_2 \log(1 - \lambda_i^{\mathbf{B}_n})], \quad (1.3)$$

where $\lambda_i^{\mathbf{B}_n}$ denotes the i -th smallest eigenvalue of \mathbf{B}_n . It is clear that \mathcal{L} is invariant with respect to transformations

$$\mathbf{z}_j^{*(l)} = \mathbf{C} \mathbf{z}_j^{(l)} + \boldsymbol{\nu}_l,$$

where \mathbf{C} is nonsingular. Therefore, we only need to obtain the asymptotic distributions of the redefined LRT in (1.3), which is addressed in the next section.

This paper also considers the test statistic \tilde{L} ,

$$\tilde{L} = \log |n_1 \mathbf{S}_1^{\mathbf{z}} (n_1 \mathbf{S}_1^{\mathbf{z}} + n_2 \mathbf{S}_2^{\mathbf{z}})^{-1}|,$$

which was proposed in [19] and discussed in [16, 17]. As \tilde{L} is the first term of L , thus it can be viewed as a lite LRT. Similar to \mathcal{L} , we redefine \tilde{L} by

$$\tilde{\mathcal{L}} = \sum_{\lambda_i^{\mathbf{B}_n} \in (0,1)} \log \lambda_i^{\mathbf{B}_n}. \quad (1.4)$$

Unlike \mathcal{L} , it is obvious that $\tilde{\mathcal{L}}$ is monotone for matrix \mathbf{B}_n . That means under the alternative $\Sigma_1 > \Sigma_2$ or $\Sigma_1 < \Sigma_2$, all the eigenvalues $\{\lambda_i^{\mathbf{B}_n} | \lambda_i^{\mathbf{B}_n} \in (0, 1)\}$ become bigger or smaller than that under the null hypothesis. Thus, $\tilde{\mathcal{L}}$ becomes bigger or smaller correspondingly. However, it is uncertain for \mathcal{L} . Therefore, $\tilde{\mathcal{L}}$ should be more powerful than \mathcal{L} in the two cases.

The rest of this paper is organized as follows. The main results are presented in Section 2, including the asymptotic normality of \mathcal{L} and $\tilde{\mathcal{L}}$ and their necessary conditions, which stand for the modification of LRT cannot be improved anymore in our model settings. In Section 3, we present the simulation results for the proposed statistics by comparison with that proposed by [14] and [7]. In Section 5, we introduce a real data application to demonstrate the application of the proposed tests. All technical details are relegated to the appendix. We note that for the high-dimensional testing problem (1.1), the exact distribution of the test statistic is difficult to obtain when the distributions are arbitrary.

2. Main results

In this section, we give the asymptotic distributions of the redefined LRT in (1.3) and (1.4). For the application, we also present consistent estimators for the fourth moments of the samples under the null hypothesis. Before presenting the main results, we give some notation and the necessary assumptions. In the following, we denote the indicator function by $\delta_{(\cdot)}$, the natural logarithm function by $\log(\cdot)$, convergence in distribution by \xrightarrow{D} , and

$$\begin{aligned} y_1 &:= y_{n_1} = p/n_1, & y_2 &:= y_{n_2} = p/n_2, & h &:= h_n = \sqrt{y_1 + y_2 - y_1 y_2}, \\ c_1 &:= n_1/(n_1 + n_2) = y_2/(y_1 + y_2), & c_2 &:= n_2/(n_1 + n_2) = y_1/(y_1 + y_2), \\ l(y_1, y_2) &= \log\left(h \frac{2c_1 h^2}{y_1 y_2}\right) \delta_{y_1 > 1} - \log\left(y_1^{\frac{c_1(1+y_2)}{y_2}} y_2^{\frac{c_1(1-y_1)}{y_1}}\right) \delta_{y_1 > 1}, \\ u(y_1, y_2) &= \log\left(\frac{y_1^{c_1}}{h^{c_1}}\right) \delta_{y_1 > 1}, & v(y_1, y_2) &= \log\left(\frac{y_1^{2c_1}}{h^{2c_1(c_1+2c_2)}}\right) \delta_{y_1 > 1}, \\ \Psi(y_1, y_2) &= c_2 y_1^2 [y_2^4 \delta_{y_2 < 1} + h^2 (2y_2^2 - h^2) \delta_{y_2 > 1}] \\ &\quad - c_1 y_2^2 [y_1^3 (y_1 + 2y_2) \delta_{y_1 < 1} + h^2 (y_1 + y_2 + y_1 y_2) \delta_{y_1 > 1}]. \end{aligned}$$

We set two assumptions of the sample that will be shown to be necessary for the proposed test statistics.

- (Moments Assumption:) $\mathbb{E}x_{11}^{(1)} = \mathbb{E}x_{11}^{(2)} = 0, \mathbb{E}(x_{11}^{(1)})^2 = \mathbb{E}(x_{11}^{(2)})^2 = 1, \mathbb{E}(x_{11}^{(1)})^4 = \Delta_1 + 3 < \infty$ and $\mathbb{E}(x_{11}^{(2)})^4 = \Delta_2 + 3 < \infty$;
- (Dimensions Assumption:) $y_1 \neq 1, y_2 \neq 1$ and $p/(n_1 + n_2) < 1$.

We are now ready to present the main results of this paper.

Theorem 2.1. *In addition to the Moments Assumption and Dimensions Assumption, we assume that as $\min\{p, n_1, n_2\} \rightarrow \infty, \lim y_1 \notin \{0, 1\}, \lim y_2 \notin \{0, 1\}$ and $\lim p/(n_1 + n_2) \in (0, 1)$. Then under the null hypothesis, we have*

$$T := \frac{\mathcal{L} - p\ell_n - \mu_n}{\nu_n} \xrightarrow{D} N(0, 1), \tag{2.1}$$

where

$$\ell_n = \log\left(\frac{y_1^{c_2} y_2^{c_1} h^{\frac{2h^2}{y_1 y_2}}}{(y_1 + y_2)^{\frac{(y_1+y_2)}{y_1 y_2}} |1 - y_1|^{\frac{c_1|1-y_1|}{y_1}} |1 - y_2|^{\frac{c_2|1-y_2|}{y_2}}}\right) - l(y_1, y_2) - l(y_2, y_1),$$

$$\begin{aligned} \mu_n = & \log \left[\frac{(y_1 + y_2)^{\frac{1}{2}} |1 - y_1|^{\frac{c_1}{2}} |1 - y_2|^{\frac{c_2}{2}}}{h} \right] - u(y_1, y_2) - u(y_2, y_1) \\ & + \frac{\Delta_1 \Psi(y_1, y_2)}{2y_1 y_2^2 (y_1 + y_2)^2} + \frac{\Delta_2 \Psi(y_2, y_1)}{2y_2 y_1^2 (y_1 + y_2)^2}, \end{aligned}$$

and

$$\begin{aligned} \nu_n^2 = & \log \frac{h^4}{|1 - y_1|^{2c_1^2} |1 - y_2|^{2c_2^2} (y_1 + y_2)^2} + 2[v(y_1, y_2) + v(y_2, y_1) \\ & + \log(h^{4c_1 c_2}) \delta_{y_1 > 1} \delta_{y_2 > 1}] \\ & + \frac{(y_1 \Delta_1 + y_2 \Delta_2)}{y_1^2 y_2^2 (y_1 + y_2)^2} [(y_1 - 1) y_2^2 \delta_{y_1 > 1} - (y_2 - 1) y_1^2 \delta_{y_2 > 1}]^2. \end{aligned}$$

According to the above theorem, we can easily conclude the following corollary under normal circumstances:

Corollary 2.2. *If $z_j^{(l)}$, $j = 1, \dots, n_l$, $l = 1, 2$ are normally distributed, then (2.1) in Theorem 2.1 reduces to:*

$$T := \frac{\mathcal{L} - p\ell_n - \mu_n}{\nu_n} \xrightarrow{D} N(0, 1),$$

where

$$\ell_n = \log \left(\frac{y_1^{c_2} y_2^{c_1} h^{\frac{2h^2}{y_1 y_2}}}{(y_1 + y_2)^{\frac{(y_1 + y_2)}{y_1 y_2}} |1 - y_1|^{\frac{c_1 |1 - y_1|}{y_1}} |1 - y_2|^{\frac{c_2 |1 - y_2|}{y_2}}} \right) - l(y_1, y_2) - l(y_2, y_1),$$

$$\mu_n = \log \left[\frac{(y_1 + y_2)^{\frac{1}{2}} |1 - y_1|^{\frac{c_1}{2}} |1 - y_2|^{\frac{c_2}{2}}}{h} \right] - u(y_1, y_2) - u(y_2, y_1),$$

and

$$\begin{aligned} \nu_n^2 = & \log \frac{h^4}{|1 - y_1|^{2c_1^2} |1 - y_2|^{2c_2^2} (y_1 + y_2)^2} \\ & + 2[v(y_1, y_2) + v(y_2, y_1) + \log(h^{4c_1 c_2}) \delta_{y_1 > 1} \delta_{y_2 > 1}]. \end{aligned}$$

Remark 2.3. *If the Dimensions Assumption in Theorem 2.1 is satisfied, the limit condition that $\lim y_1 \notin \{0, 1\}$, $\lim y_2 \notin \{0, 1\}$ and $\lim p/(n_1 + n_2) \in (0, 1)$ could be considered to hold, because there is no information for the convergence of the dimensions and sizes for any dataset. In addition, [12] showed that if $p/n_l \rightarrow 1$, Theorem 2.1 also holds for normally distributed data. However, if y_l is near 1, then the variance ν_n will be large and the LRT will become unstable, see Figure 1 for illustration.*

Remark 2.4. *When $y_1 < 1$ and $y_2 < 1$, Corollary 2.2 recovers Theorem 4.1 in [5] directly.*

The necessity of the Dimensions Assumption is clear because of the definitions of ℓ_n , μ_n and ν_n . For the Moments Assumption, we only need to consider the fourth moments of the sample. From the variance ν_n in Theorem 2.1, we know that its fourth-moment term cannot be removed except $y_1 < 1$ and $y_2 < 1$. However, if $y_1 < 1$ and $y_2 < 1$, it is not difficult to obtain that

$$\Psi(y_2, y_1) = \Psi(y_1, y_2) = -y_1^3 y_2^3,$$

which implies that the fourth-moment term of μ_n cannot be removed. Therefore, we conclude that the existence of the fourth moment of the sample is necessary for the modified LRT statistic \mathcal{L} . However, when y_1 or y_2 is close to 1, the variance ν_n will increase rapidly, resulting in poor power. For illustration, we present two 3D figures of μ_n and ν_n^2 with $\Delta_1 = \Delta_2 = 0$ in Figure 1.

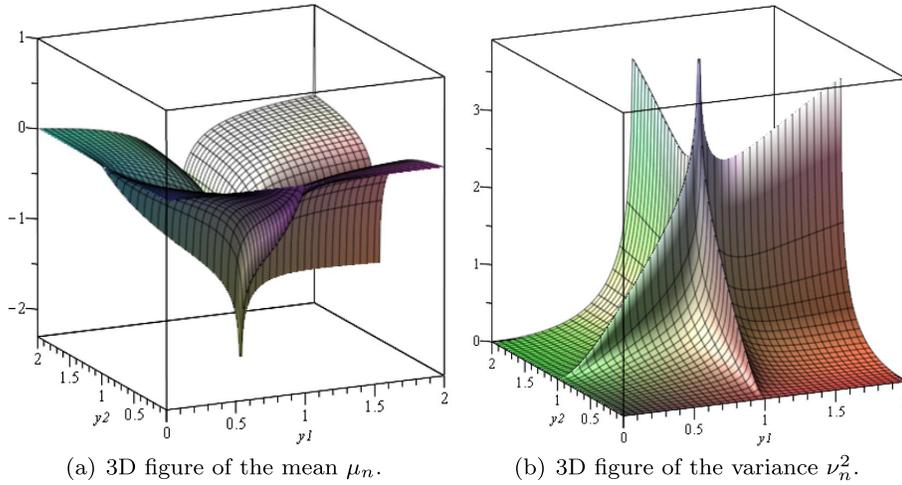


FIG 1. This figure was made using Maple software with $y_1 \in (0, 2)$ and $y_2 \in (0, 2)$. The vertical axes present the values of μ_n and ν_n , respectively.

Now, we give the asymptomatic distribution of $\tilde{\mathcal{L}}$.

Theorem 2.5. *In addition to the Moments Assumption and Dimensions Assumption, we assume that as $\min\{p, n_1, n_2\} \rightarrow \infty$, $\lim y_1 \neq 1$, $\lim y_2 \neq 1$ and $\lim p/(n_1 + n_2) < 1$. Then under the null hypothesis, we have*

$$\tilde{T} := \frac{\tilde{\mathcal{L}} - p\tilde{\ell}_n - \tilde{\mu}_n}{\tilde{\nu}_n} \xrightarrow{D} N(0, 1),$$

where

$$\tilde{\ell}_n = \log \left(\frac{y_2 h^{\frac{2h^2}{y_1 y_2}}}{(y_1 + y_2)^{\frac{(y_1 + y_2)}{y_1 y_2}} |1 - y_1|^{\frac{|1 - y_1|}{y_1}}} \right) - \log \left(\frac{h^{\frac{2h^2}{y_1 y_2}}}{y_1^{\frac{1 + y_2}{y_2}} y_2^{\frac{1 - y_1}{y_1}}} \right) \delta_{y_1 > 1}$$

$$\begin{aligned} \tilde{\mu}_n = & \log \left[\frac{(y_1 + y_2)^{\frac{1}{2}} |1 - y_1|^{\frac{1}{2}}}{h} \right] - \log \left(\frac{y_1}{h} \right) \delta_{y_1 > 1} \\ & - \frac{\Delta_1 [y_1^3 (y_1 + 2y_2) \delta_{y_1 < 1} + h^2 (y_1 + y_2 + y_1 y_2) \delta_{y_1 > 1}]}{2y_1 (y_1 + y_2)^2} \\ & + \frac{\Delta_2 [y_1^4 y_2 \delta_{y_1 < 1} + h^2 y_2 (2y_1^2 - h^2) \delta_{y_1 > 1}]}{2y_1^2 (y_1 + y_2)^2}, \end{aligned}$$

and

$$\begin{aligned} \tilde{\nu}_n^2 = & 2 \log \frac{h^2}{|1 - y_1| (y_1 + y_2)} + 2 \log \left(\frac{y_1^2}{h^2} \right) \delta_{y_1 > 1} \\ & + \frac{(y_1 \Delta_1 + y_2 \Delta_2)}{y_1^2 (y_1 + y_2)^2} [y_1^4 \delta_{y_1 < 1} + h^4 \delta_{y_1 > 1}]. \end{aligned}$$

Remark 2.6. Notice that the asymptotic distributions in Theorem 2.1 and Theorem 2.5 are obtained under the null hypothesis, which can only guarantee the Type I errors. For the powers, that is under the alternative hypothesis $\Sigma_1 \neq \Sigma_2$, the asymptotic distributions of statistics \mathcal{L} and $\tilde{\mathcal{L}}$ will depend on the eigenvalues of $\Sigma_1 \Sigma_2^{-1}$. In this case, if the dimension p is smaller than either of the two sample sizes, the power functions for T and \tilde{T} can be obtained by the CLT of the general Fisher matrices which is derived by [22]. However, if p is bigger than both of the two sample sizes, because of the lack of theoretical results about the general Beta matrix $n_1 \mathbf{S}_1 (n_1 \mathbf{S}_1 + n_2 \mathbf{A}^{1/2} \mathbf{S}_2 \mathbf{A}^{1/2})^{-1}$, the asymptotic distributions of statistics \mathcal{L} and $\tilde{\mathcal{L}}$ are also open problems and will be left for our future work. Here \mathbf{A} is any non-random symmetric matrix.

If $\Delta_l \neq 0$, or more specifically, if the samples are not normally distributed, the estimates for Δ_l are necessary for the test application. Thus, we obtain their consistent estimators using the method of moments and random matrix theory. Let

$$\begin{aligned} \hat{\Delta}_1 = & (1 - y)^2 \frac{\sum_{j=1}^{N_1} [(\mathbf{z}_j^{(1)} - \bar{\mathbf{z}}^{(1)})' (c_{11} \mathbf{S}_{1j}^z + c_{12} \mathbf{S}_{2j}^z)^{-1} (\mathbf{z}_j^{(1)} - \bar{\mathbf{z}}^{(1)}) - \frac{p}{1-y}]^2}{pN_1} \\ & - \frac{2}{1-y}, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \hat{\Delta}_2 = & (1 - y)^2 \frac{\sum_{j=1}^{N_2} [(\mathbf{z}_j^{(2)} - \bar{\mathbf{z}}^{(2)})' (c_{21} \mathbf{S}_1^z + c_{22} \mathbf{S}_{2j}^z)^{-1} (\mathbf{z}_j^{(2)} - \bar{\mathbf{z}}^{(2)}) - \frac{p}{1-y}]^2}{pN_2} \\ & - \frac{2}{1-y}, \end{aligned} \quad (2.3)$$

where $y = \frac{p}{n_1 + n_2 - 1}$, $c_{11} = \frac{n_1 - 1}{n_1 + n_2 - 1}$, $c_{12} = \frac{n_2}{n_1 + n_2 - 1}$, $c_{21} = \frac{n_1}{n_1 + n_2 - 1}$, $c_{22} = \frac{n_2 - 1}{n_1 + n_2 - 1}$ and \mathbf{S}_{lj}^z is the sample covariance matrix by removing the vector $\mathbf{z}_j^{(l)}$ from the l -th sample, $l = 1, 2$.

Theorem 2.7. Under the same assumptions of Theorem 2.1 and under the null hypothesis, we have that the estimators $\hat{\Delta}_l$, $l = 1, 2$, defined in (2.2) and (2.3), are weakly consistent and asymptotically unbiased.

Remark 2.8. Actually, $(c_{11}\mathbf{S}_{1j}^z + c_{12}\mathbf{S}_2^z)^{-1}$ in (2.2) is used to avoid the case where the sample sizes are not larger than the dimension, which gives rise to the singularity of the sample covariance matrix. Otherwise, if the dimension p is smaller than some sample size, say, $p < n_1$, we can estimate Δ_1 by replacing the term $(c_{11}\mathbf{S}_{1j}^z + c_{12}\mathbf{S}_2^z)^{-1}$ and $y = \frac{p}{n_1+n_2-1}$ in (2.2) by $(\mathbf{S}_{1j}^z)^{-1}$ and $y = \frac{p}{n_1-1}$ respectively. The estimator $\hat{\Delta}_2$ in (2.3) can be treated analogously.

The proofs of Theorem 2.1, Theorem 2.5 Theorem 2.7 are given in the appendix.

3. Results of the simulation

In this section, we compare the performance of the statistics proposed in [14] and [7] and our modified LRTs \mathcal{L} and $\tilde{\mathcal{L}}$ under various settings of sample size and dimensionality. The classical LRT statistic in [20] was shown to have poor performance for (1.1) by [5]; thus, it will not be considered in this section. Without loss of generality, we assume $\boldsymbol{\mu}_l = 0$ and set

$$\Sigma_1 = (1 + a/n_1)\Sigma_2,$$

where a is a constant. The samples are drawn from the following distributions:

- Case 1: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are both standard normal distributed and $\Sigma_2 = \mathbf{I}_p$;
- Case 2: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are from the uniform distribution $U(-\sqrt{3}, \sqrt{3})$ and $\Sigma_2 = \mathbf{I}_p$;
- Case 3: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are from the uniform distribution $U(-\sqrt{3}, \sqrt{3})$ and

$$\Sigma_2 = \text{Diag}(p^2, 1, \dots, 1);$$

- Case 4: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are from the uniform distribution $U(-\sqrt{3}, \sqrt{3})$ and

$$\Sigma_2 = (0.5\mathbf{I}_p + 0.5\mathbf{1}_p\mathbf{1}_p').$$

Here, $\mathbf{1}_p$ represents a p -dimensional vector with all entries 1. The results are obtained based on 10,000 replicates. In the tables, T and \tilde{T} denote the proposed modified LRTs, T_{lc} denotes the nonparametric test of [14] and T_{clx} denotes the maximum element test of [7].

In the first part of this section, we report the results by assuming the forth moments of the $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are known. Tables 1-4 present the empirical sizes and empirical powers of Cases 1-4. Additionally, we provide four figures (Figures 2-5) to show the divergence of powers of the four test statistics as the parameter a increases. The results indicate that the sizes T and \tilde{T} perform quite well for all cases. However, in Case 3 and Case 4, the sizes of T_{lc} and T_{clx} are not accurate, which reflects the fact that the null distributions of the test statistics T_{lc} and T_{clx} are not well approximated by their asymptotic distributions in these cases. For the empirical power, we can conclude that T and \tilde{T} are more sensitive than T_{lc} and T_{clx} when at least one of the sample sizes is smaller than the dimensions.

TABLE 1
Empirical sizes and empirical powers of the tests T , \hat{T} , T_{lc} and T_{clx} in Case 1. These results are based on the 5% significance level.

(n_1, n_2, p)	(25,35,40)			(50,70,80)			(100,140,160)			(200,280,320)		
	size		power	size		power	size		power	size		power
	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$
$y_1 > 1, y_2 > 1$												
T	0.055	0.951	1	0.053	0.951	1	0.054	0.951	1	0.051	0.946	1
\hat{T}	0.056	0.999	1	0.048	1	1	0.048	1	1	0.049	1	1
T_{lc}	0.074	0.804	1	0.057	0.489	0.999	0.054	0.208	0.954	0.054	0.093	0.571
T_{clx}	0.082	0.152	0.591	0.057	0.071	0.343	0.048	0.049	0.120	0.042	0.046	0.062
(n_1, n_2, p)	(25,35,30)			(50,70,60)			(100,140,120)			(200,280,240)		
$y_1 > 1, y_2 < 1$												
T	0.059	0.685	0.999	0.057	0.618	0.998	0.054	0.558	0.994	0.052	0.523	0.987
\hat{T}	0.058	0.997	1	0.053	0.999	1	0.050	0.999	1	0.049	1	1
T_{lc}	0.063	0.793	1	0.055	0.494	0.999	0.053	0.205	0.952	0.050	0.098	0.574
T_{clx}	0.081	0.154	0.621	0.054	0.076	0.380	0.049	0.048	0.147	0.045	0.046	0.068
(n_1, n_2, p)	(35,25,30)			(70,50,60)			(140,100,120)			(280,200,240)		
$y_1 < 1, y_2 > 1$												
T	0.060	0.109	0.075	0.055	0.167	0.282	0.050	0.213	0.515	0.050	0.248	0.645
\hat{T}	0.056	0.956	1	0.052	0.985	1	0.052	0.989	1	0.046	0.994	1
T_{lc}	0.068	0.314	0.992	0.059	0.143	0.896	0.054	0.077	0.489	0.051	0.054	0.179
T_{clx}	0.077	0.203	0.677	0.054	0.102	0.433	0.049	0.061	0.189	0.042	0.051	0.076
(n_1, n_2, p)	(25,35,20)			(50,70,40)			(100,140,80)			(200,280,160)		
$y_1 < 1, y_2 < 1$												
T_1	0.060	0.251	0.960	0.056	0.119	0.715	0.050	0.079	0.330	0.049	0.053	0.139
\hat{T}	0.057	0.996	1	0.052	0.999	1	0.049	1	1	0.050	0.999	1
T_{lc}	0.069	0.783	1	0.059	0.476	0.999	0.051	0.211	0.948	0.049	0.095	0.563
T_{clx}	0.076	0.169	0.679	0.058	0.083	0.438	0.046	0.053	0.180	0.047	0.050	0.072

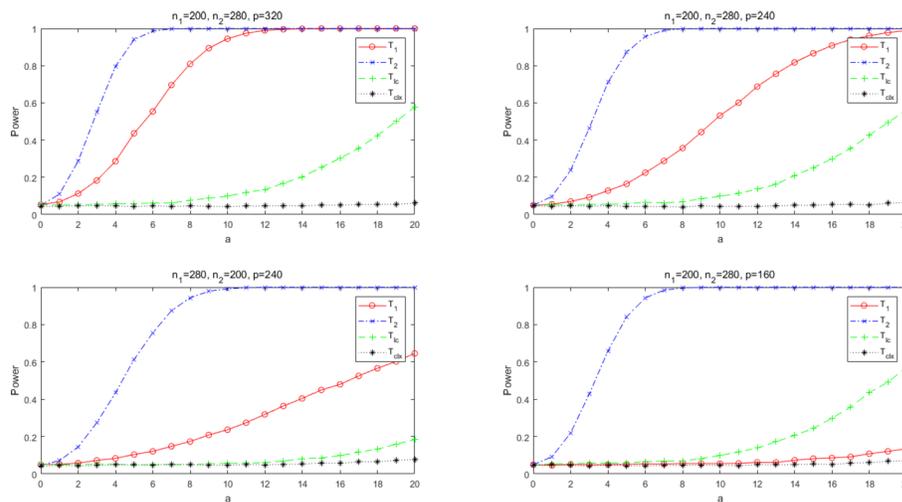


FIG 2. Graphs of the divergence of the four powers in Case 1.

TABLE 2
 Empirical sizes and empirical powers of the tests T , T , T_{lc} and T_{clx} in Case 2. These results are based on the 5% significance level.

(n_1, n_2, p) $y_1 > 1, y_2 > 1$	(25,35,40)			(50,70,80)			(100,140,160)			(200,280,320)		
	size		power	size		power	size		power	size		power
	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$
T	0.052	0.961	1	0.046	0.953	1	0.046	0.950	1	0.048	0.948	1
\bar{T}	0.054	1	1	0.050	1	1	0.049	1	1	0.048	1	1
T_{lc}	0.055	0.815	1	0.056	0.486	0.999	0.050	0.209	0.959	0.050	0.093	0.579
T_{clx}	0.191	0.858	1	0.125	0.567	1	0.086	0.207	0.996	0.070	0.095	0.663
(n_1, n_2, p) $y_1 > 1, y_2 < 1$	(25,35,30)			(50,70,60)			(100,140,120)			(200,280,240)		
	size		power	size		power	size		power	size		power
	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$
T	0.056	0.701	0.999	0.050	0.610	0.998	0.047	0.563	0.994	0.053	0.517	0.986
\bar{T}	0.057	0.999	1	0.052	1	1	0.051	1	1	0.050	1	1
T_{lc}	0.061	0.818	1	0.053	0.481	0.999	0.050	0.205	0.956	0.050	0.093	0.573
T_{clx}	0.163	0.855	1	0.112	0.581	1	0.087	0.228	0.997	0.068	0.095	0.679
(n_1, n_2, p) $y_1 < 1, y_2 > 1$	(35,25,30)			(70,50,60)			(140,100,120)			(280,200,240)		
	size		power	size		power	size		power	size		power
	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$
T	0.056	0.108	0.071	0.052	0.179	0.272	0.058	0.225	0.503	0.055	0.247	0.654
\bar{T}	0.058	0.977	1	0.053	0.993	1	0.053	0.997	1	0.051	0.998	1
T_{lc}	0.058	0.307	0.995	0.050	0.137	0.904	0.052	0.072	0.488	0.049	0.051	0.177
T_{clx}	0.169	0.755	1	0.117	0.409	0.998	0.087	0.156	0.858	0.066	0.081	0.303
(n_1, n_2, p) $y_1 < 1, y_2 < 1$	(25,35,20)			(50,70,40)			(100,140,80)			(200,280,160)		
	size		power	size		power	size		power	size		power
	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$
T	0.053	0.245	0.976	0.055	0.117	0.724	0.050	0.072	0.330	0.051	0.055	0.134
\bar{T}	0.061	0.999	1	0.056	1	1	0.050	1	1	0.047	1	1
T_{lc}	0.052	0.813	1	0.057	0.490	0.999	0.052	0.204	0.957	0.051	0.093	0.568
T_{clx}	0.146	0.845	1	0.106	0.583	1	0.078	0.237	0.996	0.064	0.098	0.706

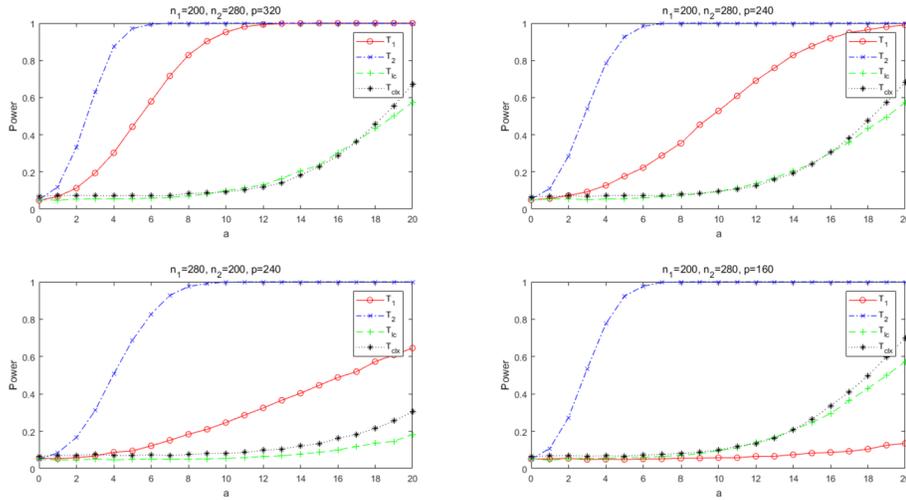


FIG 3. Graphs of the divergence of the four powers in Case 2.

TABLE 3
 Empirical sizes and empirical powers of the tests T , \hat{T} , T_{lc} and T_{clx} in Case 3. These results are based on the 5% significance level.

(n_1, n_2, p)	(25,35,40)			(50,70,80)			(100,140,160)			(200,280,320)		
	size		power	size		power	size		power	size		power
	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$
$y_1 > 1, y_2 > 1$												
T	0.051	0.960	1	0.046	0.954	1	0.046	0.949	1	0.048	0.949	1
\hat{T}	0.054	1	1	0.050	1	1	0.049	1	1	0.048	1	1
T_{lc}	0.020	0.618	0.982	0.017	0.409	0.934	0.016	0.219	0.746	0.016	0.106	0.455
T_{clx}	0.191	0.861	1	0.125	0.568	1	0.086	0.210	0.996	0.070	0.095	0.662
(n_1, n_2, p)	(25,35,30)			(50,70,60)			(100,140,120)			(200,280,240)		
$y_1 > 1, y_2 < 1$												
T	0.056	0.702	0.999	0.051	0.610	0.998	0.046	0.564	0.994	0.052	0.517	0.986
\hat{T}	0.058	0.999	1	0.053	1	1	0.052	1	1	0.049	1	1
T_{lc}	0.019	0.613	0.983	0.011	0.407	0.934	0.016	0.216	0.755	0.015	0.103	0.458
T_{clx}	0.162	0.856	1	0.112	0.579	1	0.087	0.229	0.997	0.068	0.094	0.679
(n_1, n_2, p)	(35,25,30)			(70,50,60)			(140,100,120)			(280,200,240)		
$y_1 < 1, y_2 > 1$												
T	0.057	0.106	0.071	0.052	0.179	0.274	0.057	0.223	0.502	0.054	0.249	0.648
\hat{T}	0.060	0.978	1	0.053	0.992	1	0.053	0.997	1	0.052	0.998	1
T_{lc}	0.019	0.313	0.869	0.016	0.185	0.667	0.018	0.098	0.408	0.017	0.056	0.220
T_{clx}	0.168	0.757	1	0.117	0.407	0.999	0.088	0.157	0.855	0.067	0.081	0.305
(n_1, n_2, p)	(25,35,20)			(50,70,40)			(100,140,80)			(200,280,160)		
$y_1 < 1, y_2 < 1$												
T	0.053	0.250	0.975	0.054	0.119	0.725	0.050	0.071	0.327	0.052	0.054	0.136
\hat{T}	0.060	0.999	1	0.056	1	1	0.050	1	1	0.048	1	1
T_{lc}	0.019	0.629	0.983	0.016	0.403	0.930	0.015	0.212	0.752	0.015	0.113	0.445
T_{clx}	0.147	0.847	1	0.106	0.581	1	0.079	0.237	0.996	0.063	0.098	0.706

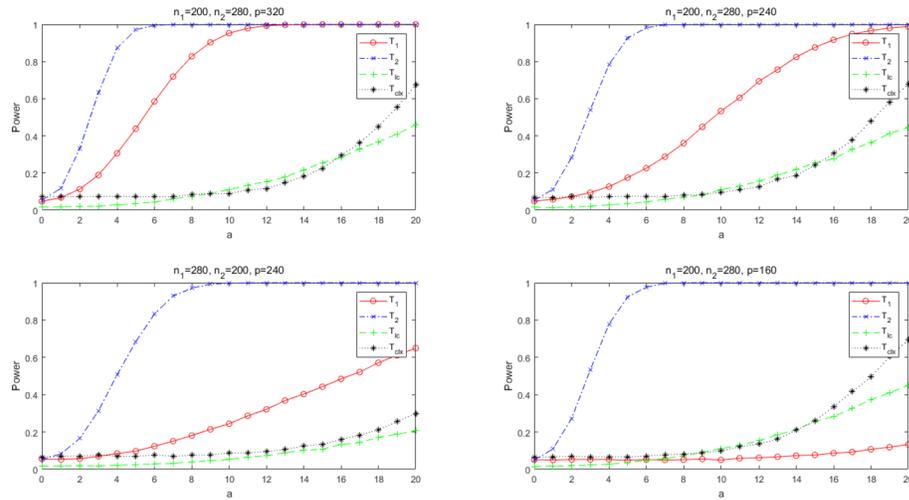


FIG 4. Graphs of the divergence of the four powers in Case 3.

TABLE 4
 Empirical sizes and empirical powers of the tests T , \hat{T} , T_{lc} and T_{clx} in Case 4. These results are based on the 5% significance level.

(n_1, n_2, p)	(25,35,40)			(50,70,80)			(100,140,160)			(200,280,320)		
	size		power	size		power	size		power	size		power
	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$	$a=0$	$a=10$	$a=20$
$y_1 > 1, y_2 > 1$												
T	0.052	0.961	1	0.046	0.954	1	0.045	0.950	1	0.047	0.949	1
\hat{T}	0.052	1	1	0.049	1	1	0.049	1	1	0.048	1	1
T_{lc}	0.083	0.505	0.909	0.081	0.348	0.793	0.087	0.240	0.587	0.082	0.179	0.386
T_{clx}	0.062	0.207	0.778	0.044	0.076	0.492	0.030	0.032	0.187	0.022	0.020	0.068
$y_1 > 1, y_2 < 1$												
T	0.055	0.701	0.999	0.050	0.611	0.998	0.046	0.565	0.994	0.053	0.518	0.987
\hat{T}	0.057	0.999	1	0.053	1	1	0.052	1	1	0.049	1	1
T_{lc}	0.089	0.517	0.918	0.083	0.359	0.786	0.083	0.235	0.596	0.081	0.172	0.395
T_{clx}	0.068	0.230	0.818	0.043	0.085	0.539	0.031	0.036	0.221	0.021	0.023	0.078
$y_1 < 1, y_2 > 1$												
T	0.057	0.107	0.069	0.051	0.178	0.276	0.057	0.224	0.506	0.054	0.248	0.649
\hat{T}	0.060	0.979	1	0.054	0.993	1	0.053	0.997	1	0.051	0.998	1
T_{lc}	0.084	0.310	0.736	0.083	0.229	0.563	0.081	0.170	0.375	0.084	0.125	0.247
T_{clx}	0.065	0.272	0.777	0.042	0.145	0.487	0.029	0.084	0.244	0.021	0.048	0.112
$y_1 < 1, y_2 < 1$												
T	0.052	0.245	0.975	0.055	0.119	0.724	0.050	0.072	0.329	0.051	0.055	0.135
\hat{T}	0.061	0.999	1	0.056	1	1	0.050	1	1	0.048	1	1
T_{lc}	0.084	0.518	0.924	0.079	0.362	0.792	0.083	0.249	0.595	0.081	0.172	0.398
T_{clx}	0.072	0.273	0.862	0.043	0.107	0.608	0.032	0.044	0.261	0.021	0.025	0.092

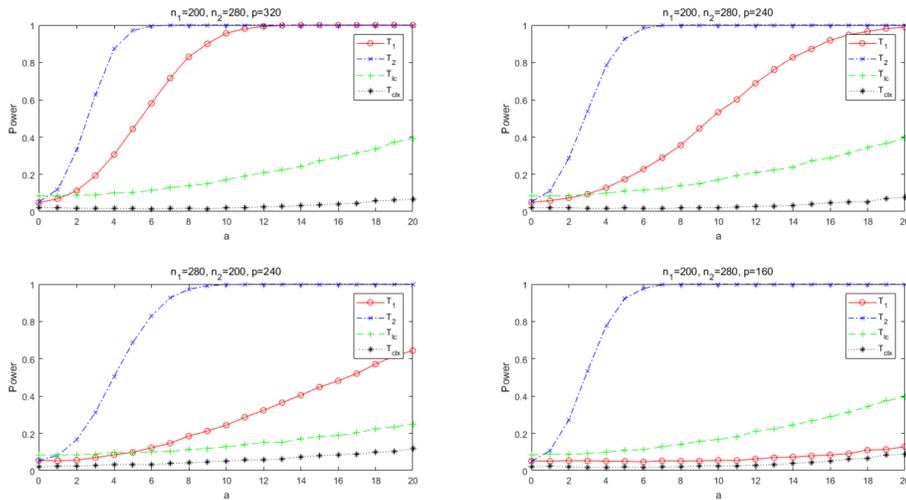


FIG 5. Graphs of the divergence of the four powers in Case 4.

TABLE 5
 Numerical results for the estimator $\hat{\Delta}_1$ of standard normal distribution. Here the true value $\Delta_1 = 0$.

(200,280)			(400,560)			(800,1120)		
p	mean	variance	p	mean	variance	p	mean	variance
2	0.0186	0.1368	4	0.0099	0.0441	8	0.0049	0.0160
10	0.0197	0.0642	20	0.0081	0.0260	40	0.0034	0.0116
20	0.0193	0.0563	40	0.0107	0.0252	80	0.0040	0.0116
100	0.0310	0.0732	200	0.0121	0.0338	400	0.0057	0.0166
180	0.0552	0.1189	360	0.0223	0.0555	720	0.0135	0.0261
220	0.0719	0.1634	440	0.0332	0.0736	880	0.0137	0.0355
240	0.0739	0.1887	480	0.0395	0.0870	960	0.0190	0.0424
300	0.1491	0.3568	600	0.0695	0.1598	1200	0.0373	0.0771

TABLE 6
 Numerical results for the estimator $\hat{\Delta}_1$ of uniform distribution $(-\sqrt{3}, \sqrt{3})$. Here the true value $\Delta_1 = -1.2$.

(200,280)			(400,560)			(800,1120)		
p	mean	variance	p	mean	variance	p	mean	variance
2	-1.2021	0.0058	4	-1.2020	0.0032	8	-1.2012	0.0015
10	-1.2009	0.0073	20	-1.2002	0.0036	40	-1.2018	0.0019
20	-1.2005	0.0081	40	-1.2001	0.0039	80	-1.1994	0.0020
100	-1.1971	0.0179	200	-1.1977	0.0091	400	-1.2001	0.0044
180	-1.1858	0.0450	360	-1.1892	0.0213	720	-1.1957	0.0105
220	-1.1704	0.0721	440	-1.1894	0.0314	880	-1.1938	0.0159
240	-1.1650	0.0902	480	-1.1829	0.0417	960	-1.1892	0.0204
300	-1.1170	0.2170	600	-1.1537	0.0974	1200	-1.1760	0.0445

Otherwise, the LRT T does not perform as well when the dimensions are large. However, the lite LRT \tilde{T} is always powerful because of its monotonicity for matrix \mathbf{B}_n , which coincides with our intuition.

Next we will show the performance of the estimator we proposed in Theorem 2.7. Here we have to explain the reason that why we did not use the estimators in the previous simulation results. Because our estimator is based on the method of moments and requires $n = n_1 + n_2$ loops and inverse process for one replication, and we need 10,000 replications for one result, which makes the running times to be excessive. In the simulation, we set x_{ij} be standard normal distributed and uniform distributed on $(-\sqrt{3}, \sqrt{3})$ to estimate the Δ_1 respectively. Under each circumstance we repeat 10,000 times and the results are reported at Table 5 and Table 6. From the numerical results, the performance of the estimator is remarkable, especially when the sample size is large. Therefore, we believe that the proposed modified LRTs must also perform well under the null hypothesis when using the estimators instead of their true values. But, under the alternative, we can not have any supporting evidence because of lack of theoretical results about the general Beta matrix.

TABLE 7

p-values of the test of equality of the two covariance matrices of daily returns from the Consumer Discretionary (C.D.) sector and Materials sector (M.) in different quarters.

(n_1, n_2, p)	C.D.	C.D.	M.	M.
	(61,62,71)	(60,61,71)	(60,58,22)	(60,61,22)
T	0	0	0.0458	0
\tilde{T}	0	0.0003	0.2214	0
T_{lc}	0	0	0.0632	0
T_{clx}	0.0048	0.0002	0.2053	0.0001

4. An example

For illustration, we apply the proposed test statistics to the daily returns of a selection of stocks issued by companies on Standard & Poor's (S&P) 500. The original data are the closing prices or the bid/ask average of these stocks for the trading days during 2012 and 2013. This dataset is derived from the Center for Research in Security Prices Daily Stock from Wharton Research Data Services. A common interest is to test whether the covariance matrices of the logarithmic daily returns for some stocks are the same over a period of time. Logarithmic daily returns are commonly used in finance. There are several theoretical and practical advantages of using logarithmic daily returns, including that we can assume that the sequences of logarithmic daily returns are independent of each other, see [10].

According to the Global Industry Classification Standard (GICS), which is an industry taxonomy developed in 1999 by MSCI and S&P for use by the global financial community, we select two sectors, Consumer Discretionary and Materials, which include 71 and 22 stocks, respectively. First, for Consumer Discretionary sector, we test whether the covariance matrices of the logarithmic daily returns of the second quarters (1 April-30 June) of 2012 and 2013 are the same. There were 63 and 64 trading days, respectively, in the second quarters of 2012 and 2013. By using the logarithmic difference transformation on the original data, we obtain the logarithmic daily returns dataset with sample sizes $n_1 = 61$ and $n_2 = 62$. Simultaneously, we use the first quarters (1 January-31 March) data of 2012 and 2013 when applying to Materials sector. That makes the sample sizes be $n_1 = 60$ and $n_2 = 58$. The p values obtained by applying the four test statistics T , \tilde{T} , T_{lc} and T_{clx} are shown in the first and third columns of Table 7. Next, we use the same procedure to test whether the covariance matrices of the logarithmic daily returns of the first quarter (1 January-31 March) and the second quarter (1 April-30 June) of 2012 are the same. The results are shown in the second and fourth columns of Table 7. The results show that all the p -values are much smaller than 0.05. Thus, there is strong evidence that the two covariance matrices are different. Therefore, we suggest caution with the assumption that the daily returns are identically distributed.

5. Conclusion and discussion

In this paper, we propose two modified LRTs for the equality of two high-dimensional covariance matrices and show the asymptotic distributions under

the Moments Assumption and the null hypothesis. Furthermore, we show that the Moments Assumption and Dimensions Assumption are necessary for our results. We also present the weakly consistent and asymptotic unbiased estimators of Δ_l for non-Gaussian distributions under the null hypothesis. According to the simulation results, the performances of these modified LRTs are remarkable under conditions where they are applicable, but the theoretical results for the alternative hypothesis are not considered in this paper because of the lack of random matrix theory. The modification of the LRT under the alternative hypothesis test will be presented in the future.

Appendix A: Technical details

In the appendix, we give the proofs of Theorem 2.1, Theorem 2.5 and Theorem 2.7. By comparing the definitions of \mathcal{L} and $\tilde{\mathcal{L}}$, it is easy to verify that the proof of Theorem 2.1 can be split into two parts, one of which is the same as the proof of Theorem 2.5 and the other of which is an analogous analysis. Hence, we first present the proof Theorem 2.5, and then give the rough proof of the other analogue part.

A.1. Proof of Theorem 2.5

The main tool used to prove the theorems is the Cauchy integral formula and Theorem 1.6 in [1], which established the central limit theorem of linear spectral statistics of random matrix \mathbf{B}_n and is presented below for convenience. Denote $\alpha_n = n_2/n_1$, $G_n(x) = p(\mathbf{F}^{\mathbf{B}_n}(x) - \mathbf{F}_{y_1, y_2}(x))$ and

$$\mathbf{F}_{y_1, y_2}(x) = \frac{(\alpha_n + 1)\sqrt{(x_r - x)(x - x_l)}}{2\pi y_1 x(1 - x)} \delta_{x \in (x_l, x_r)}$$

$x_l, x_r = \frac{y_2(h \mp y_1)^2}{(y_1 + y_2)^2}$ be the limit spectral distribution of \mathbf{B}_n with parameters α_n, y_1, y_2 ,

Lemma A.1 (Theorem 1.6 in [1]). *In addition to the Moments Assumption and the Dimensions Assumption, we further assume that:*

(1) *As $\min\{p, n_1, n_2\} \rightarrow \infty$, $y_1 \rightarrow \gamma_1 \in (0, 1) \cup (1, \infty)$, $y_2 \rightarrow \gamma_2 \in (0, 1) \cup (1, \infty)$, and $\alpha_n \rightarrow \alpha > 0$.*

(2) *Let f_1, \dots, f_k be the analytic functions on an open region containing the interval $[a_l, a_r]$, where $a_l = v^{-1}(1 - \sqrt{\gamma_1})^2$, $a_r = 1 - \alpha v^{-1}(1 - \sqrt{\gamma_2})^2$, and v is defined as $v = (1 + \frac{\gamma_1}{\gamma_2})(1 - \sqrt{\frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}})^2$.*

Then, as $\min(n_1, n_2, p) \rightarrow \infty$, the random vector

$$\left(\int f_i dG_n(x) \right), \quad i = 1, \dots, k,$$

converges weakly to a Gaussian vector $(G_{f_1}, \dots, G_{f_k})$ with mean functions

$$\mathbb{E}G_{f_i} = \frac{1}{4\pi i} \oint f_i\left(\frac{z}{\alpha + z}\right) d \log\left(\frac{(1 - \gamma_2)m_3^2(z) + 2m_3(z) + 1 - \gamma_1}{(1 + m_3(z))^2}\right)$$

$$\begin{aligned}
 & + \frac{\Delta_1}{2\pi i} \oint \gamma_1 f_i\left(\frac{z}{\alpha+z}\right) (1+m_3)^{-3} dm_3(z) \\
 & + \frac{\Delta_2}{4\pi i} \oint f_i\left(\frac{z}{\alpha+z}\right) (1-\gamma_2 m_3^2(z) (1+m_3(z))^{-2}) d \log(1-\gamma_2 m_3^2(z) (1+m_3(z))^{-2})
 \end{aligned}$$

and covariance functions

$$\begin{aligned}
 \text{Cov}(G_{f_i}, G_{f_j}) &= -\frac{1}{2\pi^2} \oint \oint \frac{f_i\left(\frac{z_1}{\alpha+z_1}\right) f_j\left(\frac{z_2}{\alpha+z_2}\right) dm_3(z_1) dm_3(z_2)}{(m_3(z_1) - m_3(z_2))^2} \\
 & - \frac{\gamma_1 \Delta_1 + \gamma_2 \Delta_2}{4\pi^2} \oint \oint \frac{f_i\left(\frac{z_1}{\alpha+z_1}\right) f_j\left(\frac{z_2}{\alpha+z_2}\right) dm_3(z_1) dm_3(z_2)}{(m_3(z_1) + 1)^2 (m_3(z_2) + 1)^2},
 \end{aligned}$$

where

$$\begin{aligned}
 m_0(z) &= \frac{(1+\gamma_1)(1-z) - \alpha z(1-\gamma_2)}{2z(1-z)(\gamma_1(1-z) + \alpha z \gamma_2)} \\
 & + \frac{\sqrt{((1-\gamma_1)(1-z) + \alpha z(1-\gamma_2))^2 - 4\alpha z(1-z)}}{2z(1-z)(\gamma_1(1-z) + \alpha z \gamma_2)} - \frac{1}{z}, \\
 m_1(z) &= \frac{\alpha}{(\alpha+z)^2} m_0\left(\frac{z}{\alpha+z}\right) - \frac{1}{\alpha+z}, \quad m_2(z) = -z^{-1}(1-\gamma_1) + \gamma_1 m_1(z), \\
 m_{mp}^{\gamma_2}(z) &= \frac{1-\gamma_2-z + \sqrt{(z-1-\gamma_2)^2 - 4\gamma_2}}{2\gamma_2 z}, \\
 m_3(z) &= \gamma_2 m_{mp}^{\gamma_2}(-m_2(z)) + (m_2(z))^{-1}(1-\gamma_2).
 \end{aligned}$$

All the above contour integrals can be evaluated on any contour enclosing the interval $[\frac{\alpha c_l}{1-c_l}, \frac{\alpha c_r}{1-c_r}]$.

Note that Lemma A.1 is proved based on the centralized sample covariance matrices, which are constructed by not subtracting the sample mean vector from each sample vector. However, [21] showed that the only difference between the two types of sample covariance matrices is normalization by p/n_l and p/N_l . It is not difficult to verify that Theorem 2.7 satisfies Lemma A.1 by choosing the kernel function to the logarithmic function. Therefore, the main task of proving Theorem 2.7 is to calculate the three integrals, which will be shown step by step.

Proof of the limit part $\tilde{\ell}_n$. When calculating the integral

$$p \int \log x \frac{(\alpha_n + 1) \sqrt{(x_r - x)(x - x_l)}}{2\pi y_1 x(1-x)} \delta_{x \in (x_l, x_r)} dx \tag{A.1}$$

to achieve the limit part $\tilde{\ell}_n$, we choose a transformation

$$x = \frac{y_2 |y_1 + h\xi|^2}{(y_1 + y_2)^2}.$$

Clearly, when x moves from $\frac{y_2(h-y_1)^2}{(y_1+y_2)^2}$ to $\frac{y_2(h+y_1)^2}{(y_1+y_2)^2}$ two times, ξ shifts along a unit circle in the positive direction. Then, we obtain that the integral (A.1) is equal to

$$p \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \left[\log \frac{y_2 |y_1 + h\xi|^2}{(y_1 + y_2)^2} \right] \cdot \frac{(\xi^2 - 1)^2}{\xi^3 |y_1 + h\xi|^2 |y_2 - h\xi|^2} d\xi \quad (\text{A.2})$$

when $y_1 > 1$ and

$$p \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \left[\log \frac{y_2 |h + y_1\xi|^2}{(y_1 + y_2)^2} \right] \cdot \frac{(\xi^2 - 1)^2}{\xi^3 |y_1 + h\xi|^2 |y_2 - h\xi|^2} d\xi \quad (\text{A.3})$$

when $y_1 < 1$. Two forms of the integral ensure that the logarithmic function returns a finite number of poles related to y_1 of the integrand. The pole related to y_2 of the integrand is h/y_2 when $y_2 > 1$. There is no difference in the integral value before and after the transformation $\xi = 1/\xi$, except that the residue point in the unit disc turns into y_2/h , which is the residue point under the assumption $y_2 < 1$. Thus, we assume $y_2 > 1$ without loss of generality. Then, we obtain that if $y_1 > 1$, (A.2) can be rewritten as

$$\begin{aligned} & p \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \left[\log \frac{y_2(y_1 + h\xi)}{(y_1 + y_2)^2} \right] \times \frac{(\xi^2 - 1)^2}{\xi^3 |y_1 + h\xi|^2 |y_2 - h\xi|^2} d\xi \\ & + p \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \left[\log \frac{y_2(y_1 + \frac{h}{\xi})}{(y_1 + y_2)^2} \right] \times \frac{(\xi^2 - 1)^2}{\xi^3 |y_1 + h\xi|^2 |y_2 - h\xi|^2} d\xi \end{aligned}$$

which is equal to

$$p \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \left[\log \frac{y_2(y_1 + h\xi)^2}{(y_1 + y_2)^2} \right] \times \frac{(\xi^2 - 1)^2}{\xi^3 (y_1 + h\xi)(y_1 + \frac{h}{\xi})(y_2 - h\xi)(y_2 - \frac{h}{\xi})} d\xi.$$

Similarly, if $y_1 < 1$, we have (A.3) equals

$$p \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \left[\log \frac{y_2(h + y_1\xi)^2}{(y_1 + y_2)^2} \right] \times \frac{(\xi^2 - 1)^2}{\xi^3 (y_1 + h\xi)(y_1 + \frac{h}{\xi})(y_2 - h\xi)(y_2 - \frac{h}{\xi})} d\xi.$$

According to Cauchy's residue theorem, we find three poles $\{0, -\frac{h}{y_1}, \frac{h}{y_2}\}$ under the settings of $y_1 > 1, y_2 > 1$. The corresponding residues are

$$\frac{p(y_1 + y_2)}{2y_1 y_2} \log \frac{y_1^2 y_2}{(y_1 + y_2)^2}, \quad \frac{p(1 - y_1)}{2y_1} \log \frac{y_2(y_1 - 1)^2}{y_1^2}, \quad \frac{p(1 - y_2)}{2y_2} \log \frac{1}{y_2},$$

respectively. In the same way, under the assumptions $y_1 < 1, y_2 > 1$, we obtain three poles $\{0, -\frac{y_1}{h}, \frac{h}{y_2}\}$ and three residues

$$\frac{p(y_1 + y_2)}{2y_1 y_2} \log \frac{h^2 y_2}{(y_1 + y_2)^2}, \quad \frac{p(1 - y_1)}{2y_1} \log \frac{h^2}{y_2(1 - y_1)^2}, \quad \frac{p(1 - y_2)}{2y_2} \log \frac{h^2}{y_2}.$$

Therefore, by combining the above results and basic calculations, we obtain the limit part (A.1) is

$$\log \left(\frac{y_2 h^{\frac{2h^2}{y_1 y_2}}}{(y_1 + y_2)^{\frac{(y_1 + y_2)}{y_1 y_2}} |1 - y_1|^{\frac{|1 - y_1|}{y_1}}} \right) - \log \left(\frac{h^{\frac{2h^2}{y_1 y_2}}}{y_1^{\frac{1+y_2}{y_2}} y_2^{\frac{1-y_1}{y_1}}} \right) \delta_{y_1 > 1},$$

which completes the proof. □

Proof of the mean part $\tilde{\mu}_n$. Because m_3 satisfies the equation

$$z = -\frac{m_3(z)(m_3(z) + 1 - y_1)}{(1 - y_2)(m_3(z) + \frac{1}{1 - y_2})},$$

we make an integral conversion $z = (1 + hr\xi)(1 + \frac{h}{r\xi})/(1 - y_2)^2$, where r is a number greater than but close to 1. According to the discussion in the last section, we assume $y_2 > 1$ without loss of generality. From the equation

$$\frac{(1 + hr\xi)(1 + \frac{h}{r\xi})}{(1 - y_2)^2} = -\frac{m_3(m_3 + 1 - y_1)}{(1 - y_2)m_3 + 1},$$

we obtain that $m_3 = -(1 + hr\xi)/(1 - y_2)$ or $m_3 = -(1 + \frac{h}{r\xi})/(1 - y_2)$. When z runs in the positive direction around the contour enclosing the interval $[\frac{\alpha c_l}{1 - c_l}, \frac{\alpha c_r}{1 - c_r}]$, m_3 runs in the opposite direction. Thus, when $y_2 > 1$, we choose the outcome $m_3 = -(1 + \frac{h}{r\xi})/(1 - y_2)$. Consequently, we have

$$\frac{z}{\alpha + z} = \frac{y_2 |1 + hr\xi|^2}{|y_2 + hr\xi|^2}.$$

Therefore, for $y_1 > 1$, we get the mean part $\tilde{\mu}_n$ is equal to

$$\lim_{r \downarrow 1} \frac{1}{4\pi i} \oint_{|\xi|=1} \left(\log \frac{y_2(1 + hr\xi)(1 + \frac{h}{r\xi})}{(y_2 + hr\xi)(y_2 + \frac{h}{r\xi})} \right) \times \left(\frac{1}{\xi - \frac{1}{r}} + \frac{1}{\xi + \frac{1}{r}} - \frac{2}{\xi + \frac{h}{y_2 r}} \right) d\xi \quad (A.4)$$

$$+ \lim_{r \downarrow 1} \frac{\Delta_1}{2\pi i} \oint -y_1 \left(\log \frac{y_2(1 + hr\xi)(1 + \frac{h}{r\xi})}{(y_2 + hr\xi)(y_2 + \frac{h}{r\xi})} \right) \frac{(1 - y_2)^2 h}{y_2^3} \frac{\xi}{(\xi + \frac{h}{y_2 r})^3} d\xi \quad (A.5)$$

$$+ \lim_{r \downarrow 1} \frac{\Delta_2}{4\pi i} \oint \left(\log \frac{y_2(1 + hr\xi)(1 + \frac{h}{r\xi})}{(y_2 + hr\xi)(y_2 + \frac{h}{r\xi})} \right) \frac{(y_2 - 1)(\xi^2 - \frac{h^2}{y_2 r^2})}{y_2(\xi + \frac{h}{y_2 r})^2} \times \left[\frac{2\xi}{(\xi^2 - \frac{h^2}{y_2 r^2})} - \frac{2}{\xi + \frac{h}{y_2 r}} \right] d\xi. \quad (A.6)$$

If $y_1 < 1$, we only need to change the logarithmic term in the above expression into the following form

$$\log \frac{y_2(1 + hr\xi)(1 + \frac{h}{r\xi})}{(h + y_2 r\xi)(h + \frac{y_2}{r\xi})},$$

and the other terms remain the same. For $y_1 > 1$, there are four poles of term (A.4)

$$\frac{1}{r}, \quad -\frac{1}{r}, \quad -\frac{h}{y_2 r}, \quad 0,$$

and four residues

$$\frac{1}{2} \log \frac{y_2(1+h)}{y_2+h}, \quad \frac{1}{2} \log \frac{y_2(1-h)}{y_2-h}, \quad \frac{1}{2} \log \frac{(y_1+y_2)^2}{y_2^2 y_1^2}, \quad 0.$$

Thus, by Cauchy’s residue theorem, we have

$$(A.4) = \frac{1}{2} \log \frac{(y_1+y_2)(y_1-1)}{y_1^2}.$$

Analogously,

$$(A.5) = -\Delta_1 \frac{h^2(y_1+y_2+y_1 y_2)}{2y_1(y_1+y_2)^2} \quad \text{and} \quad (A.6) = \Delta_2 \frac{h^2 y_2(2y_1^2-h^2)}{2y_1^2(y_1+y_2)^2}.$$

For $y_1 < 1$,

$$\begin{aligned} & \lim_{r \downarrow 1} \frac{1}{4\pi i} \oint_{|\xi|=1} \left(\log \frac{y_2(1+hr\xi)(1+\frac{h}{r\xi})}{(h+y_2 r\xi)(h+\frac{y_2}{r\xi})} \right) \times \left(\frac{1}{\xi-\frac{1}{r}} + \frac{1}{\xi+\frac{1}{r}} - \frac{2}{\xi+\frac{h}{y_2 r}} \right) d\xi \\ &= \lim_{r \downarrow 1} \frac{1}{4\pi i} \oint_{|\xi|=1} \left(\log \frac{y_2(1+\frac{h}{r\xi})}{(h+\frac{y_2}{r\xi})} \right) \times \left(\frac{1}{\xi-\frac{1}{r}} + \frac{1}{\xi+\frac{1}{r}} - \frac{2}{\xi+\frac{h}{y_2 r}} \right) d\xi \\ &+ \lim_{r \downarrow 1} \frac{1}{4\pi i} \oint_{|\xi|=1} \left(\log \frac{(\xi+hr)}{(h\xi+y_2 r)} \right) \times \frac{1}{\xi} \times \left(\frac{r}{r-\xi} + \frac{r}{r+\xi} - \frac{\frac{2y_2 r}{h}}{\frac{h}{y_2 r} + \xi} \right) d\xi \\ &= \frac{1}{2} \log \frac{(y_1+y_2)(1-y_1)}{h^2}, \\ & \lim_{r \downarrow 1} \frac{\Delta_1}{2\pi i} \oint -y_1 \left(\log \frac{y_2(1+hr\xi)(1+\frac{h}{r\xi})}{(h+y_2 r\xi)(h+\frac{y_2}{r\xi})} \right) \frac{(1-y_2)^2 h}{y_2^3} \frac{\xi}{(\xi+\frac{h}{y_2 r})^3} d\xi \\ &= \Delta_1 \frac{y_1^2(y_1+2y_2)}{-2(y_1+y_2)^2}, \end{aligned}$$

and

$$\begin{aligned} & \lim_{r \downarrow 1} \frac{\Delta_2}{4\pi i} \oint \left(\log \frac{y_2(1+hr\xi)(1+\frac{h}{r\xi})}{(h+y_2 r\xi)(h+\frac{y_2}{r\xi})} \right) \frac{(y_2-1)(\xi^2-\frac{h^2}{y_2 r^2})}{y_2(\xi+\frac{h}{y_2 r})^2} \\ & \times \left[\frac{2\xi}{(\xi^2-\frac{h^2}{y_2 r^2})} - \frac{2}{\xi+\frac{h}{y_2 r}} \right] d\xi = \Delta_2 \frac{y_1^2 y_2}{2(y_1+y_2)^2}. \end{aligned}$$

Thus, by combining the above results, the mean part $\tilde{\mu}_n$ is

$$\log \left[\frac{(y_1+y_2)^{\frac{1}{2}} |1-y_1|^{\frac{c_1}{2}}}{h} \right] - \log \left(\frac{y_1}{h} \right) \delta_{y_1 > 1}$$

$$\begin{aligned} & - \frac{\Delta_1[y_1^3(y_1 + 2y_2)\delta_{y_1 < 1} + h^2(y_1 + y_2 + y_1y_2)\delta_{y_1 > 1}]}{2y_1(y_1 + y_2)^2} \\ & + \frac{\Delta_2[y_1^4y_2\delta_{y_1 < 1} + h^2y_2(2y_1 - h^2)\delta_{y_1 > 1}]}{2y_1^2(y_1 + y_2)^2}. \end{aligned}$$

This completes the proof. □

Proof of the variance part. To calculate the variance part \tilde{v}_n

$$- \frac{1}{2\pi^2} \oint \oint \frac{f(\frac{z_1}{\alpha+z_1})f(\frac{z_2}{\alpha+z_2})dm_3(z_1)dm_3(z_2)}{(m_3(z_1) - m_3(z_2))^2} \tag{A.7}$$

$$- \frac{y_1\Delta_1 + y_2\Delta_2}{4\pi^2} \oint \oint \frac{f(\frac{z_1}{\alpha+z_1})f(\frac{z_2}{\alpha+z_2})dm_3(z_1)dm_3(z_2)}{(m_3(z_1) + 1)^2(m_3(z_2) + 1)^2}, \tag{A.8}$$

we make an analogous integral conversion

$$z_1 = (1 + hr_1\xi_1)(1 + \frac{h}{r_1\xi_1})/(1 - y_2)^2$$

and

$$z_2 = (1 + hr_2\xi_2)(1 + \frac{h}{r_2\xi_2})/(1 - y_2)^2.$$

Therefore, the relationship between ξ_l and $m_3(z_l)$, $l = 1, 2$ is

$$m_3(z_1) = -\frac{1 + \frac{h}{r_1\xi_1}}{(1 - y_2)}, \quad m_3(z_2) = -\frac{1 + \frac{h}{r_2\xi_2}}{(1 - y_2)}.$$

Without loss of generality, we assume $r_1 < r_2$. When $y_1 > 1, y_2 > 1$,

$$\begin{aligned} & - \frac{1}{2\pi^2} \oint \oint \frac{f_i(\frac{z_1}{\alpha+z_1})f_j(\frac{z_2}{\alpha+z_2})dm_3(z_1)dm_3(z_2)}{(m_3(z_1) - m_3(z_2))^2} \\ = & 2 \lim_{r_2 \downarrow 1} \oint \frac{1}{2\pi i} (\log \frac{y_2(1 + hr_2\xi_2)(1 + \frac{h}{r_2\xi_2})}{(y_2 + hr_2\xi_2)(y_2 + \frac{h}{r_2\xi_2})}) \\ & \times \{ \lim_{r_1 \downarrow 1} \oint \frac{1}{2\pi i} (\log \frac{y_2(1 + hr_1\xi_1)(1 + \frac{h}{r_1\xi_1})}{(y_2 + hr_1\xi_1)(y_2 + \frac{h}{r_1\xi_1})}) \frac{r_1}{(r_1\xi_1 - r_2\xi_2)^2} d\xi_1 \} r_2 d\xi_2 \\ = & 2 \lim_{r_2 \downarrow 1} \oint \frac{1}{2\pi i} (\log \frac{y_2(1 + hr_2\xi_2)(1 + \frac{h}{r_2\xi_2})}{(y_2 + hr_2\xi_2)(y_2 + \frac{h}{r_2\xi_2})}) \\ & \times \{ \lim_{r_1 \downarrow 1} \oint \frac{1}{2\pi i} [(\log \frac{y_2(1 + hr_1\xi_1)}{(y_2 + hr_1\xi_1)}) \frac{r_1}{(r_1\xi_1 - r_2\xi_2)^2} \\ & + (\log \frac{(1 + \frac{h}{r_1\xi_1})}{(y_2 + \frac{h}{r_1\xi_1})}) \frac{r_1}{r_2^2\xi_2^2(\frac{r_1}{r_2\xi_2} - \xi_1)^2}] d\xi_1 \} r_2 d\xi_2. \end{aligned}$$

There is only one pole $\frac{r_1}{r_2\xi_2}$ in the unit disc of the integration formula with respect to ξ_1 . Thus, by Cauchy's residue theorem, (A.7) is equal to

$$2 \lim_{r_2 \downarrow 1} \oint \frac{1}{2\pi i} \left(\log \frac{y_2(1 + hr_2\xi_2)(1 + \frac{h}{r_2\xi_2})}{(y_2 + hr_2\xi_2)(y_2 + \frac{h}{r_2\xi_2})} \right) \cdot \frac{1}{\xi_2} \left(\frac{h}{h + r_2\xi_2} - \frac{h}{h + y_2r_2\xi_2} \right) d\xi_2$$

which has three poles 0, $-\frac{h}{r_2}$, $-\frac{h}{r_2y_2}$. Thus, we finally obtain that

$$(A.7) = \frac{2y_1^2}{(y_1 + y_2)(y_1 - 1)}.$$

Similarly, if $y_1 < 1, y_2 > 1$, (A.7) = $\frac{2h^2}{(y_1 + y_2)(1 - y_1)}$. In addition, as when $y_1 > 1, y_2 > 1$,

$$\begin{aligned} & -\frac{y_1\Delta_1 + y_2\Delta_2}{4\pi^2} \oint \oint \frac{f_i(\frac{z_1}{\alpha+z_1})f_j(\frac{z_2}{\alpha+z_2})dm_3(z_1)dm_3(z_2)}{(m_3(z_1) + 1)^2(m_3(z_2) + 1)^2} \\ &= (y_1\Delta_1 + y_2\Delta_2) \left\{ \lim_{r_2 \downarrow 1} \oint \frac{1}{2\pi i} \left(\log \frac{y_2(1 + hr_2\xi_2)(1 + \frac{h}{r_2\xi_2})}{(y_2 + hr_2\xi_2)(y_2 + \frac{h}{r_2\xi_2})} \right) \frac{(1 - y_2)r_1hd\xi_1}{(y_2r_1\xi_1 + h)^2} \right\} \times \\ & \quad \left\{ \lim_{r_1 \downarrow 1} \oint \frac{1}{2\pi i} \left(\log \frac{y_2(1 + hr_1\xi_1)(1 + \frac{h}{r_1\xi_1})}{(y_2 + hr_1\xi_1)(y_2 + \frac{h}{r_1\xi_1})} \right) \frac{(1 - y_2)r_2hd\xi_2}{(y_2r_2\xi_2 + h)^2} \right\} \\ &= (y_1\Delta_1 + y_2\Delta_2) \cdot \frac{h^4}{y_1^2(y_1 + y_2)^2} \end{aligned}$$

and

$$\begin{aligned} & -\frac{y_1\Delta_1 + y_2\Delta_2}{4\pi^2} \oint \oint \frac{f_i(\frac{z_1}{\alpha+z_1})f_j(\frac{z_2}{\alpha+z_2})dm_3(z_1)dm_3(z_2)}{(m_3(z_1) + 1)^2(m_3(z_2) + 1)^2} \\ &= (y_1\Delta_1 + y_2\Delta_2) \cdot \frac{y_1^2}{(y_1 + y_2)^2} \end{aligned}$$

when $y_1 < 1, y_2 > 1$. Thus, the variance part \tilde{v}_n is equal to

$$2 \log \frac{h^2}{|1 - y_1|(y_1 + y_2)} + 2 \log \left(\frac{y_1^2}{h^2} \right) \delta_{y_1 > 1} + \frac{(y_1\Delta_1 + y_2\Delta_2)}{y_1^2(y_1 + y_2)^2} [y_1^4 \delta_{y_1 < 1} + h^4 \delta_{y_1 > 1}].$$

This completes the proof. \square

A.2. Proof of Theorem 2.1

In this part, we give the proof of Theorem 2.1.

Proof. First, calculate the limit part ℓ_n ,

$$p \int [c_1 \log x + c_2 \log(1 - x)] \frac{(1 + \alpha)\sqrt{(x_r - x)(x - x_l)}}{2\pi y_1 x(1 - x)} dx, \quad (A.9)$$

where

$$x_l, x_r = \frac{y_2(h \mp y_1)^2}{(y_1 + y_2)^2}.$$

Make the transformation

$$x = \frac{y_2|y_1 + h\xi|^2}{(y_1 + y_2)^2},$$

correspondingly,

$$1 - x = \frac{y_1|y_2 - h\xi|^2}{(y_1 + y_2)^2}.$$

To ensure the logarithmic functions are not infinity, we change the transformation into following tow forms

$$x = \frac{y_2|y_1 + h\xi|^2}{(y_1 + y_2)^2} = \frac{y_2(y_1 + h\xi)(y_1 + \frac{h}{\xi})}{(y_1 + y_2)^2},$$

when $y_1 > 1$, and

$$x = \frac{y_2|y_1 + h\xi|^2}{(y_1 + y_2)^2} = \frac{y_2(h + y_1\xi)(h + \frac{y_1}{\xi})}{(y_1 + y_2)^2},$$

when $y_1 < 1$.

Because of that the $\log(1 - x)$ part can only be affect by y_2 , with same discussion, we get

$$1 - x = \frac{y_1|y_2 - h\xi|^2}{(y_1 + y_2)^2} = \frac{y_1(y_2 - h\xi)(y_2 - \frac{h}{\xi})}{(y_1 + y_2)^2},$$

when $y_2 > 1$, and

$$1 - x = \frac{y_1|y_2 - h\xi|^2}{(y_1 + y_2)^2} = \frac{y_1(h - y_2\xi)(h - \frac{y_2}{\xi})}{(y_1 + y_2)^2}.$$

when $y_2 < 1$. Thus,

$$\begin{aligned} & p \int c_1 \log x \frac{(1 + \alpha)\sqrt{(x_r - x)(x - x_l)}}{2\pi y_1 x(1 - x)} dx \tag{A.10} \\ &= p c_1 \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \log \frac{y_2(y_1 + h\xi)(y_1 + \frac{h}{\xi})}{(y_1 + y_2)^2} \cdot \frac{(\xi^2 - 1)^2}{\xi^3 |y_1 + h\xi|^2 |y_2 - h\xi|^2} d\xi \\ &= p c_1 \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \log \frac{y_2(y_1 + h\xi)(y_1 + \frac{h}{\xi})}{(y_1 + y_2)^2} \\ &\quad \times \frac{(\xi^2 - 1)^2}{\xi^3 (y_1 + h\xi)(y_1 + \frac{h}{\xi})(y_2 - h\xi)(y_2 - \frac{h}{\xi})} d\xi. \end{aligned}$$

There are three poles

$$0, \quad -\frac{h}{y_1}, \quad \frac{h}{y_2}$$

and three residues

$$-\log\left(\frac{y_1^2 y_2}{(y_1 + y_2)^2}\right)^{\frac{c_1}{y_1 y_2 h^2}}, \log\left(\frac{y_2(y_1 - \frac{h^2}{y_1})^2}{y_1 + y_2}\right)^{\frac{c_1(y_1 - 1)}{y_1 h^2(y_1 + y_2)}},$$

$$\log\left(\frac{y_2(y_1 + \frac{h^2}{y_2})^2}{y_1 + y_2}\right)^{\frac{c_1(y_2 - 1)}{y_2 h^2(y_1 + y_2)}}$$

in (A.10), respectively. Then we get

$$(A.10) = pc_1\left(-\frac{y_1 + y_2}{y_1 y_2} \log(y_1 + y_2) + \frac{y_2 + 1}{y_2} \log y_1\right. \\ \left. + \frac{1 - y_1}{y_1} \log(y_1 - 1) + \frac{1}{y_1} \log y_2\right).$$

In the same way, (A.11) has the same poles with (A.10)

$$p \int c_2 \log(1 - x) \frac{(1 + \alpha)\sqrt{(x_r - x)(x - x_l)}}{2\pi y_1 x(1 - x)} dx \tag{A.11}$$

$$= pc_2 \frac{(y_1 + y_2)h^2 i}{4\pi} \oint \log \frac{y_1(y_2 - h\xi)(y_2 - \frac{h}{\xi})}{(y_1 + y_2)^2} \cdot \frac{(\xi^2 - 1)^2}{\xi^3 |y_1 + h\xi|^2 |y_2 - h\xi|^2} d\xi$$

$$= pc_2\left(-\frac{y_1 + y_2}{y_1 y_2} \log(y_1 + y_2) + \frac{y_1 + 1}{y_1} \log y_2 + \frac{1 - y_2}{y_2} \log(y_2 - 1) + \frac{1}{y_2} \log y_1\right).$$

Thus we have that,

$$(A.9) = p\left[c_1\left(-\frac{y_1 + y_2}{y_1 y_2} \log(y_1 + y_2) + \frac{y_2 + 1}{y_2} \log y_1 + \frac{1 - y_1}{y_1} \log(y_1 - 1) + \frac{1}{y_1} \log y_2\right)\right. \\ \left.+ c_2\left(-\frac{y_1 + y_2}{y_1 y_2} \log(y_1 + y_2) + \frac{y_1 + 1}{y_1} \log y_2 + \frac{1 - y_2}{y_2} \log(y_2 - 1) + \frac{1}{y_2} \log y_1\right)\right]$$

When the assumption becomes $y_1 > 1, y_2 < 1$, we use the following form of the transformation

$$x = \frac{y_2(y_1 + h\xi)(y_1 + \frac{h}{\xi})}{(y_1 + y_2)^2},$$

and

$$1 - x = \frac{y_1(h - y_2\xi)(h - \frac{y_2}{\xi})}{(y_1 + y_2)^2}.$$

Thus, we get

$$(A.9) = p\left[c_1\left(-\frac{y_1 + y_2}{y_1 y_2} \log(y_1 + y_2) + \frac{y_2 + 1}{y_2} \log y_1 + \frac{1 - y_1}{y_1} \log(y_1 - 1) + \frac{1}{y_1} \log y_2\right)\right. \\ \left.+ c_2\left(-\frac{y_1 + y_2}{y_1 y_2} \log(y_1 + y_2) + \frac{h^2}{y_1 y_2} \log h^2 + \frac{y_2 - 1}{y_2} \log(1 - y_2) + \log y_1\right)\right].$$

We can get the conclusion of other two cases similarly. According to the discussion above, we find out that the limit part is

$$\ell_n = \log \left(\frac{y_1^{c_2} y_2^{c_1} h^{\frac{2h^2}{y_1 y_2}}}{(y_1 + y_2)^{\frac{(y_1 + y_2)}{y_1 y_2}} |1 - y_1|^{\frac{c_1 |1 - y_1|}{y_1}} |1 - y_2|^{\frac{c_2 |1 - y_2|}{y_2}}} \right) - l(y_1, y_2) - l(y_2, y_1),$$

where

$$l(y_1, y_2) = \log(h^{\frac{2c_1 h^2}{y_1 y_2}}) \delta_{y_1 > 1} - \log(y_1^{\frac{c_1(1+y_2)}{y_2}} y_2^{\frac{c_1(1-y_1)}{y_1}}) \delta_{y_1 > 1}.$$

Next, we calculate the mean μ_n . By the transformation

$$m_3 = -(1 + hr\xi)/(1 - y_2)$$

when $y_2 < 1$, and

$$m_3 = -(1 + \frac{h}{r\xi})/(1 - y_2)$$

when $y_2 > 1$, in order to make the logarithmic function meaningful, we stipulate that

$$\frac{z}{\alpha + z} = \frac{y_2 |1 + hr\xi|^2}{|y_2 + hr\xi|^2} = \frac{y_2(1 + hr\xi)(1 + \frac{h}{r\xi})}{(y_2 + hr\xi)(y_2 + \frac{h}{r\xi})}$$

if $y_1 > 1$,

$$\frac{z}{\alpha + z} = \frac{y_2 |1 + hr\xi|^2}{|y_2 + hr\xi|^2} = \frac{y_2(h + \frac{1}{r\xi})(h + r\xi)}{(y_2 + \frac{h}{r\xi})(y_2 + hr\xi)}$$

if $y_1 < 1$,

$$1 - \frac{z}{\alpha + z} = \frac{y_1(y_2 - 1)^2}{|y_2 + hr\xi|^2} = \frac{y_1(y_2 - 1)^2}{(y_2 + hr\xi)(y_2 + \frac{h}{r\xi})}$$

if $y_2 > 1$, and

$$1 - \frac{z}{\alpha + z} = \frac{y_1(y_2 - 1)^2}{|y_2 + hr\xi|^2} = \frac{y_1(y_2 - 1)^2}{(h + y_2 r\xi)(h + \frac{y_2}{r\xi})}$$

if $y_2 < 1$. We first calculate the $y_1 > 1, y_2 > 1$ case. We cut each logarithmic function into two pieces

$$\frac{1}{4\pi i} \oint f_i(\frac{z}{\alpha + z}) d \log \left(\frac{(1 - y_2)m_3^2(z) + 2m_3(z) + 1 - y_1}{(1 + m_3(z))^2} \right) \tag{A.12}$$

$$+ \frac{\Delta_1}{2\pi i} \oint y_1 f_i(\frac{z}{\alpha + z}) (1 + m_3)^{-3} dm_3(z) \tag{A.13}$$

$$+ \frac{\Delta_2}{4\pi i} \oint f_i(\frac{z}{\alpha + z}) (1 - y_2 m_3^2(z) (1 + m_3(z))^{-2}) d \log(1 - y_2 m_3^2(z) (1 + m_3(z))^{-2}) \tag{A.14}$$

$$\begin{aligned}
&= \lim_{r \downarrow 1} \frac{1}{4\pi i} \oint_{|\xi|=1} c_1 \log \frac{y_2(1+hr\xi)(1+\frac{h}{r\xi})}{(y_2+hr\xi)(y_2+\frac{h}{r\xi})} \cdot \left(\frac{1}{\xi-\frac{1}{r}} + \frac{1}{\xi+\frac{1}{r}} - \frac{2}{\xi+\frac{h}{y_2r}} \right) d\xi \\
&+ \lim_{r \downarrow 1} \frac{1}{4\pi i} \oint_{|\xi|=1} c_2 \log \frac{y_1(y_2-1)^2}{(y_2+hr\xi)(y_2+\frac{h}{r\xi})} \cdot \left(\frac{1}{\xi-\frac{1}{r}} + \frac{1}{\xi+\frac{1}{r}} - \frac{2}{\xi+\frac{h}{y_2r}} \right) d\xi \\
&+ \lim_{r \downarrow 1} \frac{\Delta_1}{2\pi i} \oint -y_1 c_1 \left(\log \frac{y_2(1+hr\xi)(1+\frac{h}{r\xi})}{(y_2+hr\xi)(y_2+\frac{h}{r\xi})} \right) \frac{(1-y_2)^2 h}{y_2^3} \frac{\xi}{(\xi+\frac{h}{y_2r})^3} d\xi \\
&+ \lim_{r \downarrow 1} \frac{\Delta_1}{2\pi i} \oint -y_1 c_2 \left(\log \frac{y_1(y_2-1)^2}{(y_2+hr\xi)(y_2+\frac{h}{r\xi})} \right) \frac{(1-y_2)^2 h}{y_2^3} \frac{\xi}{(\xi+\frac{h}{y_2r})^3} d\xi \\
&+ \lim_{r \downarrow 1} \frac{\Delta_2}{4\pi i} \oint \left(\log \frac{y_2(1+hr\xi)(1+\frac{h}{r\xi})}{(y_2+hr\xi)(y_2+\frac{h}{r\xi})} \right) \frac{(y_2-1)(\xi^2-\frac{h^2}{y_2r^2})}{y_2(\xi+\frac{h}{y_2r})^2} \\
&\quad \times \left[\frac{2c_1\xi}{(\xi^2-\frac{h^2}{y_2r^2})} - \frac{2c_1}{\xi+\frac{h}{y_2r}} \right] d\xi \\
&+ \lim_{r \downarrow 1} \frac{\Delta_2}{4\pi i} \oint \left(\log \frac{y_1(y_2-1)^2}{(y_2+hr\xi)(y_2+\frac{h}{r\xi})} \right) \frac{(y_2-1)(\xi^2-\frac{h^2}{y_2r^2})}{y_2(\xi+\frac{h}{y_2r})^2} \\
&\quad \times \left[\frac{2c_2\xi}{(\xi^2-\frac{h^2}{y_2r^2})} - \frac{2c_2}{\xi+\frac{h}{y_2r}} \right] d\xi.
\end{aligned}$$

According to Cauchy's residue theorem, we get

$$\begin{aligned}
&(\text{A.12}) + (\text{A.13}) + (\text{A.14}) \\
&= \frac{c_1}{2} \log \frac{(y_1+y_2)(y_1-1)}{y_1^2} + \frac{c_2}{2} \log \frac{(y_1+y_2)(y_2-1)}{y_2^2} \\
&\quad + \Delta_1 c_1 \frac{y_1^2 y_2^2 - (y_1+y_2)^2}{2y_1(y_1+y_2)^2} + \Delta_1 c_2 \frac{y_1(2y_2^2 h^2 - h^4)}{2y_2^2(y_1+y_2)^2} \\
&\quad + \Delta_2 c_1 \frac{h^2 y_2(2y_1^2 - h^2)}{2y_1^2(y_1+y_2)^2} + \Delta_2 c_2 \frac{h^2(y_1+y_2+y_1 y_2)}{-2y_2(y_1+y_2)^2}.
\end{aligned}$$

The integral transforms of (A.12), (A.13) and (A.14) are different between the setting $y_2 > 1$ and the setting $y_2 < 1$. We find that, when $y_2 < 1$,

$$\begin{aligned}
d \log \left(\frac{(1-y_2)m_3^2(z) + 2m_3(z) + 1 - y_1}{(1+m_3(z))^2} \right) &= \left(\frac{1}{\xi-\frac{1}{r}} + \frac{1}{\xi+\frac{1}{r}} - \frac{2}{\xi+\frac{y_2}{hr}} \right) d\xi, \\
\frac{1}{(1+m_3)^3} dm_3(z) &= \frac{(1-y_2)^2}{h^2} \frac{1}{(\xi+\frac{y_2}{hr})^3} d\xi
\end{aligned}$$

and

$$\left(1 - \frac{y_2 m_3^2(z)}{(1+m_3(z))^2} \right) d \log \left(1 - \frac{y_2 m_3^2(z)}{(1+m_3(z))^2} \right)$$

$$= \frac{(1 - y_2)(\xi^2 - \frac{y_2}{h^2 r^2})}{(\xi + \frac{y_2}{hr})^2} \left[\frac{2\xi}{\xi^2 - \frac{y_2}{h^2 r^2}} - \frac{2}{\xi + \frac{y_2}{hr}} \right] d\xi.$$

Thus we draw a conclusion that

$$\begin{aligned} & \text{(A.12)} + \text{(A.13)} + \text{(A.14)} \\ &= \frac{c_1}{2} \log \frac{(y_1 + y_2)(y_1 - 1)}{y_1^2} + \frac{c_2}{2} \log \frac{(y_1 + y_2)(1 - y_2)}{h^2} \\ & \quad + \Delta_1 c_1 \frac{y_1^2 y_2^2 - (y_1 + y_2)^2}{2y_1(y_1 + y_2)^2} + \Delta_1 c_2 \frac{y_1 y_2^2}{2(y_1 + y_2)^2} \\ & \quad + \Delta_2 c_1 \frac{h^2 y_2(2y_1^2 - h^2)}{2y_1^2(y_1 + y_2)^2} + \Delta_2 c_2 \frac{y_2^2(2y_1 + y_2)}{-2(y_1 + y_2)^2} \end{aligned}$$

when $y_1 > 1, y_2 < 1$. Merging the conclusion of other two settings, the outcome of the mean part obviously is

$$\begin{aligned} \mu_n = \log & \left[\frac{(y_1 + y_2)^{\frac{1}{2}} |1 - y_1|^{\frac{c_1}{2}} |1 - y_2|^{\frac{c_2}{2}}}{h} \right] - u(y_1, y_2) - u(y_2, y_1) \\ & + \frac{\Delta_1 \Psi(y_1, y_2)}{2y_1 y_2^2 (y_1 + y_2)^2} + \frac{\Delta_2 \Psi(y_2, y_1)}{2y_2 y_1^2 (y_1 + y_2)^2}, \end{aligned}$$

where

$$\begin{aligned} u(y_1, y_2) &= \log \left(\frac{y_1^{c_1}}{h^{c_1}} \right) \delta_{y_1 > 1}, \quad v(y_1, y_2) = \log \left(\frac{y_1^{2c_1}}{h^{2c_1(c_1 + 2c_2)}} \right) \delta_{y_1 > 1}, \\ \Psi(y_1, y_2) &= c_2 y_1^2 [y_2^4 \delta_{y_2 < 1} + h^2(2y_2^2 - h^2) \delta_{y_2 > 1}] \\ & \quad - c_1 y_2^2 [y_1^3 (y_1 + 2y_2) \delta_{y_1 < 1} + h^2(y_1 + y_2 + y_1 y_2) \delta_{y_1 > 1}]. \end{aligned}$$

Then we complete the calculation.

At last, we show the variance part ν_n . Under the circumstance $y_2 > 1$ with the transformation discussed above, we can easily find

$$\begin{aligned} & \frac{dm_3(z_1)dm_3(z_2)}{(m_3(z_1) - m_3(z_2))^2} \\ &= \frac{(1 - y_2)^2}{\left(\frac{h}{r_1 \xi_1} - \frac{h}{r_2 \xi_2}\right)^2} \frac{hd\xi_1}{(1 - y_2)r_1 \xi_1^2} \frac{hd\xi_2}{(1 - y_2)r_2 \xi_2^2} = \frac{r_1 r_2}{(r_1 \xi_1 - r_2 \xi_2)^2} d\xi_1 d\xi_2, \\ & \frac{dm_3(z_1)}{(m_3(z_1) + 1)^2} = \frac{(1 - y_2)^2}{\left(y_2 + \frac{h}{r_1 \xi_1}\right)^2} \cdot \frac{hd\xi_1}{(1 - y_2)r_1 \xi_1^2} = \frac{(1 - y_2)r_1 hd\xi_1}{(y_2 r_1 \xi_1 + h)^2}, \\ & \frac{dm_3(z_2)}{(m_3(z_2) + 1)^2} = \frac{(1 - y_2)^2}{\left(y_2 + \frac{h}{r_2 \xi_2}\right)^2} \frac{hd\xi_2}{(1 - y_2)r_2 \xi_2^2} = \frac{(1 - y_2)r_2 hd\xi_2}{(y_2 r_2 \xi_2 + h)^2}. \end{aligned}$$

Because of the symmetry of the function $\frac{dm_3(z_1)dm_3(z_2)}{(m_3(z_1)-m_3(z_2))^2}$ related to m_3 , integral transform of $\frac{dm_3(z_1)dm_3(z_2)}{(m_3(z_1)-m_3(z_2))^2}$ keeps the same form with $y_2 > 1$ setting. When $y_2 < 1$, integral transformation is

$$\begin{aligned} \frac{dm_3(z_1)}{(m_3(z_1)+1)^2} &= \frac{(1-y_2)^2}{(y_2+hr_1\xi_1)^2} \cdot \frac{hr_1d\xi_1}{(y_2-1)} = \frac{(y_2-1)r_1hd\xi_1}{(hr_1\xi_1+y_2)^2}, \\ \frac{dm_3(z_2)}{(m_3(z_2)+1)^2} &= \frac{(1-y_2)^2}{(y_2+hr_2\xi_2)^2} \cdot \frac{hr_2d\xi_2}{(y_2-1)} = \frac{(y_2-1)r_2hd\xi_2}{(hr_2\xi_2+y_2)^2}. \end{aligned}$$

Thus by Cauchy’s residue theorem, we reach the decision that when $y_1 > 1$ and $y_2 > 1$,

$$-\frac{2}{4\pi^2} \oint \oint \frac{f_i(\frac{z_1}{\alpha+z_1})f_j(\frac{z_2}{\alpha+z_2})dm_3(z_1)dm_3(z_2)}{(m_3(z_1)-m_3(z_2))^2} \tag{A.15}$$

$$\begin{aligned} &-\frac{y_1\Delta_1+y_2\Delta_2}{4\pi^2} \oint \oint \frac{f_i(\frac{z_1}{\alpha+z_1})f_j(\frac{z_2}{\alpha+z_2})dm_3(z_1)dm_3(z_2)}{(m_3(z_1)+1)^2(m_3(z_2)+1)^2} \tag{A.16} \\ &= -2c_2^2 \log(y_2-1) - 2c_1^2 \log(y_1-1) + 4c_2 \log y_2 + 4c_1 \log y_1 \\ &\quad - 2 \log(y_1+y_2) + c\left(\frac{(y_1-y_2)h^2}{y_1y_2(y_1+y_2)}\right)^2. \end{aligned}$$

Similarly, when $y_1 > 1, y_2 < 1$, we get that

$$\begin{aligned} \text{(A.15)} + \text{(A.16)} &= -2c_1^2 \log(y_1-1) + 2c_2^2 \log \frac{h^2}{(1-y_2)} \\ &\quad + 4c_1 \log y_1 - 2 \log(y_1+y_2) + c\left(\frac{(y_1-1)y_2}{y_1(y_1+y_2)}\right)^2. \end{aligned}$$

Combining the other two cases, we come to a conclusion

$$\begin{aligned} \nu_n^2 &= 2 \log \frac{h^2}{|1-y_1|^{c_1^2}|1-y_2|^{c_2^2}(y_1+y_2)} + 2v(y_1, y_2) + 2v(y_2, y_1) \\ &\quad + 2 \log(h^{4c_1c_2})\delta_{y_1>1}\delta_{y_2>1} \\ &\quad + \frac{(y_1\Delta_1+y_2\Delta_2)}{y_1^2y_2^2(y_1+y_2)^2} [(y_1-1)y_2^2\delta_{y_1>1} - (y_2-1)y_1^2\delta_{y_2>1}]^2. \end{aligned}$$

Then we complete the proof. □

A.3. Proof of Theorem 2.7

We now prove that $\hat{\Delta}_1$ is a weakly consistent and asymptotically unbiased estimator of Δ_1 .

Proof. From the definition of Δ_1 in (2.6) and Chebyshev’s inequality, we only need to prove the following two results: Under the same assumptions in Theorem 1 and the null hypothesis,

$$\mathbb{E}\hat{\Delta}_1 \rightarrow \Delta_1 \tag{A.17}$$

and

$$\mathbb{E}(\hat{\Delta}_1 - \Delta_1)^2 \rightarrow 0. \tag{A.18}$$

We first consider (A.17). Without loss of generality, we assume the mean of $\mathbf{z}_j^{(1)}$ is zero. By using the truncation steps in [2] we may truncate and re-normalize the random variables as follows

$$|x_{ij}^{(1)}| \leq \eta_n \sqrt{n}, \quad \mathbb{E}x_{ij}^{(1)} = 0, \quad \mathbb{E}(x_{ij}^{(1)})^2 = 1 \quad \text{and} \quad \mathbb{E}(x_{ij}^{(1)})^4 = \Delta_1 + 3 + O(n^{-1}),$$

where $\eta_n \rightarrow 0$ slowly. In addition, in the following we assume the sample covariance matrix without the negative sample mean, because their difference is only a rank one matrix $\bar{\mathbf{z}}_1 \bar{\mathbf{z}}_1'$ which will not affect the results. From the proof of Theorem 1 in [4], one can conclude that under the conditions of Theorem 1, there exists a constant $M > 0$ such that for any $k > 0$

$$\mathbb{P}(\|(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\| \geq M) \leq n^{-k},$$

where $\|\cdot\|$ means the spectral norm of a matrix. Thus, in the sequel, we can assume the smallest eigenvalue of $c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}}$ is bounded away from zero uniformly. Then we have

$$\begin{aligned} & \mathbb{E}[(\mathbf{z}_j^{(1)} - \bar{\mathbf{z}}^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}(\mathbf{z}_j^{(1)} - \bar{\mathbf{z}}^{(1)}) - \frac{p}{1-y}]^2 \\ &= \mathbb{E}[(\mathbf{z}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}\mathbf{z}_j^{(1)}]^2 + \frac{p^2}{(1-y)^2} \\ & \quad - \frac{2p}{1-y}\mathbb{E}[(\mathbf{z}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}\mathbf{z}_j^{(1)}] + O(p). \end{aligned}$$

Here we use the fact that

$$\mathbb{E}[(\bar{\mathbf{z}}^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}(\bar{\mathbf{z}}^{(1)})] = O(1)$$

and

$$\mathbb{E}[(\bar{\mathbf{z}}^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}(\mathbf{z}_j^{(1)})] = O(1).$$

These results can be found in [15]. According to the independence of $\mathbf{z}_j^{(1)}$ and $\mathbf{S}_{1j}^{\mathbf{z}}$ and under the null hypothesis, we obtain that

$$\begin{aligned} & \mathbb{E}[(\mathbf{z}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}\mathbf{z}_j^{(1)}] \\ &= \mathbb{E}[(\mathbf{x}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_j^{(1)}] = \mathbb{E}tr[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}] \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E}[(\mathbf{z}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}\mathbf{z}_j^{(1)}]^2 \\ &= \Delta_1 \mathbb{E}tr[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1} \circ (c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}] \\ & \quad + 2\mathbb{E}tr[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-2}] + \{\mathbb{E}tr[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]\}^2. \end{aligned}$$

where \circ is the Hadamard product. Notice that $c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}}$ is a sample covariance matrix with dimension p and sample size $n_1 + n_2 - 1$, whose limit spectral distribution is the famous Marcenko-Pastur law with Stieltjes transform

$$s_{mp}(z, \gamma) = \frac{1 - \gamma - z + \sqrt{(z - 1 - \gamma)^2 - 4\gamma}}{2\gamma z}, \quad z \in \mathbb{C}^+. \tag{A.19}$$

Here $y := p/(n_1 + n_2 - 1) \rightarrow \gamma \in (0, \infty)$ and for any distribution function F , its Stieltjes transform is defined by

$$s(z) = \int_{-\infty}^{+\infty} \frac{1}{x - z} dF(x),$$

where $z \in \mathbb{C}^+$. Thus it is not difficult to check (see Section 3.3.2 in [3]) that

$$\frac{1}{p} \mathbb{E} \text{tr}[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}] - \lim_{z \downarrow 0} s_{mp}(z, y) \rightarrow 0 \tag{A.20}$$

and

$$\frac{1}{p} \mathbb{E} \text{tr}[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-2}] - \lim_{z \downarrow 0} \frac{\partial s_{mp}(z, y)}{\partial z} \rightarrow 0.$$

From (A.19), L'Hospital's rule and the fact that $y < 1$, we have

$$\lim_{z \downarrow 0} s_{mp}(z, y) - \frac{1}{1 - y} = 0 \tag{A.21}$$

and

$$\lim_{z \downarrow 0} \frac{\partial s_{mp}(z, y)}{\partial z} - \frac{1}{(1 - y)^3} = 0.$$

In addition, applying Lemma 4.3 of [1], we obtain that

$$\begin{aligned} & \mathbb{E} \text{tr}[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1} \circ (c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}] \\ &= \mathbb{E} \sum_{i=1}^p [(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}(i, i)]^2 = p^{-1} \{\mathbb{E} \text{tr}[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]\}^2 + o(p), \end{aligned}$$

which together with (A.20) and (A.21) implies

$$\mathbb{E} \text{tr}[(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1} \circ (c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}] = \frac{p}{(1 - y)^2} + o(p).$$

Thus, combining the above results, we conclude that

$$\begin{aligned} & \mathbb{E}[(\mathbf{z}_j^{(1)} - \bar{\mathbf{z}}^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{z}} + c_{12}\mathbf{S}_2^{\mathbf{z}})^{-1}(\mathbf{z}_j^{(1)} - \bar{\mathbf{z}}^{(1)}) - \frac{p}{1 - y}]^2 \\ &= \frac{p\Delta_1}{(1 - y)^2} + \frac{2p}{(1 - y)^3} + o(p), \end{aligned}$$

which complete the proof of (A.17).

For (A.18), by the same argument as above, we have that $\mathbb{E}(\hat{\Delta}_1 - \Delta_1)^2$ to

$$\begin{aligned} & \mathbb{E}(\hat{\Delta}_1 - \Delta_1)^2 \\ &= \frac{(1-y)^4 \mathbb{E}\{\sum_{j=1}^{N_1} [(\mathbf{x}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_j^{(1)} - \text{tr}(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]^2\}^2}{p^2 N_1^2} \\ & \quad - (\Delta_1 + \frac{2}{1-y})^2 + o(1). \end{aligned}$$

Now, by (2.1) in [2], we obtain that

$$\begin{aligned} & \frac{\sum_{j=1}^{N_1} \mathbb{E}[(\mathbf{x}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_j^{(1)} - \text{tr}(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]^4}{p^2 N_1^2} \\ &= O(\eta_n^4 N_1^{-1}) \rightarrow 0. \end{aligned} \tag{A.22}$$

Next we use the fact that

$$\begin{aligned} (c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1} &= (c_{11}\mathbf{S}_{1ji}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1} \\ & \quad - \frac{\frac{1}{n_1+n_2-1}(c_{11}\mathbf{S}_{1ji}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_i^{(1)}(\mathbf{x}_i^{(1)})'(c_{11}\mathbf{S}_{1ji}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}}{1 + \frac{1}{n_1+n_2-1}(\mathbf{x}_i^{(1)})'(c_{11}\mathbf{S}_{1ji}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_i^{(1)}} \end{aligned}$$

and

$$(\mathbf{x}_i^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_i^{(1)} = \frac{(\mathbf{x}_i^{(1)})'(c_{11}\mathbf{S}_{1ji}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_i^{(1)}}{1 + \frac{1}{n_1+n_2-1}(\mathbf{x}_i^{(1)})'(c_{11}\mathbf{S}_{1ji}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_i^{(1)}},$$

where $\mathbf{S}_{1ji}^{\mathbf{x}}$ is the sample covariance matrix by removing the vector $\mathbf{x}_j^{(1)}$ and $\mathbf{x}_i^{(1)}$. Thus, we get that

$$\begin{aligned} & \sum_{i \neq j}^{N_1} \mathbb{E}\{[(\mathbf{x}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_j^{(1)} - \text{tr}(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]^2 \\ & \quad \times [(\mathbf{x}_i^{(1)})'(c_{11}\mathbf{S}_{1i}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_i^{(1)} - \text{tr}(c_{11}\mathbf{S}_{1i}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]^2\} \\ &= N_1(N_1 - 1)\mathbb{E}\{[(\mathbf{x}_1^{(1)})'(c_{11}\mathbf{S}_{112}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_1^{(1)} - \text{tr}(c_{11}\mathbf{S}_{112}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]^2 \\ & \quad \times [(\mathbf{x}_2^{(1)})'(c_{11}\mathbf{S}_{112}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_2^{(1)} - \text{tr}(c_{11}\mathbf{S}_{112}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]^2\} + O(N_1^2) \\ & \quad = N_1(N_1 - 1)\left(\frac{p\Delta_1}{(1-y)^2} + \frac{2p}{(1-y)^3}\right)^2 + O(N_1^3), \end{aligned}$$

which together with (A.22) implies

$$\begin{aligned} & \frac{(1-y)^4 \mathbb{E}\{\sum_{j=1}^{N_1} [(\mathbf{x}_j^{(1)})'(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}\mathbf{x}_j^{(1)} - \text{tr}(c_{11}\mathbf{S}_{1j}^{\mathbf{x}} + c_{12}\mathbf{S}_2^{\mathbf{x}})^{-1}]^2\}^2}{p^2 N_1^2} \\ &= (\Delta_1 + \frac{2}{1-y})^2 + o(1). \end{aligned}$$

Then we complete the proof of Theorem 2.7. \square

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