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Errors-in-variables models with dependent measurements

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Abstract: Suppose that we observe $y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n \times m}$ in the following errors-in-variables model:

$$\begin{aligned} y &= X_0\beta^* + \epsilon \\ X &= X_0 + W \end{aligned}$$

where X_0 is an $n \times m$ design matrix with independent subgaussian row vectors, $\epsilon \in \mathbb{R}^n$ is a noise vector and W is a mean zero $n \times m$ random noise matrix with independent subgaussian column vectors, independent of X_0 and ϵ. This model is significantly different from those analyzed in the literature in the sense that we allow the measurement error for each covariate to be a dependent vector across its n observations. Such error structures appear in the science literature when modeling the trial-to-trial fluctuations in response strength shared across a set of neurons.

Under sparsity and restrictive eigenvalue type of conditions, we show that one is able to recover a sparse vector $\beta^* \in \mathbb{R}^m$ from the model given a single observation matrix X and the response vector y. We establish consistency in estimating β^* and obtain the rates of convergence in the ℓ_q norm, where $q = 1, 2$ for the Lasso-type estimator, and for $q \in [1, 2]$ for a Dantzig-type Conic programming estimator. We show error bounds which approach that of the regular Lasso and the Dantzig selector in case the errors in W are tending to 0. We analyze the convergence rates of the gradient descent methods for solving the nonconvex programs and show that the composite gradient descent algorithm is guaranteed to converge at a geometric rate to a neighborhood of the global minimizers: the size of the neighborhood is bounded by the statistical error in the ℓ_2 norm. Our analysis reveals interesting connections between computational and statistical efficiency and the concentration of measure phenomenon in random matrix theory. We provide simulation evidence illuminating the theoretical predictions.

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1. Introduction

The matrix variate normal model has a long history in psychology and social sciences. In recent years, it is becoming increasingly popular in biology and genomics, neuroscience, econometric theory, image and signal processing, wireless communication, and machine learning; see for example [15, 22, 17, 52, 5, 54, 18, 2, 26] and references therein. We call the random matrix X, which contains n rows and m columns a single data matrix, or one instance from the matrix variate normal distribution. We say that an $n \times m$ random matrix X follows a matrix normal distribution with a separable covariance matrix $\Sigma_X = A \otimes B$ and mean $M \in \mathbb{R}^{n\times m}$, which we write $X_{n\times m} \sim \mathcal{N}_{n,m}(M, A_{m\times m} \otimes B_{n\times n})$. This is equivalent to say $\mathrm{vec}\{\mathrm{X}\}$ follows a multivariate normal distribution with mean $\mathrm{vec}\{\mathrm{M}\}$ and covariance $\Sigma_X = A \otimes B$. Here, $\mathrm{vec}\{\mathrm{X}\}$ is formed by stacking the columns of X into a vector in $\mathbb{R}^{mn}$. Intuitively, A describes the covariance between columns of X, while B describes the covariance between rows of X. See [15, 22] for more characterization and examples.

In this paper, we introduce the related sum of Kronecker product models to encode the covariance structure of a matrix variate distribution. The proposed models and methods incorporate ideas from recent advances in graphical models, high-dimensional regression model with observation errors, and matrix decomposition. Let $A_{m\times m}, B_{n\times n}$ be symmetric positive definite covariance matrices. Denote the Kronecker sum of $A = (a_{ij})$ and $B = (b_{ij})$ by

$$\begin{aligned}\Sigma &= A \oplus B := A \otimes I_n + I_m \otimes B \\ &= \begin{bmatrix} a_{11}I_n + B & a_{12}I_n & \dots & a_{1m}I_n \\ a_{21}I_n & a_{22}I_n + B & \dots & a_{2m}I_n \\ \dots & & & \\ a_{m1}I_n & a_{m2}I_n & \dots & a_{mm}I_n + B \end{bmatrix}_{(mn)\times(mn)}\end{aligned}$$

where I_n is an $n \times n$ identity matrix. This covariance model arises naturally from the context of errors-in-variables regression model defined as follows.

Suppose that we observe $y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n\times m}$ in the following model:

$$y = X_0\beta^* + \epsilon \tag{1.1a}$$

$$X = X_0 + W \tag{1.1b}$$

where X_0 is an $n \times m$ design matrix with independent row vectors, $\epsilon \in \mathbb{R}^n$ is a noise vector and W is a mean zero $n \times m$ random noise matrix, independent of X_0 and ϵ, with independent column vectors $\omega^1, \dots, \omega^m$.

In particular, we are interested in the additive model of $X = X_0 + W$ such that

$$\mathrm{vec}\{\mathrm{X}\} \sim \mathcal{N}(0, \Sigma) \quad \text{where} \quad \Sigma = \mathrm{A} \oplus \mathrm{B} := \mathrm{A} \otimes \mathrm{I_n} + \mathrm{I_m} \otimes \mathrm{B} \tag{1.2}$$

where we use one covariance component $A \otimes I_n$ to describe the covariance of matrix $X_0 \in \mathbb{R}^{n\times m}$, which is considered as the *signal* matrix, and the other component $I_m \otimes B$ to describe that of the *noise matrix* $W \in \mathbb{R}^{n\times m}$, where $\mathbb{E}\omega^j \otimes \omega^j = B$ for all j, where ω^j denotes the j^{th} column vector of W. Our focus is on deriving the statistical properties of two estimators for estimating β^* in (1.1a) and (1.1b) despite the presence of the additive error W in the observation matrix X. We will show that our theory and analysis works with a model much more general than that in (1.2), which we will define in Section 1.1.

Before we go on to define our estimators, we now use an example to motiviate (1.2) and its subgaussian generalization in (1.4). Suppose that there are n patients in a particular study, for which we use X_0 to model the "systolic blood pressure" and W to model the seasonal effects. In this case, X models the fact that among the n patients we measure, each patient has its own row vector of observed set of blood pressures across time, and each column vector in W models the seasonal variation on top of the true signal at a particular day/time. Thus we consider X as measurement of X_0 with W being the observation error. That is, we model the seasonal effects on blood pressures across a set of patients in a particular study with a vector of dependent entries. Thus W is a matrix which consists of repeated independent sampling of spatially dependent vectors, if we regard the individuals as having spatial coordinates, for example, through their geographic locations. We will come back to discuss this example in Section 1.4.

1.1. The model and the method

We first need to define an independent isotropic vector with *subgaussian* marginals as in Definition 1.1. For a vector $y = (y_1, \dots, y_p)$ in $\mathbb{R}^p$, denote by $\|y\|_2 = \sqrt{\sum_j y_j^2}$ the length of y.

Definition 1.1. *Let Y be a random vector in $\mathbb{R}^p$*

1. *Y is called isotropic if for every $y \in \mathbb{R}^p$, $\mathbb{E}\left(|\langle Y, y\rangle|^2\right) = \|y\|_2^2$.*
2. *Y is ψ_2 with a constant α if for every $y \in \mathbb{R}^p$,*

$$\|\langle Y, y\rangle\|_{\psi_2} := \inf\{t : \mathbb{E}\left(\exp(\langle Y, y\rangle^2/t^2)\right) \le 2\} \le \alpha\|y\|_2. \quad (1.3)$$

The ψ_2 condition on a scalar random variable V is equivalent to the subgaussian tail decay of V, which means $\mathbb{P}(|V| > t) \le 2\exp(-t^2/c^2)$, for all $t > 0$.

Throughout this paper, we use ψ_2 vector, a vector with subgaussian marginals and subgaussian vector interchangeably.

The model. Let Z be an $n \times m$ random matrix with independent entries Z_{ij} satisfying $\mathbb{E}Z_{ij} = 0$, $1 = \mathbb{E}Z_{ij}^2 \le \|Z_{ij}\|_{\psi_2} \le K$. Let Z_1, Z_2 be independent copies of Z. Let

$$X = X_0 + W \quad (1.4)$$

such that $X_0 = Z_1 A^{1/2}$ is the design matrix with independent subgaussian row vectors, and $W = B^{1/2}Z_2$ is a random noise matrix with independent subgaussian column vectors.

Assumption (A1) allows the covariance model in (1.2) and its subgaussian variant in (1.4) to be identifiable.

(A1) We assume $\mathrm{tr}(A) = m$ is a known parameter, where $\mathrm{tr}(A)$ denotes the trace of matrix A.

In the Kronecker sum model, we could assume we know $\mathrm{tr}(B)$, in order not to assume knowing $\mathrm{tr}(A)$. Assuming one or the other is known is unavoidable as the covariance model is not identifiable otherwise. Moreover, by knowing $\mathrm{tr}(A)$, we can construct an estimator for $\mathrm{tr}(B)$:

$$\widehat{\mathrm{tr}}(B) = \tfrac{1}{m}\left(\|X\|_F^2 - n\mathrm{tr}(A)\right)_+ \quad \text{and define} \quad \widehat{\tau}_B := \tfrac{1}{n}\widehat{\mathrm{tr}}(B) \ge 0 \quad (1.5)$$

where $(a)_+ = a \vee 0$ and $\|X\|_F^2 := \sum_i \sum_j X_{ij}^2$. We first introduce the corrected Lasso estimator, adapted from those as considered in [30].

Suppose that $\widehat{\mathrm{tr}}(B)$ is an estimator for $\mathrm{tr}(B)$; for example, as constructed in (1.5). Let

$$\widehat{\Gamma} = \frac{1}{n}X^T X - \frac{1}{n}\widehat{\mathrm{tr}}(B) I_m \quad \text{and} \quad \widehat{\gamma} = \tfrac{1}{n}X^T y. \quad (1.6)$$

For a chosen penalization parameter $\lambda \ge 0$, and parameters b_0 and d, we consider the following regularized estimation with the ℓ_1-norm penalty,

$$\widehat{\beta} = \underset{\beta:\|\beta\|_1 \le b_0\sqrt{d}}{\arg\min} \frac{1}{2}\beta^T\widehat{\Gamma}\beta - \langle \widehat{\gamma}, \beta\rangle + \lambda\|\beta\|_1, \quad (1.7)$$

which is a variation of the Lasso [48] or the Basis Pursuit [12] estimator. Although in our analysis, we set $b_0 \ge \|\beta^*\|_2$ and $d = |\mathrm{supp}(\beta^*)| := \left|\{j : \beta_j^* \ne 0\}\right|$

for simplicity, in practice, both b_0 and d are understood to be parameters chosen to provide an upper bound on the ℓ_2 norm and the sparsity of the true β^*.

For a vector $\beta \in \mathbb{R}^m$, denote by $\|\beta\|_\infty := \max_j |\beta_j|$. Recently, [3] discussed the following conic programming compensated matrix uncertainly (MU) selector, which is a variant of the Dantzig selector [6, 35, 36]. Adapted to our setting, it is defined as follows. Let $\lambda, \mu, \tau > 0$,

$$\begin{aligned} \widehat{\beta} &= \arg\min \left\{ \|\beta\|_1 + \lambda t \ : \ (\beta, t) \in \Upsilon \right\} \text{ where} \\ \Upsilon &= \left\{ (\beta, t) \ : \ \beta \in \mathbb{R}^m, \left\| \widehat{\gamma} - \widehat{\Gamma}\beta \right\|_\infty \le \mu t + \omega, \|\beta\|_2 \le t \right\} \end{aligned} \tag{1.8}$$

where $\widehat{\gamma}$ and $\widehat{\Gamma}$ are as defined in (1.6) with $\mu \sim \sqrt{\frac{\log m}{n}}$, $\omega \sim \sqrt{\frac{\log m}{n}}$. We refer to this estimator as the Conic programming estimator from now on.

1.2. Gradient descent algorithms

In order to obtain fast, approximate solutions to the optimization goal as in (1.10), we adopt the computational framework of [1, 30], namely, the composite gradient descent method due to Nesterov [34] to analyze our computational and statistical errors in an integrated manner. First we denote the population and empirical loss functions by

$$\mathcal{L}(\beta) = \frac{1}{2}\beta^T \Sigma_x \beta - \beta^{*T} \Sigma_x \beta \qquad \text{and} \qquad \mathcal{L}_n(\beta) = \frac{1}{2}\beta^T \widehat{\Gamma}\beta - \widehat{\gamma}^T \beta \tag{1.9}$$

respectively. We consider regularizers that are separable across all coordinates and write

$$\rho_\lambda(\beta) = \sum_{i=1}^m \rho_\lambda(\beta_i).$$

Throughout this paper, we denote by

$$\phi(\beta) = \frac{1}{2}\beta^T \widehat{\Gamma}\beta - \widehat{\gamma}^T \beta + \rho_\lambda(\beta).$$

From the formulation (1.7), the corrected linear regression estimator is given by minimizing the penalized loss function $\phi(\beta)$ subject to the constraint that $g(\beta) \le R$:

$$\widehat{\beta} \in \underset{\beta \in \mathbb{R}^m, g(\beta) \le R}{\arg\min} \left\{ \frac{1}{2}\beta^T \widehat{\Gamma}\beta - \widehat{\gamma}^T \beta + \rho_\lambda(\beta) \right\} \tag{1.10}$$

where $g(\beta)$ is a convex function, which is allowed to be identical to $\|\beta\|_1$ and R is a second tuning parameter that is chosen to confine the solution $\widehat{\beta}$ within the ℓ_1 ball of radius R, while at the same time ensuring that β^* is a feasible solution.

The gradient descent method generates a sequence $\{\beta^t\}_{t=0}^{\infty}$ of iterates by first initializing to some parameter $\beta^0 \in \mathbb{R}^m$ and then for $t = 0, 1, 2, \ldots$, applying the recursive updates:

$$\beta^{t+1} = \underset{\beta\in\mathbb{R}^m, g(\beta)\le R}{\arg\min} \left\{ \mathcal{L}_n(\beta^t) + \langle \nabla \mathcal{L}_n(\beta^t), \beta - \beta^t \rangle + \frac{\zeta}{2} \left\|\beta - \beta^t\right\|_2^2 + \rho_\lambda(\beta) \right\} \tag{1.11}$$

where ζ is the step size parameter.

More generally, we consider loss function $\mathcal{L}_n : \mathbb{R}^m \to \mathbb{R}$ and ρ_λ which are possibly nonconvex and consider the regularized M-estimator of the form

$$\widehat{\beta} \in \underset{\beta\in\mathbb{R}^m, g(\beta)\le R}{\arg\min} \{\mathcal{L}_n(\beta; X) + \rho_\lambda(\beta)\} \tag{1.12}$$

where $\rho_\lambda : \mathbb{R}^m \to \mathbb{R}$ is a regularizer depending on a tuning parameter $\lambda > 0$. Because of this potential nonconvexity, we also include a side constraint in the form of $g(\beta) \le R$, where

$$g(\beta) := \frac{1}{\lambda} \left\{ \rho_\lambda(\beta) + \frac{\mu}{2} \|\beta\|_2^2 \right\} \tag{1.13}$$

so that this choice of g is convex for properly chosen parameter $\mu \ge 0$ for a class of weakly convex penalty functions ρ [51]; See Assumption 1 in [31] where properties of g and ρ_λ are stated in terms of the univariate function $\rho_\lambda : \mathbb{R} \to \mathbb{R}$ and the parameter $\mu \ge 0$. While our results hold for the general nonconvex penalty ρ_λ that is weakly convex in the sense that (1.13) holds for some parameter $\mu > 0$, we focus our discussion to the choice of $\rho_\lambda(\beta) = \lambda \|\beta\|_1$ and $\mu = 0$ in the present paper.

1.3. Our contributions

We provide a unified analysis of the rates of convergence for both the corrected Lasso estimator (1.7) and the Conic programming estimator (1.8), which is a Dantzig selector-type, although under slightly different conditions. We will show the rates of convergence in the ℓ_q norm for $q = 1, 2$ for estimating a sparse vector $\beta^* \in \mathbb{R}^m$ in the model (1.1a) and (1.1b) using the corrected Lasso estimator (1.7) in Theorems 3 and 6, and the Conic programming estimator (1.8) in Theorems 4 and 7 for $1 \le q \le 2$. We also show bounds on the predictive errors for the Conic programming estimator. The bounds we derive in Theorems 3 and 4 focus on cases where the errors in W are not too small in their magnitudes in the sense that $\tau_B := \mathrm{tr}(B)/n$ is bounded from below. For the extreme case when τ_B approaches 0, one hopes to recover bounds close to those for the regular Lasso or the Dantzig selector since the effect of the noise in matrix W on the procedure becomes negligible. We show in Theorems 6 and 7 that this is indeed the case. These results are new to the best of our knowledge.

Let Z_1, Z_2 be independent subgaussian random matrices with independent entries (cf. (1.4)). In Theorems 3 to 7, we consider the regression model in

(1.1a) and (1.1b) with subgaussian random design, where $X_0 = Z_1 A^{1/2}$ is a subgaussian random matrix with independent row vectors, and $W = B^{1/2} Z_2$ is an $n \times m$ random noise matrix with independent column vectors, This model is significantly different from those analyzed in the literature. For example, unlike the present work, the authors in [30] apply Theorem 16 which states a general result on statistical convergence properties of the estimator (1.7) to cases where W is composed of independent subgaussian row vectors, when the row vectors of X_0 are either independent or follow a Gaussian vector auto-regressive model. See also [35, 36, 3] for the corresponding results on the compensated MU selectors, variations on the Conic programming estimator (1.8).

The second key difference between our framework and the existing work is that we assume that only one observation matrix X with the single measurement error matrix W is available. Assuming (A1) allows us to estimate $\mathbb{E}W^T W$ as required in the estimation procedure (1.6) directly, given the knowledge that W is composed of independent column vectors. In contrast, existing work needs to assume that the covariance matrix $\Sigma_W := \frac{1}{n}\mathbb{E}W^T W$ of the independent row vectors of W or its functionals are either known a priori, or can be estimated from a dataset independent of X, or from replicated X measuring the same X_0; see for example [35, 36, 3, 30, 10]. Although the model we consider is different from those in the literature, the identifiability issue, which arises from the fact that we observe the data under an additive error model, is common. Such repeated measurements are not always available or costly to obtain in practice [10]. We will explore such tradeoffs in future work.

A noticeable exception is the work of [11], which deals with the scenario when the noise covariance is not assumed to be known. We now elaborate on their result, which is a variant of the orthogonal matching pursuit (OMP) algorithm [49, 50]. Their support recovery result, that is, recovering the support set of β^*, applies only to the case when both signal matrix and the measurement error matrix have isotropic subgaussian row vectors. In other words, they assume independence among both rows and columns in X (X_0 and W). Moreover, their algorithm requires the knowledge of the sparsity parameter d, which is the number of non-zero entries in β^*, as well as a $\beta_{\min}$ condition: $\min_{j \in \operatorname{supp} \beta^*} |\beta_j^*| = \Omega\left(\sqrt{\frac{\log m}{n}}(\|\beta^*\|_2 + 1)\right)$. Under these conditions, they recover essentially the same ℓ_2-error bounds as in the current work, and [30], where the covariance Σ_W is assumed to be known.

Finally, we present in Theorems 2 and 9 the optimization error for the gradient descent algorithms in solving (1.12) and more specifically (1.7). Let $\widehat{\beta}$ be a global optimizer of (1.12). Let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ be the largest and smallest eigenvalues, and $\kappa(A)$ be the condition number for matrix A. Let $0 < \kappa < 1$ be a contraction factor to be defined in (2.6). Similar to the work of [1, 30], we show that the geometric convergence is not guaranteed to an arbitrary precision, but only to an accuracy related to statistical precision of the problem, measured by the ℓ_2 error: $\|\widehat{\beta} - \beta^*\|_2^2 =: \varepsilon_{\text{stat}}^2$ between the global optimizer $\widehat{\beta}$ and the true parameter β^*.

More precisely, our analysis guarantees geometric convergence of the sequence $\{\beta^t\}_{t=0}^{\infty}$ to a parameter β^* up to a neighborhood of radius defined through the statistical error bound $\varepsilon_{\text{stat}}^2$

$$\delta^2 \asymp \frac{\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{d\log m}{n},$$

where κ is a contraction coefficient to be defined (2.6), so that for all $t \geq T^*(\delta)$ as in (2.12), $\alpha_\ell \asymp \lambda_{\min}(A)$ and $\alpha_u \asymp \lambda_{\max}(A)$,

$$\left\|\beta^t - \widehat{\beta}\right\|_2^2 \leq \frac{4\delta^2}{\alpha_\ell} + \frac{\alpha_\ell \varepsilon_{\text{stat}}^2}{4} + \frac{4\delta^4}{b_0^2 \alpha_\ell \lambda_{\max}(A)} = O(\varepsilon_{\text{stat}}^2)$$

for $\lambda, \zeta \geq \alpha_u$ appropriately chosen, $R = \widetilde{O}(\sqrt{\frac{n}{\log m}})$ and $n = \widetilde{\Omega}\,(d\log m)$, where the $\widetilde{O}(\cdot)$ and $\widetilde{\Omega}(\cdot)$ symbols hide spectral parameters regarding A and B. To quantify such results, we first need to introduce some conditions in Section 2. See Theorem 2 and Corollary 10 for the precise conditions and statements.

1.4. Discussion

The theory on matrix variate normal data show that having replicates will allow one to estimate more complicated graphical structures and achieve faster rates of convergence under less restrictive assumptions [56]. Our consistency results in the present work deal with only a single random matrix following the model (1.4), assuming that $\text{tr}(A)$ is known. With replicates, this assumption can be lifted off immediately. Assume there exists a replicate

$$\widetilde{X} = X_0 + \widetilde{W}, \tag{1.14}$$

then we can use $\widetilde{X} - X = \widetilde{W} - W$ to estimate B using existing methods. The rationale for considering such an option is one may have a repeated measurement of X_0 for which the errors W and $\widetilde{W}$ follow the same error distribution. Such external data or knowledge of the noise distribution is needed in order to do inference under such additive measurement error model [10].

The second key modeling question is: would each row vector in W for a particular patient across all time points be a correlated normal or subgaussian vector as well? It is our conjecture that combining the newly developed techniques, namely, the concentration of measure inequalities we have derived in the current framework with techniques from existing work [56], we can handle the case when W follows a matrix normal distribution with a separable covariance matrix $\Sigma_W = C \otimes B$, where C is an $m \times m$ positive semi-definite covariance matrix. Moreover, for this type of "seasonal effects" as the measurement errors, the time varying covariance model would make more sense to model W, which we elaborate in the second example.

In neuroscience applications, population encoding refers to the information contained in the combined activity of multiple neurons [27]. The relationship

between population encoding and correlations is complicated and is an area of active investigation, see for example [40, 13]. It becomes more often that repeated measurements (trials) simultaneously recorded across a set of neurons and over an ensemble of stimuli are available. In this context, one can use a random matrix $X_0 \sim \mathcal{N}_{n,m}(\mu, A \otimes B)$ which follows a matrix-variate normal distribution, or its subgaussian correspondent, to model the ensemble of mean response variables, e.g., the membrane potential, corresponding to the cross-trial average over a set of experiments. Here we use A to model the task correlations and B to model the baseline correlation structure among all pairs of neurons at the *signal* level. It has been observed that the onset of stimulus and task events not only change the cross-trial mean response in μ, but also alter the structure and correlation of the *noise* for a set of neurons, which correspond to the trial-to-trial fluctuations of the neuron responses. We use W to model such task-specific trial-to-trial fluctuations of a set of neurons recorded over the time-course of a variety of tasks. Models as in (1.1a) and (1.1b) are useful in predicting the response of set of neurons based on the current and past mean responses of all neurons. Moreover, we could incorporate non-i.i.d. non-Gaussian $W = [w_1, \ldots, w_m]$ with $w_t = B^{1/2}(t)z(t)$, where $z(1), \ldots, z(m)$ are independent isotropic subgaussian random vectors and $B(t) \succ 0$ for all t, to model the time-varying correlated noise as observed in the trial-to-trial fluctuations. It is possible to combine the techniques developed in the present paper with those in [57, 56] to develop estimators for A, B and the time varying $B(t)$, which is itself an interesting topic, however, beyond the scope of the current work.

In summary, oblivion in Σ_W and a general dependency condition in the data matrix X are not simultaneously allowed in existing work. In contrast, while we assume that X_0 is composed of independent subgaussian row vectors, we allow rows of W to be dependent, which brings dependency to the row vectors of the observation matrix X. In the current paper, we focus on the proof-of-the-concept on using the Kronecker sum covariance and additive model to model two way dependency in data matrix X, and derive bounds in statistical and computational convergence for (1.7) and (1.8). In some sense, we are considering a parsimonious model for fitting observation data with two-way dependencies: we use the signal matrix to encode column-wise dependency among covariates in X, and error matrix W to explain its row-wise dependency. When replicates of X or W are available, we are able to study more sophisticated models and inference problems, some of which are described earlier in this section.

We leave the investigation of this more general modeling framework and relevant statistical questions to future work. We refer to [10] for an excellent survey of the classical as well as modern developments in measurement error models. In future work, we will also extend the estimation methods to the settings where the covariates are measured with multiplicative errors which are shown to be reducible to the additive error problem as studied in the present work [36, 30]. Moreover, we are interested in applying the analysis and concentration of measure results developed in the current paper and in our ongoing work to the more general contexts and settings where measurement error models are introduced

and investigated; see for example [16, 8, 44, 24, 20, 45, 9, 7, 14, 46, 25, 28, 47, 53, 23, 29, 32, 2, 43, 41, 42] and references therein.

Notation. Let $e_1, \ldots, e_p$ be the canonical basis of $\mathbb{R}^p$. For a set $J \subset \{1, \ldots, p\}$, denote $E_J = \mathsf{span}\{e_j : j \in J\}$. For a matrix A, we use $\|A\|_2$ to denote its operator norm. For a set $V \subset \mathbb{R}^p$, we let $\operatorname{conv} V$ denote the convex hull of V. For a finite set Y, the cardinality is denoted by $|Y|$. Let B_1^p, B_2^p and S^{p-1} be the unit ℓ_1 ball, the unit Euclidean ball and the unit sphere respectively. For a matrix $A = (a_{ij})_{1 \le i,j \le m}$, let $\|A\|_{\max} = \max_{i,j} |a_{ij}|$ denote the entry-wise max norm. Let $\|A\|_1 = \max_j \sum_{i=1}^m |a_{ij}|$ denote the matrix ℓ_1 norm. The Frobenius norm is given by $\|A\|_F^2 = \sum_i \sum_j a_{ij}^2$. Let $|A|$ denote the determinant and $\operatorname{tr}(A)$ be the trace of A. The operator or ℓ_2 norm $\|A\|_2^2$ is given by $\lambda_{\max}(AA^T)$. For a matrix A, denote by $r(A)$ the effective rank $\operatorname{tr}(A)/\|A\|_2$. Let $\|A\|_F^2/\|A\|_2^2$ denote the stable rank for matrix A. We write $\operatorname{diag}(A)$ for a diagonal matrix with the same diagonal as A. For a symmetric matrix A, let $\Upsilon(A) = (v_{ij})$ where $v_{ij} = \mathbb{I}(a_{ij} \neq 0)$, where $\mathbb{I}(\cdot)$ is the indicator function. Let I be the identity matrix. For two numbers a, b, $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. For a function $g : \mathbb{R}^m \to \mathbb{R}$, we write ∇g to denote a gradient or subgradient, if it exists. We write $a \asymp b$ if $ca \le b \le Ca$ for some positive absolute constants c, C which are independent of n, m or sparsity parameters. Let $(a)_+ := a \vee 0$. We write $a = O(b)$ if $a \le Cb$ for some positive absolute constants C which are independent of n, m or sparsity parameters. The absolute constants $C, C_1, c, c_1, \ldots$ may change line by line.

2. Assumptions and preliminary results

We will now define some parameters related to the restricted and sparse eigenvalue conditions that are needed to state our main results. We also state a preliminary result in Lemma 1 regarding the relationships between the two conditions in Definitions 2.1 and 2.2.

Definition 2.1. (Restricted eigenvalue condition $\mathsf{RE}(s_0, k_0, A)$). *Let $1 \le s_0 \le p$, and let k_0 be a positive number. We say that a $q \times p$ matrix A satisfies $\mathsf{RE}(s_0, k_0, A)$ condition with parameter $K(s_0, k_0, A)$ if for any $v \neq 0$,*

$$\frac{1}{K(s_0, k_0, A)} := \min_{\substack{J \subseteq \{1,\ldots,p\},\ \|v_{J^c}\|_1 \le k_0 \|v_J\|_1 \\ |J| \le s_0}} \frac{\|Av\|_2}{\|v_J\|_2} > 0.$$

where v_J represents the subvector of $v \in \mathbb{R}^p$ confined to a subset J of $\{1, \ldots, p\}$.

It is clear that when s_0 and k_0 become smaller, this condition is easier to satisfy. We also consider the following variation of the baseline RE condition.

Definition 2.2. (Lower-RE condition) [30] *The matrix Γ satisfies a Lower-RE condition with curvature $\alpha > 0$ and tolerance $\tau > 0$ if*

$$\theta^T \Gamma \theta \ge \alpha \|\theta\|_2^2 - \tau \|\theta\|_1^2 \quad \forall \theta \in \mathbb{R}^m.$$

where $\|\theta\|_1 := \sum_j |\theta_j|$. As α becomes smaller, or as τ becomes larger, the Lower-RE condition is easier to be satisfied.

Lemma 1. *Suppose that the Lower-*RE *condition holds for* $\Gamma := A^T A$ *with* $\alpha, \tau > 0$ *such that* $\tau(1+k_0)^2 s_0 \le \alpha/2$. *Then the* $\mathsf{RE}(s_0, k_0, A)$ *condition holds for* A *with*

$$\frac{1}{K(s_0, k_0, A)} \ge \sqrt{\frac{\alpha}{2}} > 0.$$

Assume that $\mathsf{RE}((k_0+1)^2, k_0, A)$ *holds. Then the Lower-*RE *condition holds for* $\Gamma = A^T A$ *with*

$$\alpha = \frac{1}{(k_0+1)K^2(s_0, k_0, A)} > 0$$

where $s_0 = (k_0+1)^2$, *and* $\tau > 0$ *which satisfies*

$$\lambda_{\min}(\Gamma) \ge \alpha - \tau s_0/4.$$

The condition above holds for any $\tau \ge \frac{4}{(k_0+1)^3 K^2(s_0,k_0,A)} - \frac{4\lambda_{\min}(\Gamma)}{(k_0+1)^2}$.

The first part of Lemma 1 means that, if k_0 is fixed, then smaller values of τ guarantee $\mathsf{RE}(s_0, k_0, A)$ holds with larger s_0, that is, a stronger RE condition. The second part of the Lemma implies that a weak RE condition implies that the Lower-RE (LRE) holds with a large τ. On the other hand, if one assumes $\mathsf{RE}((k_0+1)^2, k_0, A)$ holds with a large value of k_0 (in other words, a strong RE condition), this would imply LRE with a small τ. In short, the two conditions are similar but require tweaking the parameters. Weaker RE condition implies LRE condition holds with a larger τ, and Lower-RE condition with a smaller τ, that is, stronger LRE implies stronger RE. We prove Lemma 1 in Section 9.

Definition 2.3. (Upper-RE condition) [30] *The matrix* Γ *satisfies an upper-*RE *condition with smoothness* $\widetilde{\alpha} > 0$ *and tolerance* $\tau > 0$ *if*

$$\theta^T \Gamma \theta \le \widetilde{\alpha} \|\theta\|_2^2 + \tau \|\theta\|_1^2 \quad \forall \theta \in \mathbb{R}^m.$$

Definition 2.4. *Define the largest and smallest d-sparse eigenvalue of a* $p \times q$ *matrix* A *to be*

$$\rho_{\max}(d, A) := \max_{t \ne 0; d-sparse} \|At\|_2^2 / \|t\|_2^2, \text{ where } d < p,$$
$$\text{and } \rho_{\min}(d, A) := \min_{t \ne 0; d-sparse} \|At\|_2^2 / \|t\|_2^2.$$

Before stating some general result for the optimization program (1.12) and its implications for the Lasso-type estimator (1.7) in terms of statistical and optimization errors, we need to introduce some more notation and the following assumptions. Let $a_{\max} = \max_i a_{ii}$ and $b_{\max} = \max_i b_{ii}$ be the maximum diagonal entries of A and B respectively. In general, under (A1), one can think of $\lambda_{\min}(A) \le 1$ and for $s \ge 1$,

$$1 \le a_{\max} \le \rho_{\max}(s, A) \le \lambda_{\max}(A), \tag{2.1}$$

where $\lambda_{\max}(A)$ denotes the maximum eigenvalue of A.

(A2) The minimal eigenvalue $\lambda_{\min}(A)$ of the covariance matrix A is bounded: $1 \geq \lambda_{\min}(A) > 0$.

(A3) Moreover, we assume that the condition number $\kappa(A)$ is upper bounded by $O\left(\sqrt{\frac{n}{\log m}}\right)$ and $\tau_B = O(\lambda_{\max}(A))$.

Throughout the rest of the paper, $s_0 \geq 32$ is understood to be the largest integer chosen such that the following inequality still holds:

$$\sqrt{s_0}\varpi(s_0) \leq \frac{\lambda_{\min}(A)}{32C}\sqrt{\frac{n}{\log m}} \quad \text{where} \quad \varpi(s_0) := \rho_{\max}(s_0, A) + \tau_B \tag{2.2}$$

where we denote by $\tau_B = \operatorname{tr}(B)/n$ and C is to be defined. Denote by

$$M_A = \frac{64C\varpi(s_0)}{\lambda_{\min}(A)} \geq 64C. \tag{2.3}$$

Throughout this paper, we denote by $\mathcal{A}_0$ the event that the modified gram matrix $\widehat{\Gamma}$ as defined in (1.6) satisfies the Lower as well as Upper RE conditions with

$$\begin{aligned} \text{curvature} \qquad & \alpha = \frac{5}{8}\lambda_{\min}(A), \text{ smoothness } \widetilde{\alpha} = \frac{11}{8}\lambda_{\max}(A) \\ \text{and tolerance} \qquad & \frac{384C^2\varpi(s_0)^2}{\lambda_{\min}(A)}\frac{\log m}{n} \leq \tau := \frac{\lambda_{\min}(A) - \alpha}{s_0} \\ & \leq \frac{396C^2\varpi^2(s_0+1)}{\lambda_{\min}(A)}\frac{\log m}{n} \end{aligned}$$

for $\alpha, \widetilde{\alpha}$ and τ as defined in Definitions 2.2 and 2.3, and $C, s_0, \varpi(s_0)$ in (2.2).

To bound the optimization errors, we show that the corrected linear regression loss function (1.9) satisfies the following *Restricted Strong Convexity (RSC)* and *Restricted Smoothness (RSM)* conditions when the sample size and effective rank of matrix B satisfy certain lower bounds (cf. Theorem 3); namely, for all vectors $\beta_0, \beta_1 \in \mathbb{R}^m$ and

$$\mathcal{T}(\beta_1, \beta_0) \quad := \quad \mathcal{L}_n(\beta_1) - \mathcal{L}_n(\beta_0) - \langle \nabla\mathcal{L}_n(\beta_0), \beta_1 - \beta_0 \rangle ,$$

we show that for some parameters $(\alpha_\ell, \tau_\ell(\mathcal{L}_n))$ and $(\alpha_u, \tau_u(\mathcal{L}_n))$,

$$\mathcal{T}(\beta_1, \beta_0) \quad \geq \quad \frac{\alpha_\ell}{2}\|\beta_1 - \beta_0\|_2^2 - \tau_\ell(\mathcal{L}_n)\|\beta_1 - \beta_0\|_1^2 \quad \text{and} \tag{2.4}$$

$$\mathcal{T}(\beta_1, \beta_0) \quad \leq \quad \frac{\alpha_u}{2}\|\beta_1 - \beta_0\|_2^2 + \tau_u(\mathcal{L}_n)\|\beta_1 - \beta_0\|_1^2 . \tag{2.5}$$

Applied to (1.12), the composite gradient descent procedure of [34] produces a sequence of iterates $\{\beta^t\}_{t=0}^{\infty}$ via the updates

$$\beta^{t+1} = \underset{\beta\in\mathbb{R}^m, g(\beta)\leq R}{\arg\min}\left\{\frac{1}{2}\left\|\beta - \left(\beta^t - \frac{\nabla\mathcal{L}_n(\beta^t)}{\zeta}\right)\right\|_2^2 + \frac{\rho_\lambda(\beta)}{\zeta}\right\}$$

where $\frac{1}{\zeta}$ is the step size. Let $\nu_\ell = 64d\tau_\ell(\mathcal{L}_n)$ and $\bar{\alpha}_\ell := \alpha_\ell - \nu_\ell$. We show that the composite gradient updates exhibit a type of globally geometric convergence in terms of the compound contraction coefficient

$$\kappa \quad = \quad \frac{1 - \frac{\bar{\alpha}_\ell}{4\zeta} + \varrho}{1 - \varrho}, \quad \text{where} \quad \varrho := \frac{2\nu(d,m,n)}{\alpha_\ell - \nu_\ell} := \frac{128d\tau_u(\mathcal{L}_n)}{\bar{\alpha}_\ell} \tag{2.6}$$

where $\nu_\ell < \alpha_\ell / C$ for some $C > 1$ to be specified. Let $\tau(\mathcal{L}_n) = \tau_\ell(\mathcal{L}_n) \vee \tau_u(\mathcal{L}_n)$. Define

$$\xi \quad := \quad \frac{2\tau(\mathcal{L}_n)}{1-\varrho}\left(\frac{\bar{\alpha}_\ell}{4\zeta} + 2\varrho + 5\right) > 10\tau(\mathcal{L}_n). \tag{2.7}$$

For simplicity, we present in Theorem 2 the case for $\rho_\lambda(\beta) = \lambda \|\beta\|_1$ only.

Theorem 2. *Consider the optimization program (1.12) for a radius R such that β^* is feasible. Let $g(\beta) = \frac{1}{\lambda}\rho_\lambda(\beta)$ where $\rho_\lambda(\beta) = \lambda\|\beta\|_1$. Suppose that the loss function $\mathcal{L}_n$ satisfies the RSC/RSM conditions (2.4) and (2.5) with parameters $(\alpha_\ell, \tau_\ell(\mathcal{L}_n))$ and $(\alpha_u, \tau_u(\mathcal{L}_n))$ respectively. Let ϱ, κ and ξ be defined as in (2.6) and (2.7) respectively. Suppose that the regularization parameter is chosen such that for $\zeta \geq \alpha_u$*

$$\lambda \quad \geq \quad \max\left\{12\left\|\nabla\mathcal{L}_n(\beta^*)\right\|_{\max}, \frac{16R\xi}{(1-\kappa)}\right\}. \tag{2.8}$$

Suppose that $\kappa < 1$. Suppose that $\widehat{\beta}$ is a global minimizer of (1.12). Then for any step size parameter $\zeta \geq \alpha_u$ and tolerance parameter

$$\delta^2 \quad \geq \quad \frac{c\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{d\log m}{n} =: \bar{\delta}^2, \qquad \text{where} \quad \varepsilon_{\text{stat}}^2 = \left\|\widehat{\beta} - \beta^*\right\|_2^2, \tag{2.9}$$

the following hold for all $t \geq T^*(\delta)$

$$\phi(\beta^t) - \phi(\widehat{\beta}) \quad \leq \quad \delta^2, \quad \text{and for} \quad \epsilon^2 = \frac{16\delta^4}{\lambda^2} \wedge 4R^2, \tag{2.10}$$

$$\left\|\beta^t - \widehat{\beta}\right\|_2^2 \quad \leq \quad \frac{2}{\bar{\alpha}_\ell}\left(\delta^2 + 4\nu\varepsilon_{\text{stat}}^2 + 4\tau(\mathcal{L}_n)\epsilon^2\right), \tag{2.11}$$

where $\nu = 64d\tau(\mathcal{L}_n)$, $\tau(\mathcal{L}_n) \asymp \frac{\log m}{n}$, and

$$T^*(\delta) = \frac{2\log(\frac{\phi(\beta^0)-\phi(\widehat{\beta})}{\delta^2})}{\log(1/\kappa)} + \log\log\left(\frac{\lambda R}{\delta^2}\right)\left(1 + \frac{\log 2}{\log(1/\kappa)}\right). \tag{2.12}$$

We prove Theorem 2 in Section B. Theorem 2 is similar in spirit to the main result Theorem 2 in [1] that deals with a convex loss function, and Theorem 3 in [31] on a similar setting to the present work. Compared to [31], we simplified the condition on λ by not imposing an upper bound. Moreover, we present refined analysis on the sample requirement and illuminate its dependence upon

the condition number $\kappa(A)$ and the tolerance parameter τ when applied to the corrected linear regression problem (1.10). It is understood throughout the paper that for the same C as in (2.3),

$$\tau \asymp \tau_0 \frac{\log m}{n}, \quad \text{where} \quad \tau_0 \asymp \frac{400C^2\varpi(s_0+1)^2}{\lambda_{\min}(A)} \approx M_A^2 \lambda_{\min}(A)/10 \tag{2.13}$$

and it is helpful to consider M_A as being upper bounded by $O(\kappa(A))$ in view of (2.1) and (A3). Toward this end, we prove in Section 5 that under event $\mathcal{A}_0 \cap \mathcal{B}_0$, the RSC and RSM conditions as stated in Theorem 2 hold with $\alpha_\ell \asymp \lambda_{\min}(A)$ and $\alpha_u \asymp \lambda_{\max}(A)$ and $\tau_\ell(\mathcal{L}_n) = \tau_u(\mathcal{L}_n) \asymp \tau$; then we have for all $t \geq T^*(\delta)$ as defined in (2.12) and for $\delta^2 \asymp \frac{\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{d\log m}{n}$,

$$\left\|\beta^t - \widehat{\beta}\right\|_2^2 \quad \leq \quad \frac{4}{\alpha_\ell}\delta^2 + \frac{\alpha_\ell}{4}\varepsilon_{\text{stat}}^2 + O\left(\frac{\delta^2\varepsilon_{\text{stat}}^2}{b_0^2}\right), \tag{2.14}$$

where $0 < \kappa < 1$ so long as $\zeta \asymp \lambda_{\max}(A)$ and $n = \Omega(\kappa(A)M_A^2 d\log m)$.

We now check the conditions on λ in Theorem 2. First, we note that both types of conditions on λ are also required in the present paper for the statistical error bounds shown in Theorems 3 and 6. We state in Theorem 16 a deterministic result from [30] on the statistical error for the corrected linear model, which requires that

$$\lambda \geq 2\left\|\nabla\mathcal{L}_n(\beta^*)\right\|_{\max} \quad \text{and} \quad \lambda \geq 4b_0\sqrt{d}\tau \asymp 4R\tau \text{ for } \tau := \tau_0\frac{\log m}{n} \tag{2.15}$$

as defined in (2.13) and $d\tau \leq \frac{\alpha_\ell}{32}$ in order to obtain the statistical error bound for the corrected linear model at the order of

$$\varepsilon_{\text{stat}}^2 = \left\|\widehat{\beta} - \beta^*\right\|_2^2 \asymp \frac{400}{\alpha_\ell^2}\lambda^2 d.$$

Under suitable conditions on the sample size n and the effective rank of matrix B to be stated in Theorem 3, we show that for the loss function (1.9), the RSC and RSM conditions hold under event $\mathcal{A}_0$ (cf. Lemma 15) following the Lower and Upper-RE conditions as derived in Lemma 15,

$$\bar{\alpha}_\ell \approx \alpha_\ell \asymp \frac{\lambda_{\min}(A)}{2}, \quad \alpha_u \asymp \frac{3\lambda_{\max}(A)}{2}, \quad \text{and} \quad \tau(\mathcal{L}_n) \asymp \tau.$$

Compared with the lower bound imposed on λ as in (2.15) that we use to derive statistical error bounds, the penalty now involves a term $\frac{\xi}{1-\kappa}$ that crucially depends on the condition number $\kappa(A)$ in (2.8); Assuming that $\zeta \geq \alpha_u$, then the second condition in (2.8) on λ implies that

$$\begin{aligned} \lambda &= \Omega(R\tau(\mathcal{L}_n)\kappa(A)) \quad \text{given} \\ \frac{\xi}{1-\kappa} &\geq 40\tau(\mathcal{L}_n)\frac{\zeta}{\bar{\alpha}_\ell} + 2\tau(\mathcal{L}_n) \asymp \tau\kappa(A), \end{aligned} \tag{2.16}$$

which now depends explicitly on the condition number $\kappa(A)$ in addition to the radius $R \asymp b_0\sqrt{d}$ and the tolerance parameter τ. This is expected given that both RSC and RSM conditions are needed in order to derive the computational convergence bounds, while for the statistical error, we only require the RSC (Lower RE) condition to hold.

Remarks. Consider the regression model in (1.1a) and (1.1b) with independent random matrices X_0, W as in (1.4), and an error vector $\epsilon \in \mathbb{R}^n$ independent of X_0, W, with independent entries ϵ_j satisfying $\mathbb{E}\epsilon_j = 0$ and $\|\epsilon_j\|_{\psi_2} \leq M_\epsilon$. Theorem 12 and its corollaries provide an upper bound on the ℓ_∞ norm of the gradient $\nabla\mathcal{L}_n(\beta^*) = \widehat{\Gamma}\beta^* - \widehat{\gamma}$ of the loss function in the corrected linear model, where $\widehat{\Gamma}$ and $\widehat{\gamma}$ are as defined in (1.6). Let

$$D_0' = \|B\|_2^{1/2} + a_{\max}^{1/2}, \quad \text{and} \quad D_{\text{oracle}} = 2(\|A\|_2^{1/2} + \|B\|_2^{1/2}). \tag{2.17}$$

Specializing to the case of corrected linear models, we have by Corollary 14, on event $\mathcal{B}_0$ as defined therein,

$$\|\nabla\mathcal{L}_n(\beta^*)\|_\infty = \left\|\widehat{\Gamma}\beta^* - \widehat{\gamma}\right\|_\infty \leq \psi\sqrt{\frac{\log m}{n}}$$

where $\psi := C_0 D_0' K\left(M_\epsilon + \tau_B^{+/2} K \|\beta^*\|_2\right)$ and $\tau_B^{+/2} = \tau_B^{1/2} + \frac{D_{\text{oracle}}}{\sqrt{m}}$ for D_0', D_{oracle} as defined in (2.17).

The bound (2.10) characterizes the excess loss $\phi(\beta^t) - \phi(\widehat{\beta})$ for solving (1.7) using the composite gradient algorithm; moreover, for any iterate β^t such that (2.10) holds, the following bound on the optimization error $\beta^t - \widehat{\beta}$ follows immediately:

$$\left\|\beta^t - \widehat{\beta}\right\|_2^2 \leq \frac{2}{\bar{\alpha}_\ell}\left(\delta^2 + 4\nu\varepsilon_{\text{stat}}^2 + \frac{64\tau_\ell(\mathcal{L}_n)\delta^4}{\lambda^2}\right),$$

where $\nu = 64d\tau(\mathcal{L}_n)$ and $4\tau(\mathcal{L}_n)\epsilon^2 = 64\tau(\mathcal{L}_n)\frac{\delta^4}{\lambda^2}$ by definition of ϵ^2 in view of (2.11). Finally, we note that Theorem 2 holds for a class of weakly convex penalties as considered in [31] with suitable adaptation of RSC and parameters and conditions to involve μ, following exactly the same sequence of arguments. Notable examples of such weakly convex penalty functions are SCAD [19] and MCP [55].

The rest of the paper is organized as follows. In Section 3, we present two main results in Theorems 3 and 4. In Section 4, we state more precise results which improve upon Theorems 3 and 4; these results are more precise in the sense that our bounds and penalty parameters now take $\text{tr}(B)$, the parameter that measures the magnitudes of errors in W, into consideration. In Section 5, we show that the RSC and RSM conditions hold for the corrected linear loss function and present our computational convergence bounds with regard to (1.7) in Theorem 9 and Corollary 10. In Section 6, we outline the proof of the main theorems. In particular, we outline the proofs for Theorems 3, 4, 6 and 7 in Section 6, 6.4 and 6.6 respectively. In Section 7, we show a deterministic result as well as its application to the random matrix $\widehat{\Gamma} - A$ for $\widehat{\Gamma}$ as in (1.6) with

regards to the upper and Lower RE conditions. In Section 8, we present results from numerical simulations designed to validate the theoretical predictions in previous sections. The technical details of proofs are collected at the end of the paper. We prove Theorem 3 in Section 10. We prove Theorem 4 in Section 11. We prove Theorems 6 and 7 in Section 12 and Section 13 respectively. We defer the proof of Theorem 2 to Section B. The paper concludes with a discussion of the results in Section 16. We list a set of symbols we use throughout the paper in Table 1. Additional proofs and theoretical results are collected in the Appendix.

TABLE 1
Symbols we used throughout the proof

Symbol	Definition
α	curvature: $\alpha := \frac{5}{8}\lambda_{\min}(A)$
α_ℓ	Lower RE/ RSC curvature parameter: $\alpha_\ell = \alpha$
α_u	Upper RE/ RSM parameter $\alpha_u \asymp \frac{3}{2}\lambda_{\max}(A)$
$\bar{\epsilon}_{\text{stat}}$	$\bar{\epsilon}_{\text{stat}} = 8\sqrt{d}\varepsilon_{\text{stat}}$ where $\varepsilon_{\text{stat}} = \left\|\widehat{\beta} - \beta^*\right\|_2$
τ_0	$\tau_0 \asymp \frac{400C^2\varpi(s_0+1)^2}{\lambda_{\min}(A)}$
$\tau = \frac{\lambda_{\min}(A)-\alpha}{s_0}$	tolerance parameter $\tau = \tau_0\frac{\log m}{n}$ in Lower/Upper RE conditions
τ_B	$\tau_B = \text{tr}(B)/n$
$s_0 \geq 1$	the largest integer chosen such that the following inequality still holds: $\sqrt{s_0}\varpi(s_0) \leq \frac{\lambda_{\min}(A)}{32C}\sqrt{\frac{n}{\log m}}$
$\varpi(s_0)$	$\rho_{\max}(s_0, A) + \tau_B$
$\tau_\ell(\mathcal{L}_n)$	tolerance parameter in RSC condition: $\tau_\ell(\mathcal{L}_n) \asymp \tau_0 \log m/n$
$\tau_u(\mathcal{L}_n)$	tolerance parameters in RSM condition: $\tau_u(\mathcal{L}_n) \asymp \tau_0 \log m/n$
ν_ℓ	$\nu_\ell = 64d\tau_\ell(\mathcal{L}_n) < \frac{\alpha_\ell}{60}$
$\nu(d,m,n)$	$\nu(d,m,n) = 64d\tau_u(\mathcal{L}_n)$
$\bar{\alpha}_\ell$	effective RSC coefficient $\bar{\alpha}_\ell = \alpha_\ell - \nu_\ell$
$\phi(\beta)$	loss function: $\phi(\beta) = \frac{1}{2}\beta^T\widehat{\Gamma}\beta - \widehat{\gamma}^T + \rho_\lambda(\beta)$
$\nabla\mathcal{L}_n(\beta)$	Gradient of the loss function $\widehat{\Gamma}\beta - \widehat{\gamma}$
ρ_n	$\rho_n = C_0K\sqrt{\frac{\log m}{n}}$
$r_{m,n}$	$r_{m,n} = 2C_0K^2\sqrt{\frac{\log m}{mn}}$
ζ	step size parameter: $\zeta \geq \alpha_u = 11\lambda_{\max}/8$
ϱ	contraction parameter $\varrho := \frac{2\nu(d,m,n)}{\bar{\alpha}_\ell} = \frac{128d\tau_u(\mathcal{L}_n)}{\bar{\alpha}_\ell} < \frac{\bar{\alpha}_\ell}{8\zeta}$
κ	contraction coefficient as $\kappa := (1 - \frac{\bar{\alpha}_\ell}{4\zeta} + \varrho)(1-\varrho)^{-1} < 1$
δ^2	tolerance parameter in computational errors $\delta^2 \geq \frac{c\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{d\log p}{n}$
M_A	$M_A = \frac{64C\varpi(s_0)}{\lambda_{\min}(A)}$ where $\varpi(s_0) = \rho_{\max}(s_0, A) + \tau_B$.
M_+	$M_+ = \frac{32C\varpi(s_0+1)}{\lambda_{\min}(A)}$ where $\varpi(s_0+1) = \rho_{\max}(s_0+1, A) + \tau_B$.
ξ	$\xi = 2(\tau_\ell(\mathcal{L}_n) \vee \tau_u(\mathcal{L}_n))\left(\frac{\bar{\alpha}_\ell}{4\zeta} + 2\varrho + 5\right)(1-\varrho)^{-1}$
$\mathcal{V}$	$\mathcal{V} = 3eM_A^3/2$

3. Main results on the statistical error

In this section, we will state our main results in Theorems 3 and 4 where we consider the regression model in (1.1a) and (1.1b) with random matrices

$X_0, W \in \mathbb{R}^{n\times m}$ as defined in (1.4). For the corrected Lasso estimator, we are interested in the case where the smallest eigenvalue of the column-wise covariance matrix A does not approach 0 too quickly and the effective rank of the row-wise covariance matrix B is bounded from below (cf. (3.2)). More precisely, (A2) thus ensures that the Lower-RE condition as in Definition 2.2 is not vacuous. (A3) ensures that (2.2) holds for some $s_0 \geq 1$. Throughout this paper, for the corrected Lasso estimator, we will use the expression

$$\tau := \frac{\lambda_{\min}(A) - \alpha}{s_0}, \quad \text{where } \alpha = \frac{5}{8}\lambda_{\min}(A) \quad \text{and} \quad s_0 \asymp \frac{4n}{M_A^2 \log m}$$

where M_A is as defined in (2.3). Let

$$D_0 = \sqrt{\tau_B} + a_{\max}^{1/2} \quad \text{and} \quad D_2 = 2(\|A\|_2 + \|B\|_2). \tag{3.1}$$

Theorem 3. (**Estimation for the corrected Lasso estimator**) *Consider the regression model in (1.1a) and (1.1b) with independent random matrices X_0, W as in (1.4), and an error vector $\epsilon \in \mathbb{R}^n$ independent of X_0, W, with independent entries ϵ_j satisfying $\mathbb{E}\epsilon_j = 0$ and $\|\epsilon_j\|_{\psi_2} \leq M_\epsilon$. Set $n = \Omega(\log m)$. Suppose $n \leq (\mathcal{V}/e) m \log m$, where $\mathcal{V}$ is a constant which depends on $\lambda_{\min}(A)$, $\rho_{\max}(s_0, A)$ and $\operatorname{tr}(B)/n$. Suppose m is sufficiently large.*

Suppose (A1), (A2) and (A3) hold. Let $C_0, c', c_2, c_3 > 0$ be some absolute constants. Suppose that $\|B\|_F^2 / \|B\|_2^2 \geq \log m$. Suppose that $c'K^4 \leq 1$ and

$$r(B) := \frac{\operatorname{tr}(B)}{\|B\|_2} \quad \geq \quad 16c'K^4 \frac{n}{\log m} \log \frac{\mathcal{V} m \log m}{n}. \tag{3.2}$$

Let b_0, ϕ be numbers which satisfy

$$\frac{M_\epsilon^2}{K^2 b_0^2} \leq \phi \leq 1. \tag{3.3}$$

Assume that the sparsity of β^* satisfies for some $0 < \phi \leq 1$

$$d := |\operatorname{supp}(\beta^*)| \leq \frac{c'\phi K^4}{40 M_+^2} \frac{n}{\log m} < n/2, \tag{3.4}$$

$$\text{where} \quad M_+ = \frac{32C\varpi(s_0+1)}{\lambda_{\min}(A)}$$

for $\varpi(s_0+1) = \rho_{\max}(s_0+1, A) + \tau_B$.

Let $\widehat{\beta}$ be an optimal solution to the corrected Lasso estimator as in (1.7) with

$$\lambda \geq 4\psi \sqrt{\frac{\log m}{n}} \quad \text{where} \quad \psi := C_0 D_2 K \left(K \|\beta^*\|_2 + M_\epsilon\right). \tag{3.5}$$

Then for any d-sparse vectors $\beta^* \in \mathbb{R}^m$, such that

$$\phi b_0^2 \leq \|\beta^*\|_2^2 \leq b_0^2, \tag{3.6}$$

we have

$$\left\|\widehat{\beta}-\beta^*\right\|_2 \le \frac{20}{\alpha}\lambda\sqrt{d} \quad \textit{and} \quad \left\|\widehat{\beta}-\beta^*\right\|_1 \le \frac{80}{\alpha}\lambda d,$$

with probability at least $1-4\exp\left(-\frac{c_3 n}{M_A^2\log m}\log\left(\frac{\mathcal{V}m\log m}{n}\right)\right)-2\exp\left(-\frac{4c_2 n}{M_A^2K^4}\right)-22/m^3$.

We give an outline of the proof of Theorem 3 in Section 6.3. We prove Theorem 3 in Section 10. We defer discussions on conditions appearing Theorem 3 in Section 3.2.

For the Conic programming estimator, we impose a restricted eigenvalue condition as formulated in [4, 38] on A and assume that the sparsity of β^* is bounded by $o(\sqrt{n/\log m})$. These conditions will be relaxed in Section 4 where we allow τ_B to approach 0.

Theorem 4. *Suppose (A1) holds. Set* $0<\delta<1$. *Suppose that* $n<m\ll\exp(n)$ *and* $1\le d_0<n$. *Let* $\lambda>0$ *be the same parameter as in* (1.8). *Suppose that* $\|B\|_F^2/\|B\|_2^2\ge\log m$. *Suppose that the sparsity of* β^* *is bounded by*

$$d_0 := |\mathrm{supp}(\beta^*)| \le c_0\sqrt{n/\log m} \tag{3.7}$$

for some constant $c_0>0$. *Suppose*

$$n \ge \frac{2000dK^4}{\delta^2}\log\left(\frac{60em}{d\delta}\right) \quad \textit{where} \tag{3.8}$$

$$d = 2d_0+2d_0a_{\max}\frac{16K^2(2d_0,3k_0,A^{1/2})(3k_0)^2(3k_0+1)}{\delta^2}. \tag{3.9}$$

Consider the regression model in (1.1a) *and* (1.1b) *with* X_0, W *as in* (1.4) *and an error vector* $\epsilon\in\mathbb{R}^n$, *independent of* X_0, W, *with independent entries* ϵ_j *satisfying* $\mathbb{E}\epsilon_j=0$ *and* $\|\epsilon_j\|_{\psi_2}\le M_\epsilon$. *Let* $\widehat{\beta}$ *be an optimal solution to the Conic programming estimator as in* (1.8) *with input* $(\widehat{\gamma},\widehat{\Gamma})$ *as defined in* (1.6). *Recall* $\tau_B := \mathrm{tr}(B)/n$. *Choose for* D_0, D_2 *as in* (3.1) *and*

$$\mu \asymp D_2K^2\sqrt{\frac{\log m}{n}} \quad \textit{and} \quad \omega \asymp D_0KM_\epsilon\sqrt{\frac{\log m}{n}}.$$

Then with probability at least $1-\frac{c'}{m^2}-2\exp(-\delta^2 n/2000K^4)$,

$$\left\|\widehat{\beta}-\beta^*\right\|_q \le CD_2K^2d_0^{1/q}\sqrt{\frac{\log m}{n}}\left(\|\beta^*\|_2+\frac{M_\epsilon}{K}\right) \tag{3.10}$$

for $2\ge q\ge 1$. *Under the same assumptions, the predictive risk admits the following bounds with the same probability as above,*

$$\frac{1}{n}\left\|X(\widehat{\beta}-\beta^*)\right\|_2^2 \le C'D_2^2K^4d_0\frac{\log m}{n}\left(\|\beta^*\|_2+\frac{M_\epsilon}{K}\right)^2$$

where $c', C_0, C, C'>0$ *are some absolute constants.*

We give an outline of the proof of Theorem 4 in Section 6 while leaving the detailed proof in Section 11.

3.1. Regarding the M_A constant

Denote by

$$M_A = \frac{64C\varpi(s_0)}{\lambda_{\min}(A)} \asymp \frac{\rho_{\max}(s_0, A) + \tau_B}{\lambda_{\min}(A)}$$

- (A3) ensures that M_A and M_+ are upper bounded by the condition number of A: $\kappa(A) := \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} = O\left(\sqrt{\frac{n}{\log m}}\right)$ given that $\tau_B := \mathrm{tr}(B)/n = O(\lambda_{\max}(A))$.
- So the condition (3.4) in Theorem 3 allows $d \asymp n/\log m$ in the optimal setting when the condition number $\kappa(A)$ is understood to be a constant. As $\kappa(A)$ increases, the conservative worst case upper bound on d needs to be adjusted correspondingly. Moreover, this adjustment is also crucial in order to ensure the composite gradient algorithm to converge in the sense of Theorem 2. We will illustrate such dependencies on $\kappa(A)$ in numerical examples in Section 8.
- The condition $\tau_B = O(\lambda_{\max}(A))$ puts an upper bound on how large the measurement error in W can be. We do not allow the measurement error to overwhelm the signal entirely. When $\tau_B \to 0$, we recover the ordinary Lasso bound in [4], which we elaborate in the next two sections.

Throughout this paper, we assume that $M_A \asymp M_+$, where recall $M_+ = \frac{32C\varpi(s_0+1)}{\lambda_{\min}(A)}$.

3.2. Discussions

Throughout our analysis, we set the parameter $b_0 \geq \|\beta^*\|_2$ and $d = |\mathrm{supp}(\beta^*)| := \left|\{j : \beta_j^* \neq 0\}\right|$ for the corrected Lasso estimator. In practice, both b_0 and d are understood to be parameters chosen to provide an upper bound on the ℓ_2 norm and the sparsity of the true β^*. The parameter $0 < \phi < 1$ is a parameter that we use to describe the gap between $\|\beta^*\|_2^2$ and its upper bound b_0^2. Denote the Signal-to-noise ratio by

$$\mathsf{S}/\mathsf{N} := K^2 \|\beta^*\|_2^2 / M_\epsilon^2, \quad \text{where } \mathsf{N} := M_\epsilon^2 \text{ and } \phi K^2 b_0^2 \leq \mathsf{S} := K^2 \|\beta^*\|_2^2 \leq K^2 b_0^2.$$

The two conditions (3.3) and (3.6) on b_0 and ϕ imply that $\mathsf{N} \leq K^2 \phi b_0^2 \leq \mathsf{S}$. Notice that this could be restrictive if ϕ is small. We will show in Section 6.3 that condition (3.3) is not needed in order for the $\ell_p, p = 1, 2$ errors as stated in the Theorem 3 to hold. It was indeed introduced so as to further simplify the expression for the condition on d as shown in (3.4). Therefore we provide slightly more general conditions on d in (6.6) in Lemma 17, where (3.3) is not required. We introduce the parameter ϕ so that the conditions on d depend on ϕ and b_0^2 rather than the true signal $\|\beta^*\|_2$ (cf. Proof of Lemmas 17 and 18). It will also become clear in the sequel from the proof of Lemma 17 (cf. (H.4)) that we could use $\|\beta^*\|_2$ rather than its the lower bound $b_0^2\phi$ in the expression for d.

However, we choose to state the condition on d as in Theorem 3 for clarity of our exposition. See also Theorem 6 and Lemma 18.

In fact, we prove that Theorem 3 holds with $\mathsf{N} = M_\epsilon^2$ and $\underline{\mathsf{S}} = \phi K^2 b_0^2$ in arbitrary orders, so long as conditions (3.2) and (3.4) or (6.6) hold. For both cases, we require that $\lambda \asymp (\|A\|_2 + \|B\|_2) K \sqrt{\mathsf{S} + \mathsf{N}} \sqrt{\frac{\log m}{n}}$ as expressed in (3.5). That is, when either the noise level M_ϵ or the signal strength $K \|\beta^*\|$ increases, we need to increase λ correspondingly; moreover, when N dominates the signal $K^2 \|\beta^*\|_2^2$, we have for $d \asymp \frac{1}{M_A^2} \frac{n}{\log m}$ as in (3.4),

$$\left\|\widehat{\beta} - \beta^*\right\|_2 / \|\beta^*\|_2 = O_P\left(D_2 K^2 \sqrt{\frac{\mathsf{N}}{\mathsf{S}}} \frac{1}{\varpi(s_0+1)}\right),$$

which eventually becomes a vacuous bound when $\mathsf{N} \gg \mathsf{S}$. This bound appears a bit crude as it does not entirely discriminate between the noise, measurement error, and the signal strength. We further elaborate on the relationships among these three elements in Section 4. We will then present an improved bound in Theorem 6.

1. The choice of λ for the Lasso estimator and parameters μ, ω for the DS-type estimator satisfy

   $$\lambda \asymp \mu \|\beta^*\|_2 + \omega.$$

   This relationship is made clear through Theorem 16 regarding the corrected Lasso estimator, which follows from Theorem 1 by [30], and Lemmas 19 and 22 for the Conic programming estimator. The penalty parameter λ is chosen to bound $\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty$ from above, which is in turn bounded in Theorem 12. See Corollaries 13 and 14, which are the key results in proving Theorems 3, 4, 6, and 7.
2. Throughout our analysis of Theorems 3 and 4, our error bounds are stated in a way assuming the errors in W are sufficiently large in the sense that these bounds are optimal only when τ_B is bounded from below by some absolute constant. For example, when $\|B\|_2$ is bounded away from 0, the lower bound on the effective rank $r(B) = \mathrm{tr}(B)/\|B\|_2$ implies that τ_B must also be bounded away from 0. More precisely, by the condition on the effective rank as in (3.2), we have

   $$\tau_B = \frac{\mathrm{tr}(B)}{n} \quad \geq \quad 16c'K^4 \frac{\|B\|_2}{\log m} \log \frac{\mathcal{V} m \log m}{n} \quad \text{where } \mathcal{V} = 3eM_A^3/2.$$

   Later, we will state our results with $\tau_B = \mathrm{tr}(B)/n > 0$ being explicitly included in the error bounds as well as the penalization parameters and sparsity constraints.
3. In view of the main Theorems 3 and 4, at this point, we do not really think one estimator is preferable to the other. While the ℓ_q error bounds we obtain for the two estimators are at the same order for $q = 1, 2$, the

conditions under which these error bounds are obtained are somewhat different. In Theorem 4, we only require that $\mathsf{RE}(2d_0, 3k_0, A^{1/2})$ holds for $k_0 = 1+\lambda$ where $\lambda \asymp 1$, while in Theorem 3 we need the minimal eigenvalue of A to be bounded from below, namely, we need to assume that (A2) holds. As mentioned earlier, (A2) ensures that the Lower-RE condition as in Definition 2.2 is not vacuous while (A3) ensures that (2.2) holds for some $s_0 \geq 1$. Th condition (3.2) on the effective rank of the row-wise covariance matrix B is also needed to establish the Lower and Upper RE conditions in Lemma 15 for the corrected Lasso estimator. Moreover, for the sparsity parameter d_0 in (3.7), we show in Lemma 34 that (A2) is a sufficient condition for a type of $\mathsf{RE}(2d_0, 3k_0)$ condition to hold on non positive definite $\widehat{\Gamma}$ as defined in (1.6). See also Theorem 26.

4. In some sense, the assumptions in Theorem 3 appear to be slightly stronger, while at the same time yielding correspondingly stronger results in the following sense: The corrected Lasso procedure can recover a sparse model using $O(\log m)$ number of measurements per nonzero component despite the measurement error in X and the stochastic noise ϵ, while the Conic programming estimator allows only $d \asymp \sqrt{n/\log m}$ to achieve the error rate at the same order as the corrected Lasso estimator. Hence, while Conic programming estimator is conceptually more adaptive by not fixing an upper bound on $\|\beta^*\|_2$ a priori, the price we pay seems to be a more stringent upper bound on the sparsity level.
5. We note that following Theorem 2 as in [3], one can show that without the relatively restrictive sparsity condition (3.7), a bound similar to that in (3.10) holds, however, with $\|\beta^*\|_2$ being replaced by $\|\beta^*\|_1$, so long as the sample size satisfies the condition as in (3.8). However, we show in Theorem 7 in Section 6.6 that this restriction on the sparsity can be relaxed for the Conic programming estimator (1.8), when we make a different choice for the parameter μ based on a more refined analysis.

Results similar to Theorems 3 and 4 have been derived in [30, 3], however, under different assumptions on the distribution of the noise matrix W. When W is a random matrix with i.i.d. subgaussian noise, our results in Theorems 3 and 4 will essentially recover the results in [30] and [3]. We compare with their results in Section 4 in case $B = \tau_B I$ after we present our improved bounds in Theorems 6 and 7. We refer to the paper of [3] for a concise summary of these and some earlier results.

Finally, one reviewer asked about the dependence of the tuning parameter on properties of A and B, namely parameters $D_0 = \sqrt{\tau_B} + a_{\max}^{1/2}$, $D_0' = \|B\|_2^{1/2} + a_{\max}^{1/2}$ and $D_2 = \|A\|_2 + \|B\|_2$. We now state in Lemma 5 a sharp bound on estimating τ_B using $\widehat{\tau}_B$ as in (1.5), which will provide a natural plug-in estimate for parameters such as D_0 that involve τ_B.

Lemma 5. *Let $m \geq 2$. Let X be defined as in (1.4) and $\widehat{\tau}_B$ be as defined in (1.5). Denote by $\tau_B = \operatorname{tr}(B)/n$ and $\tau_A = \operatorname{tr}(A)/m$. Suppose that $n \vee (r(A)r(B)) >$*

$\log m$. *Denote by* $\mathcal{B}_6$ *the event such that*

$$|\widehat{\tau}_B - \tau_B| \quad \leq \quad 2C_0K^2\sqrt{\frac{\log m}{mn}}\left(\frac{\|A\|_F}{\sqrt{m}} + \frac{\|B\|_F}{\sqrt{n}}\right) =: D_1 r_{m,m},$$

where $D_1 = \frac{\|A\|_F}{\sqrt{m}} + \frac{\|B\|_F}{\sqrt{n}}$ *and* $r_{m,m} = 2C_0K^2\sqrt{\frac{\log m}{mn}}$. *Then* $\mathbb{P}(\mathcal{B}_6) \geq 1 - \frac{3}{m^3}$.

If we replace $\sqrt{\log m}$ *with* $\log m$ *in the definition of event* $\mathcal{B}_6$, *then we can drop the condition on* n *or* $r(A)r(B) = \frac{\mathrm{tr}(A)}{\|A\|_2}\frac{\mathrm{tr}(B)}{\|B\|_2}$ *to achieve the same bound on event* $\mathcal{B}_6$.

In an earlier version of the present work by the same authors [39], we presented the rate of convergence for using the corrected gram matrix $\widehat{B} := \frac{1}{m}XX^T - \frac{\mathrm{tr}(A)}{m}I_m$ to estimate B and proved isometry properties in the operator norm once the effective rank of A is sufficiently large compared to n; one can then use such estimated $\widehat{B}$ and its operator norm in D_2 and D_0'. See Theorem 21 and Corollary 22 therein. As mentioned, we use the estimated $\widehat{\tau}_B$ (cf. Lemma 5) in D_0. The dependencies on A, $\|\beta^*\|_2$ and ϵ are known problems in the Lasso and corrected Lasso literature; see [4, 30]. For example, the RE condition as stated in Definition 2.1 and its subgaussian concentration properties as shown [38] clearly depend on unknown parameter $a_{\max}$ related to covariance matrix A. See Theorem 27 in the present paper. We prove Lemma 5 in Section C.1. Lemma 5 provides the powerful technical insight and one of the key ingredients leading to the tight analysis in Theorems 6 and 7 for the corrected Lasso estimator (1.7) as well as the Conic programming estimator (1.8) in Section 4, where we also present theory for which the dependency on $\|A\|_2$ becomes extremely mild.

4. Improved bounds when the measurement errors are small

Although the conclusions of Theorems 3 and 4 apply to cases when $\|B\|_2 \to 0$, the error bounds are not as tight as the bounds we are about to derive in this section. So far, we have used more crude approximations on the error bounds in terms of estimating $\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty$ for the sake of reducing the amount of unknown parameters we need to consider. The bounds we derive in this section take the magnitudes of the measurement errors in W into consideration. As such, we allow the error bounds to depend on the parameter τ_B explicitly, which become much tighter as τ_B becomes smaller. For the extreme case when τ_B approaches 0, one hopes to recover a bound close to the regular Lasso or the Dantzig selector as the effect of the noise on the procedure should become negligible. We show in Theorems 6 and 7 that this is indeed the case. Denote by

$$\tau_B^{+/2} := \sqrt{\tau_B} + \frac{D_{\text{oracle}}}{\sqrt{m}}, \quad \text{where} \quad D_{\text{oracle}} = 2(\|A\|_2^{1/2} + \|B\|_2^{1/2}). \tag{4.1}$$

We first state a more refined result for the Lasso-type estimator, for which we now only require that

$$\lambda \asymp (a_{\max}^{1/2} + \|B\|_2^{1/2}) K \sqrt{\mathsf{N} + \tau_B \mathsf{S}} \sqrt{\frac{\log m}{n}}.$$

That is, we replace $\sqrt{\mathsf{N}+\mathsf{S}}$ in λ (3.5) now with $\sqrt{\mathsf{N}+\tau_B\mathsf{S}}$, which leads to significant improvement on the rates of convergence for estimating β^* when $\tau_B \to 0$.

Theorem 6. *Suppose all conditions in Theorem 3 hold, except that we drop* (3.3) *and replace* (3.5) *with*

$$\lambda \geq 4\psi \sqrt{\frac{\log m}{n}}, \text{ where } \psi := C_0 D_0' K \left(M_\epsilon + \tau_B^{+/2} K \|\beta^*\|_2 \right) \tag{4.2}$$

for D_0' and $\tau_B^{+/2}$ as defined in (2.17) *and* (4.1) *respectively. Let $c', \phi, b_0, M_\epsilon$, K and M_+ be as defined in Theorem 3. Let $\tau_B^+ = (\tau_B^{+/2})^2$.*

Suppose that for $0 < \phi \leq 1$ and $C_A := \frac{1}{160 M_+^2}$,

$$d := |\mathrm{supp}(\beta^*)| \leq C_A \frac{n}{\log m} \{c' c'' D_\phi \wedge 8\} =: \bar{d}_0, \quad \text{where} \tag{4.3}$$

$$c'' = \frac{\|B\|_2 + a_{\max}}{\varpi (s_0+1)^2} \quad \text{and} \quad D_\phi = \frac{K^2 M_\epsilon^2}{b_0^2} + \tau_B^+ K^4 \phi \tag{4.4}$$

Then for any d-sparse vectors $\beta^* \in \mathbb{R}^m$, such that $\phi b_0^2 \leq \|\beta^*\|_2^2 \leq b_0^2$, we have

$$\left\| \widehat{\beta} - \beta^* \right\|_2 \leq \frac{20}{\alpha} \lambda \sqrt{d} \quad \text{and} \quad \left\| \widehat{\beta} - \beta^* \right\|_1 \leq \frac{80}{\alpha} \lambda d \tag{4.5}$$

with probability at least $1 - 4\exp\left(-\frac{c_3 n}{M_A^2 \log m} \log\left(\frac{\mathcal{V} m \log m}{n} \right) \right) - 2\exp\left(-\frac{4 c_2 n}{M_A^2 K^4} \right) - 22/m^3$.

We give an outline for the proof of Theorem 6 in Section 6.4, and show the actual proof in Section 12.

We next state in Theorem 7 an improved bounds for the Conic programming estimator (1.8), which dramatically improve upon those in Theorem 4 when τ_B is small, where an "oracle" rate for estimating β^* with the Conic programming estimator $\widehat{\beta}$ (1.8) is defined and the predictive error $\|Xv\|_2^2$ when $\tau_B = o(1)$ is derived.

Let C_0 satisfy (H.6) for c as defined in Theorem 31. Throughout the rest of the paper, we denote by:

$$\rho_n = C_0 K \sqrt{\frac{\log m}{n}} \quad \text{and} \quad r_{m,m} = 2 C_0 K^2 \sqrt{\frac{\log m}{mn}}; \tag{4.6}$$

$$\tau_B^{\dagger/2} = (\tau_B^{1/2} + \frac{3}{2} C_6 r_{m,m}^{1/2}) \quad \text{and} \quad \tau_B^{\ddagger} \asymp 2\tau_B + 3 C_6^2 r_{m,m}. \tag{4.7}$$

Theorem 7. *Let $D_0 = \sqrt{\tau_B} + a_{\max}^{1/2}$, and D_0', D_{oracle} be as defined in (2.17). Let $C_6 \geq D_{\text{oracle}}$. Let ρ_n and $r_{m,m}$ be as defined in (4.6). Suppose all conditions in Theorem 4 hold, except that we replace the condition on d as in (3.7) with the following.*

Suppose that the sample size n and the size of the support of β^* satisfy the following requirements:

$$\begin{aligned} d_0 &= O\left(\tau_B^- \sqrt{\frac{n}{\log m}}\right), \quad \text{where } \tau_B^- \leq \frac{1}{\tau_B^{1/2} + 2C_6 r_{m,m}^{1/2}}, && (4.8) \\ \text{and} \quad n &\geq \frac{2000 d K^4}{\delta^2} \log\left(\frac{60em}{d\delta}\right), \text{ where} && (4.9) \\ d &= 2d_0 + 2d_0 a_{\max} \frac{16K^2(2d_0, 3k_0, A^{1/2})(3k_0)^2(3k_0+1)}{\delta^2}. \end{aligned}$$

Let $\widehat{\tau}_B$ be as defined in defined in (1.5). Let $\widehat{\beta}$ be an optimal solution to the Conic programming estimator as in (1.8) with input $(\widehat{\gamma}, \widehat{\Gamma})$ as defined in (1.6). Suppose

$$\begin{aligned} \omega \asymp D_0 M_\epsilon \rho_n \quad \text{and} \quad \mu \asymp D_0' \widetilde{\tau}_B^{1/2} K \rho_n, \\ \text{where } \widetilde{\tau}_B^{1/2} := \widehat{\tau}_B^{1/2} + C_6 r_{m,m}^{1/2}. \end{aligned} \tag{4.10}$$

Then with probability at least $1 - \frac{c''}{m^2} - 2\exp(-\delta^2 n/2000K^4)$, for $2 \geq q \geq 1$,

$$\left\| \widehat{\beta} - \beta^* \right\|_q \leq C' D_0' K^2 d_0^{1/q} \sqrt{\frac{\log m}{n}} \left(\tau_B^{\dagger/2} \left\| \beta^* \right\|_2 + \frac{M_\epsilon}{K} \right); \tag{4.11}$$

Under the same assumptions, the predictive risk admits the following bound

$$\tfrac{1}{n} \left\| X(\widehat{\beta} - \beta^*) \right\|_2^2 \leq C''(\|B\|_2 + a_{\max}) K^2 d_0 \tfrac{\log m}{n} \left(\tau_B^{\ddagger} K^2 \left\| \beta^* \right\|_2^2 + M_\epsilon^2 \right),$$

with the same probability as above, where $c'', C', C'' > 0$ are some absolute constants.

We give an outline for the proof of Theorem 7 in Section 6.6, and show the actual proof in Section 13.

4.1. Oracle results on the Lasso-type estimator

We now discuss the improvement being made in Theorem 6 and Theorem 7.

The Signal-to-noise ratio. Let us redefine the Signal-to-noise ratio by

$$\begin{aligned} \mathsf{S/M} &:= \frac{K^2 \left\| \beta^* \right\|_2^2}{\tau_B^+ K^2 \left\| \beta^* \right\|_2^2 + M_\epsilon^2}, \quad \text{where} \\ \mathsf{S} &:= K^2 \left\| \beta^* \right\|_2^2 \quad \text{and} \quad \mathsf{M} := M_\epsilon^2 + \tau_B^+ K^2 \left\| \beta^* \right\|_2^2. \end{aligned}$$

When either the noise level M_ϵ or the measurement error strength in terms of $\tau_B^{+/2} K \|\beta^*\|_2$ increases, we need to increase the penalty parameter λ correspondingly; moreover, when $d \asymp \frac{1}{M_A^2}\frac{n}{\log m}$, we have

$$\frac{\left\|\widehat{\beta}-\beta^*\right\|_2}{\|\beta^*\|_2} = O_P\left(D_0' K^2 \sqrt{\frac{\mathsf{M}}{\mathsf{S}}}\frac{1}{\varpi(s_0+1)}\right),$$

which eventually becomes a vacuous bound when $\mathsf{M} \gg \mathsf{S}$.

Finally, suppose $B = \sigma_w^2 I$, we have $\|B\|_2^{1/2} = \sigma_w$ and $\tau_B = \sigma_w^2$. In this setting, we recover essentially the same ℓ_2 error bound as that in Corollary 1 of [30] in case $\|\beta^*\|_2 \asymp 1$, as we have on event $\mathcal{A}_0 \cap \mathcal{B}_0$,

$$\left\|\widehat{\beta}-\beta^*\right\|_2 \le \frac{C(\sigma_w + a_{\max}^{1/2})}{\lambda_{\min}(A)}\sqrt{\sigma_\epsilon^2 + \sigma_w^2\|\beta^*\|_2^2}\sqrt{\frac{d\log m}{n}}$$

where $\sigma_\epsilon^2 \asymp M_\epsilon^2$ and $K^2 \asymp 1$. However, when $\|\beta^*\|_2 = \Omega(1)$, our statistical precision appears to be sharper as we allow the term $\|\beta^*\|_2$ to be removed entirely from the RHS when $\sigma_w \to 0$ and hence recover the regular Lasso rate of convergence.

The penalization parameter. We focus now on the penalization parameter λ in (1.7). The effective rank condition in (3.2) implies that for $n = O(m\log m)$

$$\|B\|_2 \le \frac{\tau_B}{16c'K^4}\frac{\log m}{\log(3eM_A^3/2) + \log(m\log m) - \log n} \le C_B \tau_B \log m$$

where $C_B = \frac{1}{16c'K^4\log(3eM_A^3/2)}$ given that $\log(m\log m) - \log n > 0$. This bound is very crude given that in practice, we focus on cases where $n \ll m\log m$. Note that under (A1) (A2) and (A3), we have for $n = O(m\log m)$,

$$\begin{aligned}\tau_B^+ &\asymp \tau_B + \frac{\|A\|_2 + \|B\|_2}{m} \\ &\le \tau_B + \frac{1}{m}(\kappa(A)\lambda_{\min}(A) + C_B\tau_B\log m) \asymp \tau_B + O\left(\frac{\lambda_{\min}(A)}{\sqrt{m}}\right).\end{aligned}$$

Without knowing τ_B, we will use $\widehat{\tau}_B$ as defined in (1.5). Notice that we know neither D_0' nor D_{oracle} in the definition of λ, where $D_{\text{oracle}}^2 \asymp D_2$; Indeed,

$$2D_2 \le D_{\text{oracle}}^2 \le 4D_2.$$

However, assuming that we normalize the column norms of the design matrix X to be roughly at the same scale, we have for $\tau_B = O(1)$ and m sufficiently large,

$$D_0' \asymp 1 \quad \text{while} \quad D_{\text{oracle}}/\sqrt{m} = o(1) \ \text{ in case } \ \|A\|_2, \|B\|_2 \le M$$

for some large enough constant M. In summary, compared to Theorem 3, in ψ, we replace $D_2 = 2(\|A\|_2 + \|B\|_2)$ with $D_0' := \|B\|_2^{1/2} + a_{\max}^{1/2}$ so that the

dependency on $\|A\|_2$ becomes much weaker. As mentioned in Section 3.2, we may use the plug-in estimate $\|\widehat{B}\|_2$ in D_0', where $\widehat{B}$ is the corrected gram matrix $\frac{1}{m}XX^T - \frac{\mathrm{tr}(A)}{m}I_m$. Finally, the concentration of measure bound for the estimator $\widehat{\tau}_B$ as in (1.5) is stated in Lemma 5, which ensures that $\widehat{\tau}_B$ is indeed a good proxy for τ_B (cf. Lemma 23).

The sparsity parameter. The condition on d (and D_ϕ) for the Lasso estimator as defined in (4.3) suggests that as $\tau_B \to 0$, and thus $\tau_B^+ \to 0$, the constraint on the sparsity parameter d becomes slightly more stringent when $K^2M_\epsilon^2/b_0^2 \asymp 1$ and much more restrictive when $K^2M_\epsilon^2/b_0^2 = o(1)$. Moreover, suppose we require

$$M_\epsilon^2 = \Omega(\tau_B^+ K^2 \|\beta^*\|_2^2),$$

that is, the stochastic error ϵ in the response variable y as in (1.1a) does not converge to 0 as quickly as the measurement error W in (1.1b) does, then the sparsity constraint becomes essentially unchanged as $\tau_B^+ \to 0$ as we show now.

Case 1. Suppose $\tau_B \to 0$ and $M_\epsilon = \Omega(\tau_B^{+/2} K \|\beta^*\|_2)$. In this case, essentially, we require that

$$d \le \frac{c_0\lambda_{\min}^2(A)}{\varpi^2(s_0+1)}\frac{n}{\log m}\left\{\frac{c'c''K^2M_\epsilon^2}{b_0^2} \wedge 1\right\} \tag{4.12}$$

$$\text{where } D_\phi \asymp \frac{K^2M_\epsilon^2}{b_0^2} \quad \text{given that}$$

$$\tau_B^+K^4\phi \le \frac{\tau_B^+K^4\|\beta^*\|_2^2}{b_0^2} \ll \frac{K^2M_\epsilon^2}{b_0^2}$$

where c_0, c' are absolute constants and $c'' := \frac{\|B\|_2 + a_{\max}}{\varpi^2(s_0+1)} \asymp 1$ where $\varpi(s_0+1) = \rho_{\max}(s_0+1, A) + \tau_B$. In this case, the sparsity constraint becomes essentially unchanged as $\tau_B^+ \to 0$.

Case 2. Analogous to (3.4), when $M_\epsilon^2 \le \tau_B^+\phi K^2 b_0^2$, we could represent the condition on d as follows:

$$d \;\le\; C_A c'c''\tau_B^+K^4\phi\frac{n}{\log m} \le C_A c'c''D_\phi\frac{n}{\log m}$$

which is sufficient for (4.3) to hold for $\tau_B \to 0$; Indeed, by assumption that $c'K^4 \le 1$ and $M_\epsilon^2 \le \tau_B^+\phi K^2 b_0^2$, we have

$$8 > 2c'K^4\tau_B^+\phi \ge c'D_\phi \asymp c'\tau_B^+K^4\phi.$$

Hence, for $c'\tau_B^+K^4 \le 1$, we have

$$\begin{aligned} d &\le C_A(c'c''\tau_B^+K^4\phi \wedge 8)\frac{n}{\log m} \asymp C_A c''(c'\tau_B^+K^4\phi \wedge 8)\frac{n}{\log m} \\ &\le C_A c''c'\tau_B^+K^4\phi\frac{n}{\log m} \asymp C_A c''c'D_\phi\frac{n}{\log m} \end{aligned}$$

This condition, however, seems to be unnecessarily strong, when $\tau_B \to 0$ (and $M_\epsilon \to 0$ simultaneously). We focus on the following Case 2 in the present work.

For both cases, it is clear that sample size needs to satisfy

$$n = \widetilde{\Omega}\left(d \log m \frac{(\rho_{\max}(s_0+1, A) + \tau_B)^4}{\lambda_{\min}(A)^2(\|B\|_2 + a_{\max})}\right),$$

where $\widetilde{\Omega}(\cdot)$ notation hides parameters K, M_ϵ, ϕ and b_0, which we treat as absolute constants that do not change as $\tau_B \to 0$. These tradeoffs are somehow different from the behavior of the Conic programming estimator (cf (4.13)). We will provide a more detailed analysis in Sections 6.1 and 6.4.

4.2. Oracle results on the Conic programming estimator

In order to exploit the oracle bound as stated in Theorem 12 regarding $\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty$, we need to know the noise level $\tau_B := \mathrm{tr}(B)/n$ in W and then we can set

$$\mu \asymp D_0'(\tau_B^{1/2} + \frac{D_{\text{oracle}}}{\sqrt{m}})K\rho_n \quad \text{while retaining } \omega \asymp D_0 M_\epsilon \rho_n,$$
$$\text{where recall } \rho_n = C_0 K\sqrt{\frac{\log m}{n}} \text{ and } D_0 = \sqrt{\tau_B} + \sqrt{a_{\max}}.$$

This will in turn lead to improved bounds in Theorems 6 and 7.

The penalization parameter. Without knowing the parameter τ_B, we rely on the estimate from $\widehat{\tau}_B$ as in (1.5), as discussed in Section 3. For a chosen parameter $C_6 \asymp D_{\text{oracle}}$, we use $\widehat{\tau}_B^{1/2} + C_6 r_{m,m}^{1/2}$ to replace $\tau_B^{+/2} := \tau_B^{1/2} + D_{\text{oracle}}/\sqrt{m}$ and set

$$\mu \asymp C_0 D_0' K^2(\widehat{\tau}_B^{1/2} + D_{\text{oracle}} r_{m,m}^{1/2})\sqrt{\frac{\log m}{n}}$$

in view of Corollary 14, where an improved error bound over $\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty$ is stated. Without knowing D_{oracle}, we could replace it with an upper bound; for example, assuming that $D_{\text{oracle}}^2 \asymp \|A\|_2 + \|B\|_2 = O\left(\sqrt{\frac{n}{\log m}}\right)$, we could set

$$\mu \asymp C_0 D_0' K^2(\widehat{\tau}_B^{1/2} + O(m^{-1/4}))\sqrt{\frac{\log m}{n}}.$$

The sparsity parameter. Roughly speaking, for the Conic programming estimator (1.8), one can think of d_0 as being bounded:

$$d_0 = O\left(\tau_B^- \sqrt{\frac{n}{\log m}} \bigwedge \frac{n}{\log(m/d_0)}\right) \quad \text{where } \tau_B^- \asymp \tau_B^{-1/2} \tag{4.13}$$

That is, when τ_B decreases, we allow larger values of d_0; however, when $\tau_B \to 0$, the sparsity level of $d = O\left(n/\log(m/d)\right)$ starts to dominate, which enables the

Conic programming estimator to achieve results similar to the Dantzig Selector when the design matrix X_0 is a subgaussian random matrix satisfying the Restricted Eigenvalue conditions; See for example [6, 4, 38].

In particular, when $\tau_B \to 0$, Theorem 7 allows us to recover a rate close to that of the Dantzig selector with an exact recovery if $\tau_B = 0$ is known a priori; see Section 16. Moreover the constraint (3.7) on the sparsity parameter d_0 appearing in Theorem 4 can now be relaxed as in (4.8). In summary, our results in Theorem 7 are stronger than those in [3] (cf. Corollary 1) as their rates as stated therein are at the same order as ours in Theorem 4. We illustrate this dependency on τ_B in Section 8 with numerical examples, where we clearly show an advantage by taking the noise level into consideration when choosing the penalty parameters for both the Lasso and the Conic programming estimators.

5. Optimization error on the gradient descent algorithm

We now present our computational convergence bounds. First we present Lemma 8 regarding the RSC and RSM conditions on the loss function (1.7). Lemma 8 follows from Lemma 15 immediately.

Lemma 8. *Suppose all conditions as stated in Theorem 3 hold. Suppose event $\mathcal{A}_0$ holds. Then (2.4) and (2.5) hold with $\alpha_\ell = \frac{5}{8}\lambda_{\min}(A)$, $\alpha_u = \frac{11}{8}\lambda_{\max}(A)$ and*

$$\tau_\ell(\mathcal{L}_n) = \tau_u(\mathcal{L}_n) = \tau_0 \frac{\log m}{n}, \quad \text{where} \quad \tau_0 \asymp \frac{400C^2\varpi(s_0+1)^2}{\lambda_{\min}(A)}.$$

Theorem 9. *Suppose all conditions in Theorem 6 hold and let ψ be defined therein. Let $g(\beta) = \frac{1}{\lambda}\rho_\lambda(\beta)$ where $\rho_\lambda(\beta) = \lambda\|\beta\|_1$. Consider the optimization program (1.10) for a radius R such that β^* is feasible and a regularization parameter chosen such that*

$$\lambda \geq \left(\frac{16R\xi}{1-\kappa}\right) \bigvee \left(12\psi\sqrt{\frac{\log m}{n}}\right). \tag{5.1}$$

Suppose that the step size parameter $\zeta \geq \alpha_u \asymp \frac{3}{2}\lambda_{\max}(A)$. Suppose that the sparsity parameter and sample size further satisfy the following relationship:

$$d < \frac{n}{512\tau_0 \log m}\left(\frac{\lambda_{\min}(A)^2}{12\lambda_{\max}(A)} \bigwedge \frac{(\alpha_\ell)^2}{5\zeta}\right) =: \bar{d}. \tag{5.2}$$

Then on event $\mathcal{A}_0 \cap \mathcal{B}_0$, the conclusions in Theorem 2 hold, where

$$\begin{aligned}\mathbb{P}(\mathcal{A}_0 \cap \mathcal{B}_0) \geq\ & 1 - 4\exp\left(-\frac{c_3 n}{M_A^2 \log m}\log\left(\frac{\mathcal{V} m \log m}{n}\right)\right) \\ & -2\exp\left(-\frac{4c_2 n}{M_A^2 K^4}\right) - 22/m^3.\end{aligned}$$

Corollary 10. *Suppose all conditions as stated in Theorem 9 hold and event $\mathcal{A}_0 \cap \mathcal{B}_0$ defined therein holds. Consider for some constant $M \le 400\tau_0$ and $\bar{\delta}^2$ as defined in Theorem 2,*

$$\delta^2 \asymp \frac{c\varepsilon_{\mathrm{stat}}^2}{1-\kappa}\frac{d\log m}{n} =: \bar{\delta}^2 \quad \textit{and} \quad \delta^2 \le M\bar{\delta}^2 \le 400\tau_0\bar{\delta}^2.$$

Then for all $t \ge T^*(\delta)$ as in (2.12) and $R = \Omega(b_0\sqrt{d})$,

$$\left\|\beta^t - \widehat{\beta}\right\|_2^2 \le \frac{3}{\alpha_\ell}\delta^2 + \frac{\alpha_\ell}{4}\varepsilon_{\mathrm{stat}}^2 + O\left(\frac{\delta^2\varepsilon_{\mathrm{stat}}^2}{b_0^2}\right). \tag{5.3}$$

Finally, suppose we fix for $M_+ = \frac{32C\varpi(s_0+1)}{\lambda_{\min}(A)}$,

$$R \asymp \sqrt{\bar{d}}b_0 \asymp \frac{b_0}{20M_+\sqrt{6\kappa(A)}}\sqrt{\frac{n}{\log m}},$$

in view of the upper bound $\bar{d}$ (5.2). Then for all $t \ge T^*(\delta)$ as in (2.12),

$$\left\|\beta^t - \widehat{\beta}\right\|_2^2 \le \frac{3}{\alpha_\ell}\delta^2 + \frac{\alpha_\ell}{4}\varepsilon_{\mathrm{stat}}^2 + \frac{2}{\alpha_\ell}\frac{\delta^4}{b_0^2\|A\|_2}. \tag{5.4}$$

We prove Theorem 9 and Corollary 10 in Section 14.

5.1. Discussions

Throughout this section, we assume ψ (4.2) is as defined in Theorem 6. Assume that $\zeta \ge \alpha_u \ge \bar{\alpha}_\ell$. In addition, suppose that the radius $R \asymp b_0\sqrt{d}$ as we set in (1.7). Let $\bar{d}_0 \le \frac{n}{160M_+^2\log m}$ be as defined in (4.3), where recall that we require the following condition on d:

$$\begin{aligned} d &\le C_A\{c'C_\phi \wedge 8\}\frac{n}{\log m} =: \bar{d}_0, \quad \text{where } C_A = \frac{1}{160M_+^2}, \\ C_\phi &= \frac{\|B\|_2 + a_{\max}}{\varpi(s_0+1)^2}D_\phi \quad \text{and} \quad b_0^2 \ge \|\beta^*\|_2^2 \ge \phi b_0^2. \end{aligned}$$

Then by the proof of Lemma 18,

$$b_0\sqrt{\bar{d}_0} \le \frac{5s_0}{3\alpha}\sqrt{\frac{\log m}{n}}\psi =: \frac{\psi}{\tau}\sqrt{\frac{\log m}{n}}, \quad \text{where} \quad \tau = \frac{3\alpha}{5s_0}. \tag{5.5}$$

In contrast, under (5.2), the following upper bound holds on d, which is slightly more restrictive in the sense that the maximum level of sparsity allowed on β^* has decreased by a factor proportional to $\kappa(A)$ compared to the upper bound $\bar{d}_0$ (4.3) in Theorem 6; Now we require that $|\mathrm{supp}(\beta^*)| \le \bar{d}$, where for $C_A = \frac{1}{160M_+^2}$,

$$\begin{aligned} \bar{d} &\asymp \frac{n\lambda_{\min}(A)^2}{1024C^2\varpi(s_0+1)^2\log m}\frac{1}{2400\kappa(A)} \\ &\approx C_A\frac{n}{\log m}\left(\frac{\lambda_{\min}(A)}{15\lambda_{\max}(A)}\right) \asymp \bar{d}_0\frac{1}{\kappa(A)}. \end{aligned} \tag{5.6}$$

To consider the general cases as stated in Theorem 6, we consider the ideal case when we set

$$\zeta = \alpha_u = \frac{11}{8}\lambda_{\max}(A)$$

such that

$$\frac{\zeta}{\bar{\alpha}_\ell} \approx \alpha_u/(\frac{59}{60}\alpha_\ell) \asymp \kappa(A), \quad \text{where } \alpha_\ell = \frac{5}{8}\lambda_{\min}(A).$$

Following the derivation in Remark 14.1, we have

$$\frac{\xi}{1-\kappa} \leq 6\tau(\mathcal{L}_n) + \frac{80\zeta}{\bar{\alpha}_\ell}\tau(\mathcal{L}_n) \approx 200\kappa(A)\tau(\mathcal{L}_n). \tag{5.7}$$

Combining (5.5) and (5.7), it is clear that one can set

$$\lambda \;=\; \Omega\left(\kappa(A)\psi\sqrt{\frac{\log m}{n}}\right) \tag{5.8}$$

in order to satisfy the condition (5.1) on λ in Theorem 2 when we set

$$R \asymp b_0\sqrt{\bar{d}_0} \;=\; O\left(\frac{\psi}{\tau}\sqrt{\frac{\log m}{n}}\right) \tag{5.9}$$

$$\text{and hence} \quad R\tau\kappa(A) \;=\; O\left(\kappa(A)\psi\sqrt{\frac{\log m}{n}}\right).$$

This choice is potentially too conservative because we are setting R in (5.9) with respect to the upper sparsity level $\bar{d}_0$ chosen to guarantee statistical convergence, leading to a larger than necessary penalty parameter as in (5.8). Similarly, when we choose step size parameter ζ to be too large, we need to increase the penalty parameter λ correspondingly given the following lower bound: $\lambda = \Omega\left(\frac{R\xi}{1-\kappa}\right)$ where

$$\begin{aligned}\frac{R\xi}{1-\kappa} &= R\left(2\tau(\mathcal{L}_n)\left(\frac{\frac{\bar{\alpha}_\ell}{4\zeta}+2\varrho}{\frac{\bar{\alpha}_\ell}{4\zeta}-2\varrho}+\frac{5}{\frac{\bar{\alpha}_\ell}{4\zeta}-2\varrho}\right)\right)\\ &\geq 40R\tau(\mathcal{L}_n)\frac{\zeta}{\bar{\alpha}_\ell}+2R\tau(\mathcal{L}_n) \asymp R\tau(\mathcal{L}_n)\kappa(A).\end{aligned}$$

Suppose we set $\zeta = \frac{3}{2}\lambda_{\max}(A)$ and $\frac{\zeta}{\bar{\alpha}_\ell} \approx 3\kappa(A)$ as in Theorem 9. It turns out that the less conservative choice of λ as in (5.10)

$$\lambda \;\asymp\; \left(b_0\sqrt{\kappa(A)}\varpi(s_0)\bigvee\psi\right)\sqrt{\frac{\log m}{n}} \tag{5.10}$$

is sufficient, for example when $\tau_B = \Omega(1)$, for which we now set

$$R \asymp b_0\sqrt{d} \asymp \frac{b_0}{20M_+}\frac{1}{\sqrt{6\kappa(A)}}\sqrt{\frac{n}{\log m}}$$

as in Corollary 10. We will discuss the two scenarios as considered in Section 4. See the detailed discussions in Section 14.

6. Proof of theorems

In Section 6.1, we develop in Theorem 12 the crucial large deviation bound on $\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|$. This entity appears in the constraint set in the Conic programming estimator (1.8), and is directly related to the choice of λ for the corrected Lasso estimator in view of Theorem 16. Its corollaries are stated in Corollary 13 and Corollary 14. In section 6.3, we provide an outline and additional Lemmas 15 and 17 to prove Theorem 3. The full proof of Theorem 3 appears in Section 10. In Section 6.4, we give an outline illustrating the improvement for the Lasso error bounds as stated in Theorem 6. We emphasize the impact of this improvement over sparsity parameter d, which we restate in Lemma 18. In Section 6.5, we provide an outline as well as technical results for Theorem 4. In Section 6.6, we give an outline illuminating the improvement in error bounds for the Conic programming estimator as stated in Theorem 7.

6.1. Stochastic error terms

In this section, we first develop stochastic error bounds in Lemma 11, where we also define some events $\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_{10}$. Recall that $\mathcal{B}_6$ was defined in Lemma 5. Putting the bounds in Lemma 11 together with that in Lemma 5 yields Theorem 12.

Lemma 11. *Assume that the stable rank of B, $\|B\|_F^2 / \|B\|_2^2 \geq \log m$. Let Z, X_0 and W as defined in Theorem 3. Let Z_0, Z_1 and Z_2 be independent copies of Z. Let $\epsilon^T \sim YM_\epsilon/K$ where $Y := e_1^T Z_0^T$. Denote by $\mathcal{B}_4$ the event such that for $\rho_n := C_0 K \sqrt{\frac{\log m}{n}}$,*

$$\begin{aligned} \tfrac{1}{n}\left\|A^{\frac{1}{2}} Z_1^T \epsilon\right\|_\infty &\leq \rho_n M_\epsilon a_{\max}^{1/2} \\ \text{and } \tfrac{1}{n}\left\|Z_2^T B^{\frac{1}{2}} \epsilon\right\|_\infty &\leq \rho_n M_\epsilon \sqrt{\tau_B} \quad \text{where} \quad \tau_B = \frac{\operatorname{tr}(B)}{n}. \end{aligned}$$

Then $\mathbb{P}(\mathcal{B}_4) \geq 1 - 4/m^3$. Moreover, denote by $\mathcal{B}_5$ the event such that

$$\begin{aligned} \tfrac{1}{n}\left\|(Z^T B Z - \operatorname{tr}(B) I_m)\beta^*\right\|_\infty &\leq \rho_n K \|\beta^*\|_2 \frac{\|B\|_F}{\sqrt{n}} \\ \text{and } \tfrac{1}{n}\left\|X_0^T W \beta^*\right\|_\infty &\leq \rho_n K \|\beta^*\|_2 \sqrt{\tau_B} a_{\max}^{1/2}. \end{aligned}$$

Then $\mathbb{P}(\mathcal{B}_5) \geq 1 - 4/m^3$.

Finally, denote by $\mathcal{B}_{10}$ the event such that

$$\begin{aligned} \tfrac{1}{n}\left\|(Z^T B Z - \operatorname{tr}(B) I_m)\right\|_{\max} &\leq \rho_n K \frac{\|B\|_F}{\sqrt{n}} \\ \text{and } \tfrac{1}{n}\left\|X_0^T W\right\|_{\max} &\leq \rho_n K \sqrt{\tau_B} a_{\max}^{1/2}. \end{aligned}$$

Then $\mathbb{P}(\mathcal{B}_{10}) \geq 1 - 4/m^2$.

We prove Lemma 11 in Section C.2. Denote by $\mathcal{B}_0 := B_4 \cap \mathcal{B}_5 \cap \mathcal{B}_6$, which we use throughout this paper.

Theorem 12. *Suppose (A1) holds. Let* $\rho_n = C_0 K \sqrt{\frac{\log m}{n}}$. *Suppose that*

$$\|B\|_F^2 / \|B\|_2^2 \geq \log m \quad \textit{where} \quad m \geq 16.$$

Let $\widehat{\Gamma}$ *and* $\widehat{\gamma}$ *be as in* (1.6). *Let* $D_0 = \sqrt{\tau_B} + \sqrt{a_{\max}}$ *and* D_0' *be as defined in* (2.17). *Let* $D_1 = \frac{\|A\|_F}{\sqrt{m}} + \frac{\|B\|_F}{\sqrt{n}}$. *On event* $\mathcal{B}_0$, *for which* $\mathbb{P}(\mathcal{B}_0) \geq 1 - 16/m^3$,

$$\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty \quad \leq \quad \left(D_0' K \tau_B^{1/2} \|\beta^*\|_2 + \frac{2D_1 K}{\sqrt{m}} \|\beta^*\|_\infty + D_0 M_\epsilon\right) \rho_n. \quad (6.1)$$

We next state the first Corollary 13 of Theorem 12, which we use in proving Theorems 3 and 4. Here we state a somewhat simplified bound on $\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty$ for the sake of reducing the number of unknown parameters involved with a slight worsening of the statistical error bounds when $\tau_B \asymp 1$. On the other hand, the bound in (6.1) provides a significant improvement over the error bound in Corollary 13 in case $\tau_B = o(1)$.

Corollary 13. *Suppose all conditions in Theorem 12 hold. Let* $\widehat{\Gamma}$ *and* $\widehat{\gamma}$ *be as in* (1.6). *On event* $\mathcal{B}_0$, *we have for* $D_2 = 2(\|A\|_2 + \|B\|_2)$ *and some absolute constant* C_0

$$\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty \leq \psi \sqrt{\frac{\log m}{n}}, \quad \textit{where} \quad \psi = C_0 D_2 K \left(K \|\beta^*\|_2 + M_\epsilon\right)$$

is as defined in Theorem 3.

In particular, Corollary 13 ensures that for the corrected Lasso estimator, (6.5) holds with high probability for λ chosen as in (3.5). We prove Corollary 13 in Section D.

6.2. What happens when $\tau_B \to 0$*?*

Recall $D_0 = \sqrt{\tau_B} + a_{\max}^{1/2}$ and $D_0' := \sqrt{\|B\|_2} + a_{\max}^{1/2}$. When $\tau_B \to 0$, we have by Theorem 12

$$\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty \quad = \quad O\left(D_1 K \frac{1}{\sqrt{m}} \|\beta^*\|_\infty + D_0 K M_\epsilon\right) K \sqrt{\frac{\log m}{n}}$$

where $D_0 \to a_{\max}^{1/2}$ and $D_1 = \frac{\|A\|_F}{\sqrt{m}} + \frac{\|B\|_F}{\sqrt{n}} \to \|A\|_2^{1/2}$ under (A1), given that $\|B\|_F / \sqrt{n} \leq \tau_B^{1/2} \|B\|_2^{1/2} \to 0$. In this case, the error term involving $\|\beta^*\|_2$ in (4.2) vanishes, and we only need to set (cf. Theorem 16)

$$\lambda \geq 2\psi \sqrt{\frac{\log m}{n}} \quad \text{for} \quad \psi \asymp a_{\max}^{1/2} K M_\epsilon + \|A\|_2^{1/2} K^2 \|\beta^*\|_\infty / m^{1/2},$$

where the second term in ψ defined immediately above comes from the estimation error in Lemma 5; this term vanishes if we were to assume that (1) $\mathrm{tr}(B)$ is also known or (2) $\|\beta^*\|_\infty = o(M_\epsilon m^{1/2}/K)$. For both cases, by setting $\lambda \asymp 4a_{\max}^{1/2} K M_\epsilon \sqrt{\frac{\log m}{n}}$, we can recover the regular Lasso rate of

$$\left\|\widehat{\beta} - \beta^*\right\|_q = O_p(\lambda d^{1/q}), \quad \text{for} \quad q = 1, 2,$$

when the design matrix X is almost free of measurement errors.

Finally, we state a second Corollary 14 of Theorem 12. Corollary 14 is essentially a restatement of the bound in (6.1).

Corollary 14. *Suppose all conditions in Theorem 12 hold. Let $D_0, D_0', D_{\text{oracle}}$, and $\tau_B^{+/2} := \tau_B^{1/2} + \frac{D_{\text{oracle}}}{\sqrt{m}}$ be as defined in (2.17) and (4.1). On event $\mathcal{B}_0$,*

$$\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty \leq \psi\sqrt{\frac{\log m}{n}}, \quad \text{where}$$
$$\psi := C_0 K \left(D_0' \tau_B^{+/2} K \|\beta^*\|_2 + D_0 M_\epsilon\right).$$

Then $\mathbb{P}(\mathcal{B}_0) \geq 1 - 16/m^3$.

We mention in passing that Corollaries 13 and 14 are crucial in proving Theorems 3, 4, 6 and 7.

6.3. Outline for proof of Theorem 3

In this section, we state Theorem 16, and two Lemmas 15 and 17. Theorem 3 follows from Theorem 16 in view of Corollary 13, Lemmas 15 and 17. In more details, Lemma 15 checks the Lower and the Upper RE conditions on the modified gram matrix,

$$\widehat{\Gamma}_A := \tfrac{1}{n}(X^T X - \widehat{\mathrm{tr}}(B) I_m), \tag{6.2}$$

while Lemma 17 checks condition (6.4) as stated in Theorem 16 for curvature α and tolerance τ regarding the lower RE condition as derived in Lemma 15.

First, we replace (A3) with (A3') which reveals some additional information regarding the constant hidden inside the $O(\cdot)$ notation.

(A3') Suppose (A3) holds; moreover, $mn \geq 4096 C_0^2 D_2^2 K^4 \log m/\lambda_{\min}^2(A)$ for $D_2 = 2(\|A\|_2 + \|B\|_2)$, or equivalently,

$$\frac{\lambda_{\min}(A)}{\|A\|_2 + \|B\|_2} > C_K \sqrt{\frac{\log m}{mn}} \text{ for some large enough contant } C_K.$$

Lemma 15. (Lower and Upper-RE conditions) *Suppose (A1), (A2) and (A3') hold. Denote by $\mathcal{V} := 3eM_A^3/2$, where M_A is as defined in (2.3). Let $s_0 \geq 32$ be as defined in (2.2). Recall that we denote by $\mathcal{A}_0$ the event that the*

modified gram matrix $\widehat{\Gamma}$ *as defined in* (1.6) *satisfies the Lower as well as Upper* RE *conditions with*

$$\begin{aligned} &\textit{curvature} \qquad \alpha = \frac{5}{8}\lambda_{\min}(A), \ \textit{smoothness} \ \widetilde{\alpha} = \frac{11}{8}\lambda_{\max}(A) \\ &\textit{and tolerance} \qquad \frac{384C^2\varpi(s_0)^2}{\lambda_{\min}(A)}\frac{\log m}{n} \le \tau := \frac{\lambda_{\min}(A)-\alpha}{s_0} \\ &\qquad\qquad \le \frac{396C^2\varpi^2(s_0+1)}{\lambda_{\min}(A)}\frac{\log m}{n} \end{aligned}$$

for $\alpha, \widetilde{\alpha}$ *and* τ *as defined in Definitions 2.2 and 2.3, and* $C, s_0, \varpi(s_0)$ *in* (2.2). *Suppose that for some* $c' > 0$ *and* $c'K^4 < 1$,

$$\frac{\operatorname{tr}(B)}{\|B\|_2} \quge c'K^4\frac{s_0}{\varepsilon^2}\log\left(\frac{3em}{s_0\varepsilon}\right) \quad \textit{where} \ \ \varepsilon = \frac{1}{2M_A}. \tag{6.3}$$

Then $\mathbb{P}(\mathcal{A}_0) \ge 1 - 4\exp\left(-\frac{c_3 n}{M_A^2\log m}\log\left(\frac{\mathcal{V}m\log m}{n}\right)\right) - 2\exp\left(-\frac{4c_2 n}{M_A^2K^4}\right) - 6/m^3$.

The main focus of the current section is then to apply Theorem 16 to show Theorem 3. Theorem 16 follows from Theorem 1 by [30].

Theorem 16. *Consider the regression model in* (1.1a) *and* (1.1b). *Let* $d \le n/2$. *Let* $\widehat{\gamma}, \widehat{\Gamma}$ *be as constructed in* (1.6). *Suppose that the matrix* $\widehat{\Gamma}$ *satisfies the Lower-*RE *condition with curvature* $\alpha > 0$ *and tolerance* $\tau > 0$,

$$\sqrt{d}\tau \le \min\left\{\frac{\alpha}{32\sqrt{d}}, \frac{\lambda}{4b_0}\right\}, \tag{6.4}$$

where d, b_0 *and* λ *are as defined in* (1.7). *Then for any* d*-sparse vectors* $\beta^* \in \mathbb{R}^m$, *such that* $\|\beta^*\|_2 \le b_0$ *and*

$$\left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty \le \frac{1}{2}\lambda, \tag{6.5}$$

the following bounds hold:

$$\left\|\widehat{\beta} - \beta^*\right\|_2 \le \frac{20}{\alpha}\lambda\sqrt{d} \quad \textit{and} \quad \left\|\widehat{\beta} - \beta^*\right\|_1 \le \frac{80}{\alpha}\lambda d,$$

where $\widehat{\beta}$ *is an optimal solution to the corrected Lasso estimator as in* (1.7).

We include the proof of Theorem 16 for the sake of self-containment and defer it to Section G for clarity of presentation.

Lemma 17. *Let* $c', \phi, b_0, M_\epsilon, M_+$ *and* K *be as defined in Theorem 3, where we assume that* $b_0^2 \ge \|\beta^*\|_2^2 \ge \phi b_0^2$ *for some* $0 < \phi \le 1$. *Suppose all conditions in Lemma 15 hold. Suppose that* $s_0 \ge 32$ *and*

$$d := |\mathrm{supp}(\beta^*)| \le C_A \frac{n}{\log m} \{c' D_\phi \wedge 2\}, \quad \textit{where} \tag{6.6}$$
$$C_A := \frac{1}{40 M_+^2} \ \textit{and} \ D_\phi = K^4 \left(\frac{M_\epsilon^2}{K^2 b_0^2} + \phi \right) \ge K^4 \phi \ge \phi.$$

Then the following condition holds

$$d \le \frac{\alpha}{32\tau} \bigwedge \frac{1}{\tau^2} \frac{\log m}{n} \left(\frac{\psi}{b_0} \right)^2, \tag{6.7}$$

where $\psi = C_0 D_2 K (K \|\beta^*\|_2 + M_\epsilon)$ *is as defined in* (3.5), $\alpha = 5\lambda_{\min}(A)/8$, *and* τ *is as defined in Lemma 15.*

We prove Lemmas 15 and 17 in Sections F and H.1 respectively. Lemma 15 follows immediately from Corollary 25. We prove Lemmas 15 and Corollary 25 in Sections F and L respectively.

Remark 6.1. *Clearly for* d, b_0, ϕ *as bounded in Theorem 3, we have by assumption* (3.3) *the following upper and lower bound on* D_ϕ*:*

$$2K^4\phi \ge D_\phi := \left(\frac{M_\epsilon^2 K^2}{b_0^2} + K^4 \phi \right) \ge K^4 \phi.$$

In this regime, the conditions on d *as in* (6.6) *can be conveniently expressed as that in* (3.4) *instead.*

6.4. Improved bounds for the corrected Lasso estimator

The proof of Theorem 6 follows exactly the same line of arguments as in Theorem 3, except that we now use the improved bound on the error term $\left\| \widehat{\gamma} - \widehat{\Gamma} \beta^* \right\|_\infty$ given in Corollary 14, instead of that in Corollary 13. Moreover, we replace Lemma 17 with Lemma 18, the proof of which follows from Lemma 17 with d now being bounded as in (4.3) and ψ being redefined as in (6.2). The proof of Lemma 18 appears in Section H.2. See Section 12 for the proof of Theorem 6.

Lemma 18. *Let* $c', \phi, b_0, M_\epsilon$, M_+ *and* K *be as defined in Theorem 3. Suppose all conditions in Lemma 15 hold. Suppose that* (4.3) *holds:*

$$d := |\mathrm{supp}(\beta^*)| \le C_A \frac{n}{\log m} \{c' c'' D_\phi \wedge 8\}, \quad \textit{where } C_A := \frac{1}{160 M_+^2}, \tag{6.8}$$
$$c'' := \frac{\|B\|_2 + a_{\max}}{\varpi (s_0+1)^2} \le \left(\frac{D_0'}{\varpi (s_0+1)} \right)^2 \quad \textit{and} \ \ D_\phi = \frac{K^2 M_\epsilon^2}{b_0^2} + \tau_B^+ K^4 \phi.$$

Then (6.7) *holds with* ψ *as defined in Theorem 6 and* $\alpha = \frac{5}{8}\lambda_{\min}(A)$.

6.5. Outline for proof of Theorem 4

We provide an outline and state the technical lemmas needed for proving Theorem 4. Our first goal is to show that the following holds with high probability,

$$\left\|\widehat{\gamma}-\widehat{\Gamma}\beta^*\right\|_\infty \;=\; \left\|\tfrac{1}{n}X^T(y-X\beta^*)+\tfrac{1}{n}\widehat{\mathrm{tr}}(B)\beta^*\right\|_\infty \;\leq\; \mu\left\|\beta^*\right\|_2+\omega,$$

where μ,ω are chosen as in (6.9). This forms the basis for proving the ℓ_q convergence, where $q\in[1,2]$, for the Conic programming estimator (1.8). This follows immediately from Theorem 12 and Corollary 13. More explicitly, we will state it in Lemma 19.

Lemma 19. *Let $D_0=\sqrt{\tau_B}+\sqrt{a_{\max}}$ and $D_2=2(\|A\|_2+\|B\|_2)$ be as in Theorem 4. Suppose all conditions in Theorem 12 hold. Then on event $\mathcal{B}_0$ as defined therein, the pair $(\beta,t)=(\beta^*,\|\beta^*\|_2)$ belongs to the feasible set of the minimization problem (1.8) with*

$$\mu\asymp 2D_2K\rho_n \quad \textit{and} \quad \omega\asymp D_0M_\epsilon\rho_n, \qquad \textit{where} \quad \rho_n:=C_0K\sqrt{\frac{\log m}{n}}. \tag{6.9}$$

Before we proceed, we first need to introduce some notation and definitions. Let $X_0=Z_1A^{1/2}$ be defined as in (1.4). Let $k_0=1+\lambda$. First we need to define the ℓ_q-sensitivity parameter for $\Psi:=\frac{1}{n}X_0^TX_0$ following [3]:

$$\begin{aligned}\kappa_q(d_0,k_0) &= \min_{J:|J|\leq d_0}\min_{\Delta\in\mathrm{Cone}_J(k_0)}\frac{\|\Psi\Delta\|_\infty}{\|\Delta\|_q}, \quad \text{where} \\ \mathrm{Cone}_J(k_0) &= \left\{x\in\mathbb{R}^m \mid \text{ s.t. } \|x_{J^c}\|_1\leq k_0\|x_J\|_1\right\}.\end{aligned} \tag{6.10}$$

See also [21]. Let $(\widehat{\beta},\widehat{t})$ be the optimal solution to (1.8) and denote by $v=\widehat{\beta}-\beta^*$. We will state the following auxiliary lemmas, the first of which is deterministic in nature. The two lemmas reflect the two geometrical constraints on the optimal solution to (1.8). The optimal solution $\widehat{\beta}$ satisfies:

1. The vector v obeys the following cone constraint: $\|v_{S^c}\|_1\leq k_0\|v_S\|_1$, and $\widehat{t}\leq\frac{1}{\lambda}\|v\|_1+\|\beta^*\|_2$.
2. $\|\Psi v\|_\infty$ is upper bounded by a quantity at the order of $O(\mu(\|\beta^*\|_2+\|v\|_1)+\omega)$.

Lemma 20. *Let $\mu,\omega>0$ be set. Suppose that the pair $(\beta,t)=(\beta^*,\|\beta^*\|_2)$ belongs to the feasible set of the minimization problem (1.8), for which $(\widehat{\beta},\widehat{t})$ is an optimal solution. Denote by $v=\widehat{\beta}-\beta^*$. Then*

$$\|v_{S^c}\|_1 \;\leq\; (1+\lambda)\|v_S\|_1 \quad \textit{and} \quad \widehat{t}\;\leq\;\frac{1}{\lambda}\|v\|_1+\|\beta^*\|_2.$$

Lemma 21. *On event $\mathcal{B}_0\cap\mathcal{B}_{10}$,*

$$\|\Psi v\|_\infty\leq\mu_1\|\beta^*\|_2+\mu_2\|v\|_1+\omega',$$

where $\mu_1=2\mu$, $\mu_2=\mu(\frac{1}{\lambda}+1)$ and $\omega'=2\omega$ for μ,ω as defined in (6.9).

Now combining Lemma 6 of [3] and an earlier result of the two authors (cf. Theorem 27 [38]), we can show that the $\mathsf{RE}(2d_0, 3(1+\lambda), A^{1/2})$ condition and the sample requirement as in (3.8) are enough to ensure that the ℓ_q-sensitivity parameter satisfies the following lower bound for all $1 \le q \le 2$: for some contant c,

$$\kappa_q(d_0, k_0) \quad \ge \quad c d_0^{-1/q},$$

which ensures that for $v = \widehat{\beta} - \beta^*$ and $\Psi = \frac{1}{n} X_0^T X_0$,

$$\|\Psi v\|_\infty \quad \ge \quad \kappa_q(d_0, k_0) \|v\|_q \ge c d_0^{-1/q} \|v\|_q . \tag{6.11}$$

Combining (6.11) with Lemmas 19, 20 and 21 gives us both the lower and upper bounds on $\|\Psi v\|_\infty$, with the lower bound being $\kappa_q(d_0, k_0) \|v\|_q$ and the upper bound as specified in Lemma 21. Following some algebraic manipulation, this yields the bound on the $\|v\|_q$ for all $1 \le q \le 2$. We prove Theorem 4 in Section 11 and Lemmas 19, 20 and 21 in Section I. The proof of Lemma 20 follows the same line of arguments in [3] in view of Lemma 19.

6.6. Improved bounds for the DS-type estimator

Lemma 22 follows directly from Corollary 14.

Lemma 22. *Suppose all conditions in Corollary 14 hold. Let* $D_0 = \sqrt{\tau_B} + \sqrt{a_{\max}} \asymp 1$ *under (A1). Then on event* $\mathcal{B}_0$, *the pair* $(\beta, t) = (\beta^*, \|\beta^*\|_2)$ *belongs to the feasible set* Υ *of the minimization problem* (1.8) *with*

$$\mu \ge D_0' \tau_B^{+/2} K \rho_n \quad \text{and} \quad \omega \ge D_0 M_\epsilon \rho_n, \tag{6.12}$$

where $\tau_B^{+/2} := \tau_B^{1/2} + \frac{D_{\text{oracle}}}{\sqrt{m}}$ *is as defined in* (4.1).

Lemma 23. *On event* $\mathcal{B}_6$ *and (A1), the choice of* $\widetilde{\tau}_B^{1/2} := \widehat{\tau}_B^{1/2} + C_6 r_{m,m}^{1/2}$ *as in* (4.10), *where recall* $r_{m,m} = 2C_0 K^2 \sqrt{\frac{\log m}{mn}}$, *satisfies for* $m \ge 16$ *and* $C_0 \ge 1$,

$$\tau_B^{+/2} \quad \le \quad \widetilde{\tau}_B^{1/2} \le \tau_B^{1/2} + \frac{3}{2} C_6 r_{m,m}^{1/2} =: \tau_B^{\dagger/2}, \tag{6.13}$$

$$\widetilde{\tau}_B \quad \le \quad 2\tau_B + 3C_6^2 r_{m,m} \asymp \tau_B^{\ddagger}, \quad \text{and moreover} \quad \widetilde{\tau}_B^{1/2} \tau_B^{-} \le 1. \tag{6.14}$$

We next state an updated result in Lemma 24.

Lemma 24. *On event* $\mathcal{B}_0 \cap \mathcal{B}_{10}$, *the solution* $\widehat{\beta}$ *to* (1.8) *with* μ, ω *as in* (4.10) *satisfies for* $v := \widehat{\beta} - \beta^*$

$$\left\| \tfrac{1}{n} X_0^T X_0 v \right\|_\infty \le \mu_1 \|\beta^*\|_2 + \mu_2 \|v\|_1 + \omega',$$

where $\mu_1 = 2\mu$, $\mu_2 = 2\mu(1 + \frac{1}{2\lambda})$ *and* $\omega' = 2\omega$.

7. Lower and Upper RE conditions

The goal of this section is to show that for Δ defined in (7.2), the presumption in Lemmas 37 and 39 as restated in (7.1) holds with high probability (cf Theorem 26). We first state a deterministic result showing that the Lower and Upper RE conditions hold for $\widehat{\Gamma}_A$ under condition (7.1) in Corollary 25. This allows us to prove Lemma 15 in Section F. See Sections K and L, where we show that Corollary 25 follows immediately from the geometric analysis result as stated in Lemma 39.

Corollary 25. *Let $1/8 > \delta > 0$. Let $1 \le k < m/2$. Let $A_{m\times m}$ be a symmetric positive semidefinite covariance matrix. Let $\widehat{\Gamma}_A$ be an $m \times m$ symmetric matrix and $\Delta = \widehat{\Gamma}_A - A$. Let $E = \cup_{|J|\le k} E_J$, where $E_J = \mathsf{span}\{e_j : j \in J\}$. Suppose that $\forall u, v \in E \cap S^{m-1}$*

$$\left|u^T \Delta v\right| \le \delta \le \frac{3}{32}\lambda_{\min}(A). \tag{7.1}$$

Then the Lower and Upper RE *conditions hold: for all $v \in \mathbb{R}^m$,*

$$\begin{aligned} v^T\widehat{\Gamma}_A v &\ge \frac{5}{8}\lambda_{\min}(A)\,\|v\|_2^2 - \frac{3\lambda_{\min}(A)}{8k}\,\|v\|_1^2 \\ \text{and}\quad v^T\widehat{\Gamma}_A v &\le \frac{11}{8}\lambda_{\max}(A)\,\|v\|_2^2 + \frac{3\lambda_{\min}(A)}{8k}\,\|v\|_1^2 . \end{aligned}$$

Theorem 26. *Let $A_{m\times m}, B_{n\times n}$ be symmetric positive definite covariance matrices. Let $E = \cup_{|J|\le k} E_J$ for $1 \le k < m/2$. Let Z, X be $n \times m$ random matrices defined as in Theorem 3. Let $\widehat{\tau}_B$ be defined as in (1.5). Let*

$$\Delta := \widehat{\Gamma}_A - A := \tfrac{1}{n}X^T X - \widehat{\tau}_B I_m - A.$$

Suppose that for some absolute constant $c' > 0$ and $0 < \varepsilon \le \frac{1}{C}$,

$$\frac{\mathrm{tr}(B)}{\|B\|_2} \ge \left(c' K^4 \frac{k}{\varepsilon^2}\log\left(\frac{3em}{k\varepsilon}\right)\right) \bigvee \log m, \tag{7.2}$$

where $C = C_0/\sqrt{c'}$ for C_0 as chosen to satisfy (H.6).

Then with probability at least $1 - 4\exp\left(-c_2\varepsilon^2 \frac{\mathrm{tr}(B)}{K^4\|B\|_2}\right) - 2\exp\left(-c_2\varepsilon^2 \frac{n}{K^4}\right) - 6/m^3$ for $c_2 \ge 2$, we have for all $u, v \in E \cap S^{m-1}$,

$$\left|u^T \Delta v\right| \le 8C\varpi(k)\varepsilon + 4C_0 D_1 K^2 \sqrt{\frac{\log m}{mn}},$$

where $\varpi(k) = \tau_B + \rho_{\max}(k, A)$, and $D_1 \le \frac{\|A\|_F}{\sqrt{m}} + \frac{\|B\|_F}{\sqrt{n}}$,

We prove Theorem 26 in Section M.

8. Numerical results

In this section, we present results from numerical simulations designed to validate the theoretical predictions as presented in previous sections. We implemented the composite gradient descent algorithm as described in [1, 30, 31] for solving the corrected Lasso objective function (1.7) with $(\widehat{\Gamma}, \widehat{\gamma})$ as defined in (1.6). For the Conic programming estimator, we use the implementation provided by the authors [3] with the same input $(\widehat{\Gamma}, \widehat{\gamma})$ (1.6). Throughout our experiments, A is a correlation matrix with $a_{\max} = 1$. We set the following as our default parameters: $D_0' = \|B\|_2^{1/2} + 1$, $D_0 = \sqrt{\tau_B} + 1$ and $R = \|\beta^*\|_2 \sqrt{d}$, where d is the sparsity parameter, the number of non-zero entries in β^*. In one set of simulations, we also vary R.

In our simulations, we look at three different models from which A and B will be chosen. Let $\Omega = A^{-1} = (\omega_{ij})$ and $\Pi = B^{-1} = (\pi_{ij})$. Let E denote edges in Ω, and F denote edges in Π. We choose A from one of these two models:

- AR(1) model. In this model, the covariance matrix is of the form $A = \{\rho^{|i-j|}\}_{i,j}$. The graph corresponding to the precision matrix A^{-1} is a chain.
- Star-Block model. In this model the covariance matrix is block-diagonal with equal-sized blocks whose inverses correspond to star structured graphs, where $A_{ii} = 1$, for all i. We have 32 subgraphs, where in each subgraph, 16 nodes are connected to a central hub node with no other connections. The rest of the nodes in the graph are singletons. The covariance matrix for each block S in A is generated by setting $S_{ij} = \rho_A$ if $(i, j) \in E$, and $S_{ij} = \rho_A^2$ otherwise, where $\rho_A \in \{0.3, 0.5, 0.7, 0.9\}$.

We choose B from one of the following models. Recall that $\tau_B = \operatorname{tr}(B)/n$.

- For B and $B^* = B/\tau_B = \rho(B)$, we consider the AR(1) model with two parameters. First we choose the AR(1) parameter $\rho_{B^*} \in \{0.3, 0.7\}$ for the correlation matrix B^*. We then set $B = \tau_B B^*$, where $\tau_B \in \{0.3, 0.7, 0.9\}$, depending on the experimental setup.
- We also consider a second model based on $\Pi = B^{-1}$, where we use the random concentration matrix model in [57]. The graph is generated according to a type of Erdős–Rényi random graph model. Initially, we set $\Pi = cI_{n\times n}$, and c is a constant. Then we randomly select $n \log n$ edges and update Π as follows: for each new edge (i, j), a weight $w > 0$ is chosen uniformly at random from $[w_{\min}, w_{\max}]$ where $w_{\max} > w_{\min} > 0$; we subtract w from π_{ij} and π_{ji}, and increase π_{ii} and π_{jj} by w. This keeps Π positive definite. We then rescale B to have a certain desired trace parameter τ_B.

For a given β^*, we first generate matrices A and B, where A is $m \times m$ and B is $n \times n$. For the given covariance matrices A and B, we repeat the following steps to estimate β^* in the errors-in-variables model as in (1.1a) and (1.1b),

1. We first generate random matrices $X_0 \sim \mathcal{N}_{n,m}(0, A \otimes I)$ and $W \sim \mathcal{N}_{n,m}(0, I \otimes B)$ independently from the matrix variate normal distribution

as follows. Let $Z \in \mathbb{R}^{n\times m}$ be a Gaussian random ensemble with independent entries Z_{ij} satisfying $\mathbb{E}Z_{ij} = 0$, $\mathbb{E}Z_{ij}^2 = 1$. Let Z_1, Z_2 be independent copies of Z. Let $X_0 = Z_1 A^{1/2}$ and $W = B^{1/2}Z_2$, where $A^{1/2}$ and $B^{1/2}$ are the unique square root of the positive definite matrix A and $B = \tau_B B^*$ respectively.

2. We then generate $X = X_0 + W$ and $y = X_0\beta^* + \epsilon$, where ϵ_i i.i.d. $\sim N(0,1)$. We compute $\widehat{\tau}_B$, $\widehat{\gamma}$ and $\widehat{\Gamma}$ according to (1.5) and (1.6) using X, y, where by (1.5), $\widehat{\tau}_B := \frac{1}{n}\widehat{\mathrm{tr}}(B) = \frac{1}{mn}\big(\|X\|_F^2 - n\mathrm{tr}(A)\big)_+$.
3. Finally, we feed X and y to the Composite Gradient Descent algorithm as described in [1, 30] to solve the Lasso program (1.7) to recover β^*, where we set the step size parameter to be ζ. The output of this step is denoted by $\widehat{\beta}$, the estimated β^*. We then compute the relative error of $\widehat{\beta}$: $\|\widehat{\beta} - \beta^*\|/\|\beta^*\|$, where $\|\cdot\|$ denotes either the ℓ_1 or the ℓ_2 norm.

The final relative error is the average of 100 runs for each set of tuning and step-size parameters; for the Conic programming estimator, we solve (1.8) instead of (1.7) to recover β^*.

8.1. Relative error

In the first experiment, A and B are generated using the AR(1) model with parameters $\rho_A, \rho_{B^*} \in \{0.3, 0.7\}$ and trace parameter $\tau_B \in \{0.3, 0.7, 0.9\}$. We see in Figures 1 and 2 that a larger sample size is required when ρ_A, ρ_{B^*} or τ_B increases. To explain these results, we first recall the following definition of the Signal-to-noise ratio, where we take $K = M_\varepsilon \asymp 1$

$$\begin{aligned} \mathsf{S/M} &\asymp \frac{\|\beta^*\|_2^2}{\tau_B\|\beta^*\|_2^2 + 1} = \frac{1}{\tau_B + (1/\|\beta^*\|_2^2)}, \quad \text{where} \\ \mathsf{S} &:= \|\beta^*\|_2^2 \quad \text{and} \quad \mathsf{M} := 1 + \tau_B\|\beta^*\|_2^2, \end{aligned}$$

which clearly increases as $\|\beta^*\|_2^2$ increases or as the measurement error metric τ_B decreases. We keep $\|\beta^*\|_2 = 5$ throughout our simulations. The corrected Lasso recovery problem thus becomes more difficult as τ_B increases. Indeed, we observe that a larger sample size n is needed when τ_B increases from 0.3 to 0.9 in order to control the relative ℓ_2 error to stay at the same level. Moreover, in view of Theorem 6, we can express the relative error as follows: for $\alpha \asymp \lambda_{\min}(A)$ and $K \asymp 1$,

$$\frac{\left\|\widehat{\beta} - \beta^*\right\|_2}{\|\beta^*\|_2} = O_P\left(\frac{(\|B\|_2^{1/2} + 1)}{\lambda_{\min}(A)}\sqrt{\frac{\mathsf{M}}{\mathsf{S}}}\sqrt{\frac{d\log m}{n}}\right). \tag{8.1}$$

Note that when $\|\beta^*\|_2$ is large enough and $\tau_B = \Omega(1)$, the factor preceding $\sqrt{\frac{d\log m}{n}}$ on the RHS of (8.1) is proportional to $\frac{(\|B\|_2^{1/2}+1)\sqrt{\tau_B}}{\lambda_{\min}(A)}$. When we plot the relative ℓ_2 error $\|\widehat{\beta} - \beta^*\|_2/\|\beta^*\|_2$ versus the rescaled sample size $\frac{n}{d\log m}$

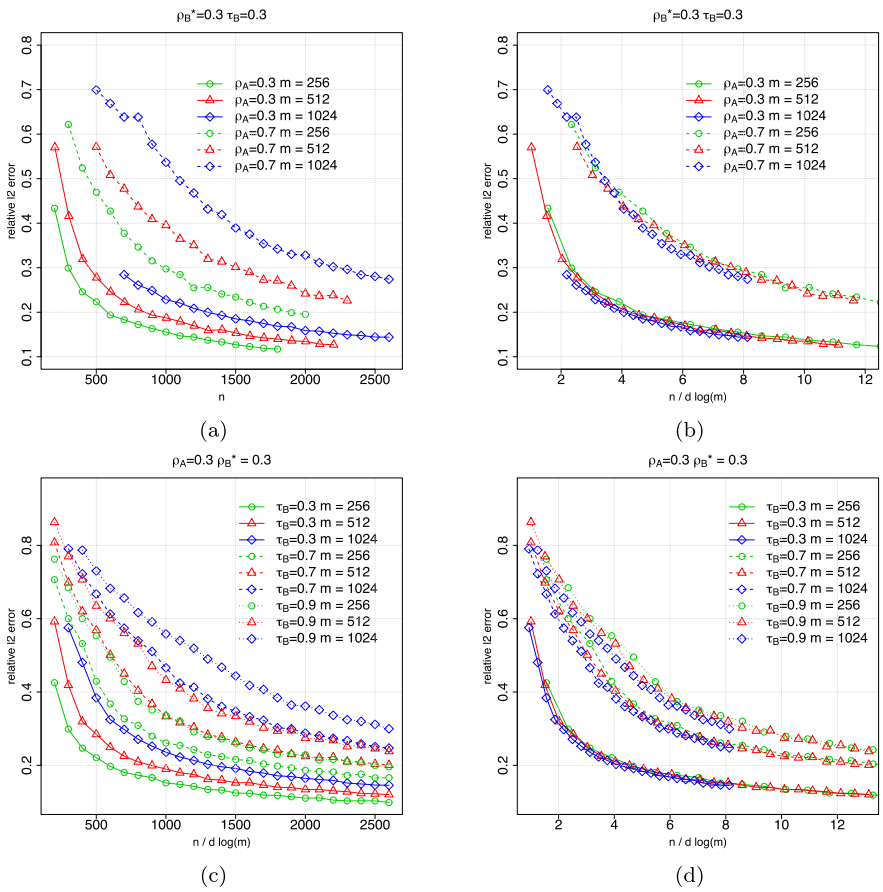
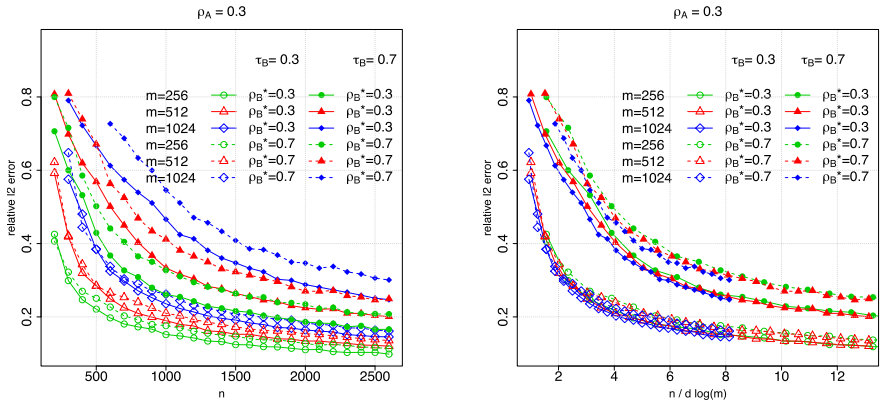


FIG 1. *Plots of the relative ℓ_2 error after running composite gradient descent algorithm on recovering β^* using the corrected Lasso objective function with sparsity parameter $d = \lfloor \sqrt{m} \rfloor$, where we vary $m \in \{256, 512, 1024\}$. We generate A and B using the* AR(1) *model with $\rho_A, \rho_{B^*} \in \{0.3, 0.7\}$ and $\tau_B = \{0.3, 0.7, 0.9\}$. In the left and right column, we plot the relative ℓ_2 error with respect to sample size n as well as the rescaled sample size $n/(d \log m)$. As n increases, we see that the statistical error decreases. In the top row, we vary the* AR(1) *parameter $\rho_A \in \{0.3, 0.7\}$, while holding (τ_B, ρ_B^*) and $\|\beta^*\|_2$ invariant. Plot (a) shows the relative ℓ_2 error versus n for $m = 256, 512, 1024$. In Plots (c) and (d), we vary the trace parameter $\tau_B \in \{0.3, 0.7, 0.9\}$, while fixing the* AR(1) *parameters $\rho_A, \rho_B^* = 0.3$. Plot (b) and (d) show the relative ℓ_2 error versus the rescaled sample size $n/(d \log m)$. The curves now align for different values of m in the rescaled plots, consistent with the theoretical prediction in Theorem 6.*

under the same S/M ratio, the two sets of curves corresponding to $\rho_A = 0.3$ and $\rho_A = 0.7$ indeed line up in Figure 1(b), as predicted by (8.1). We observe in Figure 1(b), the rescaled curves overlap well for different values of (m, d) for each ρ_A when we keep (ρ_{B^*}, τ_B) and the length $\|\beta^*\|_2 = 5$ invariant. Moreover, the upper bound on the relative ℓ_2 error (8.1) characterizes the relative positions



(a) (b)

FIG 2. *Plots of the relative ℓ_2 error after running composite gradient descent algorithm on recovering β^* using the corrected Lasso objective function with sparsity parameter $d = \lfloor \sqrt{m} \rfloor$, where we vary $m \in \{256, 512, 1024\}$. We generate A and B using the* AR(1) *model with $\rho_A = 0.3$ and $\rho_B^* \in \{0.3, 0.7\}$. We set $B = \tau_B B^*$ and vary the trace parameter $\tau_B \in \{0.3, 0.7\}$ for each value of ρ_{B^*}. The parameters τ_B and ρ_B^* affect the rate of convergence through $D_0' = \|B\|_2^{1/2} + a_{\max}^{1/2}$ and $\tau_B^{1/2}$. Plot (b) shows the relative ℓ_2 error versus the rescaled sample size $n/(d \log m)$. We observe that as τ_B increases from* 0.3 *to* 0.7*, the two sets of curves corresponding to $\rho_{B^*} = 0.3, 0.7$ become visibly more separated. As n increases, all curves converge to* 0.

of these two sets of curves in that the ratio between the ℓ_2 error corresponding to $\rho_A = 0.7$ and that for $\rho_A = 0.3$ along the y-axis roughly falls within the interval $(2, 3)$ for each n, while $\lambda_{\min}(\text{AR}(1), 0.3)/\lambda_{\min}(\text{AR}(1), 0.7) = 3$. These results are consistent with the theoretical predictions in Theorems 3 and 6.

In Figure 1(c) and (d), we also show the effect of τ_B when τ_B is chosen from $\{0.3, 0.7, 0.9\}$, while fixing the AR(1) parameters $\rho_A = 0.3$ and $\rho_{B^*} = 0.3$. As predicted by our theory, as the measurement error magnitude τ_B increases, M increases, resulting in a larger relative ℓ_2 error for a fixed sample size n.

While the effect of ρ_A as shown in (8.1) through the minimal eigenvalue of A is directly visible in Figure 1(b), the effect of $\rho_B{}^*$ is more subtle, as it is modulated by τ_B as shown in Figure 2(a) and (b). When τ_B is fixed, our theory predicts that $\|B\|_2$ plays a role in determining the ℓ_p error, $p = 1, 2$, through the penalty parameter λ in view of (8.1). The effect of ρ_{B^*}, which goes into the parameter $D_0' = \|B\|_2^{1/2} + a_{\max}^{1/2} \asymp 1$, is not changing the sample requirement or the rate of convergence as significantly as that of ρ_A when $\tau_B = 0.3$. This is shown in the bottom set of curves in Figure 2(a) and (b). On the other hand, the trace parameter τ_B plays a dominating role in determining the sample size as well as the ℓ_p error for $p = 1, 2$, especially when the length of the signal β^* is large: $\|\beta^*\|_2 = \Omega(1)$. In particular, the separation between the two sets of curves in Figure 2(b), which correspond to the two choices of $\rho_B{}^*$, is clearly modulated by τ_B and becomes more visible when $\tau_B = 0.7$.

These findings are also consistent with our theoretical prediction that in order to guarantee statistical and computational convergence, the sample size needs to grow according to the following relationship to be specified in (8.2). We will show in the proof of Theorem 9 that the condition on sparsity d as stated in (5.2) implies that as ρ_A, or τ_B, or the step size parameter ζ increases, we need to increase the sample size in order to guarantee computational convergence for the composite gradient descent algorithm given the following lower bound:

$$\begin{aligned} n &= \Omega\left(d\tau_0 \log m \left\{\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)^2}\right\} \bigvee \left\{\frac{\zeta}{(\bar{\alpha}_\ell)^2}\right\}\right), \quad \text{where} \qquad (8.2) \\ \tau_0 &\asymp \frac{(\rho_{\max}(s_0, A) + \tau_B)^2}{\lambda_{\min}(A)}. \end{aligned}$$

We illustrate the effect of the penalty and step size parameters in Section 8.2.

8.2. Corrected Lasso via GD versus Conic programming estimator

In the second experiment, both A and B are generated using the AR(1) model with parameters $\rho_A = 0.3$, $\rho_{B^*} = 0.3$, and $\tau_B \in \{0.3, 0.7\}$. We set $m = 1024$, $d = 10$ and $\|\beta^*\|_2 = 5$. We then compare the performance of the corrected Lasso estimator (1.7) using the composite gradient descent algorithm with the Conic programming estimator, which is a convex program designed and implemented by authors of [3].

We consider three choices for the step size parameter for the composite gradient descent algorithm: $\zeta_1 = \lambda_{\max}(A) + \frac{1}{2}\lambda_{\min}(A)$, $\zeta_2 = \frac{3}{2}\lambda_{\max}(A)$ and $\zeta_3 = 2\lambda_{\max}(A)$. We observe that the gradient descent algorithm consistently produces an output such that its statistical error in ℓ_2 norm is lower than the best solution produced by the Conic programming estimator, when both methods are subject to optimal tuning after we fix upon the radius $R = \sqrt{d}\,\|\beta^*\|_2$ for (1.10) and (ω, λ) in (1.8) as follows. As illustrated in our theory, one can think of the parameter λ in (1.7) and parameters μ, ω in (1.8) satisfying

$$\lambda \asymp \mu\,\|\beta^*\|_2 + \omega,$$

where we set $\omega = 0.1 D_0 \sqrt{\frac{\log m}{n}}$, where the factor 0.1 is chosen without loss of generality, as we will sweep over $f \in (0, 0.8]$ to run through a sufficiently large range of values of the tuning parameters:

- For the corrected Lasso estimator, we set $\lambda = f D_0' \widehat{\tau}_B^{1/2}\,\|\beta^*\|_2 \sqrt{\frac{\log m}{n}} + \omega$;
- For the Conic programming estimator, we use $\mu = f D_0' \widehat{\tau}_B^{1/2} \sqrt{\frac{\log m}{n}}$. We set $\lambda = 1$ in (1.8), which is independent of the Lasso penalty.

The factor f is chosen to reflect the fact that in practice, we do not know the exact value of $\|\beta^*\|_2$ or $\|\beta^*\|_1$, D_0 or D_0', or other parameters related to the spectrum properties of A, B; moreover, in practice, we wish to understand the whole-path behavior for both estimators.

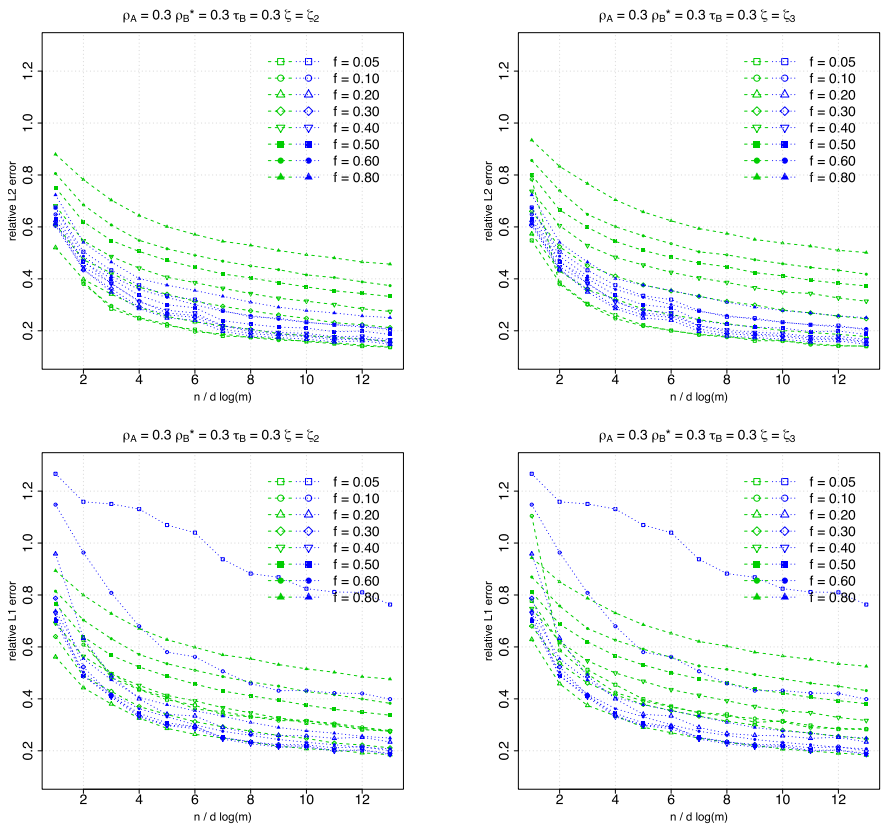


FIG 3. *Plots of the relative ℓ_1 and ℓ_2 error $\left\|\widehat{\beta} - \beta^*\right\| / \|\beta^*\|$ for the Conic programming estimator and the corrected Lasso estimator obtained via running the composite gradient descent algorithm on (approximately) recovering β^*. Set parameters $d = 10$ and $m = 1024$ while varying n. Generate A and B using the* AR(1) *model with parameters $\rho_A = 0.3$, $\rho_{B^*} = 0.3$ and $\tau_B = 0.3$. Set $\zeta \in \{\zeta_1, \zeta_2, \zeta_3\}$. We compare the performance of the corrected Lasso and the Conic programming estimators over choices of λ and μ while sweeping through $f \in (0, 0.8]$. In the top row, we plot the relative ℓ_2 error for the Conic programming estimator (blue dotted lines) and the corrected Lasso (green dashed lines) via the composite gradient descent algorithm with step size parameter set to be $\zeta_2 = \frac{3}{2}\lambda_{\max}(A)$ and $\zeta_3 = 2\lambda_{\max}(A)$; in the bottom row, we plot the relative ℓ_1 error under the same settings. We note that the composite gradient descent algorithm starts to converge even when we set the step size parameter to be $\zeta_1 = \lambda_{\max}(A) + \frac{1}{2}\lambda_{\min}(A)$.*

In Figures 3 and 4, we plot the relative error in ℓ_1 and ℓ_2 norm as n increases from 100 to 2500, while sweeping over penalty factor $f \in [0.05, 0.8]$ for $\tau_B = 0.3$ and $\tau_B = 0.7$ respectively. For both estimators, the relative ℓ_2 and ℓ_1 error versus the scaled sample size $n/(d \log m)$ are also plotted. In these figures, green dashed lines are for the corrected Lasso estimator via gradient descent algorithm, and blue dotted lines are for the Conic programming estimator. These plots allow us to observe the behaviors of the two estimators across a set of tuning

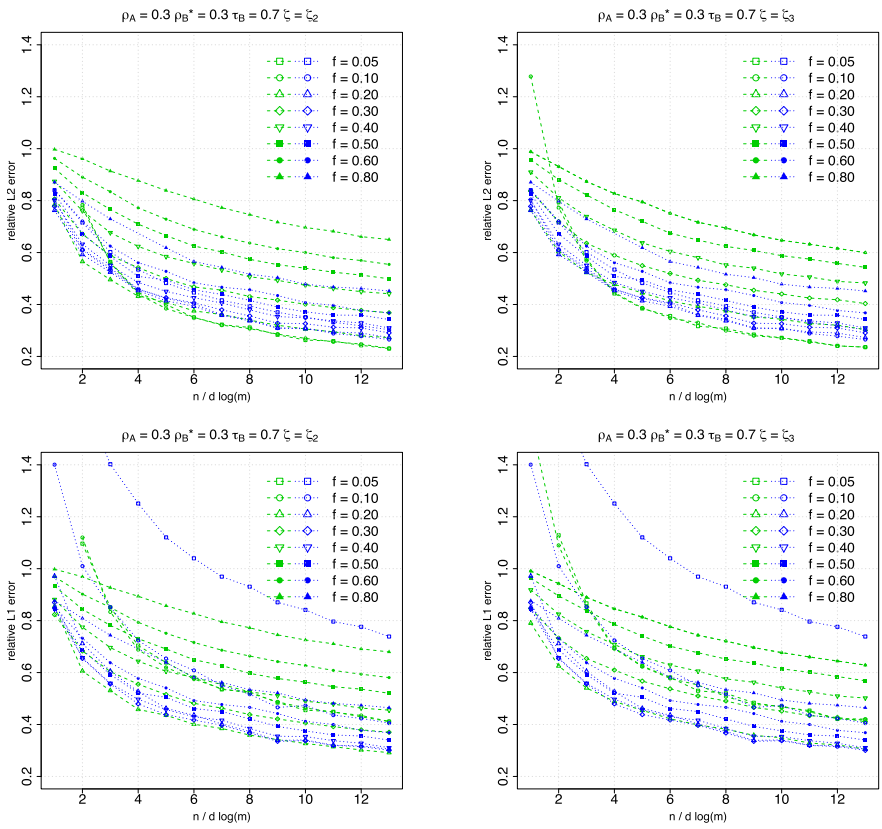


FIG 4. *Plots of the relative ℓ_1 and ℓ_2 error $\left\|\widehat{\beta} - \beta^*\right\| / \|\beta^*\|$ after running the Conic programming estimator and composite gradient descent algorithm on recovering β^* using the corrected Lasso objective function with sparsity parameter $d = 10$ and $m = 1024$ while varying n. Both A and B are generated using the* AR(1) *model with parameters $\rho_A = 0.3$, $\rho_{B^*} = 0.3$ and $\tau_B = 0.7$. We compare the performance of the corrected Lasso (green dashed lines) and the Conic programming estimators (blue dotted lines) over choices of λ and μ while sweeping through $f \in (0, 0.8]$. For the composite gradient descent algorithm, we choose ζ from $\{\zeta_1, \zeta_2, \zeta_3\}$. In the top row, we plot the ℓ_2 error for the Conic and the corrected Lasso with $\zeta_2 = \frac{3}{2}\lambda_{\max}(A)$ and $\zeta_3 = 2\lambda_{\max}(A)$, while in the bottom row, we plot the ℓ_1 error corresponding to the two step size parameters.*

parameters. Overall, we see that both methods are able to achieve low relative error in $\ell_p, p = 1, 2$ norm when λ and μ are chosen from a suitable range.

For the corrected Lasso estimator, we display results where the step size parameter ζ is set to $\zeta_2 = \frac{3}{2}\lambda_{\max}(A)$ and $\zeta_3 = 2\lambda_{\max}(A)$ in the left and right column respectively. We mention in passing that the algorithm starts to converge even when we set $\zeta = \zeta_1 = \lambda_{\max}(A) + \frac{1}{2}\lambda_{\min}(A)$ as we observe quantitively similar behavior as the displayed cases. For both estimators, we observe that we need a larger sample size n in case $\tau_B = 0.7$ in order to control the error at the same level as in case $\tau_B = 0.3$. In Figure 5, we plot the ℓ_2 and ℓ_1 error

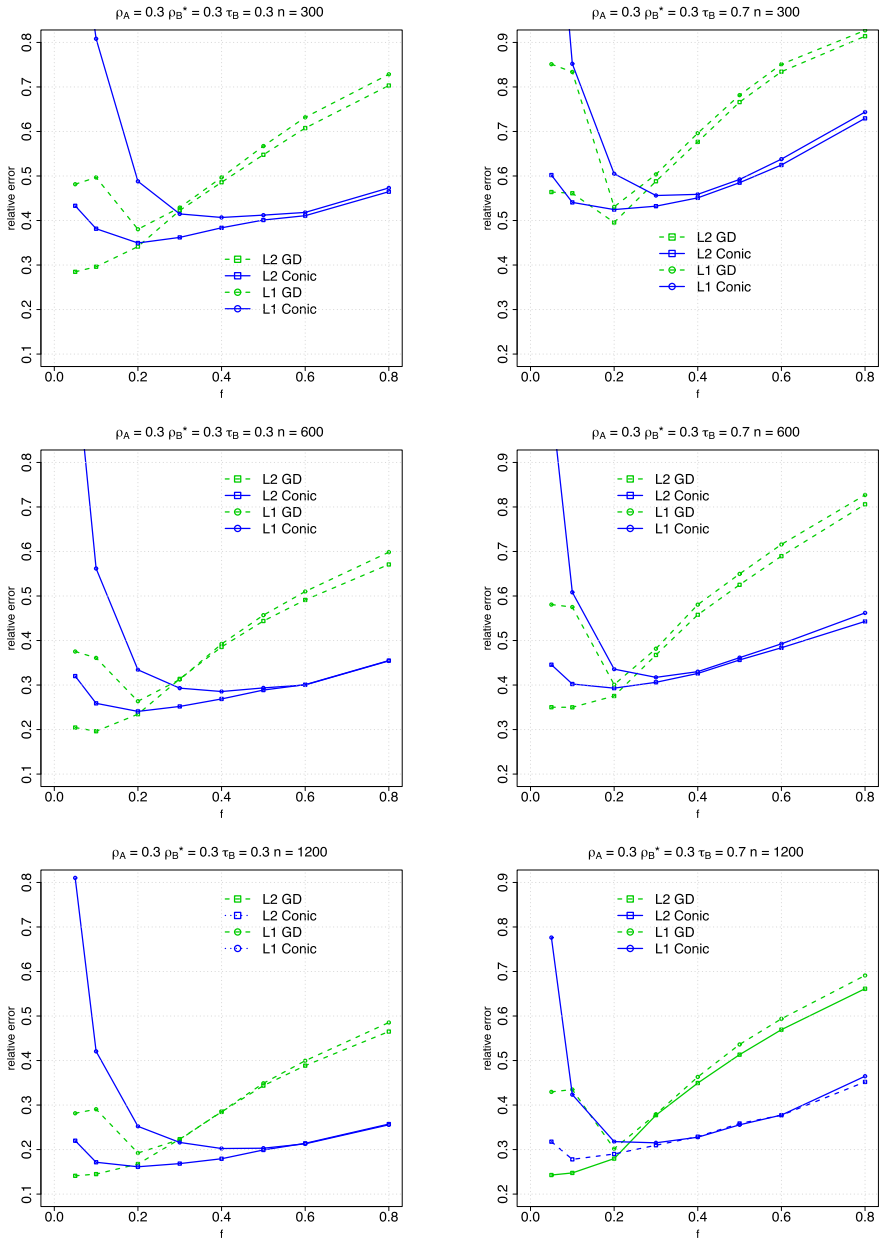


FIG 5. *Plot of the relative error in ℓ_2 and ℓ_1 norm versus the penalty factor $f \in (0, 0.8]$ as we change the sample size n. Set $m = 1024$ and $d = 10$. Both A and B are generated using the* AR(1) *model with parameters $\rho_A = 0.3$ and $\rho_{B^*} = 0.3$. We plot the relative error in ℓ_1 and ℓ_2 norm versus the penalty parameter factor $f \in (0, 0.8]$ for $n = 300, 600, 1200$ when $\zeta = \frac{3}{2}\lambda_{\max}(A)$. In the left column, $\tau_B = 0.3$. In the right column, we set $\tau_B = 0.7$.*

versus the penalty factor $f \in [0.05, 0.8]$ for sample size $n \in \{300, 600, 1200\}$. We plot results for $\tau_B = 0.3$ and $\tau_B = 0.7$ in the left and right column respectively. For these plots, we focus on cases when $n > d\kappa(A) \log m$, by choosing $n \in \{300, 600, 1200\}$; Otherwise, the gradient descent algorithm does not yet reach the sample requirement (8.2) that guarantees computational convergence. In Figure 5, we observe that the Conic programming estimator is relatively stable over the choices of μ once $f \geq 0.2$. The composite gradient algorithm favors smaller penalties such as $f \in [0.05, 0.2]$, leading to smaller relative error in the ℓ_1 and ℓ_2 norm, consistent with our theoretical predictions. These results also confirm our theoretical prediction that the Lasso and Conic programming penalty parameters λ and μ need to be adaptively chosen based on the noise level τ_B, because a larger than necessary amount of penalty will cause larger relative error in both ℓ_1 and ℓ_2 norm.

8.3. Sensitivity to tuning parameters

In the third experiment, we change the ℓ_1-ball radius $R \in \{R^*, 5R^*, 9R^*\}$ in (1.10), where $R^* = \|\beta^*\|_2\sqrt{d}$, while running through different penalties for the composite gradient descent algorithm. In the left column in Figure 6, A and B are generated using the AR(1) model with $\rho_A = 0.3$, $\rho_{B^*} = 0.3$ and $\tau_B = 0.7$. In the right column, we set $\tau_B = 0.3$, while keeping other parameters invariant.

As predicted by our theory, a larger radius demands correspondingly larger penalty to ensure consistent estimation using the composite gradient descent algorithm; this in turn will increase the relative error when R is too large, for example, when $R = \widetilde{\Omega}(\sqrt{\frac{n}{\log m}})$, where the $\widetilde{\Omega}(\cdot)$ notation hides parameters involving τ_B and $\kappa(A)$. This is observed in Figure 6. When n is sufficiently large relative to τ_B and $\kappa(A)$, the optimal ℓ_1 and ℓ_2 error become less sensitive with regard to the choice of R, so long as $R = \widetilde{O}(\sqrt{\frac{n}{\log m}})$, where $\widetilde{O}(\cdot)$ hides parameters involving τ_B and $\kappa(A)$, as shown in Figure 6.

8.4. Statistical and optimization error in gradient descent

In the last set of experiments, we study the statistical error and optimization error for each iteration within the composite gradient descent algorithm. We observe a geometric convergence of the optimization error $\|\beta^t - \widehat{\beta}\|_2$.

For each experiment, we repeat the following procedure 10 times: we start with a random initialization point β^0 and apply the composite gradient descent algorithm to compute an estimate $\widehat{\beta}$; we compute the optimization error $\log(\|\beta^t - \widehat{\beta}\|_2)$, which records the difference between β^t and $\widehat{\beta}$, where $\widehat{\beta}$ is the final solution. In all simulations, we plot the log error $\log(\|\beta^t - \widehat{\beta}\|_2)$ between the iterate β^t at time t versus the final solution $\widehat{\beta}$, as well as the statistical error $\log(\|\beta^t - \beta^*\|_2)$, which is the difference between β^t and β^* at time t. Each curve plots the results averaged over ten random instances.

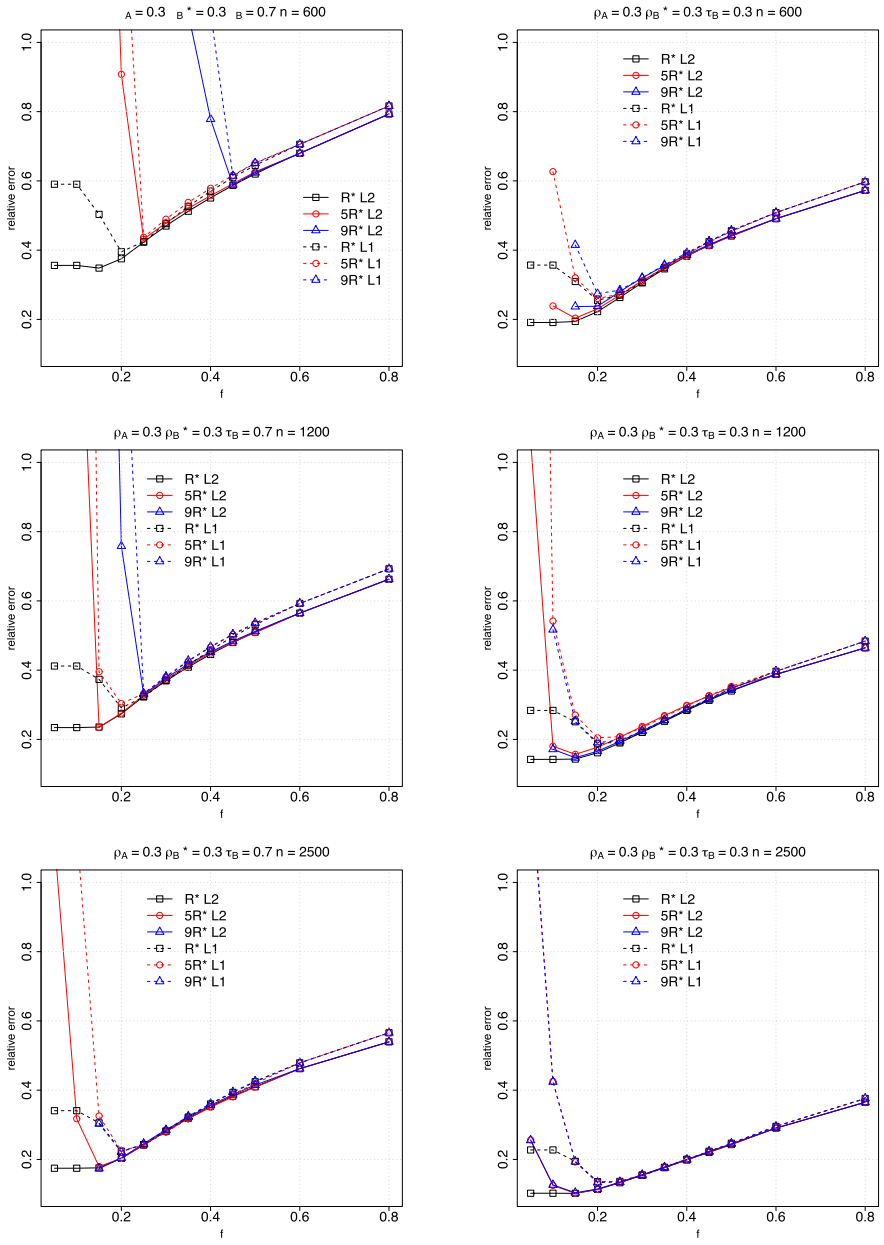
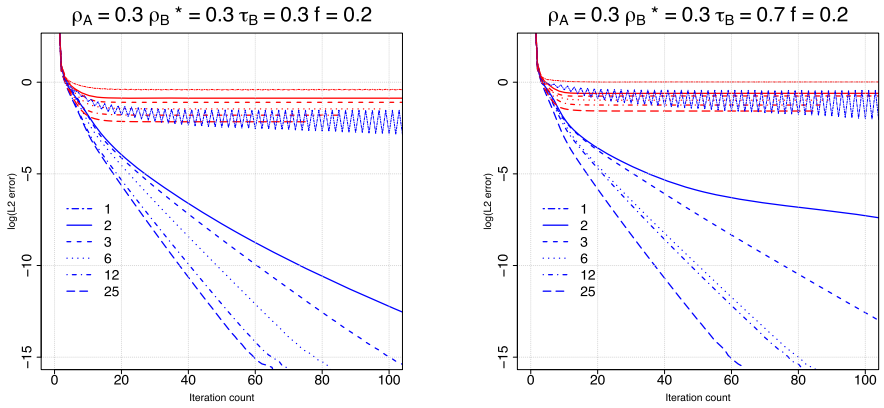


FIG 6. *Plot of the relative error in ℓ_2 and ℓ_1 norm versus the penalty factor $f \in (0, 0.8]$ as we change the radius R. Set $m = 1024$, $d = 10$ and $n \in \{600, 1200, 2500\}$. We change the ℓ_1-ball radius $R \in \{R^*, 5R^*, 9R^*\}$, where $R^* = \|\beta^*\|_2\sqrt{d}$, while running through different penalties for the composite gradient descent algorithm. In the left column, A and B are generated using the* AR(1) *model with $\rho_A = 0.3$, $\rho_{B^*} = 0.3$ and $\tau_B = 0.7$. In the right column, we set $\tau_B = 0.3$, while keeping other parameters invariant.*

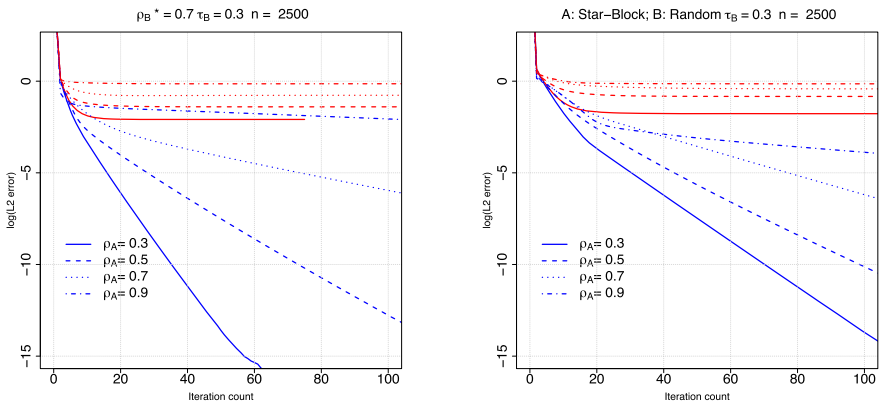


(a) (b)

FIG 7. *Plots of the statistical error* $\log(\|\beta^t - \beta^*\|_2)$*, and the optimization error* $\log(\|\beta^t - \widehat{\beta}\|_2)$ *versus iteration number* t*, generated by running the composite gradient descent algorithm on the corrected Lasso objective function. Each curve represents an average over 10 random trials, each with a different initialization point of* β^0*. In Plots (a) and (b),* B *is generated using the* AR(1) *model with* $\rho_{B^*} = 0.3$ *and* A *is generated using the* AR(1) *model with* $\rho_A = 0.3$*. We set* $\tau_B = 0.3, 0.7$ *in Plot (a) and (b) respectively. We set* $n = \lceil \rho d \log m \rceil$*, where we vary* $\rho \in \{1, 2, 3, 6, 12, 25\}$*.*

In the first experiment, both A and B are generated using the AR(1) model with parameters $\rho_A = 0.3$ and $\rho_{B^*} = 0.3$. We set $m = 1024$, $d = 10$ and $\tau_B \in \{0.3, 0.7\}$. These results are shown in Figure 7. Within each plot, the red curves show the statistical error and the blue curves show the optimization error. We can see the optimization error $\|\beta^t - \widehat{\beta}\|_2$ decreases exponentially for each iteration, obeying a geometric convergence. To illuminate the dependence of convergence rate on the sample size n, we study the optimization error $\log(\|\beta^t - \widehat{\beta}\|_2)$ when $n = \lceil \rho d \log m \rceil$, where we vary $\rho \in \{1, 2, 3, 6, 12, 25\}$. When $n = d \log m$, the composite gradient algorithm fails to converge since the sample size is too small for the RSC/RSM conditions to hold, resulting in the oscillatory behavior of the algorithm for a constant step size. As the factor ρ increases, the lower and upper RE curvature α and smoothness parameter $\widetilde{\alpha}$ become more concentrated around $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively, and the tolerance parameter τ decreases at the rate of $\frac{\log m}{n}$. Hence we observe faster rates of convergence for $\rho = 25, 12, 6$ compared to $\rho = 2, 3$. This is well aligned with our theoretical prediction that once $n = \Omega(\kappa(A) \frac{\tau_0}{\lambda_{\min}(A)} d \log m)$ (cf. (8.2)), we expect to observe a geometric convergence of the computational error $\|\beta^t - \widehat{\beta}\|_2$.

For the statistical error, we first observe the geometric contraction, and then the curves flatten out after a certain number of iterations, confirming the claim that β^t converges to β^* only up to a neighborhood of radius defined through the statistical error bound $\varepsilon^2_{\text{stat}}$; that is, the geometric convergence is not guaranteed to an arbitrary precision, but only to an accuracy related to statistical precision



(a) (b)

FIG 8. *Plots of the statistical error* $\log(\|\beta^t - \beta^*\|_2)$*, and the optimization error when we change the topology. In the last experiment, we have* $m = 1024$*,* $d = 10$ *and* $n = 2500$*. In (a),* A *is generated using the* AR(1) *model with four choices of* $\rho_A \in \{0.3, 0.5, 0.7, 0.9\}$ *and* B *is generated using* AR(1) *model with* $\rho_{B^*} = 0.7$ *and* $\tau_B = 0.3$*. In (b),* A *follows the Star-Block model and* B *follows the random graph model. We show four choices of* $\rho_A \in \{0.3, 0.5, 0.7, 0.9\}$

of the problem measured by ℓ_2 error: $\|\widehat{\beta} - \beta^*\|_2^2 =: \varepsilon_{\text{stat}}^2$ between the global optimizer $\widehat{\beta}$ and the true parameter β^*.

In the second experiment, A is generated from the Star-Block model, where we have 32 subgraphs and each subgraph has 16 edges; B is generated using the random graph model with $n \log n$ edges and adjusted to have $\tau_B = 0.3$. We set $m = 1024$, $n = 2500$ and $d = 10$. We then choose $\rho_A \in \{0.3, 0.5, 0.7, 0.9\}$. The results are shown in Figure 8(b). As we increase ρ_A, we need larger sample size to control the statistical error. Hence for a fixed n, the statistical error is bigger for $\rho_A = 0.7$, compared to cases where $\rho_A = 0.5$ or $\rho_A = 0.3$, for which we have $\kappa(A) = 42.06$ and $\kappa(A) = 10.2$ (for $\rho_A = 0.3$) respectively; Moreover, the rates of convergence are faster for the latter two compared to $\rho_A = 0.7$, where $\kappa(A) = 169.4$. When $\rho_A = 0.9$, the composite gradient descent algorithm fails to converge as $\rho(A)$ is too large (hence not plotted here) with respect to the sample size we fix upon. In Figure 8(a), we show results of A being generated using the AR(1) model with four choices of $\rho_A \in \{0.3, 0.5, 0.7, 0.9\}$ and B being generated using the AR(1) model with $\rho_{B^*} = 0.7$ and $\tau_B = 0.3$. We observe quantitively similar behavior as in Figure 8(b).

9. Proof of Lemma 1

Proof of Lemma 1. Part I: Suppose that the Lower-RE condition holds for $\Gamma := A^T A$. Let $x \in \text{Cone}(s_0, k_0)$. Then

$$\|x\|_1 \leq (1 + k_0) \|x_{T_0}\|_1 \leq (1 + k_0)\sqrt{s_0} \|x_{T_0}\|_2 .$$

Thus for $x \in \mathrm{Cone}(s_0, k_0) \cap S^{p-1}$ and $\tau(1+k_0)^2 s_0 \le \alpha/2$, we have

$$\begin{aligned} \|Ax\|_2 = (x^T A^T A x)^{1/2} &\ge \left(\alpha \|x\|_2^2 - \tau \|x\|_1^2\right)^{1/2} \\ &\ge \left(\alpha \|x\|_2^2 - \tau(1+k_0)^2 s_0 \|x_{T_0}\|_2^2\right)^{1/2} \\ &\ge \left(\alpha - \tau(1+k_0)^2 s_0\right)^{1/2} \ge \sqrt{\frac{\alpha}{2}}. \end{aligned}$$

Thus the $\mathsf{RE}(s_0, k_0, A)$ condition holds with

$$\frac{1}{K(s_0, k_0, A)} := \min_{x \in \mathrm{Cone}(s_0, k_0)} \frac{\|Ax\|_2}{\|x_{T_0}\|_2} \ge \sqrt{\frac{\alpha}{2}}$$

where we use the fact that for any $J \in \{1, \ldots, p\}$ such that $|J| \le s_0$, $\|x_J\|_2 \le \|x_{T_0}\|_2$. We now show the other direction.

Part II. Assume that $\mathsf{RE}(4R^2, 2R-1, A)$ holds for some integer $R > 1$. Assume that for some $R > 1$

$$\|x\|_1 \le R \|x\|_2 .$$

Let $(x_i^*)_{i=1}^p$ be non-increasing arrangement of $(|x_i|)_{i=1}^p$. Then

$$\begin{aligned} \|x\|_1 &\le R \left(\sum_{j=1}^{s} (x_j^*)^2 + \sum_{j=s+1}^{\infty} \left(\frac{\|x\|_1}{j} \right)^2 \right)^{1/2} \\ &\le R \left(\|x_J^*\|_2^2 + \|x\|_1^2 \frac{1}{s} \right)^{1/2} \le R \left(\|x_J^*\|_2 + \|x\|_1 \frac{1}{\sqrt{s}} \right) \end{aligned}$$

where $J := \{1, \ldots, s\}$. Choose $s = 4R^2$. Then

$$\|x\|_1 \le R \|x_J^*\|_2 + \frac{1}{2} \|x\|_1 .$$

Thus we have

$$\|x\|_1 \le 2R \|x_J^*\|_2 \le 2R \|x_J^*\|_1 \quad \text{and hence} \tag{9.1}$$

$$\|x_{J^c}^*\|_1 \le (2R-1) \|x_J^*\|_1 . \tag{9.2}$$

Then $x \in \mathrm{Cone}(4R^2, 2R-1)$. Then for all $x \in S^{p-1}$ such that $\|x\|_1 \le R \|x\|_2$, we have for $k_0 = 2R-1$ and $s_0 := 4R^2$,

$$x^T \Gamma x \ge \frac{\|x_{T_0}\|_2^2}{K^2(s_0, k_0, A)} \ge \frac{\|x\|_2^2}{\sqrt{s_0} K^2(s_0, k_0, A)} =: \alpha \|x\|_2^2$$

where we use the fact that $(1+k_0)\|x_{T_0}\|_2^2 \ge \|x\|_2^2$ by Lemma 33 with x_{T_0} as defined therein. Otherwise, suppose that $\|x\|_1 \ge R \|x\|_2$. Then for a given $\tau > 0$,

$$\alpha \|x\|_2^2 - \tau \|x\|_1^2 \le \left(\frac{1}{\sqrt{s_0} K^2(s_0, k_0, A)} - \tau R^2\right) \|x\|_2^2 . \tag{9.3}$$

Thus we have by the choice of τ as in (2.1) and (9.3)

$$\begin{aligned} x^T\Gamma x \geq \lambda_{\min}(\Gamma)\,\|x\|_2^2 &\geq \left(\frac{1}{\sqrt{s_0}K^2(s_0,k_0,A)} - \tau R^2\right)\|x\|_2^2 \\ &\geq \alpha\,\|x\|_2^2 - \tau\,\|x\|_1^2. \end{aligned}$$

The Lemma thus holds. □

10. Proof of Theorem 3

Throughout this proof, we assume that $\mathcal{A}_0 \cap \mathcal{B}_0$ holds. First we note that it is sufficient to have (3.2) in order for (6.3) to hold. Condition (3.2) guarantees that for $\mathcal{V} = 3eM_A^3/2$,

$$\begin{aligned} r(B) := \frac{\mathrm{tr}(B)}{\|B\|_2} &\geq 16c'K^4\frac{n}{\log m}\log\frac{\mathcal{V}m\log m}{n} \\ &\geq 16c'K^4\frac{n}{\log m}\log\left(\frac{3emM_A^3\log m}{2n}\right) \\ &= c'K^4\frac{1}{\varepsilon^2}\frac{4}{M_A^2}\frac{n}{\log m}\log\left(\frac{6emM_A}{\frac{4}{M_A^2}(n/\log m)}\right) \\ &\geq c'K^4\frac{1}{\varepsilon^2}s_0\log\left(\frac{6emM_A}{s_0}\right) \\ &= c'K^4\frac{s_0}{\varepsilon^2}\log\left(\frac{3em}{s_0\varepsilon}\right) \end{aligned} \tag{10.1}$$

where $\varepsilon = \frac{1}{2M_A} \leq \frac{1}{128C}$, and the last inequality holds given that $k\log(cm/k)$ on the RHS of (10.1) is a monotonically increasing function of k,

$$s_0 \leq \frac{4n}{M_A^2\log m} \text{ and } M_A = \frac{64C(\rho_{\max}(s_0,A)+\tau_B)}{\lambda_{\min}(A)} \geq 64C.$$

Next we check that the choice of d as in (3.4) ensures that (6.6) holds for D_ϕ defined there. Indeed, for $c'K^4 \leq 1$, we have

$$d \leq C_A(c'K^4 \wedge 1)\frac{\phi n}{\log m} \leq C_A\,(c'D_\phi \wedge 1)\,\frac{n}{\log m}.$$

By Lemma 15, we have on event $\mathcal{A}_0$, the modified gram matrix $\widehat{\Gamma}_A := \frac{1}{n}(X^TX - \widehat{\mathrm{tr}}(B)I_m)$ satisfies the Lower RE conditions with α and τ as in (10.2). Theorem 3 follows from Theorem 16, so long as we can show that condition (6.4) holds for $\lambda \geq 4\psi\sqrt{\frac{\log m}{n}}$, where the parameter ψ is as defined (3.5),

$$\text{curvature } \alpha = \frac{5}{8}\lambda_{\min}(A) \text{ and tolerance } \tau = \frac{\lambda_{\min}(A)-\alpha}{s_0} = \frac{3\alpha}{5s_0}. \tag{10.2}$$

Combining (10.2) and (6.4), we need to show (6.7) holds. This is precisely the content of Lemma 17. This is the end of the proof for Theorem 3 □

11. Proof of Theorem 4

For the set $\mathrm{Cone}_J(k_0)$ as in (F.3),

$$\kappa_{\mathsf{RE}}(d_0,k_0) := \min_{J:|J|\le d_0}\min_{\Delta\in \mathrm{Cone}_J(k_0)} \frac{|\Delta^T\Psi\Delta|}{\|\Delta_J\|_2^2} = \left(\frac{1}{K(d_0,k_0,(1/\sqrt{n})Z_1A^{1/2})}\right)^2.$$

Recall the following Theorem 27 from [38].

Theorem 27. *([38]) Set $0<\delta<1$, $k_0>0$, and $0<d_0<p$. Let $A^{1/2}$ be an $m\times m$ matrix satisfying $\mathsf{RE}(d_0,3k_0,A^{1/2})$ condition as in Definition 2.1. Set*

$$d \;=\; d_0 + d_0\max_j \left\|A^{1/2}e_j\right\|_2^2 \frac{16K^2(d_0,3k_0,A^{1/2})(3k_0)^2(3k_0+1)}{\delta^2}.$$

Let Ψ be an $n\times m$ matrix whose rows are independent isotropic ψ_2 random vectors in $\mathbb{R}^m$ with constant α. Suppose the sample size satisfies

$$n \ge \frac{2000d\alpha^4}{\delta^2}\log\left(\frac{60em}{d\delta}\right). \tag{11.1}$$

Then with probability at least $1-2\exp(-\delta^2n/2000\alpha^4)$, $\mathsf{RE}(d_0,k_0,(1/\sqrt{n})\Psi A^{1/2})$ condition holds for matrix $(1/\sqrt{n})\Psi A$ with

$$0<K(d_0,k_0,(1/\sqrt{n})\Psi A^{1/2}) \le \frac{K(d_0,k_0,A^{1/2})}{1-\delta}. \tag{11.2}$$

Proof of Theorem 4. Suppose $\mathsf{RE}(2d_0,3k_0,A^{1/2})$ holds. Then for d as defined in (3.9) and $n=\Omega(dK^4\log(m/d))$, we have with probability at least $1-2\exp(\delta^2n/2000K^4)$, $\mathsf{RE}(2d_0,k_0,\frac{1}{\sqrt{n}}Z_1A^{1/2})$ condition holds with

$$\kappa_{\mathsf{RE}}(2d_0,k_0) \;=\; \left(\frac{1}{K(2d_0,k_0,(1/\sqrt{n})Z_1A^{1/2})}\right)^2 \ge \left(\frac{1}{2K(2d_0,k_0,A^{1/2})}\right)^2$$

by Theorem 27.

The rest of the proof follows from [3] Theorem 1 and thus we only provide a sketch. In more details, in view of the lemmas shown in Section 6, we need

$$\kappa_q(d_0,k_0)\ge cd_0^{-1/q}$$

to hold for some constant c for $\Psi := \frac{1}{n}X_0^TX_0$. It is shown in Appendix C in [3] that under the $\mathsf{RE}(2d_0,k_0,\frac{1}{\sqrt{n}}Z_1A^{1/2})$ condition, for any $d_0\le m/2$ and $1\le q\le 2$,

$$\begin{aligned}\kappa_1(d_0,k_0) &\ge cd_0^{-1}\kappa_{\mathsf{RE}}(d_0,k_0) \quad \text{and}\\ \kappa_q(d_0,k_0) &\ge c(q)d_0^{-1/q}\kappa_{\mathsf{RE}}(2d_0,k_0),\end{aligned} \tag{11.3}$$

where $c(q)>0$ depends on k_0 and q. The theorem is thus proved following exactly the same line of arguments as in the proof of Theorem 1 in [3] in view of

the ℓ_q sensitivity condition derived immediately above, in view of Lemmas 19, 20 and 21. Indeed, for $v := \widehat{\beta} - \beta^*$, we have by definition of ℓ_q sensitivity as in (6.10),

$$\begin{aligned} c(q)d_0^{-1/q}\kappa_{\mathsf{RE}}(2d_0, k_0)\left\|v\right\|_q &\leq \kappa_q(d_0, k_0)\left\|v\right\|_q \\ &\leq \left\|\tfrac{1}{n}X_0^T X_0 v\right\|_\infty \\ &\leq \mu_1\left\|\beta^*\right\|_2 + \mu_2\left\|v\right\|_1 + \omega \\ &\leq \mu_1\left\|\beta^*\right\|_2 + \mu_2(2+\lambda)\left\|v_S\right\|_1 + \omega \\ &\leq \mu_1\left\|\beta^*\right\|_2 + \mu_2(2+\lambda)d_0^{1-1/q}\left\|v_S\right\|_q + \omega \\ &\leq \mu_1\left\|\beta^*\right\|_2 + \mu_2(2+\lambda)d_0^{1-1/q}\left\|v\right\|_q + \omega. \end{aligned} \tag{11.4}$$

Thus we have for $d_0 = c_0\sqrt{n/\log m}$, where c_0 is sufficiently small,

$$d_0^{-1/q}(c(q)\kappa_{\mathsf{RE}}(2d_0, k_0) - \mu_2(2+\lambda)d_0)\left\|v\right\|_q \leq \mu_1\left\|\beta^*\right\|_2 + \omega$$

and hence

$$\begin{aligned} \left\|v\right\|_q &\leq C(4D_2\rho_n K\left\|\beta^*\right\|_2 + 2D_0 M_\epsilon \rho_n)d_0^{1/q} \\ &\leq 4CD_2\rho_n(K\left\|\beta^*\right\|_2 + M_\epsilon)d_0^{1/q} \end{aligned}$$

for some constant

$$C = 1/\left(c(q)\kappa_{\mathsf{RE}}(2d_0, k_0) - \mu_2(2+\lambda)d_0\right) \geq 1/\left(2c(q)\kappa_{\mathsf{RE}}(2d_0, k_0)\right),$$

where

$$\begin{aligned} \mu_2(2+\lambda)d_0 &= 2D_2K\rho_n(\frac{1}{\lambda}+1)(2+\lambda)c_0\sqrt{n/\log m} \\ &= 2c_0C_0D_2K^2(2+\lambda)(\frac{1}{\lambda}+1) \end{aligned}$$

is sufficiently small and thus (3.10) holds. The prediction error bound follows exactly the same line of arguments as in [3] which we omit here. See proof of Theorem 7 in Section 6.6 for details. □

12. Proof of Theorem 6

Throughout this proof, we assume that $\mathcal{A}_0 \cap \mathcal{B}_0$ holds. The proof is also identical to the proof of Theorem 3 up till (10.2), except that we replace the condition on d as in the theorem statement by (4.3). Theorem 6 follows from Theorem 16, so long as we can show that condition (6.4) holds for α and $\tau = \frac{\lambda_{\min}(A)-\alpha}{s_0}$ as defined in (10.2), and $\lambda \geq 2\psi\sqrt{\frac{\log m}{n}}$, where the parameter ψ is as defined (6.2). Combining (10.2) and (6.4), we need to show (6.7) holds. This is precisely the content of Lemma 18. This is the end of the proof for Theorem 6.

13. Proof of Theorem 7

Throughout this proof, we assume that $\mathcal{B}_0 \cap \mathcal{B}_{10}$ holds. The rest of the proof follows that of Theorem 4, except for the last part. Let μ_1, μ_2, ω be as defined in Lemma 21. We have for $\mu_2 := 2\mu(1 + \frac{1}{2\lambda})$, where $\mu = D_0' K \rho_n \widetilde{\tau}_B^{1/2}$ and $d_0 = c_0 \tau_B^- \sqrt{n/\log m}$,

$$\begin{aligned} \mu_2(2+\lambda)d_0 &= 2C_0 D_0' K^2 \widetilde{\tau}_B^{1/2} (\frac{1}{2\lambda} + 1)(2+\lambda) c_0 \tau_B^- \\ &\le 2c_0 C_0 D_0' K^2 (2+\lambda)(\frac{1}{2\lambda} + 1) \le \frac{1}{2} c(q) \kappa_{\mathsf{RE}}(2d_0, k_0), \end{aligned} \tag{13.1}$$

which holds when c_0 is sufficiently small, where $\tau_B^- \widetilde{\tau}_B^{1/2} \le 1$ by (6.14). Hence

$$\mu_2 d_0 \le \frac{c(q)\kappa_{\mathsf{RE}}(2d_0, k_0)}{2(2+\lambda)}.$$

Thus for c_0 sufficiently small, $\mu_1 = 2\mu$, we have by (11.3), (13.1), (11.4) and (6.13),

$$\begin{aligned} & d_0^{-1/q} \frac{1}{2} (c(q)\kappa_{\mathsf{RE}}(2d_0, k_0)) \left\| v \right\|_q \\ &= d_0^{-1/q} (c(q)\kappa_{\mathsf{RE}}(2d_0, k_0) - \mu_2(2+\lambda)d_0) \left\| v \right\|_q \\ &\le (\kappa_q(d_0, k_0) - \mu_2(2+\lambda) d_0^{1-1/q}) \left\| v \right\|_q \le \mu_1 \left\| \beta^* \right\|_2 + \omega \\ &\le 2D_0' \rho_n K^2 ((\tau_B^{1/2} + (3/2) C_6 r_{m,m}^{1/2}) \left\| \beta^* \right\|_2 + M_\epsilon / K) \end{aligned} \tag{13.2}$$

and thus (4.11) holds, following the proof in Theorem 4. The prediction error bound follows exactly the same line of arguments as in [3], which we now include for the sake completeness. Following (4.11), we have by (13.2),

$$\left\| v \right\|_1 \le C_{11} d_0 (\mu_1 \left\| \beta^* \right\|_2 + \omega)$$

where

$$\begin{aligned} C_{11} &= 2/\left(c(q)\kappa_{\mathsf{RE}}(2d_0, k_0)\right) \quad \text{and hence} \\ \mu_2 \left\| v \right\|_1 &\le C_{11} \mu_2 d_0 (\mu_1 \left\| \beta^* \right\|_2 + \omega) \\ &\le C_{11} \frac{1}{2(2+\lambda)} \left(c(q)\kappa_{\mathsf{RE}}(2d_0, k_0)\right) (\mu_1 \left\| \beta^* \right\|_2 + \omega) \\ &= \frac{1}{2+\lambda} (\mu_1 \left\| \beta^* \right\|_2 + \omega). \end{aligned}$$

Thus we have by (13.2), (6.14) and the bounds immediately above,

$$\begin{aligned} & \tfrac{1}{n} \left\| X(\widehat{\beta} - \beta^*) \right\|_2^2 \le \left\| v \right\|_1 \left\| \tfrac{1}{n} X_0^T X_0 v \right\|_\infty \\ &\le C_{11} d_0 (\mu_1 \left\| \beta^* \right\|_2 + \omega) (\mu_1 \left\| \beta^* \right\|_2 + \mu_2 \left\| v \right\|_1 + 2\omega) \end{aligned}$$

$$\begin{aligned}
&\leq C_{11} d_0 (\mu_1 \|\beta^*\|_2 + \omega)(1 + \frac{1}{2+\lambda})(\mu_1 \|\beta^*\|_2 + 2\omega) \\
&= C'(D_0')^2 K^4 d_0 \frac{\log m}{n} \left(\tilde{\tau}_B^{1/2} \|\beta^*\|_2 + \frac{M_\epsilon}{K} \right)^2 \\
&\leq C''(\|B\|_2 + a_{\max}) K^2 d_0 \frac{\log m}{n} \left((2\tau_B + 3C_6^2 r_{m,m}) K^2 \|\beta^*\|_2^2 + M_\epsilon^2 \right),
\end{aligned}$$

where $(D_0')^2 \leq 2\|B\|_2 + 2a_{\max}$. The theorem is thus proved. □

14. Proof of Theorem 9

Suppose that event $\mathcal{A}_0 \cap \mathcal{B}_0$ holds. The condition on d in (5.2) implies that

$$n > 512 d \tau_0 \log m \left\{ \frac{12\lambda_{\max}(A)}{\lambda_{\min}(A)^2} \right\} \bigvee \left\{ \frac{4\zeta}{(\bar{\alpha}_\ell)^2} \right\}, \quad \text{where} \tag{14.1}$$

$$\tau_0 \asymp \frac{400 C^2 \varpi (s_0+1)^2}{\lambda_{\min}(A)}. \tag{14.2}$$

To see this, note that the following holds by the first bound in (5.2):

$$\begin{aligned}
\nu_\ell = \frac{64 d \tau_0 \log m}{n} &\leq 64 d \tau_0 \log m \frac{\lambda_{\min}(A)}{256 d \tau_0 \log m * 24\kappa(A)} \\
&= \frac{\lambda_{\min}(A)}{96\kappa(A)} \leq \frac{\alpha_\ell}{60},
\end{aligned} \tag{14.3}$$

where $\alpha_\ell = \frac{5}{8}\lambda_{\min}(A)$ by Lemma 8, and hence $\bar{\alpha}_\ell \geq \frac{59\alpha_\ell}{60}$. Thus we have

$$\frac{\alpha_\ell^2}{5\zeta} \leq \frac{\bar{\alpha}_\ell^2}{4\zeta}, \quad \text{where} \quad \zeta \geq \alpha_u > \lambda_{\max}(A) \geq \kappa(A)\bar{\alpha}_\ell.$$

Now, by definition of $\nu(d,m,n)$ and the second bound on n in (14.1),

$$2\nu(d,m,n) = 128 d \tau_u(\mathcal{L}_n) := \frac{128 d \tau_0 \log m}{n} \leq \frac{(\bar{\alpha}_\ell)^2}{16\zeta}$$

Then

$$2\varrho := \frac{4\nu(d,m,n)}{\bar{\alpha}_\ell} = \frac{256 d \tau_u(\mathcal{L}_n)}{\bar{\alpha}_\ell} \leq \frac{\bar{\alpha}_\ell}{8\zeta}.$$

That is, we actually need to have for $2\varrho \leq \frac{\bar{\alpha}_\ell}{8\zeta}$

$$\begin{aligned}
\frac{\xi}{1-\kappa} &= \frac{1}{1-\varrho} 2\tau(\mathcal{L}_n) \left(\frac{\bar{\alpha}_\ell}{4\zeta} + 2\varrho + 5 \right) \frac{1-\varrho}{\frac{\bar{\alpha}_\ell}{4\zeta} - 2\varrho} \\
&= 2\tau(\mathcal{L}_n) \left(\frac{\bar{\alpha}_\ell}{4\zeta} + 2\varrho + 5 \right) \frac{1}{\frac{\bar{\alpha}_\ell}{4\zeta} - 2\varrho}
\end{aligned}$$

$$= 2\tau(\mathcal{L}_n)\left(\frac{\frac{\bar\alpha_\ell}{4\zeta}+2\varrho}{\frac{\bar\alpha_\ell}{4\zeta}-2\varrho}+\frac{40\zeta}{\bar\alpha_\ell}\right) \le 2\tau(\mathcal{L}_n)\left(3+\frac{40\zeta}{\bar\alpha_\ell}\right)$$
$$\le 6\tau(\mathcal{L}_n)+\frac{80\zeta}{\bar\alpha_\ell}\tau(\mathcal{L}_n),$$

where we use the second bound in (14.1), and hence

$$\frac{\bar\alpha_\ell}{4\zeta}+2\varrho \le \frac{3}{2}\frac{\bar\alpha_\ell}{4\zeta} \quad \text{and}$$
$$(1-\kappa)(1-\varrho)=\frac{\bar\alpha_\ell}{4\zeta}-2\varrho \ge \frac{1}{2}\frac{\bar\alpha_\ell}{4\zeta}.$$

Finally, putting all bounds in (2.6), we have $0<\kappa<1$. Thus the conclusion of Theorem 2 hold. □

14.1. Proof of Corollary 10

Suppose that event $\mathcal{A}_0\cap\mathcal{B}_0$ holds. We first show that

$$4\nu\varepsilon_{\text{stat}}^2+4\tau(\mathcal{L}_n)\epsilon^2 \asymp 64\tau(\mathcal{L}_n)\left(4d\varepsilon_{\text{stat}}^2+\frac{\delta^4}{\lambda^2}\right) \quad \text{in case} \quad \delta^2\le M\bar\delta^2.$$

Recall that $\xi\ge 10\tau_\ell(\mathcal{L}_n)$ by definition of ξ in (2.7). The condition (5.1) on λ as stated in Theorem 2 indicates that

$$\lambda \ge \frac{160b_0\sqrt{d}\tau_\ell(\mathcal{L}_n)}{1-\kappa} \quad \text{where} \quad R\asymp b_0\sqrt{d}. \tag{14.4}$$

We first show that for the choice of λ and R as in (14.4),

$$\left\|\widehat\Delta^t\right\|_2^2 \le \frac{2}{\bar\alpha_\ell}\left(\delta^2+64\tau\left(4d\varepsilon_{\text{stat}}^2+\frac{\delta^4}{\lambda^2}\right)\right)$$
$$\le \frac{3}{\alpha_\ell}\delta^2+\frac{\alpha_\ell\varepsilon_{\text{stat}}^2}{4}+\frac{2}{\alpha_\ell}O\left(\frac{\delta^2}{\tau_0}\frac{M\varepsilon_{\text{stat}}^2}{400b_0^2}\right).$$

Then (5.3) holds.

For the second term on the RHS of (2.11), we have by (14.1),

$$n \ge \frac{256d\tau_0\log m}{\bar\alpha_\ell}\frac{8\zeta}{\bar\alpha_\ell}, \quad \text{where} \quad \tau_0\asymp\frac{400C^2\varpi(s_0+1)^2}{\lambda_{\min}(A)}. \tag{14.5}$$

Thus

$$4\nu_\ell = 256d\tau_0\frac{\log m}{n}\le\bar\alpha_\ell\frac{\bar\alpha_\ell}{8\zeta} \quad \text{and} \quad \frac{2}{\bar\alpha_\ell}4\nu\varepsilon_{\text{stat}}^2\le\frac{\bar\alpha_\ell}{4\zeta}\varepsilon_{\text{stat}}^2\le\frac{\alpha_\ell\varepsilon_{\text{stat}}^2}{4\zeta}.$$

Consider the choice of $\bar\eta=\delta^2$, where $M\bar\delta^2\ge\bar\eta=\delta^2\ge\frac{c\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{d\log m}{n}=:\bar\delta^2$.

Thus we have for (14.4),

$$\frac{2\delta^2}{\lambda} \leq \frac{Mc\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{d\log m}{n}\frac{1-\kappa}{160b_0\sqrt{d}\tau_\ell(\mathcal{L}_n)} = \frac{Mc\varepsilon_{\text{stat}}^2}{160b_0}\frac{\sqrt{d}}{\tau_0} < R$$

and hence $\epsilon = \frac{4\delta^2}{\lambda}$.

Then for the last term on the RHS of (2.11), we have for $\tau_\ell(\mathcal{L}_n) \asymp \tau$,

$$\begin{aligned} 4\tau_\ell(\mathcal{L}_n)\epsilon^2 &= 16\tau_\ell(\mathcal{L}_n)\min\left(\frac{2\delta^2}{\lambda}, R\right)^2 \\ &= 64\tau\frac{\delta^4}{\lambda^2} \leq \frac{\delta^4(1-\kappa)^2}{400b_0^2\tau_0}\frac{n}{d\log m} \\ &\leq \delta^2\frac{cM\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{(1-\kappa)^2}{400b_0^2\tau_0} \\ &= \frac{c\delta^2}{\tau_0}\frac{M\varepsilon_{\text{stat}}^2(1-\kappa)}{400b_0^2} = O\left(\frac{c\delta^2}{\tau_0}\frac{M\varepsilon_{\text{stat}}^2}{400b_0^2}\right) \end{aligned}$$

where $\delta^2 \leq \frac{Mc\varepsilon_{\text{stat}}^2}{1-\kappa}\frac{d\log m}{n}$.

Finally, suppose we fix

$$R \asymp \frac{b_0}{20M_+\sqrt{6\kappa(A)}}\sqrt{\frac{n}{\log m}}$$

in view of the upper bound $\bar{d}$ (5.6). Then in order for

$$\lambda \geq 16R\frac{\xi}{1-\kappa}$$

to hold, we need to set

$$\lambda \geq 640R\tau(\mathcal{L}_n)\kappa(A),$$

because of the following lower bound $\frac{\xi}{1-\kappa} \geq 40\tau(\mathcal{L}_n)\kappa(A)$ as shown in (2.16).

Then (5.4) holds given that the last term on the RHS of (2.11) is now bounded by

$$\begin{aligned} \frac{2}{\bar{\alpha}_\ell}4\tau_\ell(\mathcal{L}_n)\epsilon^2 &= \frac{2}{\bar{\alpha}_\ell}16\tau_\ell(\mathcal{L}_n)\min\left(\frac{2\delta^2}{\lambda}, R\right)^2 \leq \frac{60}{59}\frac{2}{\alpha_\ell}\frac{64\delta^4}{640^2R^2\kappa(A)^2\tau_\ell(\mathcal{L}_n)} \\ &\leq \frac{60}{59}\frac{2}{\alpha_\ell}\frac{\delta^4}{6400\kappa(A)\tau_0}\left(\frac{20M_+\sqrt{6}}{b_0}\right)^2 \\ &= \frac{60}{59}\frac{12}{\alpha_\ell}\frac{\delta^4M_+^2}{16b_0^2\kappa(A)\tau_0} \leq \frac{60}{59}\frac{3\delta^4}{4\alpha_\ell}\frac{1024}{400b_0^2\left\|A\right\|_2} \leq \frac{2\delta^4}{b_0^2\alpha_\ell\left\|A\right\|_2}. \end{aligned}$$

□

Remark 14.1. *First we obtain an upper bound on* $\frac{\xi}{1-\kappa}$ *for* $\zeta = \alpha_u \asymp \frac{3\lambda_{\max}(A)}{2}$ *and* $\frac{59}{60}\frac{5}{8}\lambda_{\min}(A) \leq \bar{\alpha}_\ell$

$$\begin{aligned}\frac{\xi}{1-\kappa} &\leq 6\tau(\mathcal{L}_n) + \frac{80\zeta}{\bar{\alpha}_\ell}\tau(\mathcal{L}_n) \\ &\leq 6\tau(\mathcal{L}_n) + \frac{80\zeta}{\frac{59}{60}\frac{5}{8}\lambda_{\min}(A)}\tau(\mathcal{L}_n) \\ &\approx 200\tau(\mathcal{L}_n)\kappa(A).\end{aligned}$$

Now we obtain an upper bound using $R \leq b_0\sqrt{d}$ for $d \leq \bar{d}$ as in (5.6),

$$\begin{aligned}R\frac{\xi}{1-\kappa} &\leq 200\kappa(A)\tau b_0\sqrt{d} \leq 200\kappa(A)\tau_0\sqrt{\frac{\log m}{n}}\frac{b_0}{20M_+\sqrt{6\kappa(A)}} \\ &= 200\kappa(A)\tau_0\frac{\lambda_{\min}(A)}{\varpi(s_0+1)}\sqrt{\frac{\log m}{n}}\frac{b_0}{640C\sqrt{6\kappa(A)}} \\ &\leq 10b_0\frac{\tau_0}{M_+\sqrt{6}}\sqrt{\kappa(A)}\sqrt{\frac{\log m}{n}} \\ &= \frac{125b_0}{\sqrt{6}}C\varpi(s_0+1)\sqrt{\kappa(A)}\sqrt{\frac{\log m}{n}},\end{aligned}$$

where we use (5.2) and the fact that $\frac{\tau_0}{M_+} = 12.5C\varpi(s_0+1)$. We now discuss the implications of this bound on the choice of λ in Section 5.1. We consider two cases.

- When $\tau_B = \Omega(1)$. It is sufficient to have for $\|\beta^*\|_2 \leq b_0$ and $\tau_B \asymp 1$,

  $$\lambda \geq 16Cb_0\left(50\sqrt{\kappa(A)}\varpi(s_0+1)\bigvee\left(D_0'K(K\tau_B^{1/2} + \frac{M_\epsilon}{b_0})\right)\right)\sqrt{\frac{\log m}{n}}$$

  following the discussions in Section 4, where the first and the second term on the RHS are at the same order except that the new lower bound involves the condition number $\kappa(A)$, while the original bound in Theorem 6 involves only $D_0' = \|B\|_2^{1/2} + a_{\max}^{1/2}$.
- When $\tau_B = o(1)$ and $M_\epsilon = \Omega(\tau_B^{+/2}K\|\beta^*\|_2)$. Now d satisfies (4.12) and hence

  $$b_0\sqrt{d} \leq \frac{1}{4\sqrt{5}M_+}\sqrt{\frac{n}{\log m}}\left\{\frac{\sqrt{c'}D_0'KM_\epsilon}{\varpi(s_0+1)} \wedge b_0\right\}.$$

  Now combining this with the condition on d as in (5.2) implies that it is sufficient to set R such that

  $$\begin{aligned}R\frac{\xi}{1-\kappa} &\asymp \frac{\kappa(A)\tau}{M_+}\left(\frac{b_0}{\sqrt{\kappa(A)}}\bigwedge\frac{D_0'KM_\epsilon}{\varpi(s_0+1)}\right)\sqrt{\frac{n}{\log m}} \\ &= \kappa(A)\varpi(s_0+1)\left(\frac{b_0}{\sqrt{\kappa(A)}}\bigwedge\frac{D_0'KM_\epsilon}{\varpi(s_0+1)}\right)\sqrt{\frac{\log m}{n}} \\ &\approx \left(b_0\varpi(s_0+1)\sqrt{\kappa(A)} \wedge \kappa(A)D_0'KM_\epsilon\right)\sqrt{\frac{\log m}{n}} =: \bar{U}.\end{aligned}$$

Hence it is sufficient to have for $\psi \asymp D_0' K \left(M_\epsilon + K\tau_B^{+/2} \|\beta^*\|_2 \right)$ as in (4.2),

$$\lambda \quad \geq \quad (\bar{U} \vee \psi) \sqrt{\frac{\log m}{n}}.$$

15. Proof of Theorem 12

Clearly the condition on the stable rank of B guarantees that

$$n \geq r(B) = \frac{\mathrm{tr}(B)}{\|B\|_2} = \frac{\mathrm{tr}(B)\,\|B\|_2}{\|B\|_2^2} \geq \|B\|_F^2 / \|B\|_2^2 \geq \log m.$$

Thus the conditions in Lemmas 11 and 5 hold. First notice that

$$\begin{aligned} \widehat{\gamma} \quad &= \quad \tfrac{1}{n}\left(X_0^T X_0 \beta^* + W^T X_0 \beta^* + X_0^T \epsilon + W^T \epsilon\right) \\ (\tfrac{1}{n} X^T X - \tfrac{\widehat{\mathrm{tr}}(B)}{n} I_m)\beta^* &= \\ &\tfrac{1}{n}(X_0^T X_0 + W^T X_0 + X_0^T W + W^T W - \tfrac{\widehat{\mathrm{tr}}(B)}{n} I_m)\beta^*. \end{aligned}$$

Thus

$$\begin{aligned} \left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty \quad &\leq \quad \left\|\widehat{\gamma} - \tfrac{1}{n}\left(X^T X - \widehat{\mathrm{tr}}(B) I_m\right)\beta^*\right\|_\infty \\ &= \quad \tfrac{1}{n}\left\|X_0^T \epsilon + W^T \epsilon - \left(W^T W + X_0^T W - \widehat{\mathrm{tr}}(B) I_m\right)\beta^*\right\|_\infty \\ &\leq \quad \tfrac{1}{n}\left\|X_0^T \epsilon + W^T \epsilon\right\|_\infty + \tfrac{1}{n}\left\|(W^T W - \widehat{\mathrm{tr}}(B) I_m)\beta^*\right\|_\infty \\ &\quad + \left\|\tfrac{1}{n} X_0^T W \beta^*\right\|_\infty \\ &\leq \quad \tfrac{1}{n}\left\|X_0^T \epsilon + W^T \epsilon\right\|_\infty + \tfrac{1}{n}(\left\|(Z^T B Z - \mathrm{tr}(B) I_m)\beta^*\right\|_\infty) \\ &\quad + \tfrac{1}{n}\left\|X_0^T W \beta^*\right\|_\infty + \tfrac{1}{n}\left|\widehat{\mathrm{tr}}(B) - \mathrm{tr}(B)\right| \|\beta^*\|_\infty \\ &=: \quad U_1 + U_2 + U_3 + U_4. \end{aligned}$$

By Lemma 11 we have on $\mathcal{B}_4$ for $D_0 := \sqrt{\tau_B} + a_{\max}^{1/2}$,

$$U_1 = \tfrac{1}{n}\left\|X_0^T \epsilon + W^T \epsilon\right\|_\infty = \tfrac{1}{n}\left\|A^{\frac{1}{2}} Z_1^T \epsilon + Z_2^T B^{\frac{1}{2}} \epsilon\right\|_\infty \leq \rho_n M_\epsilon D_0,$$

and on event $\mathcal{B}_5$ for $D_0' := \sqrt{\|B\|_2} + a_{\max}^{1/2}$,

$$\begin{aligned} U_2 + U_3 &= \tfrac{1}{n}\left\|(Z^T B Z - \mathrm{tr}(B) I_m)\beta^*\right\|_\infty + \tfrac{1}{n}\left\|X_0^T W \beta^*\right\|_\infty \\ &\leq \quad \rho_n K \|\beta^*\|_2 \left(\frac{\|B\|_F}{\sqrt{n}} + \sqrt{\tau_B} a_{\max}^{1/2} \right) \leq K \rho_n \|\beta^*\|_2 \tau_B^{1/2} D_0', \end{aligned}$$

where recall $\|B\|_F \leq \sqrt{\mathrm{tr}(B)}\,\|B\|_2^{1/2}$. Denote by $\mathcal{B}_0 := \mathcal{B}_4 \cap \mathcal{B}_5 \cap \mathcal{B}_6$. We have on $\mathcal{B}_0$ and under (A1), by Lemmas 11 and 5 and D_1 defined therein,

$$\begin{aligned} \left\|\widehat{\gamma} - \widehat{\Gamma}\beta^*\right\|_\infty \quad &\leq \quad U_1 + U_2 + U_3 + U_4 \\ &\leq \quad \rho_n M_\epsilon D_0 + D_0' \tau_B^{1/2} K \rho_n \|\beta^*\|_2 + \tfrac{1}{n}\left|\widehat{\mathrm{tr}}(B) - \mathrm{tr}(B)\right| \|\beta^*\|_\infty \end{aligned}$$

$$\leq D_0 M_\epsilon \rho_n + D_0' K \tau_B^{1/2} \|\beta^*\|_2 \rho_n + D_1 \|\beta^*\|_\infty r_{m,m}$$
$$\leq D_0 M_\epsilon \rho_n + D_0' K \tau_B^{1/2} \|\beta^*\|_2 \rho_n + 2 D_1 K \frac{1}{\sqrt{m}} \rho_n.$$

Finally, we have by the union bound, $\mathbb{P}(\mathcal{B}_0) \geq 1 - 16/m^3$. This is the end of the proof of Theorem 12. □

16. Conclusion

In this paper, we provide a unified analysis on the rates of convergence for both the corrected Lasso estimator (1.7) and the Conic programming estimator (1.8). As n increases or as the measurement error metric τ_B decreases, we see performance gains over the entire paths for both ℓ_1 and ℓ_2 error for both estimators as expected. When we focus on the lowest ℓ_2 error along the paths as we vary the penalty factor $f \in [0.05, 0.8]$, the corrected Lasso via the composite gradient descent algorithm performs slightly better than the Conic programming estimator as shown in Figure 5.

For the Lasso estimator, when we require that the stochastic error ϵ in the response variable y as in (1.1a) does not approach 0 as quickly as the measurement error W in (1.1b) does, then the sparsity constraint becomes essentially unchanged as $\tau_B \to 0$. These tradeoffs are somehow different from the behavior of the Conic programming estimator versus the Lasso estimator; however, we believe the differences are minor. Eventually, as $\tau_B \to 0$, the relaxation on d as in (4.13) enables the Conic programming estimator to achieve bounds which are essentially identical to the Dantzig Selector when the design matrix X_0 is a subgaussian random matrix satisfying the Restricted Eigenvalue conditions; See for example [6, 4, 38].

When $\tau_B \to 0$ and $M_\epsilon = \Omega(\tau_B^+ K \|\beta^*\|_2)$, we set

$$\lambda \geq 2\psi \sqrt{\frac{\log m}{n}}, \quad \text{where} \quad \psi := 4 C_0 D_0' K M_\epsilon, \tag{16.1}$$

so as to recover the regular lasso bounds in ℓ_q loss for $q = 1, 2$ in (4.5) in Theorem 6. Moreover, suppose that $\mathrm{tr}(B)$ is given, then one can drop the second term in ψ as in (4.2) involving $\|\beta^*\|_2$ entirely and hence recover the lasso bound as well.

Finally, we note that the bounds corresponding to the Upper RE condition as stated in Corollary 25, Theorem 26 and Lemma 15 are not needed for Theorems 3 and 6. They are useful to ensure algorithmic convergence and to bound the optimization error for the gradient descent-type of algorithms as considered in [1, 30], when one is interested in approximately solving the nonconvex optimization function (1.7). Our Theorem 9 illustrates this result. Our theory in Theorem 9 predicts the dependencies of the computational and statistical rates of convergence for the corrected Lasso via gradient descent algorithm on the condition number $\kappa(A)$, the trace parameter τ_B and the radius R as

$$\lambda \asymp \frac{R\xi}{1-\kappa} \asymp \tau_0 \kappa(A) \frac{R \log m}{n}, \quad \text{where} \quad \tau_0 \asymp \frac{(\rho_{\max}(s_0, A) + \tau_B)^2}{\lambda_{\min}(A)}$$

depends on τ_B, sparse and minimal eigenvalues of A. Therefore, we need to increase the penalty when we increase the ℓ_1-ball radius R in (1.10) in order to ensure algorithmic and statistical convergence as predicted in Theorem 9. This is well-aligned with the observation in Figure 6. Our numerical results validate such algorithmic and statistical convergence properties.

Appendix A: Outline

We prove Theorem 2 in Section B. In Sections C, we present variations of the Hanson-Wright inequality as recently derived in [37] (cf. Lemma 32). We prove Lemma 11 in Section C.2. In Sections H and I, we prove the technical lemmas for Theorems 3 and 4 respectively. In Section J, we prove the Lemmas needed for Proof of Theorem 7. In order to prove Corollary 25, we need to first state some geometric analysis results Section K. We prove Corollary 25 in Section L and Theorem 26 in Section M.

Appendix B: Proof of Theorem 2

Let us first define the following shorthand notation

$$\widehat{\Delta}^t \;=\; \beta^t - \widehat{\beta} \quad \text{and} \quad \delta^t \;=\; \phi(\beta^t) - \phi(\widehat{\beta}).$$

The proof of the theorem requires two technical Lemmas 28 and 30. Both are stated under assumption (B.1), which is stated in terms of a given tolerance $\bar{\eta} > 0$ and integer $T > 0$ such that

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \bar{\eta}, \;\; \forall t \geq T, \tag{B.1}$$

where the distance between β^t and the global optimizer $\widehat{\beta}$ is measured in terms of the objective function ϕ, namely, $\delta^t = \phi(\beta^t) - \phi(\widehat{\beta})$.

We first show Lemma 28, which ensures that the vector $\widehat{\Delta}^t := \beta^t - \widehat{\beta}$ satisfies a certain cone-type condition. The proof is omitted, as it is a shortened proof of Lemma 1 of [31].

Lemma 28. *(***Iterated Cone Bound***)* *Under the conditions of Theorem 2, suppose there exists a pair* $(\bar{\eta}, T)$ *such that* (B.1) *holds. Then for any iteration* $t \geq T$*, we have*

$$\left\|\beta^t - \widehat{\beta}\right\|_1 \leq 4\sqrt{d}\left\|\beta^t - \widehat{\beta}\right\|_2 + 8\sqrt{d}\left\|\widehat{\beta} - \beta^*\right\|_2 + 2 \cdot \min\left(\frac{2\bar{\eta}}{\lambda}, R\right).$$

We next state the following auxiliary result on the loss function. We use Lemma 29 in the proof of Lemma 28 and Corollary 10.

Lemma 29. *Denote by* $\tau_\ell(\mathcal{L}_n) := \tau_0 \frac{\log m}{n}$ *and* $\nu_\ell = 64 d \tau_\ell(\mathcal{L}_n)$*. Let* $\bar{\epsilon}_{\text{stat}} = 8\sqrt{d}\varepsilon_{\text{stat}}$*, where* $\varepsilon_{\text{stat}} = \left\|\widehat{\beta} - \beta^*\right\|_2$ *and* $\epsilon = 2 \cdot \min\left(\frac{2\bar{\eta}}{\lambda}, R\right)$*. Under the assumptions of Lemma 28, we have for* $\widehat{\Delta}^t := \beta^t - \widehat{\beta}$ *and* $t > T$*,*

$$\mathcal{T}(\widehat{\beta}, \beta^t) \;\; \geq \;\; \frac{\alpha_\ell - \nu_\ell}{2}\left\|\widehat{\Delta}^t\right\|_2^2 - 2\tau_\ell(\mathcal{L}_n)(\bar{\epsilon}_{\text{stat}} + \epsilon)^2 \quad \textit{and} \tag{B.2}$$

$$\phi(\beta^t) - \phi(\widehat{\beta}) \quad \geq \quad \mathcal{T}(\beta^t, \widehat{\beta}) \geq \frac{\alpha_\ell - \nu_\ell}{2} \left\| \widehat{\Delta}^t \right\|_2^2 - 2\tau_\ell(\mathcal{L}_n)(\bar{\epsilon}_{\text{stat}} + \epsilon)^2. \quad \text{(B.3)}$$

Lemma 30. (Lemma 3 of Loh-Wainwright (2015)) *Suppose the RSC and RSM conditions as stated in (2.4) and (2.5) hold with parameters $(\alpha_\ell, \tau_\ell(\mathcal{L}_n))$ and $(\alpha_u, \tau_u(\mathcal{L}_n))$ respectively. Under the conditions of Theorem 2, suppose there exists a pair $(\bar{\eta}, T)$ such that (B.1) holds. Then for any iteration $t \geq T$, we have for $0 < \kappa < 1$,*

$$\begin{aligned} \phi(\beta^t) - \phi(\widehat{\beta}) \quad &\leq \quad \kappa^{t-T}(\phi(\beta^T) - \phi(\widehat{\beta})) + \frac{\xi}{1-\kappa}(\bar{\epsilon}_{\text{stat}}^2 + \epsilon^2) \quad \textit{for} \\ \bar{\epsilon}_{\text{stat}} \quad &:= \quad 8\sqrt{d}\varepsilon_{\text{stat}} \quad \textit{and} \quad \epsilon = 2 \cdot \min\left(\frac{2\bar{\eta}}{\lambda}, R\right), \end{aligned}$$

where the quantities κ and ψ are as defined in Theorem 2 (cf. (2.6) and (2.7)).

Proof of Theorem 2. We are now ready to put together the final argument for the theorem. First notice that (2.11) follows from (2.10) directly in view of (B.3) and Lemma 28, where we set $\bar{\eta} = \delta^2$, $\bar{\epsilon}_{\text{stat}} = 8\sqrt{d}\varepsilon_{\text{stat}}$ and $\epsilon = 2\min\left(\frac{2\delta^2}{\lambda}, R\right)$.

Following (B.3), we have for $\nu_\ell = 64d\tau_\ell(\mathcal{L}_n)$,

$$\frac{\alpha_\ell - \nu_\ell}{2} \left\| \widehat{\Delta}^t \right\|_2^2 \quad \leq \quad \phi(\beta^t) - \phi(\widehat{\beta}) + 2\tau_\ell(\mathcal{L}_n)(\bar{\epsilon}_{\text{stat}} + \epsilon)^2,$$

and thus

$$\begin{aligned} \left\| \widehat{\Delta}^t \right\|_2^2 \quad &\leq \quad \frac{2}{\bar{\alpha}_\ell}(\phi(\beta^t) - \phi(\widehat{\beta}) + \frac{4}{\bar{\alpha}_\ell}\tau_\ell(\mathcal{L}_n)(\bar{\epsilon}_{\text{stat}} + \epsilon)^2 \\ &\leq \quad \frac{2}{\bar{\alpha}_\ell}\left(\delta^2 + 2\tau_\ell(\mathcal{L}_n)(2\bar{\epsilon}_{\text{stat}}^2 + 2\epsilon^2)\right) \\ &\leq \quad \frac{2}{\bar{\alpha}_\ell}\left(\delta^2 + 2\tau_\ell(\mathcal{L}_n)(128d\varepsilon_{\text{stat}}^2 + 2\epsilon^2)\right) \\ &\leq \quad \frac{2}{\bar{\alpha}_\ell}\left(\delta^2 + 4\nu_\ell\varepsilon_{\text{stat}}^2 + 4\tau_\ell(\mathcal{L}_n)\epsilon^2\right). \end{aligned} \quad \text{(B.4)}$$

The remainder of the proof follows an argument in [1]. We first prove the following inequality:

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \delta^2, \qquad \forall t \geq T^*(\delta).$$

We divide the iterations $t \geq 0$ into a series of epochs $[T_\ell, T_{\ell+1}]$ and defend the tolerances $\bar{\eta}_0 > \bar{\eta}_1 > \ldots$ such that

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \bar{\eta}_\ell, \qquad \forall t \geq T_\ell.$$

In the first iteration, we apply Lemma 30 with $\bar{\eta}_0 := \phi(\beta^0) - \phi(\widehat{\beta})$ to obtain

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \kappa^t(\phi(\beta^0) - \phi(\widehat{\beta})) + \frac{\xi}{1-\kappa}(\bar{\epsilon}_{\text{stat}}^2 + 4R^2) \quad \text{for any iteration } t \geq 0.$$

Set

$$\bar{\eta}_1 := \frac{2\xi}{1-\kappa}(\bar{\epsilon}^2_{\text{stat}} + 4R^2) \quad \text{and} \quad T_1 := \left\lceil \frac{\log(2\bar{\eta}_0/\bar{\eta}_1)}{\log(1/\kappa)} \right\rceil.$$

Then we have for any iteration $t \geq T_1$

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \bar{\eta}_1 := \frac{4\xi}{1-\kappa} \max\left\{\bar{\epsilon}^2_{\text{stat}}, 4R^2\right\}.$$

The same argument can be now be applied in a recursive manner. Suppose that for some $\ell \geq 1$, we are given a pair $(\bar{\eta}_\ell, T_\ell)$ such that

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \bar{\eta}_\ell, \ \ \forall t \geq T_\ell. \tag{B.5}$$

We now define

$$\bar{\eta}_{\ell+1} := \frac{2\xi}{1-\kappa}(\bar{\epsilon}^2_{\text{stat}} + \epsilon^2_\ell) \quad \text{and} \quad T_{\ell+1} := \left\lceil \frac{\log(2\bar{\eta}_\ell/\bar{\eta}_{\ell+1})}{\log(1/\kappa)} \right\rceil + T_\ell.$$

We can apply Lemma 30 to obtain for any iteration $t \geq T_\ell$ and $\varepsilon_\ell := 2\min\{\frac{\bar{\eta}_\ell}{\lambda}, R\}$,

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \kappa^{t-T_\ell}(\phi(\beta^{T_\ell}) - \phi(\widehat{\beta})) + \frac{\xi}{1-\kappa}(\bar{\epsilon}^2_{\text{stat}} + \varepsilon^2_\ell),$$

which implies that for all $t \geq T_{\ell+1}$,

$$\phi(\beta^t) - \phi(\widehat{\beta}) \leq \bar{\eta}_{\ell+1} \leq \frac{4\xi}{1-\kappa} \max\{\bar{\epsilon}^2_{\text{stat}}, \varepsilon^2_\ell\}$$

by our choice of $\{\eta_\ell, T_\ell\}_{\ell \geq 1}$. Finally, we use the recursion

$$\bar{\eta}_{\ell+1} \leq \frac{4\xi}{1-\kappa} \max(\bar{\epsilon}^2_{\text{stat}}, \varepsilon^2_\ell) \quad \text{and} \quad T_\ell \leq \ell + \frac{\log(2^\ell \bar{\eta}_0/\bar{\eta}_\ell)}{\log(1/\kappa)} \tag{B.6}$$

to establish the recursion that

$$\bar{\eta}_{\ell+1} \leq \frac{\bar{\eta}_\ell}{4^{2^{\ell-1}}} \quad \text{and} \quad \varepsilon_{\ell+1} := \frac{\bar{\eta}_{\ell+1}}{\lambda} \leq \frac{R}{4^{2^\ell}} \ \ \forall \ell = 1, 2, \ldots. \tag{B.7}$$

Taking these statements as given, we need to have

$$\bar{\eta}_\ell \leq \delta^2.$$

It is sufficient to establish that

$$\frac{\lambda R}{4^{2^{\ell-1}}} \leq \delta^2.$$

Thus we find that the error drops below δ^2 after at most

$$\ell_\delta \geq \log\left(\log(R\lambda/\delta^2)/\log(4)\right)/\log 2 + 1 = \log\log(R\lambda/\delta^2)$$

epochs. Combining the above bound on ℓ_δ with the recursion (B.6)

$$T_\ell \le \ell + \frac{\log(2^\ell \bar{\eta}_0/\bar{\eta}_\ell)}{\log(1/\kappa)},$$

we conclude that

$$\phi(\beta^t) - \phi(\widehat{\beta}) \le \delta^2$$

is guaranteed to hold for all iterations

$$t > \ell_\delta \left(1 + \frac{\log 2}{\log(1/\kappa)}\right) + \frac{\log(\bar{\eta}_0/\delta^2)}{\log(1/\kappa)}.$$

To establish (B.7), we start with $\ell = 0$ and establish that for $\bar{\epsilon}_{\text{stat}} = 8\sqrt{d}\epsilon_{\text{stat}} = o(\sqrt{d}) = o(R)$

$$\frac{\bar{\eta}_1}{\lambda} := \frac{4\xi}{(1-\kappa)\lambda}\max(\bar{\epsilon}_{\text{stat}}^2, 4R^2) = \frac{16R\xi}{(1-\kappa)\lambda}R \le \frac{R}{4} \quad \text{(B.8)}$$

$$\text{and thus} \quad \varepsilon_1 := 2\min\{\frac{\bar{\eta}_1}{\lambda}, R\} = R/2 \le \varepsilon_0 = R. \quad \text{(B.9)}$$

Assume that $\bar{\epsilon}_{\text{stat}} \le \varepsilon_1$ (otherwise, we are done at the first iteration). First, we obtain for $\ell = 1$,

$$\begin{aligned} \bar{\eta}_2 &\le \frac{4\xi}{1-\kappa}\max(\bar{\epsilon}_{\text{stat}}^2, \varepsilon_1^2) = \frac{4\xi}{1-\kappa}\varepsilon_1^2 = \frac{4\xi}{1-\kappa}\left(\frac{2\bar{\eta}_1}{\lambda}\right)^2 \\ &\le \frac{16\xi}{1-\kappa}\frac{\bar{\eta}_1^2}{\lambda^2} \le \frac{16\xi R}{1-\kappa}\frac{\bar{\eta}_1}{4\lambda} \le \frac{\bar{\eta}_1}{4}, \\ \text{and} \quad \frac{\bar{\eta}_2}{\lambda} &\le \frac{\bar{\eta}_1}{4\lambda} \le \frac{R}{16}, \end{aligned}$$

where in the last three steps, we use the fact that $\lambda \ge \frac{16R\xi}{(1-\kappa)}$ and (B.8). Thus (B.6) holds for $\ell = 1$.

Now assume that (B.7) holds for $d \le \ell$. In the induction step, we again use the assumption that $\varepsilon_\ell := 2\frac{\bar{\eta}_\ell}{\lambda} \ge \bar{\epsilon}_{\text{stat}}$ and (B.6) to obtain

$$\begin{aligned} \bar{\eta}_{\ell+1} &\le \frac{4\xi}{1-\kappa}\max(\bar{\epsilon}_{\text{stat}}^2, \varepsilon_\ell^2) = \frac{16\xi}{1-\kappa}\frac{\bar{\eta}_\ell^2}{\lambda^2} \\ &\le \frac{16\xi}{1-\kappa}\frac{R}{4^{2(\ell-1)}}\frac{\bar{\eta}_\ell}{\lambda} = \frac{16R\xi}{1-\kappa}\frac{1}{\lambda}\frac{\bar{\eta}_\ell}{4^{2(\ell-1)}} \\ &\le \frac{\bar{\eta}_\ell}{4^{2(\ell-1)}}. \end{aligned}$$

Finally, by the induction assumption

$$\frac{\bar{\eta}_\ell}{\lambda} \le \frac{R}{4^{2^{\ell-1}}},$$

we use the bound immediately above to obtain

$$\frac{\bar{\eta}_{\ell+1}}{\lambda} \le \frac{\bar{\eta}_\ell}{4^{2(\ell-1)}}\frac{1}{\lambda} \le \frac{R}{4^{2^{\ell-1}}}\frac{1}{4^{2(\ell-1)}} \le \frac{R}{4^{2^\ell}}.$$

The rest of the proof follows from that of Corollary 10. This is the end of the proof for Theorem 2. □

It remains to prove Lemma 29.

Proof of Lemma 29. Using the RSC condition, we have for $\tau_\ell(\mathcal{L}_n) := \tau_0 \frac{\log m}{n}$ and $\nu_\ell = 64 d \tau_\ell(\mathcal{L}_n) \leq \frac{\alpha_\ell}{48}$,

$$\begin{aligned} \mathcal{T}(\widehat{\beta}, \beta^t) &\geq \frac{\alpha_\ell}{2} \left\| \widehat{\Delta}^t \right\|_2^2 - \tau_\ell(\mathcal{L}_n) \left\| \widehat{\Delta}^t \right\|_1^2 \\ &\geq \frac{\alpha_\ell}{2} \left\| \widehat{\Delta}^t \right\|_2^2 - \tau_\ell(\mathcal{L}_n) \left(2 * 16 d \left\| \widehat{\Delta}^t \right\|_2^2 + 2(\bar{\epsilon}_{\text{stat}} + \epsilon)^2 \right) \\ &\geq \frac{1}{2} \bar{\alpha}_\ell \left\| \widehat{\Delta}^t \right\|_2^2 - 2\tau_\ell(\mathcal{L}_n)(\bar{\epsilon}_{\text{stat}} + \epsilon)^2 \end{aligned} \tag{B.10}$$

and by Lemma 28, for any iteration $t \geq T$,

$$\begin{aligned} \left\| \widehat{\Delta}^t - \widehat{\beta} \right\|_1 &\leq 4\sqrt{d} \left\| \beta^t - \widehat{\beta} \right\|_2 + 8\sqrt{d} \left\| \widehat{\beta} - \beta^* \right\|_2 + 2 \cdot \min\left(\frac{2\bar{\eta}}{\lambda}, R \right) \\ &\leq 4\sqrt{d} \left\| \widehat{\Delta}^t \right\|_2 + (\bar{\epsilon}_{\text{stat}} + \epsilon). \end{aligned}$$

By convexity of function g, we have

$$g(\beta^t) - g(\widehat{\beta}) - \langle \nabla g(\widehat{\beta}), \beta^t - \widehat{\beta} \rangle \geq 0. \tag{B.11}$$

Thus

$$\begin{aligned} &\phi(\beta^t) - \phi(\widehat{\beta}) - \langle \nabla \phi(\widehat{\beta}), \beta^t - \widehat{\beta} \rangle \\ &= \mathcal{L}_n(\beta^t) - \mathcal{L}_n(\widehat{\beta}) - \langle \nabla \mathcal{L}_n(\widehat{\beta}), \beta^t - \widehat{\beta} \rangle \\ &\quad + \lambda(g(\beta^t) - g(\widehat{\beta}) - \langle \nabla g(\widehat{\beta}), \beta^t - \widehat{\beta} \rangle). \end{aligned}$$

Moreover, by the first order optimality condition for $\widehat{\beta}$, we have for all feasible $\beta^t \in \Omega$

$$\langle \nabla \phi(\widehat{\beta}), \beta^t - \widehat{\beta} \rangle \geq 0,$$

and thus

$$\phi(\beta^t) - \phi(\widehat{\beta}) \geq \mathcal{L}_n(\beta^t) - \mathcal{L}_n(\widehat{\beta}) - \langle \nabla \mathcal{L}_n(\widehat{\beta}), \beta^t - \widehat{\beta} \rangle = \mathcal{T}(\beta^t, \widehat{\beta}),$$

where similar to (B.10), we have

$$\begin{aligned} \mathcal{T}(\beta^t, \widehat{\beta}) &\geq \alpha_1 \left\| \widehat{\Delta}^t \right\|_2^2 - \tau_\ell(\mathcal{L}_n) \left\| \widehat{\Delta}^t \right\|_1^2 \\ &\geq (\alpha_1 - 32 d \tau_\ell(\mathcal{L}_n)) \left\| \widehat{\Delta}^t \right\|_2^2 - 2\tau_\ell(\mathcal{L}_n)(\bar{\epsilon}_{\text{stat}} + \epsilon)^2 \\ &= \frac{1}{2} \bar{\alpha}_\ell \left\| \widehat{\Delta}^t \right\|_2^2 - 2\tau_\ell(\mathcal{L}_n)(\bar{\epsilon}_{\text{stat}} + \epsilon)^2, \end{aligned}$$

and by Lemma 28,

$$\begin{aligned}\left\|\widehat{\Delta}^t\right\|_1^2 &\leq 32d\left\|\widehat{\Delta}^t\right\|_2^2 + 2\left(8\sqrt{d}\varepsilon_{\text{stat}} + 2\cdot\min\left(\frac{2\bar{\eta}}{\lambda}, R\right)\right)^2 \\ &\leq 32d\left\|\widehat{\Delta}^t\right\|_2^2 + 2(\bar{\epsilon}_{\text{stat}} + \epsilon)^2. \end{aligned}$$

□

Appendix C: Some auxiliary results

We first need to state the following form of the Hanson-Wright inequality as recently derived in Rudelson and Vershynin [37], and an auxiliary result in Lemma 32 which may be of independent interests.

Theorem 31. *Let $X = (X_1, \ldots, X_m) \in \mathbb{R}^m$ be a random vector with independent components X_i which satisfy $\mathbb{E}(X_i) = 0$ and $\|X_i\|_{\psi_2} \leq K$. Let A be an $m \times m$ matrix. Then, for every $t > 0$,*

$$\mathbb{P}\left(\left|X^T AX - \mathbb{E}\left(X^T AX\right)\right| > t\right) \leq 2\exp\left[-c\min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|_2}\right)\right].$$

We note that following the proof of Theorem 31, it is clear that the following holds: Let $X = (X_1, \ldots, X_m) \in \mathbb{R}^m$ be a random vector as defined in Theorem 31. Let Y, Y' be independent copies of X. Let A be an $m \times m$ matrix. Then, for every $t > 0$,

$$\mathbb{P}\left(\left|Y^T AY'\right| > t\right) \leq 2\exp\left[-c\min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|_2}\right)\right]. \tag{C.1}$$

We next need to state Lemma 32, which we prove in Section N.

Lemma 32. *Let $u, w \in S^{n-1}$. Let $A \succ 0$ be an $m \times m$ symmetric positive definite matrix. Let Z be an $n \times m$ random matrix with independent entries Z_{ij} satisfying $\mathbb{E}Z_{ij} = 0$ and $\|Z_{ij}\|_{\psi_2} \leq K$. Let Z_1, Z_2 be independent copies of Z.*

Then for every $t > 0$,

$$\begin{aligned}&\mathbb{P}\left(\left|u^T Z_1 A^{1/2} Z_2^T w\right| > t\right) \\ &\quad\leq 2\exp\left(-c\min\left(\frac{t^2}{K^4\text{tr}(A)}, \frac{t}{K^2\|A\|_2^{1/2}}\right)\right) \quad \textit{and} \\ &\mathbb{P}\left(\left|u^T ZAZ^T w - \mathbb{E}u^T ZAZ^T w\right| > t\right) \\ &\quad\leq 2\exp\left(-c\min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|_2}\right)\right),\end{aligned}$$

where c is the same constant as defined in Theorem 31.

C.1. Proof of Lemma 5

First we write

$$\begin{aligned} XX^T - \mathrm{tr}(A)I_n &= \left(Z_1A^{1/2} + B^{1/2}Z_2\right)\left(Z_1A^{1/2} + B^{1/2}Z_2\right)^T - \mathrm{tr}(A)I_n \\ &= \left(Z_1A^{1/2} + B^{1/2}Z_2\right)\left(Z_2^TB^{1/2} + A^{1/2}Z_1^T\right) - \mathrm{tr}(A)I_n \\ &= Z_1A^{1/2}Z_2^TB^{1/2} + B^{1/2}Z_2Z_2^TB^{1/2} \\ &\quad + B^{1/2}Z_2A^{1/2}Z_1^T + Z_1AZ_1^T - \mathrm{tr}(A)I_n. \end{aligned}$$

Thus we have for $\check{\mathrm{tr}}(B) := \frac{1}{m}\left(\|X\|_F^2 - n\mathrm{tr}(A)\right)$,

$$\begin{aligned} &\tfrac{1}{n}(\check{\mathrm{tr}}(B) - \mathrm{tr}(B)) := \tfrac{1}{mn}\left(\|X\|_F^2 - n\mathrm{tr}(A) - m\mathrm{tr}(B)\right) \\ &= \frac{1}{mn}(\mathrm{tr}(XX^T) - n\mathrm{tr}(A) - m\mathrm{tr}(B)) \\ &= \frac{2}{mn}\mathrm{tr}(Z_1A^{1/2}Z_2^TB^{1/2}) + \left(\frac{\mathrm{tr}(B^{1/2}Z_2Z_2^TB^{1/2})}{mn} - \frac{\mathrm{tr}(B)}{n}\right) \\ &\quad + \frac{\mathrm{tr}(Z_1AZ_1^T)}{mn} - \frac{\mathrm{tr}(A)}{m}. \end{aligned}$$

By constructing a new matrix $A_n = I_n \otimes A$, which is block diagonal with n identical submatrices A along its diagonal, we prove the following large deviation bound: for $t_1 = C_0K^2\|A\|_F\sqrt{n\log m}$ and $n > \log m$,

$$\begin{aligned} &\mathbb{P}\left(\left|\mathrm{tr}(Z_1AZ_1^T) - n\mathrm{tr}(A)\right| \geq t_1\right) \\ &= \mathbb{P}\left(\left|\mathrm{vec}\left\{\mathrm{Z_1}\right\}^{\mathrm{T}}(\mathrm{I}\otimes\mathrm{A})\mathrm{vec}\left\{\mathrm{Z_1}\right\} - \mathrm{ntr(A)}\right| \geq t_1\right) \\ &\leq \exp\left(-c\min\left(\frac{t_1^2}{K^4\|A_n\|_F^2}, \frac{t_1}{K^2\|A_n\|_2}\right)\right) \\ &\leq 2\exp\left(-c\min\left(\frac{(C_0K^2\sqrt{n\log m}\,\|A\|_F)^2}{nK^4\|A\|_F^2}, \frac{C_0K^2\sqrt{n\log m}\,\|A\|_F}{K^2\|A\|_2}\right)\right) \\ &\leq 2\exp\left(-4\log m\right), \end{aligned}$$

where the first inequality holds by Theorem 31 and the second inequality holds given that $\|A_n\|_F^2 = n\|A\|_F^2$ and $\|A_n\|_2 = \|A\|_2$.

Similarly, by constructing a new matrix $B_m = I_m \otimes B$, which is block diagonal with m identical submatrices B along its diagonal, we prove the following large deviation bound: for $t_2 = C_0K^2\|B\|_F\sqrt{m\log m}$ and $m \geq 2$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\mathrm{tr}(Z_2^T B Z_2) - m\mathrm{tr}(B)\right| \geq t_2\right) = \\
&\qquad \mathbb{P}\left(\left|\mathrm{vec}\left\{\,\mathrm{Z}_2\,\right\}^{\mathrm{T}}\left(\mathrm{I}_{\mathrm{m}} \otimes \mathrm{B}\right)\mathrm{vec}\left\{\,\mathrm{Z}_2\,\right\} - \mathrm{mtr(B)}\right| \geq t_2\right) \\
&\leq \quad \exp\left(-c\min\left(\frac{t_2^2}{K^4 m \left\|B\right\|_F^2}, \frac{t_2}{K^2\left\|B\right\|_2}\right)\right) \\
&\leq \quad 2\exp\left(-c\min\left(\frac{(C_0K^2\sqrt{m\log m}\left\|B\right\|_F)^2}{K^4 m\left\|B\right\|_F^2}, \frac{C_0K^2\sqrt{m\log m}\left\|B\right\|_F}{K^2\left\|B\right\|_2}\right)\right) \\
&\leq \quad 2\exp\left(-4\log m\right).
\end{aligned}$$

Finally, we have by (C.1) for $t_0 = C_0K^2\sqrt{\mathrm{tr}(A)\mathrm{tr}(B)\log m}$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\mathrm{vec}\left\{\,\mathrm{Z}_1\,\right\}^{\mathrm{T}}\mathrm{B}^{1/2}\otimes\mathrm{A}^{1/2}\mathrm{vec}\left\{\,\mathrm{Z}_2\,\right\}\right| > t_0\right) \\
&\leq \quad 2\exp\left(-c\min\left(\frac{t_0^2}{K^4\left\|B^{1/2}\otimes A^{1/2}\right\|_F^2}, \frac{t_0}{K^2\left\|B^{1/2}\otimes A^{1/2}\right\|_2}\right)\right) \\
&= \quad 2\exp\left(-c\min\left(\frac{(C_0\sqrt{\mathrm{tr}(A)\mathrm{tr}(B)\log m})^2}{\mathrm{tr}(A)\mathrm{tr}(B)}, \frac{C_0\sqrt{\mathrm{tr}(A)\mathrm{tr}(B)\log m}}{\left\|B\right\|_2^{1/2}\left\|A\right\|_2^{1/2}}\right)\right) \\
&\leq \quad 2\exp(-4\log m),
\end{aligned}$$

where we use the fact that $r(A)r(B) \geq \log m$, $\left\|B^{1/2}\otimes A^{1/2}\right\|_2 = \left\|B\right\|_2^{1/2}\left\|A\right\|_2^{1/2}$ and

$$\begin{aligned}
\left\|B^{1/2}\otimes A^{1/2}\right\|_F^2 &= \mathrm{tr}((B^{1/2}\otimes A^{1/2})(B^{1/2}\otimes A^{1/2})) \\
&= \mathrm{tr}(B\otimes A) = \mathrm{tr}(A)\mathrm{tr}(B).
\end{aligned}$$

Thus we have with probability $1 - 6/m^4$,

$$\begin{aligned}
&\tfrac{1}{n}\left|\check{\mathrm{tr}}(B) - \mathrm{tr}(B)\right| = \tfrac{1}{mn}\left|\mathrm{tr}(XX^T) - f\mathrm{tr}(A) - m\mathrm{tr}(B)\right| \\
&\leq \quad \frac{2}{mn}\left|\mathrm{vec}\left\{\,\mathrm{Z}_1\,\right\}^{\mathrm{T}}\left(\mathrm{B}^{1/2}\otimes\mathrm{A}^{1/2}\right)\mathrm{vec}\left\{\,\mathrm{Z}_2\,\right\}\right| \\
&\qquad + \left|\frac{\mathrm{tr}(Z_2^T B Z_2)}{mn} - \frac{\mathrm{tr}(B)}{n}\right| + \left|\frac{\mathrm{tr}(Z_1 A Z_1^T)}{mn} - \frac{\mathrm{tr}(A)}{m}\right| \\
&\leq \quad \frac{1}{mn}(2t_0 + t_1 + t_2) = \frac{\sqrt{\log m}}{\sqrt{mn}}C_0K^2\left(\frac{\left\|A\right\|_F}{\sqrt{m}} + 2\sqrt{\tau_A\tau_B} + \frac{\left\|B\right\|_F}{\sqrt{n}}\right) \\
&\leq \quad 2C_0\frac{\sqrt{\log m}}{\sqrt{mn}}K^2 D_1 =: D_1 r_{m,m},
\end{aligned}$$

where recall $r_{m,m} = 2C_0K^2\frac{\sqrt{\log m}}{\sqrt{mn}}$, $D_1 = \frac{\left\|A\right\|_F}{\sqrt{m}} + \frac{\left\|B\right\|_F}{\sqrt{n}}$, and

$$2\sqrt{\tau_A\tau_B} \leq \tau_A + \tau_B \leq \frac{\left\|A\right\|_F}{\sqrt{m}} + \frac{\left\|B\right\|_F}{\sqrt{n}}.$$

To see this, recall

$$\begin{aligned} m\tau_A &= \sum_{i=1}^{m} \lambda_i(A) \le \sqrt{m}(\sum_{i=1}^{m} \lambda_i^2(A))^{1/2} = \sqrt{m}\,\|A\|_F \quad \text{and} \quad \text{(C.2)} \\ n\tau_B &= \sum_{i=1}^{n} \lambda_i(B) \le \sqrt{n}(\sum_{i=1}^{n} \lambda_i^2(B))^{1/2} = \sqrt{n}\,\|B\|_F \end{aligned}$$

where $\lambda_i(A), i = 1, \ldots, m$ and $\lambda_i(B), i = 1, \ldots, n$ denote the eigenvalues of positive semidefinite covariance matrices A and B respectively.

Denote by $\mathcal{B}_6$ the following event

$$\left\{ \tfrac{1}{n} \left| \check{\mathrm{tr}}(B) - \mathrm{tr}(B) \right| \le D_1 r_{m,m} \right\}.$$

Clearly $\widehat{\mathrm{tr}}(B) := (\check{\mathrm{tr}}(B))_+$ by definition (1.5). As a consequence, on $\mathcal{B}_6$, $\widehat{\mathrm{tr}}(B) = \check{\mathrm{tr}}(B) > 0$ when $\tau_B > D_1 r_{m,m}$; hence

$$\tfrac{1}{n} \left| \widehat{\mathrm{tr}}(B) - \mathrm{tr}(B) \right| = \tfrac{1}{n} \left| \check{\mathrm{tr}}(B) - \mathrm{tr}(B) \right| \le D_1 r_{m,m}.$$

Otherwise, it is possible that $\check{\mathrm{tr}}(B) < 0$. However, suppose we set

$$\widehat{\tau}_B := \tfrac{1}{n} \widehat{\mathrm{tr}}(B) := \tfrac{1}{n} (\check{\mathrm{tr}}(B) \vee 0),$$

then we can also guarantee that

$$|\widehat{\tau}_B - \tau_B| = |\tau_B| \le D_1 r_{m,m} \quad \text{in case} \quad \tau_B \le D_1 r_{m,m}.$$

The lemma is thus proved. □

C.2. Proof of Lemma 11

Following Lemma 32, we have for all $t > 0$, $B \succ 0$ being an $n \times n$ symmetric positive definite matrix, and $v, w \in \mathbb{R}^m$

$$\mathbb{P}\left(\left| v^T Z_1^T B^{1/2} Z_2 w \right| > t \right) \le 2 \exp\left[-c \min\left(\frac{t^2}{K^4 \mathrm{tr}(B)}, \frac{t}{K^2 \|B\|_2^{1/2}} \right) \right] \quad \text{(C.3)}$$

and

$$\mathbb{P}\left(\left| v^T Z^T B Z w - \mathbb{E} v^T Z^T B Z w \right| > t \right) \le 2 \exp\left(-c \min\left(\frac{t^2}{K^4 \|B\|_F^2}, \frac{t}{K^2 \|B\|_2} \right) \right).$$

Proof of Lemma 11. Let $e_1, \ldots, e_m \in \mathbb{R}^m$ be the canonical basis spanning $\mathbb{R}^m$. Let $x_1, \ldots, x_m, x_1', \ldots, x_m' \in \mathbb{R}^n$ be the column vectors Z_1, Z_2 respectively. Let

$Y \sim e_1^T Z_0^T$. Let $w_i = \frac{A^{1/2}e_i}{\|A^{1/2}e_i\|_2}$ for all i. Clearly the condition on the stable rank of B guarantees that

$$n \geq r(B) = \frac{\mathrm{tr}(B)}{\|B\|_2} = \frac{\mathrm{tr}(B)\,\|B\|_2}{\|B\|_2^2} \geq \|B\|_F^2 / \|B\|_2^2 \geq \log m.$$

By (C.1), we obtain for $t' = C_0 M_\epsilon K \sqrt{\mathrm{tr}(B)\log m}$

$$\begin{aligned}
&\mathbb{P}\left(\exists j, \left|\epsilon^T B^{1/2} Z_2 e_j\right| > t'\right) = \\
&\qquad \mathbb{P}\left(\exists j, \frac{M_\epsilon}{K}\left|e_1^T Z_0^T B^{1/2} Z_2 e_j\right| > C_0 M_\epsilon K \sqrt{\log m}\mathrm{tr}(B)^{\frac{1}{2}}\right) \\
&\leq \quad \exp(\log m)\mathbb{P}\left(\left|Y^T B^{1/2} x_j'\right| > C_0 K^2 \sqrt{\log m}\mathrm{tr}(B)^{\frac{1}{2}}\right) \leq 2/m^3
\end{aligned}$$

where the last inequality holds by the union bound, given that $\frac{\mathrm{tr}(B)}{\|B\|_2} \geq \log m$; Similarly, for all j and $t = C_0 K^2 \sqrt{\log m}\mathrm{tr}(B)^{1/2}$,

$$\begin{aligned}
\mathbb{P}\left(\left|Y^T B^{1/2} x_j'\right| > t\right) &\leq 2\exp\left(-c\min\left(\frac{t^2}{K^4\mathrm{tr}(B)}, \frac{t}{K^2\|B\|_2^{1/2}}\right)\right), \\
&\leq 2\exp\left(-c\min\left(C_0^2\log m, \frac{C_0\log^{1/2} m\sqrt{\mathrm{tr}(B)}}{\|B\|_2^{1/2}}\right)\right) \\
&\leq 2\exp\left(-c\min(C_0^2, C_0)\log m\right) \leq 2\exp\left(-4\log m\right).
\end{aligned}$$

Let $v, w \in S^{m-1}$. Thus we have by Lemma 32, for $t_0 = C_0 M_\epsilon K\sqrt{n\log m}$, $\tau = C_0 K^2\sqrt{n\log m}$, $w_j = \frac{A^{1/2}e_j}{\|A^{1/2}e_j\|_2}$ and $n \geq \log m$,

$$\begin{aligned}
&\mathbb{P}\left(\exists j, \left|\epsilon^T Z_1 w_j\right| > t_0\right) \leq \mathbb{P}\left(\exists j, \frac{M_\epsilon}{K}\left|Y^T Z_1 w_j\right| > C_0 M_\epsilon K\sqrt{n\log m}\right) \\
&\leq \quad m\mathbb{P}\left(\left|Y^T Z_1 w_j\right| > C_0 K^2\sqrt{n\log m}\right) \\
&= \quad \exp(\log m)\mathbb{P}\left(\left|e_1^T Z_0^T Z_1 w_j\right| > \tau\right) \leq 2\exp\left(-c\min\left(\frac{\tau^2}{nK^4}, \frac{\tau}{K^2}\right)\right) =: V
\end{aligned}$$

where

$$\begin{aligned}
V &\leq 2\exp\left(-c\min\left(\frac{(C_0K^2\sqrt{n\log m})^2}{nK^4}\frac{C_0K^2\sqrt{n\log m}}{K^2}\right) + \log m\right) \\
&\leq 2m\exp\left(-c\min\left(C_0^2\log m, C_0\log^{1/2} m\sqrt{n}\right)\right) \\
&\leq 2m\exp\left(-c\min(C_0^2, C_0)\log m\right) \leq 2\exp\left(-3\log m\right).
\end{aligned}$$

Therefore we have with probability at least $1 - 4/m^3$,

$$\begin{aligned}
\left\|Z_2^T B^{\frac{1}{2}}\epsilon\right\|_\infty &:= \max_{j=1,\ldots,m} \langle\, \epsilon^T B^{1/2} Z_2, e_j \,\rangle \le t' = C_0 M_\epsilon K \sqrt{\mathrm{tr}(B)\log m} \\
\left\|A^{\frac{1}{2}} Z_1^T \epsilon\right\|_\infty &:= \max_{j=1,\ldots,m} \langle\, A^{1/2} e_j, Z_1^T \epsilon \,\rangle \\
&\le \max_{j=1,\ldots,m} \left\|A^{1/2} e_j\right\|_2 \max_{j=1,\ldots,m} \langle\, w_j, Z_1^T \epsilon \,\rangle \\
&\le a_{\max}^{1/2} t_0 = a_{\max}^{1/2} C_0 M_\epsilon K \sqrt{n \log m}.
\end{aligned}$$

The "moreover" part follows exactly the same arguments as above. Denote by $\bar{\beta}^* := \beta^* / \|\beta^*\|_2 \in E \cap S^{m-1}$ and $w_i := A^{1/2} e_i / \left\|A^{1/2} e_i\right\|_2$. By (C.3)

$$\begin{aligned}
&\mathbb{P}\left(\exists i,\ \langle\, w_i, Z_1^T B^{1/2} Z_2 \bar{\beta}^* \,\rangle \ge C_0 K^2 \sqrt{\log m}\mathrm{tr}(B)^{1/2}\right) \\
&\le \sum_{i=1}^m \mathbb{P}\left(\ \langle\, w_i, Z_1^T B^{1/2} Z_2 \bar{\beta}^* \,\rangle \ge C_0 K^2 \sqrt{\log m \mathrm{tr}(B)}\right) \\
&\le 2\exp\left(-c\min\left(C_0^2 \log m, C_0 \log m\right) + \log m\right) \le 2/m^3.
\end{aligned}$$

Now for $t = C_0 K^2 \sqrt{\log m}\, \|B\|_F$ and $\|B\|_F / \|B\|_2 \ge \sqrt{\log m}$,

$$\begin{aligned}
&\mathbb{P}\left(\exists e_i :\ \langle\, e_i, (Z^T B Z - \mathrm{tr}(B) I_m)\bar{\beta}^* \,\rangle \ge C_0 K^2 \sqrt{\log m}\, \|B\|_F\right) \\
&\le 2m \exp\left[-c\min\left(\frac{t^2}{K^4 \|B\|_F^2}, \frac{t}{K^2 \|B\|_2}\right)\right] \le 2/m^3.
\end{aligned}$$

By the two inequalities immediately above, we have with probability at least $1 - 4/m^3$,

$$\begin{aligned}
\left\|X_0^T W \beta^*\right\|_\infty &= \left\|A^{1/2} Z_1^T B^{1/2} Z_2 \beta^*\right\|_\infty \\
&\le \|\beta^*\|_2 \max_{e_i} \left\|A^{1/2} e_i\right\|_2 \left(\sup_{w_i} \langle\, w_i, Z_1^T B^{1/2} Z_2 \bar{\beta}^* \,\rangle\ \right) \\
&\le C_0 K^2 \|\beta^*\|_2 \sqrt{\log m} a_{\max}^{1/2} \sqrt{\mathrm{tr}(B)}
\end{aligned}$$

and

$$\begin{aligned}
\left\|(Z^T B Z - \mathrm{tr}(B) I_m)\beta^*\right\|_\infty &= \left\|(Z^T B Z - \mathrm{tr}(B) I_m)\bar{\beta}^*\right\|_\infty \|\beta^*\|_2 \\
&= \|\beta^*\|_2 \left(\sup_{e_i} \langle\, e_i, (Z^T B Z - \mathrm{tr}(B) I_m)\bar{\beta}^* \,\rangle\ \right) \\
&\le C_0 K^2 \|\beta^*\|_2 \sqrt{\log m}\, \|B\|_F .
\end{aligned}$$

The last two bounds follow exactly the same arguments as above, except that we replace β^* with $e_j, j = 1, \ldots, m$ and apply the union bounds to m^2 instead of m events, and thus $\mathbb{P}(\mathcal{B}_{10}) \ge 1 - 4/m^2$. □

Appendix D: Proof of Corollary 13

Now following (6.1), we have on event $\mathcal{B}_0$,

$$\left\|\widehat{\gamma}-\widehat{\Gamma}\beta^*\right\|_\infty \;\leq\; \rho_n\left(\left(\frac{3}{4}D_2+D_2\frac{1}{\sqrt{m}}\right)K\left\|\beta^*\right\|_2+D_0M_\epsilon\right)$$

where $2D_1\leq 2\left\|A\right\|_2+2\left\|B\right\|_2=D_2$, and for $(D_0')^2\leq 2\left\|B\right\|_2+2a_{\max}$,

$$\begin{aligned} D_0\leq D_0' &\leq \sqrt{2(\left\|B\right\|_2+a_{\max})}\leq 2(a_{\max}+\left\|B\right\|_2)=D_2,\\ \text{and}\quad D_0'\tau_B^{1/2} &\leq (\left\|B\right\|_2^{1/2}+a_{\max}^{1/2})\tau_B^{1/2}\leq \tau_B+\frac{1}{2}(\left\|B\right\|_2+a_{\max})\leq\frac{3}{4}D_2 \end{aligned}$$

given that under (A1) : $\tau_A=1$, $\left\|A\right\|_2\geq a_{\max}\geq a_{\max}^{1/2}\geq 1$. Hence the lemma holds for $m\geq 16$ and $\psi=C_0D_2K\left(K\left\|\beta^*\right\|_2+M_\epsilon\right)$. □

Appendix E: Proof of Corollary 14

Suppose that event $\mathcal{B}_0$ holds. Recall $D_0'=\left\|B\right\|_2^{1/2}+a_{\max}^{1/2}$. Denote by $\rho_n:=C_0K\sqrt{\frac{\log m}{n}}$. By (6.1) and the fact that $2D_1:=2(\frac{\left\|A\right\|_F}{\sqrt{m}}+\frac{\left\|B\right\|_F}{\sqrt{n}})\leq 2(\left\|A\right\|_2^{1/2}+\left\|B\right\|_2^{1/2})(\sqrt{\tau_A}+\sqrt{\tau_B})\leq D_{\text{oracle}}D_0'$,

$$\begin{aligned} \left\|\widehat{\gamma}-\widehat{\Gamma}\beta^*\right\|_\infty &\leq D_0'K\tau_B^{1/2}\left\|\beta^*\right\|_2\rho_n+2D_1K\frac{1}{\sqrt{m}}\left\|\beta^*\right\|_\infty\rho_n+D_0M_\epsilon\rho_n\\ &\leq D_0'K\left\|\beta^*\right\|_2\rho_n\left(\tau_B^{1/2}+\frac{D_{\text{oracle}}}{\sqrt{m}}\right)+D_0M_\epsilon\rho_n \end{aligned}$$

The corollary is thus proved. □

Appendix F: Proof of Lemma 15

In view of Remark F.1, Condition (6.3) implies that (7.2) in Theorem 26 holds for $k=s_0$ and $\varepsilon=\frac{1}{2M_A}$. Now, by Theorem 26, we have $\forall u,v\in E\cap S^{m-1}$, under (A1) and (A3), condition (7.1) holds under event $\mathcal{A}_0$, and so long as $mn\geq 4096C_0^2D_2^2K^4\log m/\lambda_{\min}(A)^2$,

$$\begin{aligned} \left|u^T\Delta v\right| &\leq 8C\varpi(s_0)\varepsilon+2C_0D_2K^2\sqrt{\frac{\log m}{mn}}=:\delta\text{ with}\\ \delta &\leq \frac{\lambda_{\min}(A)}{16}+\frac{\lambda_{\min}(A)}{32}=\frac{3}{32}\lambda_{\min}(A)\leq\frac{1}{8}, \end{aligned}$$

which holds for all

$$\varepsilon\leq\frac{1}{2}\frac{\lambda_{\min}(A)}{64C\varpi(s_0)}:=\frac{1}{2M_A}\leq\frac{1}{128C}$$

with $\mathbb{P}\left(\mathcal{A}_0\right)\geq 1-4\exp\left(-c_2\varepsilon^2\frac{\text{tr}(B)}{K^4\left\|B\right\|_2}\right)-2\exp\left(-c_2\varepsilon^2\frac{n}{K^4}\right)-6/m^3$.

Hence, by Corollary 25, $\forall \theta \in \mathbb{R}^m$,

$$\theta^T \widehat{\Gamma}_A \theta \geq \alpha \|\theta\|_2^2 - \tau \|\theta\|_1^2 \quad \text{and} \quad \theta^T \widehat{\Gamma}_A \theta \leq \widetilde{\alpha} \|\theta\|_2^2 + \tau \|\theta\|_1^2,$$

where $\alpha = \frac{5}{8}\lambda_{\min}(A)$ and $\widetilde{\alpha} = \frac{11}{8}\lambda_{\max}(A)$ and $\tau = \frac{3}{8}\frac{\lambda_{\min}(A)}{s_0}$.

Now for $s_0 \geq 32$ as defined in (2.2), we have

$$s_0 \leq \frac{n}{\log m} \frac{\lambda_{\min}^2(A)}{1024 C^2 \varpi(s_0)^2} \tag{F.1}$$

$$\text{and} \quad s_0 + 1 \geq \frac{n}{\log m} \frac{\lambda_{\min}^2(A)}{1024 C^2 \varpi^2(s_0+1)} \tag{F.2}$$

given that $\tau_B + \rho_{\max}(s_0+1, A) = O(\lambda_{\max}(A))$ in view of (2.1) and (A3). Thus

$$\begin{aligned} \frac{384 C^2 \varpi(s_0)^2}{\lambda_{\min}(A)} \frac{\log m}{n} \leq \tau &= \frac{3}{8}\frac{\lambda_{\min}(A)}{s_0} \\ &\leq \frac{33}{32(s_0+1)} \frac{3}{8} \frac{\lambda_{\min}(A)}{s_0} \\ &\leq \frac{396 C^2 \varpi^2 (s_0+1)}{\lambda_{\min}(A)} \frac{\log m}{n}. \end{aligned}$$

The lemma is thus proved in view of Remark F.1. □

Remark F.1. *Clearly the condition on* $\operatorname{tr}(B)/\|B\|_2$ *as stated in Lemma 15 ensures that we have for* $\varepsilon = \frac{1}{2M_A}$ *and* $s_0 \asymp \frac{4n}{M_A^2 \log m}$,

$$\begin{aligned} \varepsilon^2 \frac{\operatorname{tr}(B)}{K^4 \|B\|_2} &\geq \frac{\varepsilon^2}{K^4} c' K^4 \frac{s_0}{\varepsilon^2} \log\left(\frac{3em}{s_0 \varepsilon}\right) \\ &\geq c' s_0 \log\left(\frac{6emM_A}{s_0}\right), \end{aligned}$$

and hence

$$\begin{aligned} \exp\left(-c_2 \varepsilon^2 \frac{\operatorname{tr}(B)}{K^4 \|B\|_2}\right) &\leq \exp\left(-c' c_2 s_0 \log\left(\frac{6emM_A}{s_0}\right)\right) \\ &\asymp \exp\left(-c_3 \frac{4n}{M_A^2 \log m} \log\left(\frac{3eM_A^3 m \log m}{2n}\right)\right). \end{aligned}$$

F.1. Comparing the two type of RE *conditions in Theorems 3 and 4*

We define $\operatorname{Cone}(d_0, k_0)$, where $0 < d_0 < m$ and k_0 is a positive number, as the set of vectors in $\mathbb{R}^m$ which satisfy the following cone constraint:

$$\operatorname{Cone}(d_0, k_0) = \{x \in \mathbb{R}^m \mid \exists I \in \{1, \ldots, m\}, |I| = d_0 \text{ s.t. } \|x_{I^c}\|_1 \leq k_0 \|x_I\|_1\}.$$

For each vector $x \in \mathbb{R}^m$, let T_0 denote the locations of the d_0 largest coefficients of x in absolute values. The following elementary estimate [38] will be used in conjunction with the RE condition.

Lemma 33. *For each vector $x \in \mathrm{Cone}(d_0, k_0)$, let T_0 denotes the locations of the d_0 largest coefficients of x in absolute values. Then*

$$\|x_{T_0}\|_2 \geq \frac{\|x\|_2}{\sqrt{1+k_0}}. \tag{F.3}$$

Lemma 34. *Suppose all conditions in Lemma 15 hold. Let $k_0 := 1+\lambda$. Suppose that $d_0 = o\left(s_0/64(1+3\lambda/4)^2\right)$. Now suppose that*

$$\tau(1+3k_0)^2 2d_0 = 2\tau(4+3\lambda)^2 d_0 \leq \alpha/2.$$

Then on event $\mathcal{A}_0$, we have $\mathsf{RE}^2(2d_0, 3k_0, \widehat{\Gamma}_A)$ condition holds on $\widehat{\Gamma}_A$ in the sense that

$$\min_{x \in \mathrm{Cone}(2d_0, 3k_0)} \frac{x^T \widehat{\Gamma}_A x}{\|x_{T_0}\|_2^2} \geq \frac{\alpha}{2}. \tag{F.4}$$

Under (A2) and (A3), we could set d_0 such that for some large enough constant C_A,

$$d_0 \leq \frac{n}{C_A \kappa(A)^2 \log m} = O\left(\frac{\lambda_{\min}^2(A)}{\varpi^2(s_0+1)} \frac{n}{\log m}\right) \tag{F.5}$$

where $\kappa(A) := \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$, so that $d_0 = O(s_0)$ and (F.4) holds.

Proof. Now following the proof Lemma 1, Part I. We have on $\mathcal{A}_0$, the Lower-RE condition holds for Γ_A. Thus for $x \in \mathrm{Cone}(2d_0, 3k_0) \cap S^{m-1}$ and $\tau(1+3k_0)^2 2d_0 \leq \alpha/2$,

$$\|x\|_1^2 \leq (1+3k_0)^2 \|x_{T_0}\|_1^2 \leq (1+3k_0)^2 2d_0 \|x_{T_0}\|_2^2.$$

Thus

$$\begin{aligned} x^T \widehat{\Gamma}_A x &\geq \left(\alpha \|x\|_2^2 - \tau \|x\|_1^2\right) \\ &\geq \left(\alpha \|x\|_2^2 - \tau(1+3k_0)^2 2d_0 \|x_{T_0}\|_2^2\right) \\ &\geq \left(\alpha - \tau(1+3k_0)^2 2d_0\right) \|x_{T_0}\|_2^2 \geq \frac{\alpha}{2} \|x_{T_0}\|_2^2. \end{aligned}$$

Thus (F.4) holds. Now (F.5) follows from (F.1), which holds by definition of s_0 as in (2.2), where s_0 is tightly bounded in the sense that both (F.1) and (F.2) need to hold. □

Remark F.2. *We note that (F.4) can be understood to be the $\mathsf{RE}(2d_0, 3k_0)$ condition on $\widehat{\Gamma}_A$. In view of Lemma 15, it is clear that for $d_0 \asymp \sqrt{n/\log m}$, it holds that*

$$4d_0(4+3\lambda)^2 = o(s_0)$$

given that $\tau s_0 = O(\alpha)$ on event $\mathcal{A}_0$; indeed, we have by Lemma 15 the Lower-RE condition holds for $\widehat{\Gamma}_A := A^T A - \widehat{\mathrm{tr}}(B) I_m$, with $\alpha, \tau > 0$ such that

$$\textit{curvature } \alpha = \frac{5}{8}\lambda_{\min}(A) \quad \textit{and tolerance} \quad \tau := \frac{3}{8}\frac{\lambda_{\min}(A)}{s_0},$$

where recall $s_0 \geq 32$ *is as defined in* (2.2)*; moreover, we replaced the parameter* $M_A \asymp \frac{\rho_{\max}(s_0,A)+\tau_B}{\lambda_{\min}(A)}$ *with* $\kappa(A)$ *in view of* (2.1) *and (A3).*

Appendix G: Proof of Theorem 16

Denote by $\beta = \beta^*$. Let $S := \operatorname{supp} \beta$, $d = |S|$ and

$$\upsilon = \widehat{\beta} - \beta,$$

where $\widehat{\beta}$ is as defined in (1.7).

We first show Lemma 35, followed by the proof of Theorem 16.

Lemma 35. *[4, 30] Suppose that* (6.5) *holds. Suppose that there exists a parameter* ψ *such that*

$$\sqrt{d}\tau \leq \frac{\psi}{b_0}\sqrt{\frac{\log m}{n}} \quad \text{and} \quad \lambda \geq 4\psi\sqrt{\frac{\log m}{n}},$$

where b_0, λ *are as defined in* (1.7)*. Then*

$$\|\upsilon_{S^c}\|_1 \leq 3\|\upsilon_S\|_1.$$

Proof. By the optimality of $\widehat{\beta}$, we have

$$\begin{aligned}
\lambda\|\beta\|_1 - \lambda\left\|\widehat{\beta}\right\|_1 &\geq \frac{1}{2}\widehat{\beta}\widehat{\Gamma}\widehat{\beta} - \frac{1}{2}\beta\widehat{\Gamma}\beta - \langle\widehat{\gamma}, \upsilon\rangle \\
&= \frac{1}{2}\upsilon\widehat{\Gamma}\upsilon + \langle\upsilon, \widehat{\Gamma}\beta\rangle - \langle\upsilon, \widehat{\gamma}\rangle \\
&= \frac{1}{2}\upsilon\widehat{\Gamma}\upsilon - \langle\upsilon, \widehat{\gamma} - \widehat{\Gamma}\beta\rangle.
\end{aligned}$$

Hence, we have for $\lambda \geq 4\psi\sqrt{\frac{\log m}{n}}$,

$$\begin{aligned}
\frac{1}{2}\upsilon\widehat{\Gamma}\upsilon &\leq \langle\upsilon, \widehat{\gamma} - \widehat{\Gamma}\beta\rangle + \lambda\left(\|\beta\|_1 - \left\|\widehat{\beta}\right\|_1\right) \qquad \text{(G.1)} \\
&\leq \lambda\left(\|\beta\|_1 - \left\|\widehat{\beta}\right\|_1\right) + \left\|\widehat{\gamma} - \widehat{\Gamma}\beta\right\|_\infty \|\upsilon\|_1.
\end{aligned}$$

Hence

$$\begin{aligned}
\upsilon\widehat{\Gamma}\upsilon &\leq \lambda\left(2\|\beta\|_1 - 2\left\|\widehat{\beta}\right\|_1\right) + 2\psi\sqrt{\frac{\log m}{n}}\|\upsilon\|_1 \qquad \text{(G.2)} \\
&\leq \lambda\left(2\|\beta\|_1 - 2\left\|\widehat{\beta}\right\|_1 + \frac{1}{2}\|\upsilon\|_1\right) \\
&\leq \lambda\frac{1}{2}\left(5\|\upsilon_S\|_1 - 3\|\upsilon_{S^c}\|_1\right), \qquad \text{(G.3)}
\end{aligned}$$

where by the triangle inequality, and $\beta_{S^c} = 0$, we have

$$\begin{aligned}
&2\|\beta\|_1 - 2\left\|\widehat{\beta}\right\|_1 + \frac{1}{2}\|\upsilon\|_1 \\
&= 2\|\beta_S\|_1 - 2\left\|\widehat{\beta}_S\right\|_1 - 2\|\upsilon_{S^c}\|_1 + \frac{1}{2}\|\upsilon_S\|_1 + \frac{1}{2}\|\upsilon_{S^c}\|_1
\end{aligned}$$

$$\leq \quad 2\|v_S\|_1 - 2\|v_{S^c}\|_1 + \frac{1}{2}\|v_S\|_1 + \frac{1}{2}\|v_{S^c}\|_1$$
$$\leq \quad \frac{1}{2}\left(5\|v_S\|_1 - 3\|v_{S^c}\|_1\right). \tag{G.4}$$

We now give a lower bound on the LHS of (G.1), applying the lower-RE condition as in Definition 2.2,

$$\begin{aligned} v^T\widehat{\Gamma}v &\geq \alpha\|v\|_2^2 - \tau\|v\|_1^2 \geq -\tau\|v\|_1^2 \\ \text{and hence} \quad -v^T\widehat{\Gamma}v &\leq \|v\|_1^2\tau \leq \|v\|_1 2b_0\sqrt{d}\tau \\ &\leq \|v\|_1 2b_0\frac{\psi}{b_0}\sqrt{\frac{\log m}{n}} = \|v\|_1 2\psi\sqrt{\frac{\log m}{n}} \\ &\leq \frac{1}{2}\lambda(\|v_S\|_1 + \|v_{S^c}\|_1), \end{aligned} \tag{G.5}$$

where we use the assumption that

$$\sqrt{d}\tau \leq \frac{\psi}{b_0}\sqrt{\frac{\log m}{n}} \quad \text{and} \quad \|v\|_1 \leq \left\|\widehat{\beta}\right\|_1 + \|\beta\|_1 \leq 2b_0\sqrt{d},$$

which holds by the triangle inequality and the fact that both $\widehat{\beta}$ and β have ℓ_1 norm being bounded by $b_0\sqrt{d}$. Hence by (G.3) and (G.5)

$$0 \quad \leq \quad -v\widehat{\Gamma}v + \frac{5}{2}\lambda\|v_S\|_1 - \frac{3}{2}\lambda\|v_{S^c}\|_1 \tag{G.6}$$
$$\leq \quad \frac{1}{2}\lambda\|v_S\|_1 + \frac{1}{2}\lambda\|v_{S^c}\|_1 + \frac{5}{2}\lambda\|v_S\|_1 - \frac{3}{2}\lambda\|v_{S^c}\|_1$$
$$\leq \quad 3\lambda\|v_S\|_1 - \lambda\|v_{S^c}\|_1. \tag{G.7}$$

Thus we have

$$\|v_{S^c}\|_1 \leq 3\|v_S\|_1,$$

and the lemma holds. □

Proof of Theorem 16. Following the conclusion of Lemma 35, we have

$$\|v\|_1 \leq 4\|v_S\|_1 \leq 4\sqrt{d}\|v\|_2. \tag{G.8}$$

Moreover, we have by the lower-RE condition as in Definition 2.2

$$v^T\widehat{\Gamma}v \quad \geq \quad \alpha\|v\|_2^2 - \tau\|v\|_1^2 \geq (\alpha - 16d\tau)\|v\|_2^2 \geq \frac{1}{2}\alpha\|v\|_2^2, \tag{G.9}$$

where the last inequality follows from the assumption that $16d\tau \leq \alpha/2$.

Combining the bounds in (G.9), (G.8) and (G.2), we have

$$\begin{aligned} \frac{1}{2}\alpha\|v\|_2^2 &\leq v^T\widehat{\Gamma}v \leq \lambda\left(2\|\beta\|_1 - 2\left\|\widehat{\beta}\right\|_1\right) + 2\psi\sqrt{\frac{\log m}{n}}\|v\|_1 \\ &\leq \frac{5}{2}\lambda\|v_S\|_1 \leq 10\lambda\sqrt{d}\|v\|_2. \end{aligned}$$

And thus we have $\|v\|_2 \leq 20\lambda\sqrt{d}$. The theorem is thus proved. □

Appendix H: Proofs for the Lasso-type estimator

Let

$$M_+ \quad = \quad \frac{32C\varpi(s_0+1)}{\lambda_{\min}(A)} \text{ and } \quad \varpi(s_0+1) = \rho_{\max}(s_0+1, A) + \tau_B =: D.$$

By definition of s_0, we have $s_0 M_A^2 \leq \frac{4n}{\log m}$ and

$$(s_0+1) \geq \frac{n}{M_+^2 \log m}$$

$$\text{given that} \quad \sqrt{s_0+1}\varpi(s_0+1) \quad \geq \quad \frac{\lambda_{\min}(A)}{32C}\sqrt{\frac{n}{\log m}}. \tag{H.1}$$

To prove the first inequality in (6.4) and (6.7), we need to show that

$$d \leq \frac{\alpha}{32\tau} = \frac{\alpha}{32}\frac{s_0}{\lambda_{\min}(A)-\alpha} = \frac{5s_0}{96}.$$

The first inequality in (6.4) holds so long as

$$d \quad \leq \quad \frac{1}{20}\frac{1}{M_+^2}\frac{n}{\log m} \leq \frac{s_0+1}{20} \leq \frac{5(s_0+1)}{100} \leq \frac{5s_0}{96}, \tag{H.2}$$

where the last inequality holds so long as $s_0 \geq 24$. To prove the second inequality in (6.7), we need to show that

$$d \leq \frac{1}{\tau^2}\frac{\log m}{n}\left(\frac{\psi}{b_0}\right)^2, \quad \text{where} \quad \tau = \frac{3}{5}\frac{\alpha}{s_0} \quad \text{for} \quad \alpha = \frac{5}{8}\lambda_{\min}(A),$$

which in turn ensures that the second inequality in (6.4) holds for $\lambda \geq 4\psi$, for ψ appropriately chosen. We use the following inequality in the proof of Lemma 17 and Lemma 18:

$$\begin{aligned}\frac{s_0+1}{\alpha^2} \quad &\geq \quad \frac{64}{25\lambda_{\min}(A)^2}\frac{1}{M_+^2}\frac{n}{\log m} \geq \left(\frac{8}{5}\frac{1}{32C\varpi(s_0+1)}\right)^2\frac{n}{\log m} \\ &= \quad \left(\frac{1}{20CD}\right)^2\frac{n}{\log m} \geq \left(\frac{1}{10CD_2}\right)^2\frac{n}{\log m},\end{aligned} \tag{H.3}$$

where we use the fact that $D = \varpi(s_0+1) = \rho_{\max}(s_0+1, A) + \tau_B \leq \|A\|_2 + \|B\|_2 := D_2/2$.

H.1. Proof of Lemma 17

Let $C_A = \frac{1}{40M_+^2}$. The first inequality in (6.7) holds in view of (H.2). Recall that $b_0^2 \geq \|\beta^*\|_2^2 \geq \phi b_0^2$ by definition of $0 < \phi \leq 1$. Let $C = C_0/\sqrt{c'}$. By (6.6) and (H.3),

$$\begin{aligned} d \quad &\leq \quad C_A c' D_\phi \frac{n}{\log m} \leq \frac{1}{40M_+^2}\left(\frac{C_0 D_2}{CD_2}\right)^2 D_\phi \frac{n}{\log m} \\ &\leq \quad \frac{25}{9}\frac{32}{33}\frac{32}{33}\frac{n}{M_+^2 \log m}\left(\frac{1}{10CD_2}\right)^2 C_0^2 D_2^2 D_\phi \end{aligned}$$

$$\leq \frac{25}{9}\frac{32}{33}\frac{32(s_0+1)}{33}\frac{(s_0+1)}{\alpha^2}\frac{\log m}{n}\left(\frac{\psi}{b_0}\right)^2$$
$$\leq \frac{25}{9}\frac{(s_0)^2}{\alpha^2}\frac{\log m}{n}\left(\frac{\psi}{b_0}\right)^2,$$

where

$$C_0^2 D_2^2 D_\phi = C_0^2 D_2^2\left(\frac{K^2 M_\epsilon^2}{b_0^2} + K^4\phi\right)$$
$$\leq C_0^2 D_2^2 \frac{K^2}{b_0^2}(M_\epsilon + K\|\beta^*\|_2)^2 = \left(\frac{\psi}{b_0}\right)^2, \tag{H.4}$$

for $\psi = C_0 D_2 K(K\|\beta^*\|_2 + M_\epsilon)$ as defined in (3.5). We have shown that (6.7) indeed holds, and the lemma is thus proved. □

H.2. Proof of Lemma 18

Let $C_A = \frac{1}{160M_+^2}$. The proof for $d \leq \frac{\alpha}{32\tau} = \frac{5s_0}{96}$ follows from (H.2). In order to show the second inequality, we follow the same line of arguments except that we need to replace one inequality (H.4) with (H.5). By definition of D_0', we have $\|B\|_2 + a_{\max} \leq (D_0')^2 \leq 2(\|B\|_2 + a_{\max})$. Let $D = \varpi(s_0+1)$.

By (6.8), (H.1) and (H.3), we have for $c'' \leq \left(\frac{D_0'}{D}\right)^2$,

$$d \leq C_A c' c'' D_\phi \frac{n}{\log m} \leq \frac{1}{160M_+^2}\frac{n}{\log m}\left(\frac{C_0 D_0'}{CD}\right)^2 D_\phi$$
$$\leq \frac{25}{9}\frac{32^2}{33^2}\left(\frac{1}{20CD}\right)^2 (C_0^2 (D_0')^2 D_\phi)\frac{n}{M_+^2 \log m}$$
$$\leq \frac{25}{9}\frac{32^2}{33^2}\frac{(s_0+1)^2}{\alpha^2}\frac{\log m}{n}\left(\frac{\psi}{b_0}\right)^2 \leq \frac{25}{9}\frac{(s_0)^2}{\alpha^2}\frac{\log m}{n}\left(\frac{\psi}{b_0}\right)^2,$$

where assuming that $s_0 \geq 32$, we have the following inequality by definition of s_0 and $\alpha = \frac{5}{8}\lambda_{\min}(A)$,

$$\frac{s_0+1}{\alpha^2}\frac{\log m}{n} \geq \left(\frac{8}{5}\frac{1}{32C\varpi(s_0+1)}\right)^2 \geq \left(\frac{1}{20CD}\right)^2.$$

We now replace (H.4) with

$$C_0^2 (D_0')^2 D_\phi = C_0^2 (D_0')^2 \frac{K^4}{b_0^2}\left(\frac{M_\epsilon^2}{K^2} + \tau_B^+ \phi b_0^2\right)$$
$$\leq C_0^2 (D_0')^2 \frac{K^2}{b_0^2}\left(M_\epsilon + \tau_B^{+/2} K\|\beta^*\|_2\right)^2 \leq \left(\frac{\psi}{b_0}\right)^2, \tag{H.5}$$
$$\text{where} \quad D_\phi := \frac{K^2 M_\epsilon^2}{b_0^2} + \tau_B^+ K^4 \phi \leq \frac{K^4}{b_0^2}\left(\frac{M_\epsilon^2}{K^2} + \tau_B^+ \|\beta^*\|_2^2\right)$$

and $\psi = C_0 D_0' K \left(K \tau_B^{+/2} \|\beta^*\|_2 + M_\epsilon K \right)$ is now as defined in (4.2). The lemma is thus proved. □

Remark H.1. *Throughout this paper, we assume that C_0 is a large enough constant such that for c as defined in Theorem 31,*

$$c \min\{C_0^2, C_0\} \geq 4. \tag{H.6}$$

By definition of s_0, we have for $\varpi^2(s_0) \geq 1$,

$$\begin{aligned} s_0 \varpi^2(s_0) &\leq \frac{c' \lambda_{\min}^2(A)}{1024 C_0^2} \frac{n}{\log m}, \quad \textit{and hence} \\ s_0 &\leq \frac{c' \lambda_{\min}^2(A)}{1024 C_0^2} \frac{n}{\log m} \leq \frac{\lambda_{\min}^2(A)}{1024 C_0^2} \frac{n}{\log m} =: \check{s}_0. \end{aligned}$$

Remark H.2. *The proof shows that one can take $C = C_0/\sqrt{c'}$, and take*

$$\mathcal{V} = 3eM_A^3/2 = \frac{3e64^3 C^3 \varpi^3(s_0)}{2\lambda_{\min}^3(A)} \leq \frac{3e64^3 C_0^3 \varpi^3(\check{s}_0)}{2(c')^{3/2} \lambda_{\min}^3(A)}.$$

Hence a sufficient condition on $r(B)$ is:

$$r(B) \geq 16 c' K^4 \frac{n}{\log m} \left(3 \log \frac{64 C_0 \varpi(\check{s}_0)}{\sqrt{c'} \lambda_{\min}(A)} + \log \frac{3em \log m}{2n} \right). \tag{H.7}$$

Appendix I: Proofs for the Conic programming estimator

We next provide proof for Lemmas 19 to 21 in this section.

I.1. Proof of Lemma 19

Suppose event $\mathcal{B}_0$ holds. Then by the proof of Corollary 13,

$$\begin{aligned} \left\| \tfrac{1}{n} X^T (y - X\beta^*) + \tfrac{1}{n} \widehat{\mathrm{tr}}(B) \beta^* \right\|_\infty &= \left\| \widehat{\gamma} - \widehat{\Gamma} \beta^* \right\|_\infty \\ &\leq 2 C_0 D_2 K^2 \|\beta^*\|_2 \sqrt{\frac{\log m}{n}} + C_0 D_0 K M_\epsilon \sqrt{\frac{\log m}{n}} \\ &=: \mu \|\beta^*\|_2 + \omega. \end{aligned}$$

The lemma follows immediately for the chosen μ, ω as in (6.9) given that $(\beta^*, \|\beta^*\|_2) \in \Upsilon$. □

I.2. Proof of Lemma 20

By optimality of $(\widehat{\beta}, \widehat{t})$, we have

$$\left\| \widehat{\beta} \right\|_1 + \lambda \left\| \widehat{\beta} \right\|_2 \leq \left\| \widehat{\beta} \right\|_1 + \lambda \widehat{t} \leq \|\beta^*\|_1 + \lambda \|\beta^*\|_2 .$$

Thus we have for $S := \operatorname{supp}(\beta^*)$,

$$\left\|\widehat{\beta}\right\|_1 = \left\|\widehat{\beta}_{S^c}\right\|_1 + \left\|\widehat{\beta}_S\right\|_1 \quad \leq \quad \|\beta^*\|_1 + \lambda(\|\beta^*\|_2 - \left\|\widehat{\beta}\right\|_2).$$

Now by the triangle inequality,

$$\begin{aligned}\left\|\widehat{\beta}_{S^c}\right\|_1 = \|v_{S^c}\|_1 &\leq \|\beta^*_S\|_1 - \left\|\widehat{\beta}_S\right\|_1 + \lambda(\|\beta^*\|_2 - \left\|\widehat{\beta}\right\|_2) \\ &\leq \|v_S\|_1 + \lambda(\|\beta^*\|_2 - \left\|\widehat{\beta}\right\|_2) \\ &\leq \|v_S\|_1 + \lambda(\|\beta^*\|_2 - \left\|\widehat{\beta}_S\right\|_2) \\ &= \|v_S\|_1 + \lambda \|v_S\|_2 \leq (1+\lambda)\|v_S\|_1 .\end{aligned}$$

The lemma thus holds given

$$\widehat{t} \quad \leq \quad \frac{1}{\lambda}(\|\beta^*\|_1 - \left\|\widehat{\beta}\right\|_1) + \|\beta^*\|_2 \leq \frac{1}{\lambda}\|v\|_1 + \|\beta^*\|_2 . \qquad \square$$

I.3. Proof of Lemma 21

Recall the following shorthand notation:

$$D_0 \quad = \quad (\sqrt{\tau_B} + \sqrt{a_{\max}}) \quad \text{and } D_2 \; = \; 2(\|A\|_2 + \|B\|_2).$$

First we rewrite an upper bound for $v = \widehat{\beta} - \beta^*$, $D = \operatorname{tr}(B)$ and $\widehat{D} = \widehat{\operatorname{tr}}(B)$,

$$\begin{aligned}\|X_0^T X_0 v\|_\infty &= \left\|(X-W)^T X_0(\widehat{\beta}-\beta^*)\right\|_\infty \\ &\leq \left\|X^T X_0(\widehat{\beta}-\beta^*)\right\|_\infty + \|W^T X_0 v\|_\infty \\ &\leq \left\|X^T(X\widehat{\beta}-y) - \widehat{D}\widehat{\beta}\right\|_\infty + \|X^T\epsilon\|_\infty + \left\|(X^TW - D)\widehat{\beta}\right\|_\infty \\ &+ \left\|(\widehat{D}-D)\widehat{\beta}\right\|_\infty + \|W^T X_0 v\|_\infty ,\end{aligned}$$

where

$$\begin{aligned}\left\|X^T X_0(\widehat{\beta}-\beta^*)\right\|_\infty &\leq \left\|X^T(X_0\widehat{\beta} - y + \epsilon)\right\|_\infty \\ &= \left\|X^T((X-W)\widehat{\beta} - y)\right\|_\infty + \|X^T\epsilon\|_\infty \\ &\leq \left\|X^T(X\widehat{\beta}-y) - \widehat{D}\widehat{\beta}\right\|_\infty + \|X^T\epsilon\|_\infty \\ &\quad + \left\|(X^TW - D)\widehat{\beta}\right\|_\infty + \left\|(\widehat{D}-D)\widehat{\beta}\right\|_\infty .\end{aligned}$$

On event $\mathcal{B}_0$, we have by Lemma 20 and the fact that $\widehat{\beta} \in \Upsilon$,

$$I := \left\|\widehat{\gamma} - \widehat{\Gamma}\widehat{\beta}\right\|_\infty \quad = \quad \left\|\tfrac{1}{n}X^T(y - X\widehat{\beta}) + \tfrac{1}{n}\widehat{D}\widehat{\beta}\right\|_\infty \leq \mu\widehat{t} + \omega$$

$$\begin{aligned} &\leq \mu(\frac{1}{\lambda}\|v\|_1 + \|\beta^*\|_2) + \omega \\ &= 2D_2K\rho_n(\frac{1}{\lambda}\|v\|_1 + \|\beta^*\|_2) + D_0\rho_n M_\epsilon; \end{aligned}$$

and on event $\mathcal{B}_4$,

$$\begin{aligned} II &:= \tfrac{1}{n}\left\|X^T\epsilon\right\|_\infty \leq \tfrac{1}{n}(\left\|X_0^T\epsilon\right\|_\infty + \left\|W^T\epsilon\right\|_\infty) \\ &\leq \rho_n M_\epsilon(a_{\max}^{1/2} + \sqrt{\tau_B}) = D_0\rho_n M_\epsilon. \end{aligned}$$

Thus on event $\mathcal{B}_0$, we have

$$\begin{aligned} I + II &\leq 2D_2K\rho_n(\frac{1}{\lambda}\|v\|_1 + \|\beta^*\|_2) + 2D_0\rho_n M_\epsilon \\ &= \mu(\frac{1}{\lambda}\|v\|_1 + \|\beta^*\|_2) + 2\omega. \end{aligned}$$

Now on event $\mathcal{B}_6$, we have for $2D_1 \leq D_2$

$$\begin{aligned} IV := \left\|(\widehat{D} - D)\widehat{\beta}\right\|_\infty &\leq \left|\widehat{D} - D\right| \left\|\widehat{\beta}\right\|_\infty \\ &\leq 2D_1K\frac{1}{\sqrt{m}}\rho_n(\|\beta^*\|_\infty + \|v\|_\infty) \\ &\leq D_2K\frac{1}{\sqrt{m}}\rho_n(\|\beta^*\|_2 + \|v\|_1). \end{aligned}$$

On event $\mathcal{B}_5 \cap \mathcal{B}_{10}$, we have

$$\begin{aligned} III &:= \tfrac{1}{n}\left\|(X^TW - D)\widehat{\beta}\right\|_\infty \\ &\leq \tfrac{1}{n}\left\|(X^TW - D)\beta^*\right\|_\infty + \tfrac{1}{n}\left\|(X^TW - D)v\right\|_\infty \\ &\leq \tfrac{1}{n}\left\|X_0^TW\beta^*\right\|_\infty + \tfrac{1}{n}\left\|(W^TW - D)\beta^*\right\|_\infty \\ &+ \tfrac{1}{n}\left(\left\|(Z^TBZ - \mathrm{tr}(B)I_m)\right\|_{\max} + \left\|X_0^TW\right\|_{\max}\right)\|v\|_1 \\ &\leq \rho_n K\left(\frac{\|B\|_F}{\sqrt{n}} + \sqrt{\tau_B}a_{\max}^{1/2}\right)(\|v\|_1 + \|\beta^*\|_2) \end{aligned}$$

and

$$\begin{aligned} V &= \tfrac{1}{n}\left\|W^TX_0v\right\|_\infty \\ &\leq \tfrac{1}{n}\left\|W^TX_0\right\|_{\max}\|v\|_1 \leq \rho_n K\sqrt{\tau_B}a_{\max}^{1/2}\|v\|_1. \end{aligned}$$

Thus we have on $\mathcal{B}_0 \cap \mathcal{B}_{10}$,

$$\begin{aligned} &III + IV + V \leq \\ &\quad \rho_n K\left(\|B\|_2 + \tau_B + a_{\max} + \frac{2}{\sqrt{m}}(\|A\|_2 + \|B\|_2)\right)(\|v\|_1 + \|\beta^*\|_2) \\ &\leq \rho_n K\left(4\|B\|_2 + 3\|A\|_2\right)(\|v\|_1 + \|\beta^*\|_2) \end{aligned}$$

$$\begin{aligned} &\leq 2D_2K\rho_n(\|v\|_1+\|\beta^*\|_2)\\ &\leq \mu(\|v\|_1+\|\beta^*\|_2), \end{aligned}$$

where $D_0 \leq D_2$ and $\tau_A = 1$, and

$$\begin{aligned} \left\|\tfrac{1}{n}X_0^TX_0v\right\|_\infty &\leq I+II+III+IV+V\\ &\leq \mu(\frac{1}{\lambda}\|v\|_1+\|\beta^*\|_2)+2D_0M_\epsilon\rho_n+\mu(\|v\|_1+\|\beta^*\|_2)\\ &\leq 2\mu\|\beta^*\|_2+\mu(\frac{1}{\lambda}+1)\|v\|_1+2\omega. \end{aligned}$$

The lemma thus holds. □

Appendix J: Proof for Theorem 7

We prove Lemmas 22 to 24 in this section.

J.1. Proof of Lemma 22

Suppose event $\mathcal{B}_0$ holds. Then by the proof of Corollary 14, we have for $D_0' = \|B\|_2^{1/2}+a_{\max}^{1/2}$,

$$\left\|\widehat{\gamma}-\widehat{\Gamma}\beta^*\right\|_\infty \leq D_0'\tau_B^{+/2}K\rho_n\|\beta^*\|_2+D_0M_\epsilon\rho_n,$$

where $\tau_B^{+/2} = \sqrt{\tau_B}+\frac{D_{\text{oracle}}}{\sqrt{m}}$ and $D_{\text{oracle}} = 2(\|B\|_2^{1/2}+\|A\|_2^{1/2})$. The lemma follows immediately for μ,ω as chosen in (6.12). □

J.2. Proof of Lemma 23

Suppose event $\mathcal{B}_6$ holds. We first show (6.13) and (6.14).

Recall $r_{m,m} := 2C_0K^2\sqrt{\frac{\log m}{mn}} \geq 2C_0K^2\frac{\log^{1/2}m}{m}$. By Lemma 5, we have on event $\mathcal{B}_6$,

$$|\widehat{\tau}_B-\tau_B| \leq D_1r_{m,m}.$$

Moreover, we have under (A1),

$$1=\tau_A\leq D_1 := \frac{\|A\|_F}{m^{1/2}}+\frac{\|B\|_F}{n^{1/2}}\leq \|A\|_2+\|B\|_2\leq (\frac{D_{\text{oracle}}}{2})^2,$$

in view of (C.2). Hence

$$\sqrt{D_1}\leq\frac{D_{\text{oracle}}}{2}=\|B\|_2^{1/2}+\|A\|_2^{1/2}.$$

By definition and construction, we have $\tau_B, \widehat{\tau}_B \geq 0$,

$$\begin{aligned} \left|\widehat{\tau}_B^{1/2} - \tau_B^{1/2}\right| &\leq \widehat{\tau}_B^{1/2} + \tau_B^{1/2}, \\ \text{and} \quad \left|\widehat{\tau}_B^{1/2} - \tau_B^{1/2}\right|^2 &\leq \left|(\widehat{\tau}_B^{1/2} + \tau_B^{1/2})(\widehat{\tau}_B^{1/2} - \tau_B^{1/2})\right| = |\widehat{\tau}_B - \tau_B|. \end{aligned}$$

Thus,

$$\left|\widehat{\tau}_B^{1/2} - \tau_B^{1/2}\right| \leq \sqrt{|\widehat{\tau}_B - \tau_B|} \leq \sqrt{D_1} r_{m,m}^{1/2} \leq \frac{D_{\text{oracle}}}{2} r_{m,m}^{1/2}$$

and for $C_6 \geq D_{\text{oracle}} \geq 2\sqrt{D_1}$ and $D_{\text{oracle}} = 2(\|A\|_2^{1/2} + \|B\|_2^{1/2})$,

$$\widehat{\tau}_B^{1/2} - \frac{D_{\text{oracle}}}{2} r_{m,m}^{1/2} \leq \tau_B^{1/2} \leq \widehat{\tau}_B^{1/2} + \frac{D_{\text{oracle}}}{2} r_{m,m}^{1/2}. \tag{J.1}$$

Thus we have for $\tau_B^{+/2}$ as defined in (4.1), (J.1) and the fact that

$$r_{m,m}^{1/2} \geq \sqrt{2C_0} K \frac{(\log m)^{1/4}}{\sqrt{m}} \geq 2/\sqrt{m} \ \text{ for } m \geq 16 \text{ and } C_0 \geq 1,$$

the following inequalities hold: for $K \geq 1$,

$$\begin{aligned} \tau_B^{+/2} &:= \tau_B^{1/2} + D_{\text{oracle}} m^{-1/2} \\ &\leq \widehat{\tau}_B^{1/2} + \frac{D_{\text{oracle}}}{2} r_{m,m}^{1/2} + \frac{D_{\text{oracle}}}{2} r_{m,m}^{1/2} \\ &\leq \widehat{\tau}_B^{1/2} + D_{\text{oracle}} r_{m,m}^{1/2} \leq \widetilde{\tau}_B^{1/2}, \end{aligned} \tag{J.2}$$

where the last inequality holds by the choice of $\widetilde{\tau}_B^{1/2} \geq \widehat{\tau}_B^{1/2} + D_{\text{oracle}} r_{m,m}^{1/2}$ as in (4.10). Moreover, by (J.1),

$$\begin{aligned} \widetilde{\tau}_B^{1/2} &:= \widehat{\tau}_B^{1/2} + C_6 r_{m,m}^{1/2} \leq \tau_B^{1/2} + \frac{D_{\text{oracle}}}{2} r_{m,m}^{1/2} + C_6 r_{m,m}^{1/2} \\ &\leq \tau_B^{1/2} + \frac{3}{2} C_6 r_{m,m}^{1/2}, \\ \text{and} \quad \widetilde{\tau}_B &:= (\widehat{\tau}_B^{1/2} + C_6 r_{m,m}^{1/2})^2 \leq 2\widehat{\tau}_B + 2C_6^2 r_{m,m} \\ &\leq 2\tau_B + 2D_1 r_{m,m} + 2C_6^2 r_{m,m} \\ &\leq 2\tau_B + \frac{D_{\text{oracle}}^2}{2} r_{m,m} + 2C_6^2 r_{m,m} \leq 2\tau_B + 3C_6^2 r_{m,m}. \end{aligned}$$

Thus (6.13) and (6.14) hold given that $2D_1 \leq D_{\text{oracle}}^2/2 \leq C_6^2/2$.

Finally, we have for τ_B^- as defined in (4.8),

$$\widetilde{\tau}_B^{1/2} \tau_B^- \leq (\tau_B^{1/2} + \frac{3}{2} C_6 r_{m,m}^{1/2}) \tau_B^- \leq \frac{\tau_B^{1/2} + \frac{3}{2} C_6 r_{m,m}^{1/2}}{\tau_B^{1/2} + 2C_6 r_{m,m}^{1/2}} \leq 1. \qquad \square$$

Remark J.1. *The set Υ in our setting is equivalent to the following: for μ, ω as defined in (4.10) and $\beta \in \mathbb{R}^m$,*

$$\Upsilon = \left\{(\beta, t) \ : \ \left\|\tfrac{1}{n} X^T (y - X\beta) + \tfrac{1}{n} \widehat{\text{tr}}(B)\beta\right\|_\infty \leq \mu t + \omega, \|\beta\|_2 \leq t\right\}. \tag{J.3}$$

J.3. Proof of Lemma 24

For the rest of the proof, we will follow the notation in the proof for Lemma 21. Notice that the bounds as stated in Lemma 20 remain true with ω, μ chosen as in (6.12), so long as $(\beta^*, \|\beta^*\|_2) \in \Upsilon$. This indeed holds by Lemma 22: for ω and μ (4.10) as chosen in Theorem 7, we have by (J.2),

$$\mu \asymp D_0' \widetilde{\tau}_B^{1/2} K \rho_n \geq D_0' K \rho_n \tau_B^{+/2}, \quad \text{where} \quad \tau_B^{+/2} = (\sqrt{\tau_B} + \frac{D_{\text{oracle}}}{\sqrt{m}}),$$

which ensures that $(\beta^*, \|\beta^*\|_2) \in \Upsilon$ by Lemma 22.

On event $\mathcal{B}_0$, we have by Lemma 20 and the fact that $\widehat{\beta} \in \Upsilon$ as in (J.3)

$$\begin{aligned} I + II &:= \left\| \widehat{\gamma} - \widehat{\Gamma}\widehat{\beta} \right\|_\infty + \tfrac{1}{n} \left\| X^T \epsilon \right\|_\infty \\ &\leq \left\| \tfrac{1}{n} X^T (y - X\widehat{\beta}) + \tfrac{1}{n} \widehat{D}\widehat{\beta} \right\|_\infty + \omega \leq \mu \widehat{t} + 2\omega \\ &\leq \mu (\frac{1}{\lambda} \|v\|_1 + \|\beta^*\|_2) + 2\omega, \end{aligned}$$

for ω, μ as chosen in (4.10). Now on event $\mathcal{B}_6$, we have under (A1),

$$\begin{aligned} IV := \left\| (\widehat{D} - D)\widehat{\beta} \right\|_\infty &\leq \left| \widehat{D} - D \right| \left\| \widehat{\beta} \right\|_\infty \leq 2 D_1 K \frac{1}{\sqrt{m}} \rho_n (\|\beta^*\|_\infty + \|v\|_\infty) \\ &\leq D_0' \frac{D_{\text{oracle}}}{\sqrt{m}} K \rho_n (\|\beta^*\|_2 + \|v\|_1), \end{aligned}$$

where $2D_1 \leq D_{\text{oracle}} D_0'$ for $1 \leq D_0' := \|B\|_2^{1/2} + a_{\max}^{1/2}$, for $a_{\max} \geq \tau_A = 1$ and $D_{\text{oracle}} = 2\left(\|B\|_2^{1/2} + \|A\|_2^{1/2} \right)$. Hence

$$\begin{aligned} III + IV + V &\leq \rho_n K \sqrt{\tau_B} \left(\|B\|_2^{1/2} + a_{\max}^{1/2} \right) (\|v\|_1 + \|\beta^*\|_2) \\ &\quad + 2 D_1 K \frac{1}{\sqrt{m}} \rho_n (\|\beta^*\|_2 + \|v\|_1) + \rho_n K \sqrt{\tau_B} a_{\max}^{1/2} \|v\|_1 \\ &\leq D_0' K \rho_n (\|v\|_1 + \|\beta^*\|_2)(\sqrt{\tau_B} + \frac{D_{\text{oracle}}}{\sqrt{m}}) + \rho_n K \sqrt{\tau_B} a_{\max}^{1/2} \|v\|_1 \\ &\leq D_0' K \rho_n \tau_B^{+/2} (\|v\|_1 + \|\beta^*\|_2) + D_0' K \rho_n \sqrt{\tau_B} \|v\|_1 \\ &\leq C_0 D_0' K^2 \sqrt{\frac{\log m}{n}} (\tau_B^{1/2} + \frac{D_{\text{oracle}}}{\sqrt{m}})(2\|v\|_1 + \|\beta^*\|_2) \\ &\leq \mu (2\|v\|_1 + \|\beta^*\|_2), \end{aligned}$$

for μ as defined in (4.10) in view of (J.2).

Thus we have

$$\begin{aligned} I + II + III + IV + V &\leq \mu (\frac{1}{\lambda} \|v\|_1 + \|\beta^*\|_2) + 2\omega + \mu (2\|v\|_1 + \|\beta^*\|_2) \\ &= 2\mu ((1 + \frac{1}{2\lambda}) \|v\|_1 + \|\beta^*\|_2) + 2\omega, \end{aligned}$$

and the improved bound as stated in the Lemma thus holds. □

Appendix K: Some geometric analysis results

Let us define the following set of vectors in $\mathbb{R}^m$:

$$\mathrm{Cone}(s) := \{v : \|v\|_1 \le \sqrt{s}\,\|v\|_2\}$$

For each vector $x \in \mathbb{R}^m$, let T_0 denote the locations of the s largest coefficients of x in absolute values. Any vector $x \in S^{m-1}$ satisfies:

$$\left\|x_{T_0^c}\right\|_\infty \le \|x_{T_0}\|_1 / s \quad \le \quad \frac{\|x_{T_0}\|_2}{\sqrt{s}}. \tag{K.1}$$

We need to state the following result from [33]. Let S^{m-1} be the unit sphere in $\mathbb{R}^m$, for $1 \le s \le m$,

$$U_s \;:=\; \{x \in \mathbb{R}^m : |\operatorname{supp}(x)| \le s\}. \tag{K.2}$$

The sets U_s is an union of the s-sparse vectors. The following three lemmas are well-known and mostly standard; See [33] and [30].

Lemma 36. *For every $1 \le s \le m$ and every $I \subset \{1, \dots, m\}$ with $|I| \le s$,*

$$\sqrt{|I|}B_1^m \cap S^{m-1} \subset 2\operatorname{conv}(U_s \cap S^{m-1}) =: 2\operatorname{conv}\left(\bigcup_{|J|\le s} E_J \cap S^{m-1}\right)$$

and moreover, for $\rho \in (0,1]$,

$$\begin{aligned}\sqrt{|I|}B_1^m \cap \rho B_2^m \quad &\subset \quad (1+\rho)\operatorname{conv}(U_s \cap B_2^m) \\ &=: \quad (1+\rho)\operatorname{conv}\left(\bigcup_{|J|\le s} E_J \cap S^{m-1}\right).\end{aligned}$$

Proof. Fix $x \in \mathbb{R}^m$. Let x_{T_0} denote the subvector of x confined to the locations of its s largest coefficients in absolute values; moreover, we use it to represent its 0-extended version $x' \in \mathbb{R}^m$ such that $x'_{T^c} = 0$ and $x'_{T_0} = x_{T_0}$. Throughout this proof, T_0 is understood to be the locations of the s largest coefficients in absolute values in x.

Moreover, let $(x_i^*)_{i=1}^m$ be non-increasing rearrangement of $(|x_i|)_{i=1}^m$. Denote by

$$\begin{aligned} L \;&=\; \sqrt{s}B_1^m \cap \rho B_2^m \quad \text{and} \\ R \;&=\; 2\operatorname{conv}\left(\bigcup_{|J|\le s} E_J \cap B_2^m\right) = 2\operatorname{conv}\left(E \cap B_2^m\right). \end{aligned}$$

Any vector $x \in \mathbb{R}^m$ satisfies:

$$\left\|x_{T_0^c}\right\|_\infty \le \|x_{T_0}\|_1 / s \quad \le \quad \frac{\|x_{T_0}\|_2}{\sqrt{s}}. \tag{K.3}$$

It follows that for any $\rho > 0$, $s \geq 1$ and for all $z \in L$, we have the i^{th} largest coordinate in absolute value in z is at most $\sqrt{s}/i$, and

$$\begin{aligned}\sup_{z\in L} \langle x, z\rangle &\leq \max_{\|z\|_2\leq\rho} \langle x_{T_0}, z\rangle + \max_{\|z\|_1\leq\sqrt{s}} \langle x_{T_0^c}, z\rangle \\ &\leq \rho\|x_{T_0}\|_2 + \|x_{T_0^c}\|_\infty \sqrt{s} \\ &\leq \|x_{T_0}\|_2 (\rho+1),\end{aligned}$$

where clearly $\max_{\|z\|_2\leq\rho} \langle x_{T_0}, z\rangle = \rho\sum_{i=1}^s (x_i^{*2})^{1/2}$. And denote by $S^J := S^{m-1} \cap E_J$,

$$\begin{aligned}\sup_{z\in R} \langle x, z\rangle &= (1+\rho) \max_{J:|J|\leq s} \max_{z\in S^J} \langle x, z\rangle \\ &= (1+\rho)\|x_{T_0}\|_2,\end{aligned}$$

given that for a convex function $\langle x, z\rangle$, the maximum happens at an extreme point; and in this case, it happens for z such that z is supported on T_0, such that $z_{T_0} = \frac{x_{T_0}}{\|x_{T_0}\|_2}$ and $z_{T_0^c} = 0$. □

Lemma 37. *Let $1/5 > \delta > 0$. Let $E = \cup_{|J|\leq s} E_J$ for $0 < s < m/2$ and $k_0 > 0$. Let Δ be a $m \times m$ matrix such that*

$$\left|u^T \Delta v\right| \leq \delta, \quad \forall u, v \in E \cap S^{m-1} \tag{K.4}$$

Then for all $v \in \left(\sqrt{s}B_1^m \cap B_2^m\right)$,

$$\left|v^T \Delta v\right| \leq 4\delta. \tag{K.5}$$

Proof. First notice that

$$\max_{v\in\left(\sqrt{s}B_1^m\cap B_2^m\right)} \left|v^T\Delta v\right| \leq \max_{w,u\in\left(\sqrt{s}B_1^m\cap B_2^m\right)} \left|w^T\Delta u\right|. \tag{K.6}$$

Now that we have decoupled u and w on the RHS of (K.6), we first fix u.

Then for any fixed $u \in S^{m-1}$ and matrix $\Delta \in \mathbb{R}^{m\times m}$, $f(w) = \left|w^T\Delta u\right|$ is a convex function of w, and hence for $w \in \left(\sqrt{s}B_1^m \cap B_2^m\right) \subset 2\,\mathrm{conv}\left(\bigcup_{|J|\leq s} E_J \cap S^{m-1}\right)$,

$$\begin{aligned}\max_{w\in\left(\sqrt{s}B_1^m\cap B_2^m\right)} \left|w^T\Delta u\right| &\leq 2 \max_{w\in\mathrm{conv}\,(E\cap S^{m-1})} \left|w^T\Delta u\right| \\ &= 2 \max_{w\in E\cap S^{m-1}} \left|w^T\Delta u\right|,\end{aligned}$$

where the maximum occurs at an extreme point of the set $\mathrm{conv}(E \cap S^{m-1})$ because of the convexity of the function $f(w)$.

Clearly the RHS of (K.6) is bounded by

$$\begin{aligned}\max_{u,w\in\left(\sqrt{s}B_1^m\cap B_2^m\right)}\left|w^T\Delta u\right| &= \max_{u\in\left(\sqrt{s}B_1^m\cap B_2^m\right)}\max_{w\in\left(\sqrt{s}B_1^m\cap B_2^m\right)}\left|w^T\Delta u\right| \\ &\le 2\max_{u\in\left(\sqrt{s}B_1^m\cap B_2^m\right)}\max_{w\in\left(E\cap S^{m-1}\right)}\left|w^T\Delta u\right| \\ &= 2\max_{u\in\left(\sqrt{s}B_1^m\cap B_2^m\right)} g(u),\end{aligned}$$

where the function g of $u\in\left(\sqrt{s}B_1^m\cap B_2^m\right)$ is defined as

$$g(u)=\max_{w\in\left(E\cap S^{m-1}\right)}\left|w^T\Delta u\right|;$$

$g(u)$ is convex since it is the maximum of a function $f_w(u):=\left|w^T\Delta u\right|$ which is convex in u for each $w\in(E\cap S^{m-1})$.

Thus we have for

$$u\in(\sqrt{s}B_1^m\cap B_2^m)\subset 2\,\mathrm{conv}\left(\bigcup_{|J|\le s}E_J\cap S^{m-1}\right)=:2\,\mathrm{conv}\left(E\cap S^{m-1}\right),$$

$$\begin{aligned}\max_{u\in\left(\sqrt{s}B_1^m\cap B_2^m\right)} g(u) &\le 2\max_{u\in\mathrm{conv}\,(E\cap S^{m-1})} g(u) \\ &= 2\max_{u\in E\cap S^{m-1}} g(u) \qquad \text{(K.7)} \\ &= 2\max_{u\in E\cap S^{m-1}}\max_{w\in E\cap S^{m-1}}\left|w^T\Delta u\right|\le 4\delta, \qquad \text{(K.8)}\end{aligned}$$

where (K.7) holds given that the maximum occurs at an extreme point of the set $\mathrm{conv}\,(E\cap B_2^m)$, because of the convexity of the function $g(u)$. □

Corollary 38. *Suppose all conditions in Lemma 37 hold. Then* $\forall v\in\mathrm{Cone}(s)$,

$$\left|v^T\Delta v\right| \le 4\delta\left\|v\right\|_2^2. \qquad \text{(K.9)}$$

Proof. It is sufficient to show that $\forall v\in\mathrm{Cone}(s)\cap S^{m-1}$,

$$\left|v^T\Delta v\right| \le 4\delta.$$

Denote by $\mathrm{Cone}:=\mathrm{Cone}(s)$. Clearly this set of vectors satisfy:

$$\mathrm{Cone}\cap S^{m-1}\subset\left(\sqrt{s}B_1^m\cap B_2^m\right).$$

Thus (K.9) follows from (K.5). □

Remark K.1. *Suppose we relax the definition of* $\mathrm{Cone}(s)$ *to be:*

$$\mathrm{Cone}(s):=\{v:\left\|v\right\|_1\le 2\sqrt{s}\left\|v\right\|_2\}.$$

Clearly, $\mathrm{Cone}(s,1)\subset\mathrm{Cone}(s)$. *given that* $\forall u\in\mathrm{Cone}(s,1)$, *we have*

$$\left\|u\right\|_1\le 2\left\|u_{T_0}\right\|_1\le 2\sqrt{s}\left\|u_{T_0}\right\|_2\le 2\sqrt{s}\left\|u\right\|_2.$$

Lemma 39. *Suppose all conditions in Lemma 37 hold. Then for all $v \in \mathbb{R}^m$,*

$$\left|v^T \Delta v\right| \le 4\delta(\|v\|_2^2 + \frac{1}{s}\|v\|_1^2). \tag{K.10}$$

Proof. The lemma follows given that $\forall v \in \mathbb{R}^m$, one of the following must hold:

$$\text{if } v \in \text{Cone}(s) \quad \left|v^T \Delta v\right| \le 4\delta \|v\|_2^2; \tag{K.11}$$

$$\text{otherwise} \quad \left|v^T \Delta v\right| \le \frac{4\delta}{s}\|v\|_1^2, \tag{K.12}$$

leading to the same conclusion in (K.10).

We have shown (K.11) in Lemma 37. Let $\text{Cone}(s)^c$ be the complement set of $\text{Cone}(s)$ in $\mathbb{R}^m$. That is, we focus now on the set of vectors such that

$$\text{Cone}(s)^c := \{v : \|v\|_1 \ge \sqrt{s}\|v\|_2\}$$

and show that for $u = \sqrt{s}\frac{v}{\|v\|_1}$,

$$\frac{\left|v^T \Delta v\right|}{\|v\|_1^2} := \frac{1}{s}\left|u^T \Delta u\right| \le \frac{1}{s}\delta.$$

Now, the last inequality holds by Lemma 37 given that

$$u \in (\sqrt{s}B_1^m \cap B_2^m) \subset 2\,\text{conv}\left(\bigcup_{|J|\le s} E_J \cap B_2^m\right)$$

and thus

$$\frac{\left|v^T \Delta v\right|}{\|v\|_1^2} \le \frac{1}{s}\sup_{u \in \sqrt{s}B_1^m \cap B_2^m}\left|u^T \Delta u\right| \le \frac{1}{s}4\delta. \qquad \square$$

Appendix L: Proof of Corollary 25

First we show that for all $v \in \mathbb{R}^m$, (L.1) holds. It is sufficient to check that the condition (K.4) in Lemma 37 holds. Then, (L.1) follows from Lemma 39: for $v \in \mathbb{R}^m$,

$$\left|v^T \Delta v\right| \le 4\delta(\|v\|_2^2 + \frac{1}{k}\|v\|_1^2) \le \frac{3}{8}\lambda_{\min}(A)(\|v\|_2^2 + \frac{1}{k}\|v\|_1^2). \tag{L.1}$$

The Lower and Upper RE conditions thus immediately follow. The Corollary is thus proved. $\square$

Appendix M: Proof of Theorem 26

We first state the following preliminary results in Lemmas 40 and 41; their proofs appear in Section O. Throughout this section, the choice of $C = C_0/\sqrt{c'}$ satisfies the conditions on C in Lemmas 40 and 41, where recall $\min\{C_0, C_0^2\} \geq 4/c$ for c as defined in Theorem 31. For a set $J \subset \{1, \ldots, m\}$, denote $F_J = A^{1/2}E_J$, where recall $E_J = \mathsf{span}\{e_j : j \in J\}$. Let Z be an $n \times m$ random matrix with independent entries Z_{ij} satisfying $\mathbb{E}Z_{ij} = 0$, $1 = \mathbb{E}Z_{ij}^2 \leq \|Z_{ij}\|_{\psi_2} \leq K$. Let Z_1, Z_2 be independent copies of Z.

Lemma 40. *Suppose all conditions in Theorem 26 hold. Let*

$$E = \bigcup_{|J|=k} E_J \cap S^{m-1}.$$

Suppose that for some $c' > 0$ and $\varepsilon \leq \frac{1}{C}$, where $C = C_0/\sqrt{c'}$,

$$r(B) := \frac{\mathrm{tr}(B)}{\|B\|_2} \quge c'kK^4 \frac{\log(3em/k\varepsilon)}{\varepsilon^2}. \tag{M.1}$$

Then for all vectors $u, v \in E \cap S^{m-1}$, on event $\mathcal{B}_1$,

$$\left|u^T Z^T BZv - \mathbb{E}u^T Z^T BZv\right| \leq 4C\varepsilon \mathrm{tr}(B),$$

where $\mathbb{P}(\mathcal{B}_1) \geq 1 - 2\exp\left(-c_2\varepsilon^2 \frac{\mathrm{tr}(B)}{K^4\|B\|_2}\right)$ for $c_2 \geq 2$.

Lemma 41. *Suppose that $\varepsilon \leq 1/C$, where C is as defined in Lemma 40. Suppose that (M.1) holds. Let*

$$E = \bigcup_{|J|=k} E_J \quad \text{and} \quad F = \bigcup_{|J|=k} F_J. \tag{M.2}$$

Then on event $\mathcal{B}_2$, where $\mathbb{P}(\mathcal{B}_2) \geq 1 - 2\exp\left(-c_2\varepsilon^2 \frac{\mathrm{tr}(B)}{K^4\|B\|_2}\right)$ for $c_2 \geq 2$, we have for all vectors $u \in E \cap S^{m-1}$ and $w \in F \cap S^{m-1}$,

$$\left|w^T Z_1^T B^{1/2} Z_2 u\right| \leq \frac{C\varepsilon \mathrm{tr}(B)}{(1-\varepsilon)^2 \|B\|_2^{1/2}} \leq 4C\varepsilon \mathrm{tr}(B)/\|B\|_2^{1/2}.$$

In fact, the same conclusion holds for all $y, w \in F \cap S^{m-1}$; and in particular, for $B = I$, we have the following.

Corollary 42. *Suppose all conditions in Lemma 40 hold. Suppose that $F = A^{1/2}E$ for E as defined in Lemma 40. Let*

$$n \quge c'kK^4 \frac{\log(3em/k\varepsilon)}{\varepsilon^2}. \tag{M.3}$$

Then on event $\mathcal{B}_3$, where $\mathbb{P}(\mathcal{B}_3) \geq 1 - 2\exp\left(-c_2\varepsilon^2 n \frac{1}{K^4}\right)$, we have for all vectors $w, y \in F \cap S^{m-1}$ and $\varepsilon \leq 1/C$ for C is as defined in Lemma 40,

$$\left|y^T\left(\tfrac{1}{n}Z^T Z - I\right)w\right| \leq 4C\varepsilon. \tag{M.4}$$

We prove Lemmas 40 and 41 and Corollary 42 in Section O. We are now ready to prove Theorem 26.

Proof of Theorem 26. Let

$$\begin{aligned}\Delta := \widehat{\Gamma}_A - A &:= \tfrac{1}{n}X^TX - \tfrac{1}{n}\widehat{\mathrm{tr}}(B)I_m - A \\ &= (\tfrac{1}{n}X_0^TX_0 - A) + \tfrac{1}{n}\big(W^TX_0 + X_0^TW\big) + \tfrac{1}{n}\big(W^TW - \widehat{\mathrm{tr}}(B)I_m\big),\end{aligned}$$

where recall $X_0 = Z_1A^{1/2}$. Notice that

$$\begin{aligned}\left|u^T(\widehat{\Gamma}_A - A)v\right| &= \left|u^T(X^TX - \widehat{\mathrm{tr}}(B)I_m - A)v\right| \\ &\le \left|u^T(\tfrac{1}{n}X_0^TX_0 - A)v\right| + \left|u^T\tfrac{1}{n}(W^TX_0 + X_0^TW)v\right| + \\ &\quad \left|u^T(\tfrac{1}{n}W^TW - \tfrac{\widehat{\mathrm{tr}}(B)}{f}I_m)v\right| \\ &\le \left|u^TA^{1/2}\tfrac{1}{n}Z_1^TZ_1A^{1/2}v - u^TAv\right| + \left|u^T\tfrac{1}{n}(W^TX_0 + X_0^TW)v\right| \\ &\quad + \left|u^T(\tfrac{1}{n}Z_2^TBZ_2 - \tau_BI_m)v\right| + \tfrac{1}{n}\left|\widehat{\mathrm{tr}}(B) - \mathrm{tr}(B)\right|\left|u^Tv\right| \\ &=: I + II + III + IV.\end{aligned}$$

For $u \in E \cap S^{m-1}$, define $h(u) := \frac{A^{1/2}u}{\left\|A^{1/2}u\right\|_2}$. The conditions in (M.1) and (M.3) hold for k.

We first bound the middle term as follows. Fix $u, v \in E \cap S^{m-1}$. Then on event $\mathcal{B}_2$, for $\Upsilon = Z_1^TB^{1/2}Z_2$,

$$\begin{aligned}\left|u^T(W^TX_0 + X_0^TW)v\right| &= \left|u^TZ_2^TB^{1/2}Z_1A^{1/2}v + u^TA^{1/2}Z_1^TB^{1/2}Z_2v\right| \\ &\le \left|u^T\Upsilon^Th(v)\right|\left\|A^{1/2}v\right\|_2 + \left|h(u)^T\Upsilon v\right|\left\|A^{1/2}u\right\|_2 \\ &\le 2\max_{w\in F\cap S^{m-1}, v\in E\cap S^{m-1}}\left|w^T\Upsilon v\right|\rho_{\max}^{1/2}(k,A) \\ &\le 8C\varepsilon\mathrm{tr}(B)\left(\frac{\rho_{\max}(k,A)}{\|B\|_2}\right)^{1/2}.\end{aligned}$$

We now use Lemma 40 to bound both I and III. We have for C as defined in Lemma 40, on event $\mathcal{B}_1 \cap \mathcal{B}_3$,

$$\left|u^T(Z_2^TBZ_2 - \mathrm{tr}(B)I_m)v\right| \le 4C\varepsilon\mathrm{tr}(B).$$

Moreover, by Corollary 42, we have on event $\mathcal{B}_3$, for all $u, v \in E \cap S^{m-1}$,

$$\begin{aligned}\left|u^T(\tfrac{1}{n}X_0^TX_0 - A)v\right| &= \left|u^TA^{1/2}Z^TZA^{1/2}v - u^TAv\right| \\ &= \left|h(u)^T(\tfrac{1}{n}Z^TZ - I)h(v)\right|\left\|A^{1/2}u\right\|_2\left\|A^{1/2}v\right\|_2 \\ &\le \tfrac{1}{n}\max_{w,y\in F\cap S^{m-1}}\left|w^T(Z^TZ - I)y\right|\rho_{\max}(k,A) \\ &\le 4C\varepsilon\rho_{\max}(k,A).\end{aligned}$$

Thus we have on event $\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$ and for $\tau_B := \mathrm{tr}(B)/n$,

$$\begin{aligned} I + II + III &\leq 4C\varepsilon \left(\rho_{\max}(k, A) + 2\tau_B \left(\frac{\rho_{\max}(k, A)}{\|B\|_2} \right)^{1/2} + \tau_B \right) \\ &\leq 8C\varepsilon \left(\tau_B + \rho_{\max}(k, A) \right). \end{aligned}$$

On event $\mathcal{B}_6$, we have for D_1 as defined in Lemma 5,

$$IV \leq |\widehat{\tau}_B - \tau_B| \leq 2C_0 D_1 K^2 \sqrt{\frac{\log m}{mn}}.$$

The theorem thus holds by the union bound. □

Appendix N: Proof of Lemma 32

Lemma 43 is a well-known fact.

Lemma 43. *Let* $A_{uw} := (u \otimes w) \otimes A$, *where* $u, w \in \mathbb{S}^{m-1}$ *for* $m \geq 2$. *Then* $\|A_{uw}\|_2 \leq \|A\|_2$ *and* $\|A_{uw}\|_F \leq \|A\|_F$.

Proof of Lemma 32. Let $z_1, \ldots, z_n, z'_1, \ldots, z'_n \in \mathbb{R}^m$ be the row vectors Z_1, Z_2 respectively. Notice that we can write the quadratic form as follows:

$$\begin{aligned} u^T Z_1 A^{1/2} Z_2^T w &= \sum_{i,j=1,m} u_i w_j z_i A^{1/2} z'_j \\ &= \mathrm{vec}\left\{ \mathrm{Z}_1^{\mathrm{T}} \right\}^{\mathrm{T}} \left((\mathrm{u} \otimes \mathrm{w}) \otimes \mathrm{A}^{1/2} \right) \mathrm{vec}\left\{ \mathrm{Z}_2^{\mathrm{T}} \right\} \\ &=: \mathrm{vec}\left\{ \mathrm{Z}_1^{\mathrm{T}} \right\}^{\mathrm{T}} \mathrm{A}_{\mathrm{uw}}^{1/2} \mathrm{vec}\left\{ \mathrm{Z}_2^{\mathrm{T}} \right\}, \\ u^T Z A Z^T w &= \mathrm{vec}\left\{ \mathrm{Z}^{\mathrm{T}} \right\}^{\mathrm{T}} \left((\mathrm{u} \otimes \mathrm{w}) \otimes \mathrm{A} \right) \mathrm{vec}\left\{ \mathrm{Z}^{\mathrm{T}} \right\} \\ &=: \mathrm{vec}\left\{ \mathrm{Z}^{\mathrm{T}} \right\}^{\mathrm{T}} \mathrm{A}_{\mathrm{uw}} \mathrm{vec}\left\{ \mathrm{Z}^{\mathrm{T}} \right\} \end{aligned}$$

where clearly by independence of Z_1, Z_2,

$$\begin{aligned} \mathbb{E}\mathrm{vec}\left\{ \mathrm{Z}_1^{\mathrm{T}} \right\}^{\mathrm{T}} \left((\mathrm{u} \otimes \mathrm{w}) \otimes \mathrm{A}^{1/2} \right) \mathrm{vec}\left\{ \mathrm{Z}_2^{\mathrm{T}} \right\} &= 0, \quad \text{and} \\ \mathbb{E}\mathrm{vec}\left\{ \mathrm{Z} \right\}^{\mathrm{T}} \left((\mathrm{u} \otimes \mathrm{u}) \otimes \mathrm{A} \right) \mathrm{vec}\left\{ \mathrm{Z} \right\} &= \mathrm{tr}\big((u \otimes u) \otimes A \big) = \mathrm{tr}(A). \end{aligned}$$

Thus we invoke (C.1) and Lemma 43 to show the concentration bounds on event $\{ |u^T Z_1 A^{1/2} Z_2^T w| > t \}$:

$$\begin{aligned} \mathbb{P}\left(\left| u^T Z_1 A^{1/2} Z_2^T w \right| > t \right) &\leq 2 \exp\left(- \min\left(\frac{t^2}{K^4 \left\| A_{uw}^{1/2} \right\|_F^2}, \frac{t}{K^2 \left\| A_{uw}^{1/2} \right\|_2} \right) \right) \\ &\leq 2 \exp\left(- \min\left(\frac{t^2}{K^4 \mathrm{tr}(A)}, \frac{t}{K^2 \left\| A^{1/2} \right\|_2} \right) \right). \end{aligned}$$

Similarly, we have by Theorem 31 and Lemma 43,

$$\begin{aligned}
&\mathbb{P}\left(\left|u^T ZAZ^T w - \mathbb{E}u^T ZAZ^T w\right| > t\right) \\
&\quad\leq\quad 2\exp\left(-c\min\left(\frac{t^2}{K^4\left\|A_{uw}\right\|_F^2}, \frac{t}{K^2\left\|A_{uw}\right\|_2}\right)\right) \\
&\quad\leq\quad 2\exp\left(-c\min\left(\frac{t^2}{K^4\left\|A\right\|_F^2}, \frac{t}{K^2\left\|A\right\|_2}\right)\right).
\end{aligned}$$

The Lemma thus holds. □

Appendix O: Proof of Lemmas 40 and 41 and Corollary 42

Throughout the following proof, we denote by $r(B) = \frac{\mathrm{tr}(B)}{\|B\|_2}$. Let $\varepsilon \leq \frac{1}{C}$ where C is large enough so that $cc'C^2 \geq 4$, and hence the choice of $C = C_0/\sqrt{c'}$ satisfies our need.

Proof of Lemma 40. First we prove concentration bounds for all pairs of $u, v \in \Pi'$, where $\Pi' \subset \mathbb{S}^{m-1}$ is an ε-net of E. Let $t = CK^2\varepsilon\mathrm{tr}(B)$. We have by Lemma 32, and the union bound,

$$\begin{aligned}
&\mathbb{P}\left(\exists u, v \in \Pi',\ \left|u^T Z^T BZv - \mathbb{E}u^T Z^T BZv\right| > t\right) \\
&\quad\leq\quad 2\left|\Pi'\right|^2 \exp\left[-c\min\left(\frac{t^2}{K^4\left\|B\right\|_F^2}, \frac{t}{K^2\left\|B\right\|_2}\right)\right] \\
&\quad\leq\quad 2\left|\Pi'\right|^2 \exp\left[-c\min\left(C^2, \frac{CK^2}{\varepsilon}\right)\frac{\varepsilon^2 r(B)}{K^4}\right] \\
&\quad\leq\quad 2\exp\left(-c_2\varepsilon^2 r(B)/K^4\right),
\end{aligned}$$

where we use the fact that $\|B\|_F^2 \leq \|B\|_2 \mathrm{tr}(B)$ and

$$|\Pi'| \leq \binom{m}{k}(3/\varepsilon)^k \leq \exp(k\log(3em/k\varepsilon)),$$

while

$$\begin{aligned}
c\min\left(C^2, \frac{CK^2}{\varepsilon}\right)\varepsilon^2\frac{r(B)}{K^4} \quad&=\quad cC^2\varepsilon^2\frac{\mathrm{tr}(B)}{\|B\|_2 K^4} \\
&\geq\quad cC_0^2 k\log\left(\frac{3em}{k\varepsilon}\right) \geq 4k\log\left(\frac{3em}{k\varepsilon}\right).
\end{aligned}$$

Denote by $\mathcal{B}_2$ the event such that for $\Lambda := \frac{1}{\mathrm{tr}(B)}(Z^T BZ - I)$,

$$\sup_{u,v\in\Pi'}\left|v^T\Lambda u\right| \quad\leq\quad C\varepsilon =: r'_{k,n}$$

holds. A standard approximation argument shows that under $\mathcal{B}_2$ and for $\varepsilon \leq 1/2$,

$$\sup_{x,y\in\mathbb{S}^{m-1}\cap E} \left|y^T \Lambda x\right| \leq \frac{r'_{k,n}}{(1-\varepsilon)^2} \leq 4C\varepsilon. \tag{O.1}$$

The lemma is thus proved. □

Proof of Lemma 41. By Lemma 32, we have for $t = C\varepsilon \mathrm{tr}(B)/\left\|B\right\|_2^{1/2}$ for $C = C_0/\sqrt{c'}$,

$$\begin{aligned}
\mathbb{P}\left(\left|w^T Z_1^T B^{1/2} Z_2 u\right| > t\right) &\leq \exp\left(-c\min\left(\frac{C^2 \frac{\mathrm{tr}(B)^2}{\|B\|_2}\varepsilon^2}{K^4 \mathrm{tr}(B)}, \frac{C\varepsilon \mathrm{tr}(B)}{K^2\left\|B\right\|_2}\right)\right) \\
&\leq 2\exp\left(-c\min\left(\frac{C^2\varepsilon^2 r_B}{K^4}, \frac{C\varepsilon r_B}{K^2}\right)\right) \\
&\leq 2\exp\left(-c\min\left(C^2, \frac{CK^2}{\varepsilon}\right)\varepsilon^2 r_B/K^4\right).
\end{aligned}$$

Choose an ε-net $\Pi' \subset S^{m-1}$ such that

$$\Pi' = \bigcup_{|J|=k} \Pi'_J \quad \text{where} \quad \Pi'_J \subset E_J \cap S^{m-1} \tag{O.2}$$

is an ε-net for $E_J \cap S^{m-1}$ and

$$|\Pi'| \leq \binom{m}{k}(3/\varepsilon)^k \leq \exp(k\log(3em/k\varepsilon)).$$

Similarly, choose ε-net Π of $F \cap S^{m-1}$ of size at most $\exp(k\log(3em/k\varepsilon))$. By the union bound and Lemma 32, and for $K^2 \geq 1$,

$$\begin{aligned}
&\mathbb{P}\left(\exists w \in \Pi, u \in \Pi' \ s.t. \ \left|w^T Z_1^T B^{1/2} Z_2 u\right| \geq C\varepsilon \mathrm{tr}(B)/\|B\|_2^{1/2}\right) \\
&\quad\leq |\Pi'|\,|\Pi|\, 2\exp\left(-c\min\left(CK^2/\varepsilon, C^2\right)\varepsilon^2 r_B/K^4\right) \\
&\quad\leq \exp\left(2k\log(3em/k\varepsilon)\right) 2\exp\left(-cC^2\varepsilon^2 r_B/K^4\right) \\
&\quad\leq 2\exp\left(-c_2\varepsilon^2 r_B/K^4\right),
\end{aligned}$$

where C is large enough such that $cc'C^2 := C' > 4$ and for $\varepsilon \leq \frac{1}{C}$,

$$c\min\left(CK^2/\varepsilon, C^2\right)\varepsilon^2 \frac{\mathrm{tr}(B)}{\|B\|_2 K^4} \geq C'k\log(3em/k\varepsilon) \geq 4k\log(3em/k\varepsilon).$$

Denote by $\Upsilon := Z_1^T B^{1/2} Z_2$. A standard approximation argument shows that if

$$\sup_{w\in\Pi, u\in\Pi'} \left|w^T \Upsilon u\right| \leq C\varepsilon \frac{\mathrm{tr}(B)}{\|B\|_2^{1/2}} =: r_{k,n},$$

an event which we denote by $\mathcal{B}_2$, then for all $u \in E$ and $w \in F$,

$$\left| w^T Z_1^T B^{1/2} Z_2 u \right| \leq \frac{r_{k,n}}{(1-\varepsilon)^2}. \tag{O.3}$$

The lemma thus holds for $c_2 \geq C'/2 \geq 2$. □

Proof of Corollary 42. Clearly (M.4) implies that (M.1) holds for $B = I$. Clearly (M.3) holds following the analysis of Lemma 40 by setting $B = I$, while replacing event $\mathcal{B}_1$ with $\mathcal{B}_3$, which denotes an event such that

$$\sup_{u,v \in \Pi} \tfrac{1}{n} \left| v^T (Z^T Z - I) u \right| \quad \leq \quad C\varepsilon.$$

The rest of the proof follows by replacing E with F everywhere. The corollary thus holds. □

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