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On consistency of least square estimators in the simple linear EV model with negatively orthant dependent errors*

Xuejun Wang[†,‡] **and Shuhe Hu**[§]

School of Mathematical Sciences, Anhui University, Hefei, 230601, P.R. China
e-mail: wxjahdx@126.com; hushuhe@263.net

Abstract: In this paper, we mainly study the asymptotic properties of least square (LS, for short) estimators in the simple linear errors-in-variables (EV, for short) regression model with negatively orthant dependent (NOD, for short) errors. Under some suitable conditions, the strong consistency, weak consistency and complete consistency of the LS estimators in the EV regression model with NOD errors are obtained, which generalize or improve the corresponding ones for independent random variables and negatively associated random variables in some sense.

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†Corresponding author.

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1. Introduction

It is well known that the simple errors-in-variables (EV, for short) regression model was proposed by Deaton [1] to correct for the effects of sampling error and is somewhat more practical than the ordinary regression model. For more details about the EV regression model, one can refer to Fuller [2], Fusek and Fusková [3], Carroll et al. [4], Hslao et al. [5], and so on.

In this article, we consider the following simple linear EV model:

$$\eta_i = \theta + \beta x_i + \varepsilon_i, \qquad \xi_i = x_i + \delta_i, \quad 1 \le i \le n, \tag{1.1}$$

where θ, β are unknown parameters, $x_1, x_2, \cdots, x_n$ are unknown constants, $(\varepsilon_1, \delta_1)$, $(\varepsilon_2, \delta_2)$, $\cdots$, $(\varepsilon_n, \delta_n)$ are random vectors and ξ_i, η_i, $i = 1, 2, \cdots, n$ are observable. From (1.1) we have

$$\eta_i = \theta + \beta \xi_i + \nu_i, \qquad \nu_i = \varepsilon_i - \beta \delta_i, \quad 1 \le i \le n. \tag{1.2}$$

Consider formally (1.2) as a usual regression model of η_i on ξ_i, we get the least square (LS, for short) estimators of θ and β as

$$\hat{\beta}_n = \frac{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)(\eta_i - \bar{\eta}_n)}{\sum_{i=1}^n (\xi_i - \bar{\xi}_n)^2}, \qquad \hat{\theta}_n = \bar{\eta}_n - \hat{\beta}_n \bar{\xi}_n, \tag{1.3}$$

where $\bar{\xi}_n = n^{-1} \sum_{i=1}^n \xi_i$, and other similar notations, such as $\bar{\eta}_n$, $\bar{\delta}_n$ and $\bar{x}_n$, are defined in the same way.

The limiting behaviors for the LS estimators of θ and β in the EV model have been studied by many authors since the EV model was proposed by Deaton [1]. For the case that the errors are sequences of independent random variables, one can refer to [6]-[13]. Under the case that the errors are sequences of dependent random variables, Fazekas and Kukush [14] studied the asymptotic properties of an estimator in nonlinear functional EV models with α-mixing error terms; Miao et al. [15] studied the strong consistency of LS estimators in the EV regression model with negatively associated (NA, for short) errors; Miao et al. [16] established the consistency of LS estimators in the EV regression model with martingale difference errors; Wang [17] studied the moderate deviation principles for the least-square estimators of the unknown parameters in EV regression models with α-mixing errors, and so on.

We are interested in the results of Liu and Chen [11]. For the case $(\varepsilon_1, \delta_1)$, $(\varepsilon_2, \delta_2)$, $\cdots$ are independent and identically distributed random variables, Liu and Chen [11] provided the sufficient and necessary conditions for $\hat{\beta}_n$ being strong and weak consistent estimator of β, and the sufficient and necessary conditions for $\hat{\theta}_n$ being a weak consistent estimator of θ.

Miao et al. [15] generalized the results of Liu and Chen [11] for independent and identically distributed random variables to the case of NA setting, and proved that for some $\tau > 0$,

$$\frac{\sqrt{S_n}}{n^\tau}(\hat{\beta}_n - \beta) \to 0 \quad a.s. \tag{1.4}$$

where $S_n = \sum_{i=1}^n (x_i - \bar{x}_n)^2$. However, the following sufficient conditions are needed:

(i) $\{\varepsilon_i, i \geq 1\}$ and $\{\delta_i, i \geq 1\}$ are both strictly stationary sequences of NA random variables and independent with each other such that $E\varepsilon_1 = 0$ and $E\delta_1 = 0$;

(ii) $E|\varepsilon_1|^q < \infty$ and $E|\delta_1|^q < \infty$ for some $q \geq 2$;

(iii) there exists some $\tau > 0$ such that

$$\max_{1\le i\le n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n} n^{\tau - 1/q}} = O(1), \qquad \frac{n^\tau}{\sqrt{S_n}} = O(1); \tag{1.5}$$

(iv) for the case $q = 2$, assume that

$$\frac{\sqrt{S_n}}{n^{1-\tau+\gamma}} \to \infty, \quad \text{for some } \gamma > 0; \tag{1.6}$$

for the case $q > 2$, assume that

$$\frac{n^{1-\tau}}{\sqrt{S_n}} \to 0. \tag{1.7}$$

It is easily checked that (1.6) implies (1.7). So we want to ask that whether condition (1.6) could be replaced by (1.7) for the case $q = 2$. Furthermore, the condition of strict stationarity seems too strong. Could it be replaced by a weaker condition, such as identical distribution? The answers are positive. Please see Theorem 2.1 in Section 2.

On the other hand, it is also very desirable to extend the result of Miao et al. [15] for NA setting to a more general setting. The main purpose of the paper is to investigate the strong consistency of LS estimators in the EV regression model with negatively orthant dependent (NOD, for short) errors, which generalizes and improves the corresponding one of Miao et al. [15]. In addition, we will study the weak consistency and complete consistency of LS estimators in the EV regression model with NOD errors, which were not considered in Miao et al. [15].

The paper is organized as follows: main results of the paper are presented in Section 2, including the strong consistency, weak consistency and complete consistency of LS estimators in the EV regression model with NOD errors. Some basic properties for NOD random variables are provided in Section 3. In Section 4, we provide the proofs of the main results.

Throughout the paper, let C be a positive constant not depending on n, which may be different in various places. $a_n = O(b_n)$ stands for $|a_n| \leq C|b_n|$ for all $n \geq 1$ and some $C > 0$, and "$\xrightarrow{P}$" stands for convergence in probability. Denote $\log x = \ln \max(x, e)$, $x^+ = \max(X, 0)$ and $x^- = \max(-X, 0)$.

2. Main results

Before we state the main results, we introduce the concept of negatively orthant dependence as follows.

Definition 2.1. *A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively orthant dependent (NOD, for short) if*

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1, x_2, \cdots, x_n \in \mathbf{R}$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise NOD if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is NOD.

The concept of NOD random variables was introduced by Joag-Dev and Proschan [18]. Obviously, independent random variables are NOD. Joag-Dev and Proschan [18] pointed out that NA random variables are NOD. They also presented an example in which $X = (X_1, X_2, X_3, X_4)$ possesses NOD, but does not possess NA. So we can see that NOD is weaker than NA. A number of limit theorems for NOD random variables have been established by many authors. We refer to [19]-[28] for instance.

Now, we present the main results of the paper.

The model (1.1) to be studied can be exactly described as follows:

$$\begin{cases} \eta_i = \theta + \beta x_i + \varepsilon_i, \ \xi_i = x_i + \delta_i, & 1 \leq i \leq n; \\ E\varepsilon_i = E\delta_i = 0, & 1 \leq i \leq n. \end{cases} \tag{2.1}$$

Here ξ_i, η_i, $i = 1, 2, \cdots, n$ are observable, while θ, β are unknown parameters, and $x_1, x_2, \cdots, x_n$ are unknown constants. In what follows, we assume that the two error sequences $\{\varepsilon_i, i \geq 1\}$ and $\{\delta_i, i \geq 1\}$ are independent with each other, where $\{\varepsilon_i, i \geq 1\}$ and $\{\delta_i, i \geq 1\}$ are both mean zero NOD random variables with identical distribution. Denote $S_n = \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$ for each $n \geq 1$.

Based on the notations above, we can get that

$$\hat{\beta}_n - \beta = \frac{\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)\varepsilon_i + \sum_{i=1}^{n} (x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) - \beta \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2}{\sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2} \tag{2.2}$$

and

$$\hat{\theta}_n - \theta = (\beta - \hat{\beta}_n)\bar{x}_n + (\beta - \hat{\beta}_n)\bar{\delta}_n + \bar{\varepsilon}_n - \beta\bar{\delta}_n. \tag{2.3}$$

These relations above will play an important role to prove the main results of this paper.

2.1. Strong consistency

In this subsection, we will present the strong consistency of LS estimators $\hat{\beta}_n$ and $\hat{\theta}_n$.

Theorem 2.1. *Under the model (2.1), let $E|\varepsilon_1|^{2p} < \infty$ and $E|\delta_1|^{2p} < \infty$ for some $p \geq 1$. Suppose that there exists some $\tau > 0$ such that*

$$\max_{1\leq i\leq n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n} n^{\tau - 1/p}} = O(1), \tag{2.4}$$

$$\frac{n^\tau}{\sqrt{S_n}} = O(1) \tag{2.5}$$

and

$$\frac{n^{1-\tau}}{\sqrt{S_n}} \to 0. \tag{2.6}$$

Assume further that $\tau > 1/p - 1/2$ if $1 \leq p \leq 2$. Then

$$\frac{\sqrt{S_n}}{n^\tau}(\hat{\beta}_n - \beta) \to 0 \quad a.s.. \tag{2.7}$$

Remark 2.1. It is easily seen that (2.6) implies (2.5) when $0 < \tau \leq 1/2$ and (2.5) implies (2.6) when $\tau > 1/2$.

Remark 2.2. Combining Theorem 2.1 with the corresponding one of Miao et al. [15], we have the following generalizations or improvements:

(i) NA errors are extended to NOD errors;
(ii) in Miao et al. [15], the two sequences $\{\varepsilon_i, i \geq 1\}$ and $\{\delta_i, i \geq 1\}$ are assumed to be strictly stationary. However, in Theorem 2.1, we only need the assumption of identical distribution, which is weaker than strict stationarity;
(iii) for $p = 1$ ($q = 2$ in Miao et al. [15]), the condition (1.6) is weakened by (2.6).

Remark 2.3. The result of Theorem 2.1 generalizes the corresponding one of Liu and Chen [11].

Theorem 2.2. *Suppose that the conditions of Theorem 2.1 are satisfied. Assume further that there exist some $0 < \nu < \min(1 - 1/p, 1/2)$ and $p > 1$ such that*

$$\frac{n^{\tau+\nu}}{\sqrt{S_n}}|\bar{x}_n| = O(1). \tag{2.8}$$

Then

$$n^\nu(\hat{\theta}_n - \theta) \to 0 \quad a.s.. \tag{2.9}$$

2.2. Weak consistency

In this subsection, we will provide the weak consistency of LS estimators $\hat{\beta}_n$ and $\hat{\theta}_n$ under much weaker conditions than those in Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Under the model (2.1), assume that $E|\varepsilon_1|^{2p} < \infty$ and $E|\delta_1|^{2p} < \infty$ for some $p > 1$. Let $\{b_n, n \geq 1\}$ be a sequence of positive real numbers such that $b_n \to \infty$ as $n \to \infty$. If*

$$\frac{b_n}{\sqrt{S_n}} = O(1), \qquad \lim_{n\to\infty} \frac{\sqrt{S_n} b_n}{n} = \infty, \tag{2.10}$$

then

$$\frac{\sqrt{S_n}}{b_n}(\hat{\beta}_n - \beta) \xrightarrow{P} 0. \tag{2.11}$$

Theorem 2.4. *Suppose that the conditions of Theorem 2.3 are satisfied. Assume further that*

$$\frac{n\bar{x}_n^2}{b_n^2 S_n} \to 0, \qquad \frac{n^{3/2}|\bar{x}_n|}{b_n S_n} \to 0, \quad \text{as } n \to \infty. \tag{2.12}$$

Then

$$\frac{\sqrt{n}}{b_n}(\hat{\theta}_n - \theta) \xrightarrow{P} 0. \tag{2.13}$$

Remark 2.4. If we take $b_n = n^\tau$ for some $\tau > 0$, then the conditions (2.10) are equivalent to (2.5) and (2.6).

2.3. Complete consistency

Theorem 2.5. *Under the model (2.1), let $E|\varepsilon_1|^{4p} < \infty$ and $E|\delta_1|^{4p} < \infty$ for some $p > 1$. Suppose that there exists some $\tau > 0$ such that (2.4)–(2.6) hold. Then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P\left(\left|\frac{\sqrt{S_n}}{n^\tau}(\hat{\beta}_n - \beta)\right| > \varepsilon\right) < \infty,$$

i.e.

$$\frac{\sqrt{S_n}}{n^\tau}(\hat{\beta}_n - \beta) \to 0 \quad \textit{completely.} \tag{2.14}$$

Theorem 2.6. *Suppose that the conditions of Theorem 2.5 are satisfied. Assume further that there exists some $0 < \nu < \min(1 - 1/p, 1/2)$ such that (2.8) holds. Then*

$$n^\nu(\hat{\theta}_n - \theta) \to 0 \quad \textit{completely.} \tag{2.15}$$

Remark 2.5. Comparing Theorem 2.1 with Theorem 2.5, and Theorem 2.2 with Theorem 2.6, respectively, the strong consistency is improved to complete consistency.

3. Properties for NOD random variables

In order to prove the main results of the paper, we will present some basic properties for NOD random variables. The first one is well known and can be found, for example, in Taylor et al. [20].

Lemma 3.1. *Let random variables $X_1, X_2, \cdots, X_n$ be NOD.*

(i) If $f_1, f_2, \cdots, f_n$ are all nondecreasing (or all nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ are also NOD.
(ii) If $X_1, X_2, \cdots, X_n$ are nonnegative, then

$$E(X_1X_2 \cdots X_n) \le EX_1 \cdot EX_2 \cdots EX_n.$$

Lemma 3.2. *Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ be both NOD random variables. If random variables $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are independent, then $X_1 + Y_1, X_2 + Y_2, \cdots, X_n + Y_n$ are also NOD random variables.*

Proof. Denote $Z_k = X_k + Y_k$, $1 \le k \le n$. For any $z_1, z_2, \cdots, z_n \in \mathbf{R}$, we have

$$\begin{aligned} &P(Z_1 > z_1, Z_2 > z_2, \cdots, Z_n > z_n) \\ &= E\{E[I(Z_1 > z_1, Z_2 > z_2, \cdots, Z_n > z_n) \mid X_1, X_2, \cdots X_n]\} \\ &= E\{E[I(Y_1 > z_1 - X_1) \cdot I(Y_2 > z_2 - X_2) \cdots I(Y_n > z_n - X_n) \mid X_1, X_2, \cdots X_n]\}. \end{aligned} \tag{3.1}$$

Note that the indicator functions above are nondecreasing of $Y_1, Y_2, \cdots Y_n$. Since $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are independent, and $Y_1, Y_2, \cdots, Y_n$ are NOD random variables, we have by Lemma 3.1 (ii) that

$$\begin{aligned} &E\{E[I(Y_1 > z_1 - X_1) \cdot I(Y_2 > z_2 - X_2) \cdots I(Y_n > z_n - X_n) \mid X_1, X_2, \cdots X_n]\} \\ &\quad \le E\{E[I(Y_1 > z_1 - X_1) \mid X_1] \\ &\qquad \cdot E[I(Y_2 > z_2 - X_2) \mid X_2] \cdots E[I(Y_n > z_n - X_n) \mid X_n]\}. \end{aligned} \tag{3.2}$$

Noting that $E[I(Y_1 > z_1 - X_1) \mid X_1], E[I(Y_2 > z_2 - X_2) \mid X_2], \cdots, E[I(Y_n > z_n - X_n) \mid X_n]$ are nondecreasing functions of $X_1, X_2, \cdots X_n$ respectively, and $X_1, X_2, \cdots, X_n$ are NOD random variables, we have by Lemma 3.1 (ii) again that

$$\begin{aligned} &E\{E[I(Y_1 > z_1 - X_1) \mid X_1] \\ &\quad \cdot E[I(Y_2 > z_2 - X_2) \mid X_2] \cdots E[I(Y_n > z_n - X_n) \mid X_n]\}. \\ &\qquad \le E\{E[I(Y_1 > z_1 - X_1) \mid X_1]\} \\ &\qquad\quad \cdot E\{E[I(Y_2 > z_2 - X_2) \mid X_2]\} \cdots E\{E[I(Y_n > z_n - X_n) \mid X_n]\}. \\ &\qquad = P(X_1 + Y_1 > z_1) \cdot P(X_2 + Y_2 > z_2) \cdots P(X_n + Y_n > z_n) \\ &\qquad = P(Z_1 > z_1) \cdot P(Z_2 > z_2) \cdots P(Z_n > z_n), \end{aligned} \tag{3.3}$$

which together with (3.1) and (3.2) yields that

$$P(Z_1 > z_1, Z_2 > z_2, \cdots, Z_n > z_n) \le P(Z_1 > z_1) \cdot P(Z_2 > z_2) \cdots P(Z_n > z_n). \tag{3.4}$$

Similarly, we have

$$P(Z_1 \le z_1, Z_2 \le z_2, \cdots, Z_n \le z_n) \le P(Z_1 \le z_1) P(Z_2 \le z_2) \cdots P(Z_n \le z_n). \tag{3.5}$$

Hence, by the definition of NOD random variables and the inequalities (3.4) and (3.5), we can see that $Z_1, Z_2, \cdots, Z_n$ are NOD random variables. That is to say, $X_1+Y_1, X_2+Y_2, \cdots, X_n+Y_n$ are NOD random variables. The proof is completed. □

Combining Lemma 3.1 and Lemma 3.2, we can get the following important property for NOD random variables, which will be used to prove the main results of the paper.

Corollary 3.1. *Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ be both NOD random variables. If random variables $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are independent, then for any $\beta \in \mathbf{R}$, $X_1 + \beta Y_1, X_2 + \beta Y_2, \cdots, X_n + \beta Y_n$ are also NOD random variables.*

The next one is the Rosenthal type inequality for NOD random variables. The first inequality can be found in Asadian et al. [21] and the second one can be found in Wu [29].

Lemma 3.3. *Let $p \ge 2$ and $\{X_n, n \ge 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for every $n \ge 1$. Then there exists a positive constant C_p depending only on p such that for every $n \ge 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \le C_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}$$

and

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \le C \log^p n \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}.$$

With the above lemmas accounted for, we can get the following strong convergence and complete convergence for weighted sums of NOD random variables, which will be applied to prove the main results of the paper. The proof is similar to that of Jing and Liang [30]. For convenience of the reader, we will present the proofs of Lemmas 3.4 and 3.5 in Appendix A.

Lemma 3.4. *Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables, which is stochastically dominated by a random variable X, namely, there exists a positive constant C such that*

$$P(|X_n| > x) \le CP(|X| > x)$$

for all $x \ge 0$ and $n \ge 1$. Assume that $E|X|^p < \infty$ for some $p > 0$ and $EX_n = 0$ if $p > 1$. Let $\{b_{ni}, i \ge 1, n \ge 1\}$ be an array of constants satisfying

$$\max_{1\le i\le n} |b_{ni}| = O(n^{-1/p}) \tag{3.6}$$

and

$$\begin{cases} \sum_{i=1}^{n} |b_{ni}|^p = O\left(n^{-\delta}\right) \text{ for some } \delta > 0, & \text{if } 0 < p \leq 2, \\ \sum_{i=1}^{n} b_{ni}^2 = o\left((\log n)^{-1}\right), & \text{if } p > 2. \end{cases} \tag{3.7}$$

Then

$$T_n \doteq \sum_{i=1}^{n} b_{ni} X_i \to 0 \quad a.s.. \tag{3.8}$$

Lemma 3.5. *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables, which is stochastically dominated by a random variable X such that $E|X|^{2p} < \infty$ for some $p > 0$. Assume further that $EX_n = 0$ if $p > 1$. Let $\{b_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying*

$$\max_{1 \leq i \leq n} |b_{ni}| = O(n^{-1/p}) \tag{3.9}$$

and

$$\begin{cases} \sum_{i=1}^{n} |b_{ni}|^p = O(n^{-\delta}) \text{ for some } \delta > 0, & \text{if } 0 < p \leq 1, \\ \sum_{i=1}^{n} b_{ni}^2 = o((\log n)^{-1}), & \text{if } p > 1. \end{cases} \tag{3.10}$$

Then

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni} X_i\right| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{3.11}$$

The last one is the Marcinkiewicz-Zygmund type strong law of large numbers for NOD random variables.

Lemma 3.6. *Let $1/2 < \alpha \leq 1$ and $\alpha p > 1$. Let $\{X_n, n \geq 1\}$ be a sequence of mean zero NOD random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty \quad \text{for all } \varepsilon > 0, \tag{3.12}$$

and thus,

$$\frac{1}{n^{\alpha}} \sum_{i=1}^{n} X_i \to 0 \quad a.s. \tag{3.13}$$

Proof. Similar to the proof of Theorem 1.1 of Zhang [31], one can get (3.12), and (3.13) follows from (3.12) immediately. The proof is completed. □

4. Proofs of the main results

Proof of Theorem 2.1

Proof. In view of (2.2), to prove the main result (2.7), it suffices to show that

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 \to 0 \quad a.s., \tag{4.1}$$

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)\varepsilon_i \to 0 \quad a.s., \tag{4.2}$$

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} (x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) \to 0 \quad a.s. \tag{4.3}$$

and

$$\frac{1}{S_n} \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2 \to 1 \quad a.s.. \tag{4.4}$$

To prove (4.1), we first prove that

$$J_{1n} \doteq \frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0)\right] \to 0 \quad a.s. \tag{4.5}$$

and

$$J_{2n} \doteq \frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_i < 0)\right] \to 0 \quad a.s.. \tag{4.6}$$

It is easily seen that $\{\delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0), i \geq 1\}$ and $\{\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_k < 0), i \geq 1\}$ are still NOD random variables by Lemma 3.1. Applying Lemma 3.4 with $X_i = \delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0)$ and $b_{ni} = \frac{1}{\sqrt{S_n} n^{\tau}}$, we have by $E|\delta_1|^{2p} < \infty$, (2.5) and (2.6) that:

(i) $E|X_1|^p \leq CE|\delta_1|^{2p} < \infty$;

(ii) $\max_{1\leq i\leq n} b_{ni} = \frac{1}{\sqrt{S_n} n^{\tau}} \leq \frac{n^{1-\tau}}{\sqrt{S_n}} \cdot \frac{1}{n^{1/p}} = O(n^{-1/p})$;

(iii) if $p > 2$, then

$$\sum_{i=1}^{n} b_{ni}^2 = \frac{n^{2-2\tau}}{S_n} \cdot \frac{1}{n} = o((\log n)^{-1});$$

if $1 < p \leq 2$, then

$$\sum_{i=1}^{n} |b_{ni}|^p = \frac{n}{S_n^{p/2} n^{p\tau}} = n^{1-p} \cdot \left(\frac{n^{1-\tau}}{\sqrt{S_n}}\right)^p = O\left(n^{-(p-1)}\right);$$

if $p = 1$, then

$$\sum_{i=1}^{n} |b_{ni}| = \frac{n^{1-\tau}}{\sqrt{S_n}} = n^{1-2\tau} \cdot \frac{n^{\tau}}{\sqrt{S_n}} = O\left(n^{-(2\tau-1)}\right),$$

where $\tau > 1/2$, since $\tau > 1/p - 1/2$ if $1 \le p \le 2$. That is to say, the conditions of Lemma 3.4 are satisfied. Hence, (4.5) follows by Lemma 3.4 immediately. Similarly, we can also get (4.6).

Note that $E\delta_1^2 < \infty$ by $E|\delta_1|^{2p} < \infty$ for $p \ge 1$, we have

$$\begin{aligned} \frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} \left(\delta_i - \bar{\delta}_n\right)^2 &\le \frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} \left(\delta_i^2 - E\delta_i^2\right) + \frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} E\delta_i^2 \\ &= J_{1n} + J_{2n} + \frac{n^{1-\tau}}{\sqrt{S_n}} E\delta_1^2. \end{aligned} \tag{4.7}$$

Hence (4.1) follows by (4.5)–(4.7) and (2.6) immediately.

Similar to the proof of (4.1), we can get that

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n} (\varepsilon_i - \bar{\varepsilon}_n)^2 \to 0 \quad a.s.. \tag{4.8}$$

Note that

$$\begin{aligned} \left| \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)\varepsilon_i \right| &= \left| \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)(\varepsilon_i - \bar{\varepsilon}_n) \right| \\ &\le \frac{1}{2} \left[\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 + \sum_{i=1}^{n} (\varepsilon_i - \bar{\varepsilon}_n)^2 \right], \end{aligned} \tag{4.9}$$

which together with (4.1) and (4.8) yield (4.2).

It is easily checked that for any $r > 0$,

$$\begin{aligned} \sum_{i=1}^{n} |x_i - \bar{x}_n| \cdot \left|\delta_i - \bar{\delta}_n\right| &\le \sqrt{\sum_{i=1}^{n} (x_i - \bar{x}_n)^2} \cdot \sqrt{\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2} \\ &= \sqrt{r \sum_{i=1}^{n} (x_i - \bar{x}_n)^2} \cdot \sqrt{\frac{1}{r} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2} \\ &\le \frac{r \sum_{i=1}^{n} (x_i - \bar{x}_n)^2 + \frac{1}{r} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2}{2} \\ &= \frac{r}{2} S_n + \frac{1}{2r} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2, \end{aligned}$$

which implies that

$$\left| \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2 - S_n \right| = \left| 2 \sum_{i=1}^{n} (x_i - \bar{x}_n)(\delta_i - \bar{\delta}_n) + \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 \right|$$

$$\leq 2\sum_{i=1}^{n}|x_i-\bar{x}_n|\cdot\left|\delta_i-\bar{\delta}_n\right|+\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2$$
$$\leq rS_n+\frac{1+r}{r}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2. \tag{4.10}$$

Hence, we have by (4.10), (2.5) and (4.1) that

$$\left|\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2-1\right|\leq r+\frac{1+r}{r}\frac{1}{S_n}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2$$
$$=r+\frac{1+r}{r}\cdot\frac{n^{\tau}}{\sqrt{S_n}}\cdot\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\to r\quad a.s. \tag{4.11}$$

Since $r>0$ is arbitrary, it follows by (4.11) that

$$\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2-1\to 0\quad a.s.,$$

which implies (4.4).

Finally, we will prove (4.3). Denote

$$b_{ni}=\frac{x_i-\bar{x}_n}{\sqrt{S_n}n^{\tau}},\qquad X_i=\varepsilon_i-\beta\delta_i.$$

It follows by Corollary 3.1 that $\{X_i,i\geq 1\}$ is still a sequence of NOD random variables. In addition, we have by $E|\varepsilon_1|^{2p}<\infty$, $E|\delta_1|^{2p}<\infty$ and (2.4) that:

(i) $E|X_1|^p=E|\varepsilon_1-\beta\delta_1|^p\leq CE|\varepsilon_1|^p+CE|\delta_1|^p<\infty$;

(ii) $\max\limits_{1\leq i\leq n}|b_{ni}|=O\left(n^{-1/p}\right)$;

(iii) if $p>2$, then

$$\sum_{i=1}^{n}b_{ni}^2=\frac{1}{n^{2\tau}}=o\left((\log n)^{-1}\right);$$

if $1\leq p\leq 2$, noting that $\tau>1/p-1/2$, we have

$$\sum_{i=1}^{n}|b_{ni}|^p=\frac{1}{S_n^{p/2}n^{p\tau}}\sum_{i=1}^{n}|x_i-\bar{x}_n|^p$$
$$\leq\frac{1}{S_n^{p/2}n^{p\tau}}\cdot n\left(\frac{1}{n}\sum_{i=1}^{n}|x_i-\bar{x}_n|^2\right)^{p/2}$$
$$=n^{1-p/2-p\tau}\;=\;O\left(n^{-(p\tau+p/2-1)}\right).$$

That is to say, the conditions of Lemma 3.4 are satisfied. Hence, (4.3) follows by Lemma 3.4 immediately. This completes the proof of the theorem. □

Proof of Theorem 2.2

Proof. According to the relation (2.3), to prove (2.9), it suffices to show that

$$n^{\nu}(\bar{\varepsilon}_n - \beta\bar{\delta}_n) \to 0 \quad a.s., \tag{4.12}$$

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{x}_n \to 0 \quad a.s. \tag{4.13}$$

and

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{\delta}_n \to 0 \quad a.s.. \tag{4.14}$$

Firstly, we will prove (4.12). It is easily seen that

$$n^{\nu}(\bar{\varepsilon}_n - \beta\bar{\delta}_n) = \frac{1}{n^{1-\nu}}\sum_{i=1}^{n}(\varepsilon_i - \beta\delta_i).$$

Applying Lemma 3.4 with $b_{ni} = \frac{1}{n^{1-\nu}}$, $X_i = \varepsilon_i - \beta\delta_i$, and noting that $0 < \nu < \min(1 - 1/p, 1/2)$, $p > 1$, we can see that:

(i) $E|X_1|^p = E|\varepsilon_1 - \beta\delta_1|^p \le CE|\varepsilon_1|^p + CE|\delta_1|^p < \infty$;

(ii) $\max\limits_{1\le i\le n} |b_{ni}| = n^{-(1-\nu)} = O\left(n^{-1/p}\right)$;

(iii) if $p > 2$, then

$$\sum_{i=1}^{n} b_{ni}^2 = \frac{1}{n^{1-2\nu}} = o\left((\log n)^{-1}\right);$$

if $1 < p \le 2$, noting that $p - 1 - p\nu > 0$, we have

$$\sum_{i=1}^{n} |b_{ni}|^p = \frac{n}{n^{p(1-\nu)}} = O\left(n^{-(p-1-p\nu)}\right).$$

That is to say, the conditions of Lemma 3.4 are satisfied. Hence, (4.12) follows by Lemma 3.4 immediately.

Next, we will prove (4.13). It can be checked that

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{x}_n = \frac{\sqrt{S_n}}{n^{\tau}}(\beta - \hat{\beta}_n) \cdot \frac{n^{\tau+\nu}}{\sqrt{S_n}}\bar{x}_n. \tag{4.15}$$

Hence (4.13) follows by (4.15), (2.8) and Theorem 2.1 immediately.

Finally, we will prove (4.14). Similar to the proof of (4.12), we can get that

$$\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i \to 0 \quad a.s.. \tag{4.16}$$

Note that

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{\delta}_n = \frac{\sqrt{S_n}}{n^{\tau}}(\beta - \hat{\beta}_n) \cdot \frac{n^{\tau}}{\sqrt{S_n}} \cdot \frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i. \tag{4.17}$$

It follows by (4.17), (2.5), Theorem 2.1 and (4.16) that (4.14) holds. This completes the proof of the theorem. □

Proof of Theorem 2.3

Proof. The proof is similar to that of Theorem 2.1. In view of (2.2), to prove the main result (2.11), it suffices to show that

$$\frac{1}{\sqrt{S_n} b_n} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 \xrightarrow{P} 0, \tag{4.18}$$

$$\frac{1}{\sqrt{S_n} b_n} \sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)\varepsilon_i \xrightarrow{P} 0, \tag{4.19}$$

$$\frac{1}{\sqrt{S_n} b_n} \sum_{i=1}^{n} (x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) \xrightarrow{P} 0 \tag{4.20}$$

and

$$\frac{1}{S_n} \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2 \xrightarrow{P} 1. \tag{4.21}$$

First, we will prove (4.18). Note that

$$\begin{aligned}
&\frac{1}{\sqrt{S_n} b_n} \sum_{i=1}^{n} \left(\delta_i - \bar{\delta}_n\right)^2 \\
&\quad \le \frac{1}{\sqrt{S_n} b_n} \sum_{i=1}^{n} \delta_i^2 \\
&\quad = \frac{n}{\sqrt{S_n} b_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i \ge 0) - E\delta_i^2 I(\delta_i \ge 0)\right] \\
&\qquad + \frac{n}{\sqrt{S_n} b_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_i < 0)\right] + \frac{n}{\sqrt{S_n} b_n} E\delta_1^2.
\end{aligned} \tag{4.22}$$

It is easily seen that $\{\delta_i^2 I(\delta_i \ge 0) - E\delta_i^2 I(\delta_i \ge 0), i \ge 1\}$ and $\{\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_k < 0), i \ge 1\}$ are both sequences of NOD random variables by Remark 3.1. Applying Lemma 3.6 with $\alpha = 1$, we have by $E|\delta_1|^{2p} < \infty$ and (2.10) that

$$\frac{n}{\sqrt{S_n} b_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i \ge 0) - E\delta_i^2 I(\delta_i \ge 0)\right] \xrightarrow{P} 0, \tag{4.23}$$

$$\frac{n}{\sqrt{S_n} b_n} \cdot \frac{1}{n} \sum_{i=1}^{n} \left[\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_i < 0)\right] \xrightarrow{P} 0 \tag{4.24}$$

and

$$\frac{n}{\sqrt{S_n} b_n} E\delta_1^2 \to 0, \quad \text{as} \quad n \to \infty. \tag{4.25}$$

Hence, (4.18) follows by (4.22)–(4.25) immediately.

Next, we will prove (4.19). Similar to the proof of (4.18), one has

$$\frac{1}{\sqrt{S_n}b_n}\sum_{i=1}^{n}(\varepsilon_i-\bar{\varepsilon}_n)^2 \xrightarrow{P} 0, \tag{4.26}$$

By (4.9), (4.18) and (4.26), we can get (4.19) immediately.

In the following, we will prove (4.20). It follows by Lemma 3.1, Lemma 3.3, $E\delta_1^2<\infty$ and $E\varepsilon_1^2<\infty$ that

$$\begin{aligned}
&\frac{1}{S_nb_n^2}E\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)(\varepsilon_i-\beta\delta_i)\right]^2\\
&\quad\le \frac{2}{S_nb_n^2}E\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)^+(\varepsilon_i-\beta\delta_i)\right]^2+\frac{2}{S_nb_n^2}E\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)^-(\varepsilon_i-\beta\delta_i)\right]^2\\
&\quad\le \frac{C}{S_nb_n^2}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2E(\varepsilon_i-\beta\delta_i)^2 \;\le\; \frac{C}{b_n^2}\left[E\varepsilon_1^2+\beta^2E\delta_1^2\right]\\
&\quad\to 0,\quad \text{as } n\to\infty,
\end{aligned}$$

which yields (4.20).

Finally, we will prove (4.21). It follows by (4.10) that

$$\begin{aligned}
\left|\sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2-S_n\right| &\le 2\sum_{i=1}^{n}|x_i-\bar{x}_n|\cdot\left|\delta_i-\bar{\delta}_n\right|+\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\\
&\le 2\sqrt{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2\cdot\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2}+\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\\
&= 2\sqrt{S_n\cdot\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2}+\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2,
\end{aligned}$$

which implies that

$$\left|\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2-1\right|\le 2\sqrt{\frac{1}{S_n}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2}+\frac{1}{S_n}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2. \tag{4.27}$$

Note that

$$\frac{1}{S_n}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\le\frac{b_n}{\sqrt{S_n}}\cdot\frac{1}{\sqrt{S_n}b_n}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2,$$

which together with (2.10) and (4.18) yields that

$$\frac{1}{S_n}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\xrightarrow{P}0. \tag{4.28}$$

Combining (4.27) and (4.28), we can get (4.21) immediately. This completes the proof of the theorem. □

Proof of Theorem 2.4

Proof. According to the relation (2.3), to prove (2.13), it suffices to show that

$$\frac{\sqrt{n}}{b_n}(\bar{\varepsilon}_n - \beta\bar{\delta}_n) \xrightarrow{P} 0, \tag{4.29}$$

$$\frac{\sqrt{n}}{b_n}(\beta - \hat{\beta}_n)\bar{x}_n \xrightarrow{P} 0 \tag{4.30}$$

and

$$\frac{\sqrt{n}}{b_n}(\beta - \hat{\beta}_n)\bar{\delta}_n \xrightarrow{P} 0. \tag{4.31}$$

By Markov's inequality, Lemma 3.3 and $E\varepsilon_1^2 < \infty$, we have for any $\varepsilon > 0$ that

$$P\left(\left|\frac{\sqrt{n}}{b_n}\bar{\varepsilon}_n\right| \geq \varepsilon\right) \leq \frac{C}{nb_n^2}E\left(\sum_{i=1}^n \varepsilon_i\right)^2 \leq \frac{C}{b_n^2} \to 0, \quad \text{as} \quad n \to \infty,$$

which implies that

$$\frac{\sqrt{n}}{b_n}\bar{\varepsilon}_n \xrightarrow{P} 0. \tag{4.32}$$

Similarly, we have

$$\frac{\sqrt{n}}{b_n}\bar{\delta}_n \xrightarrow{P} 0. \tag{4.33}$$

Combining (4.32) and (4.33), we can get (4.29).

It follows by (2.10) and (2.11) that

$$\beta - \hat{\beta}_n \xrightarrow{P} 0, \tag{4.34}$$

which together with (4.33) yields (4.31).

Finally, we will prove (4.30). From the equality (2.2) and the fact (4.21), to prove (4.30), it suffices to show

$$\frac{\sqrt{n}|\bar{x}_n|}{b_n S_n}\left[\sum_{i=1}^n(\delta_i - \bar{\delta}_n)\varepsilon_i + \sum_{i=1}^n(x_i - \bar{x}_n)(\varepsilon_i - \beta\delta_i) - \beta\sum_{i=1}^n(\delta_i - \bar{\delta}_n)^2\right] \xrightarrow{P} 0. \tag{4.35}$$

Similar to the proofs of (4.18) and (4.19), one has

$$\frac{\sqrt{n}|\bar{x}_n|}{b_n S_n}\sum_{i=1}^n(\delta_i - \bar{\delta}_n)^2 \leq \frac{\sqrt{n}|\bar{x}_n|}{b_n S_n}\sum_{i=1}^n \delta_i^2 = \frac{n^{3/2}|\bar{x}_n|}{b_n S_n}\cdot\frac{1}{n}\sum_{i=1}^n \delta_i^2 \xrightarrow{P} 0 \tag{4.36}$$

and

$$\frac{\sqrt{n}|\bar{x}_n|}{b_n S_n}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)\varepsilon_i \xrightarrow{P} 0. \tag{4.37}$$

It follows by Lemma 3.1, Lemma 3.3, $E\delta_1^2<\infty$ and $E\varepsilon_1^2<\infty$ again that

$$\begin{aligned}
&\frac{n\bar{x}_n^2}{b_n^2S_n^2}E\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)(\varepsilon_i-\beta\delta_i)\right]^2\\
&\quad\le \frac{2n\bar{x}_n^2}{b_n^2S_n^2}E\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)^+(\varepsilon_i-\beta\delta_i)\right]^2+\frac{2n\bar{x}_n^2}{b_n^2S_n^2}E\left[\sum_{i=1}^{n}(x_i-\bar{x}_n)^-(\varepsilon_i-\beta\delta_i)\right]^2\\
&\quad\le \frac{Cn\bar{x}_n^2}{b_n^2S_n^2}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2E(\varepsilon_i-\beta\delta_i)^2 \ \le\ \frac{Cn\bar{x}_n^2}{b_n^2S_n}\left[E\varepsilon_1^2+\beta^2E\delta_1^2\right]\\
&\quad\to 0,\quad \text{as } n\to\infty,
\end{aligned}$$

which yields that

$$\frac{\sqrt{n}|\bar{x}_n|}{b_n S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)(\varepsilon_i-\beta\delta_i) \xrightarrow{P} 0. \tag{4.38}$$

By (4.36)–(4.38), we can get (4.35) immediately. The proof is completed. □

Proof of Theorem 2.5

Proof. The proof is similar to that of Theorem 2.1. In view of (2.2), to prove the main result (2.14), it suffices to show that

$$\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2\to 0 \ \text{ completely,} \tag{4.39}$$

$$\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)\varepsilon_i\to 0 \ \text{ completely,} \tag{4.40}$$

$$\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}(x_i-\bar{x}_n)(\varepsilon_i-\beta\delta_i)\to 0 \ \text{ completely} \tag{4.41}$$

and

$$\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2\to 1 \ \ a.s.. \tag{4.42}$$

To prove (4.39), we firstly prove that

$$J_{1n}\doteq\frac{1}{\sqrt{S_n}n^{\tau}}\sum_{i=1}^{n}\left(\delta_i^2I(\delta_i\ge 0)-E\delta_i^2I(\delta_i\ge 0)\right)\to 0 \ \text{ completely} \tag{4.43}$$

and

$$J_{2n} \doteq \frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} \left(\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_i < 0)\right) \to 0 \ \text{ completely.} \tag{4.44}$$

It is easily seen that $\{\delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0), i \geq 1\}$ and $\{\delta_i^2 I(\delta_i < 0) - E\delta_i^2 I(\delta_k < 0), i \geq 1\}$ are still NOD random variables by Lemma 3.1. Applying Lemma 3.5 with $X_i = \delta_i^2 I(\delta_i \geq 0) - E\delta_i^2 I(\delta_i \geq 0)$ and $b_{ni} = \frac{1}{\sqrt{S_n}n^{\tau}}$, we have by $E|\delta_1|^{4p} < \infty$ and (2.6) that:

(i) $E|X_1|^{2p} \leq CE|\delta_1|^{4p} < \infty$;
(ii) $\max\limits_{1\leq i\leq n} b_{ni} = \frac{1}{\sqrt{S_n}n^{\tau}} \leq \frac{n^{1-\tau}}{\sqrt{S_n}} \cdot \frac{1}{n^{1/p}} = O(n^{-1/p})$;
(iii) $\sum_{i=1}^{n} b_{ni}^2 = \frac{n^{2-2\tau}}{S_n} \cdot \frac{1}{n} = o((\log n)^{-1})$.

That is to say, the conditions of Lemma 3.5 are satisfied. Hence, (4.43) follows by Lemma 3.5 immediately. Similarly, we can also get (4.44).

Note that $E\delta_1^2 < \infty$ by $E|\delta_1|^{4p} < \infty$ for $p > 1$, we have

$$\begin{aligned}\frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} \left(\delta_i - \bar{\delta}_n\right)^2 &\leq \frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} \left(\delta_i^2 - E\delta_i^2\right) + \frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} E\delta_i^2 \\ &= J_{1n} + J_{2n} + \frac{n^{1-\tau}}{\sqrt{S_n}} E\delta_1^2.\end{aligned} \tag{4.45}$$

It follows by (2.6) again that for any $\varepsilon > 0$,

$$\frac{n^{1-\tau}}{\sqrt{S_n}} E\delta_1^2 < \frac{\varepsilon}{2} \ \text{ for all } n \text{ large enough.} \tag{4.46}$$

By (4.43)–(4.46), we can get that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} \left(\delta_i - \bar{\delta}_n\right)^2 > \varepsilon\right) < \infty,$$

which implies (4.39).

Similar to the proof of (4.39), we can get that

$$\frac{1}{\sqrt{S_n}n^{\tau}} \sum_{i=1}^{n} (\varepsilon_i - \bar{\varepsilon}_n)^2 \to 0 \ \text{ completely.} \tag{4.47}$$

Note that

$$\left|\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)\varepsilon_i\right| \leq \frac{1}{2}\left[\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 + \sum_{i=1}^{n} (\varepsilon_i - \bar{\varepsilon}_n)^2\right],$$

which together with (4.39) and (4.47) yield (4.40).

It is easily checked that for any $r > 0$,

$$\sum_{i=1}^{n} |x_i - \bar{x}_n| \cdot \left|\delta_i - \bar{\delta}_n\right| \leq \sqrt{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2} \cdot \sqrt{\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2}$$
$$\leq \frac{r}{2} S_n + \frac{1}{2r}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2,$$

which implies that

$$\left|\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 - S_n\right| = \left|2\sum_{i=1}^{n}(x_i - \bar{x}_n)(\delta_i - \bar{\delta}_n) + \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2\right|$$
$$\leq r S_n + \frac{1+r}{r}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2. \tag{4.48}$$

It follows by (4.39) and Borel-Cantelli lemma that

$$\frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \to 0 \quad a.s. \tag{4.49}$$

Hence, we have by (2.5), (4.48) and (4.49) that

$$\left|\frac{1}{S_n}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 - 1\right| \leq r + \frac{1+r}{r} \cdot \frac{n^{\tau}}{\sqrt{S_n}} \cdot \frac{1}{\sqrt{S_n} n^{\tau}} \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \to r \quad a.s. \tag{4.50}$$

Since $r > 0$ is arbitrary, we can get (4.42) immediately by (4.50).

Finally, we will prove (4.41). Denote

$$b_{ni} = \frac{x_i - \bar{x}_n}{\sqrt{S_n} n^{\tau}}, \quad X_i = \varepsilon_i - \beta\delta_i.$$

It follows by Corollary 3.1 that $\{X_i, i \geq 1\}$ is still a sequence of NOD random variables. In addition, we have by $E|\varepsilon_1|^{4p} < \infty$, $E|\delta_1|^{4p} < \infty$ and (2.4) that

$$E|X_1|^{2p} \;=\; E|\varepsilon_1 - \beta\delta_1|^{2p} \;\leq\; CE|\varepsilon_1|^{2p} + CE|\delta_1|^{2p} < \infty,$$
$$\max_{1\leq i\leq n} |b_{ni}| \;=\; O\left(n^{-1/p}\right), \quad \sum_{i=1}^{n} b_{ni}^2 \;=\; \frac{1}{n^{2\tau}} \;=\; o\left((\log n)^{-1}\right).$$

That is to say, the conditions of Lemma 3.5 are satisfied. Hence, (4.41) follows by Theorem Lemma 3.5 immediately. This completes the proof of the theorem. □

Proof of Theorem 2.6

Proof. According to the relation (2.3), to prove (2.15), it suffices to show that

$$n^{\nu}(\bar{\varepsilon}_n - \beta\bar{\delta}_n) \to 0 \;\text{ completely}, \tag{4.51}$$

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{x}_n \to 0 \text{ completely} \tag{4.52}$$

and

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{\delta}_n \to 0 \text{ completely.} \tag{4.53}$$

Firstly, we will prove (4.51). It is easily seen that

$$n^{\nu}(\bar{\varepsilon}_n - \beta\bar{\delta}_n) = \frac{1}{n^{1-\nu}}\sum_{i=1}^{n}(\varepsilon_i - \beta\delta_i).$$

Applying Lemma 3.5 with $b_{ni} = \frac{1}{n^{1-\nu}}$, $X_i = \varepsilon_i - \beta\delta_i$, and noting that $0 < \nu < \min(1 - 1/p, 1/2)$, $p > 1$, we can see that the conditions of Lemma 3.5 are satisfied. Hence, (4.51) follows by Lemma 3.5 immediately.

Next, we will prove (4.52). It can be checked that

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{x}_n = \frac{\sqrt{S_n}}{n^{\tau}}(\beta - \hat{\beta}_n) \cdot \frac{n^{\tau+\nu}}{\sqrt{S_n}}\bar{x}_n. \tag{4.54}$$

We have by (4.54), (2.8) and Theorem 2.5 that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(n^{\nu}|(\beta - \hat{\beta}_n)\bar{x}_n| > \varepsilon\right) \le \sum_{n=1}^{\infty} P\left(\frac{\sqrt{S_n}}{n^{\tau}}|\beta - \hat{\beta}_n| > C\right) < \infty,$$

which implies (4.52).

Finally, we will prove (4.53). Similar to the proof of (4.51), we can get that

$$\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i \to 0 \text{ completely.} \tag{4.55}$$

Notinge that

$$n^{\nu}(\beta - \hat{\beta}_n)\bar{\delta}_n = \frac{\sqrt{S_n}}{n^{\tau}}(\beta - \hat{\beta}_n) \cdot \frac{n^{\tau}}{\sqrt{S_n}} \cdot \frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i, \tag{4.56}$$

we have by (4.56), (2.5), Theorem 2.5 and (4.55) that for any $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(n^{\nu}|(\beta - \hat{\beta}_n)\bar{\delta}_n| > \varepsilon\right) &\le \sum_{n=1}^{\infty} P\left(\left|\frac{\sqrt{S_n}}{n^{\tau}}(\beta - \hat{\beta}_n)\right| > C\right) \\ &\quad + \sum_{n=1}^{\infty} P\left(\left|\frac{1}{n^{1-\nu}}\sum_{i=1}^{n}\delta_i\right| > C\right) \\ &< \infty,\end{aligned}$$

which implies (4.53). This completes the proof of the theorem. ∎

Appendix A: Appendix section

To prove Lemmas 3.4 and 3.5, we need the following Kolmogorov-type exponential inequality for NOD random variables, which can be found in Shen [26] for instance.

Lemma A.1. *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with zero means and finite second moments. Denote $S_n = \sum_{i=1}^n X_i$ and $B_n^2 = \sum_{i=1}^n EX_i^2$ for each $n \geq 1$. Then for all $x > 0$ and $y > 0$,*

$$P(|S_n| \geq x) \\ \leq 2P\left(\max_{1\leq i\leq n} |X_i| \geq y\right) + 2\exp\left\{-\frac{x^2}{2(xy+B_n^2)}\left[1+\frac{2}{3}\log\left(1+\frac{xy}{B_n^2}\right)\right]\right\}.$$

By the integration by parts, we can get the following property for stochastic domination. For the proof, one can refer to Wu [32], or Shen et al. [33].

Lemma A.2. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C_1 [E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)],$$
$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$, where C is a positive constant.

Now we turn to prove Lemmas 3.4 and 3.5.

Proof of Lemmas 3.4

Proof. Without loss of generality, we may assume that $b_{ni} > 0$ (note that $b_{ni} = b_{ni}^+ - b_{ni}^-$). For any $\epsilon > 0$, we choose small $\eta > 0$ and large $N \geq 1$ (to be specialized later).

Denote for $1 \leq i \leq n$ and $n \geq 1$ that

$$\begin{aligned} X_{ni}(1) &= -b_{ni}^{-1} n^{-\eta} I\left(b_{ni}X_i < -n^{-\eta}\right) + X_i I\left(|b_{ni}X_i| \leq n^{-\eta}\right) \\ &\quad + b_{ni}^{-1} n^{-\eta} I\left(b_{ni}X_i > n^{-\eta}\right), \\ X_{ni}(2) &= \left(X_i - b_{ni}^{-1}n^{-\eta}\right) I\left(n^{-\eta} < b_{ni}X_i < \epsilon/N\right), \\ X_{ni}(3) &= \left(X_i + b_{ni}^{-1}n^{-\eta}\right) I\left(-n^{-\eta} > b_{ni}X_i > -\epsilon/N\right), \\ X_{ni}(4) &= \left(X_i - b_{ni}^{-1}n^{-\eta}\right) I\left(b_{ni}X_i \geq \epsilon/N\right) + \left(X_i + b_{ni}^{-1}n^{-\eta}\right) I\left(b_{ni}X_i \leq -\epsilon/N\right), \\ S_n(l) &= \sum_{i=1}^n b_{ni}X_{ni}(l), \quad l = 1,2,3,4. \end{aligned}$$

For fixed $n \geq 1$, it is easily seen that $T_n = \sum_{i=1}^n b_{ni}X_i = S_n(1) + S_n(2) + S_n(3) + S_n(4)$ and $\{b_{ni}X_{ni}(1) - Eb_{ni}X_{ni}(1), 1 \leq i \leq n\}$ are still NOD random variables by the definition of $X_{ni}(1)$ and Lemma 3.1.

We consider two cases.

Case 1. $p > 2$

In order to prove $T_n = o(1)$ $a.s.$, we only need to show $S_n(l) = o(1)$ $a.s.$ for $l = 1, 2, 3, 4$.

To prove $S_n(1) = o(1)$ $a.s.$, it suffices to show $ES_n(1) \to 0$ and $S_n(1) - ES_n(1) \to 0$ $a.s.$ as $n \to \infty$.

Take $\eta > 0$ small enough such that $0 < \eta < \frac{1-2/p}{p-1}$, which implies that $(p-1)\eta - 1 + 2/p < 0$. Hence, by $EX_n = 0$, Markov's inequality and Lemma A.2, we can get that

$$\begin{aligned}
|ES_n(1)| &\le n^{-\eta} \sum_{i=1}^{n} P\left(|b_{ni}X_i| > n^{-\eta}\right) + \left|\sum_{i=1}^{n} b_{ni} EX_i I\left(|b_{ni}X_i| \le n^{-\eta}\right)\right| \\
&= n^{-\eta} \sum_{i=1}^{n} P\left(|b_{ni}X_i| > n^{-\eta}\right) + \left|\sum_{i=1}^{n} b_{ni} EX_i I\left(|b_{ni}X_i| > n^{-\eta}\right)\right| \\
&\le n^{(p-1)\eta} \sum_{i=1}^{n} E|b_{ni}X_i|^p + n^{(p-1)\eta} \sum_{i=1}^{n} E|b_{ni}X_i|^p I\left(|b_{ni}X_i| > n^{-\eta}\right) \\
&\le C n^{(p-1)\eta} \left(\max_{1\le i\le n} |b_{ni}|\right)^{p-2} \sum_{i=1}^{n} b_{ni}^2 \\
&\le C n^{(p-1)\eta - 1 + 2/p} (\log n)^{-1} \to 0, \quad \text{as} \quad n \to \infty, \qquad \text{(A.1)}
\end{aligned}$$

which implies that $ES_n(1) \to 0$ as $n \to \infty$.

To prove $S_n(1) - ES_n(1) \to 0$ $a.s.$, we only need to show that for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(|S_n(1) - ES_n(1)| > \epsilon\right) < \infty. \qquad \text{(A.2)}$$

By the definition of $X_{ni}(1)$, we can see that

$$\max_{1\le i\le n} |b_{ni}X_{ni}(1) - Eb_{ni}X_{ni}(1)| \le 2n^{-\eta} \qquad \text{(A.3)}$$

and

$$\begin{aligned}
B_n^2 &\doteq \sum_{i=1}^{n} E\left[b_{ni}X_{ni}(1) - Eb_{ni}X_{ni}(1)\right]^2 \\
&\le \sum_{i=1}^{n} E\left|b_{ni}X_{ni}(1)\right|^2 \ \le \ \sum_{i=1}^{n} E\left|b_{ni}X_i\right|^2 \\
&\le CEX^2 \sum_{i=1}^{n} b_{ni}^2 \ = \ o((\log n)^{-1}). \qquad \text{(A.4)}
\end{aligned}$$

Applying Lemma A.1 with $x = \epsilon$ and $y = 2n^{-\eta}$, we have by (A.3), (A.4) and Lemma A.1 that

$$\sum_{n=1}^{\infty} P\left(|S_n(1) - ES_n(1)| > \epsilon\right)$$

$$
\begin{aligned}
&\leq 2\sum_{n=1}^{\infty} P\left(\max_{1\leq i\leq n}|b_{ni}(X_{ni}(1)-EX_{ni}(1))|>2n^{-\eta}\right)\\
&\quad + C\sum_{n=1}^{\infty}\exp\left\{-\frac{\epsilon^2}{2(2\epsilon n^{-\eta}+o((\log n)^{-1}))}\right\}\\
&\leq C\sum_{n=1}^{\infty}\exp\{-2\log n\}<\infty, \qquad \text{(A.5)}
\end{aligned}
$$

which implies (A.2). This completes the proof of $S_n(1)=o(1)$ $a.s.$.

Next, we will estimate $S_n(2)$. From the definition of $X_{ni}(2)$, we can see that $0\leq b_{ni}X_{ni}(2)<\epsilon/N$. Hence, $|S_n(2)|=\sum_{i=1}^{n} b_{ni}X_{ni}(2)>\epsilon$ implies that there are at least N i's such that $b_{ni}X_{ni}(2)\neq 0$. Therefore, by Markov's inequality and Lemma A.2, we can get that

$$
\begin{aligned}
P(|S_n(2)|>\epsilon) &\leq P\left(\text{there are at least } N\ i's \text{ such that } b_{ni}X_{ni}(2)\neq 0\right)\\
&\leq \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} P\left(b_{n1}X_{ni_1}(2)\neq 0,\cdots,b_{ni_N}X_{ni_N}(2)\neq 0\right)\\
&\leq \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} P\left(b_{n1}X_{i_1}>n^{-\eta},\cdots,b_{ni_N}X_{i_N}>n^{-\eta}\right)\\
&\leq \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} P\left(b_{n1}X_{i_1}>n^{-\eta}\right)\cdots P\left(b_{ni_N}X_{i_N}>n^{-\eta}\right)\\
&\leq \left[\sum_{i=1}^{n} P\left(b_{ni}X_i>n^{-\eta}\right)\right]^N\\
&\leq \left[\sum_{i=1}^{n} P\left(|b_{ni}X_i|>n^{-\eta}\right)\right]^N\\
&\leq C\left[n^{p\eta}\sum_{i=1}^{n} E|b_{ni}X_i|^p\right]^N\\
&\leq C\left[n^{p\eta}\left(\max_{1\leq i\leq n}|b_{ni}|\right)^{p-2}\sum_{i=1}^{n} b_{ni}^2\right]^N\\
&\leq Cn^{-(1-2/p-p\eta)N}(\log n)^{-N}. \qquad \text{(A.6)}
\end{aligned}
$$

Taking η small enough and $N>1$ large enough such that $1-2/p-p\eta>0$ and $(1-2/p-p\eta)N\geq 1$, we can see that

$$\sum_{n=1}^{\infty} P(|S_n(2)|>\epsilon)<\infty,$$

which implies that $S_n(2)=o(1)$ $a.s.$.

Note that $-\epsilon/N<b_{ni}X_{ni}(3)\leq 0$, hence, $|S_n(3)|=-\sum_{i=1}^{n} b_{ni}X_{ni}(3)>\epsilon$ implies that there are at least N i's such that $b_{ni}X_{ni}(3)\neq 0$. Similar to the proof of $S_n(2)=o(1)$ $a.s.$, we can get $S_n(3)=o(1)$ $a.s.$ immediately.

Finally, we will show that $S_n(4) = o(1)$ *a.s.*. Note that $E|X|^p < \infty$ is equivalent to

$$\sum_{i=1}^{\infty} P\left(|X_i| \geq ci^{1/p}\right) < \infty \quad \text{for any} \quad c > 0.$$

Hence, by Borel-Cantelli Lemma, we have for any $c > 0$ that

$$P\left(|X_i| \geq ci^{1/p}, i.o.\right) = 0 \quad \Leftrightarrow \quad P\left(|X_i| < ci^{1/p}, \text{ for } i \text{ large enough}\right) = 1,$$

which implies that for any $c > 0$,

$$\sum_{i=1}^{\infty} |X_i| I(|X_i| > ci^{1/p}) < \infty \quad a.s.$$

Noting that

$$\begin{aligned} |X_{ni}(4)| &= \left(|X_i| - b_{ni}^{-1} n^{-\eta}\right) I\left(|b_{ni} X_i| \geq \varepsilon/N\right) \\ &\leq |X_i| I\left(|X_i| \geq Ci^{1/p}\right), \end{aligned}$$

we have

$$\begin{aligned} |S_n(4)| &\leq \max_{1\leq i\leq n} |b_{ni}| \sum_{i=1}^{n} |X_{ni}(4)| \\ &\leq Cn^{-1/p} \sum_{i=1}^{\infty} |X_i| I(|X_i| > Ci^{1/p}) \to 0 \ a.s., \end{aligned} \tag{A.7}$$

which yields that $S_n(4) = o(1)$ *a.s.*.

Case 2. $0 < p \leq 2$

Firstly, we will show that $ES_n(1) = o(1)$. If $p > 1$, similar to the process of (A.1), we can get $ES_n(1) = o(1)$ immediately. If $0 < p \leq 1$, then for any $\eta > 0$, we have by Lemma A.2 that

$$\begin{aligned} |ES_n(1)| &\leq n^{-\eta} \sum_{i=1}^{n} P\left(|b_{ni} X_i| > n^{-\eta}\right) + \sum_{i=1}^{n} E\left|b_{ni} X_i\right| I\left(|b_{ni} X_i| \leq n^{-\eta}\right) \\ &\leq n^{(p-1)\eta} \sum_{i=1}^{n} E|b_{ni} X_i|^p + n^{(p-1)\eta} \sum_{i=1}^{n} E|b_{ni} X_i|^p I\left(|b_{ni} X_i| \leq n^{-\eta}\right) \\ &\leq Cn^{(p-1)\eta-\delta} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Thus, for all $0 < p \leq 2$, we have $ES_n(1) = o(1)$. Note that $|b_{ni} X_{ni}(1)| \leq n^{-\eta}$ and

$$B_n^2 \doteq \sum_{i=1}^{n} E\left[b_{ni} X_{ni}(1) - E b_{ni} X_{ni}(1)\right]^2$$

$$\leq \sum_{i=1}^{n} E\left|b_{ni}X_{ni}(1)\right|^2 \leq Cn^{-(2-p)\eta-\delta}$$
$$= o((\log n)^{-1}).$$

Therefore, (A.5) remains true, which implies that $S_n(1) - ES_n(1) \to 0$ $a.s.$ and hence $S_n(1) \to 0$ $a.s.$.

Similar to the proofs of (A.6) and (A.7), we can get that $S_n(l) \to 0$ $a.s.$ for $l = 2, 3, 4$ immediately. This completes the proof of the lemma. □

Proof of Lemmas 3.5

Proof. The proof is similar to that of Lemma 3.4. Without loss of generality, we may assume that $b_{ni} > 0$. For any $\varepsilon > 0$, we choose positive integer N (to be specified later) and small positive constant q such that $q < \frac{1-1/p}{p}$ if $p > 1$ and $q < \delta/p$ if $0 < p \leq 1$.

Denote for $1 \leq i \leq n$ and $n \geq 1$ that

$$\begin{aligned}
X_{ni}(1) &= X_i I(|b_{ni}X_i| \leq n^{-q}) - b_{ni}^{-1} n^{-q} I(b_{ni}X_i < -n^{-q}) \\
&\quad + b_{ni}^{-1} n^{-q} I(b_{ni}X_i > n^{-q}) \\
X_{ni}(2) &= (X_i - b_{ni}^{-1}n^{-q}) I(n^{-q} < b_{ni}X_i \leq \varepsilon/N), \\
X_{ni}(3) &= (X_i + b_{ni}^{-1}n^{-q}) I(-\varepsilon/N \leq b_{ni}X_i < -n^{-q}), \\
X_{ni}(4) &= (X_i + b_{ni}^{-1}n^{-q}) I(b_{ni}X_i < -\varepsilon/N) + (X_i - b_{ni}^{-1}n^{-q}) I(b_{ni}X_i > \varepsilon/N).
\end{aligned}$$

It is easy to check that $X_{ni}(1) + X_{ni}(2) + X_{ni}(3) + X_{ni}(4) = X_i$, which implies that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_i\right| > 4\varepsilon\right) &\leq \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(1)\right| > \varepsilon\right) \\
&\quad + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(2)\right| > \varepsilon\right) \\
&\quad + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(3)\right| > \varepsilon\right) \\
&\quad + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} b_{ni}X_{ni}(4)\right| > \varepsilon\right) \\
&\doteq I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Hence, to prove (3.11), it suffices to show that $I_1 < \infty$, $I_2 < \infty$, $I_3 < \infty$ and $I_4 < \infty$.

Noting that $\max_{1 \leq i \leq n} |b_{ni}(X_{ni}(1) - EX_{ni}(1))| \leq 2n^{-q}$ for every $n \geq 1$, we have

$$P\left(\max_{1 \leq i \leq n} |b_{ni}(X_{ni}(1) - EX_{ni}(1))| \leq 2n^{-q}\right) = 1 \quad \text{for every } n \geq 1. \tag{A.8}$$

If $0 < p \le 1$, we have by Markov's inequality, Lemma A.2 and (3.10) that

$$\begin{aligned} B_n^2 &\doteq \sum_{i=1}^n E[b_{ni}(X_{ni}(1) - EX_{ni}(1))]^2 \le \sum_{i=1}^n E(b_{ni}X_{ni}(1))^2 \\ &\le Cn^{-(2-p)q} \sum_{i=1}^n |b_{ni}|^p E|X|^p \le Cn^{-(2-p)q-\delta} = o((\log n)^{-1}). \end{aligned} \tag{A.9}$$

If $p > 1$, note that $|X_{ni}(1)| \le |X_i|$, we have by Lemma A.2 and (3.10) again that

$$\begin{aligned} B_n^2 &\doteq \sum_{i=1}^n E[b_{ni}(X_{ni}(1) - EX_{ni}(1))]^2 \\ &\le C\sum_{i=1}^n b_{ni}^2 EX^2 = o((\log n)^{-1}). \end{aligned} \tag{A.10}$$

Note that for fixed $n \ge 1$, $\{b_{ni}(X_{ni}(1) - EX_{ni}(1)), 1 \le i \le n\}$ are still NOD by Lemma 3.1. Applying Lemma A.1 with $x = \varepsilon$ and $y = 2n^{-q}$, we have by (A.8)–(A.10) that

$$\begin{aligned} &\sum_{n=1}^\infty P\left(\left|\sum_{i=1}^n b_{ni}(X_{ni}(1) - EX_{ni}(1))\right| > \varepsilon\right) \\ &\quad \le 2\sum_{n=1}^\infty P\left(\max_{1\le i\le n} |b_{ni}(X_{ni}(1) - EX_{ni}(1))| > 2n^{-q}\right) \\ &\qquad + C\sum_{n=1}^\infty \exp\left\{-\frac{\varepsilon^2}{2(2\varepsilon n^{-q} + o((\log n)^{-1}))}\right\} \\ &\quad \le C\sum_{n=1}^\infty \exp\{-2\log n\} < \infty. \end{aligned} \tag{A.11}$$

To show that $I_1 < \infty$, it remains to show

$$\left|\sum_{i=1}^n b_{ni}EX_{ni}(1)\right| \to 0 \quad \text{as } n \to \infty. \tag{A.12}$$

If $0 < p \le 1$, we have by Markov's inequality, Lemma A.2 and (3.10) that

$$\begin{aligned} &\left|\sum_{i=1}^n b_{ni}EX_{ni}(1)\right| \\ &\quad \le n^{-(1-p)q}\sum_{i=1}^n E|b_{ni}X_i|^p + \sum_{i=1}^n E|b_{ni}X_i|^p |b_{ni}X_i|^{1-p} I(|b_{ni}X_i| \le n^{-q}) \\ &\quad \le Cn^{-(1-p)q}\sum_{i=1}^n |b_{ni}|^p E|X|^p \le Cn^{-(1-p)q-\delta} \to 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{A.13}$$

If $p > 1$, noting that $EX_i = 0$ and $q < \frac{1-1/p}{p} < \frac{2-2/p}{2p-1}$, we have by Markov's inequality, Lemma A.2 and (3.9)–(3.10) that

$$\begin{aligned}\left|\sum_{i=1}^n b_{ni}EX_{ni}(1)\right| &\le n^{(2p-1)q}\sum_{i=1}^n E|b_{ni}X_i|^{2p} + n^{(2p-1)q}\sum_{i=1}^n E|b_{ni}X_i|^{2p}\\ &\le Cn^{(2p-1)q}\left(\max_{1\le i\le n}|b_{ni}|\right)^{2p-2}\sum_{i=1}^n b_{ni}^2\\ &\le Cn^{(2p-1)q-(2-2/p)}(\log n)^{-1} \to 0 \quad \text{as} \quad n\to\infty. \qquad \text{(A.14)}\end{aligned}$$

From (A.12)–(A.14), we have proved that $I_1 < \infty$.

Next, we will prove that $I_2 < \infty$. Since $0 < b_{ni}X_{ni}(2) \le \varepsilon/N$, we can see that

$$\left|\sum_{i=1}^n b_{ni}X_{ni}(2)\right| = \sum_{i=1}^n b_{ni}X_{ni}(2) > \varepsilon$$

implies that there are at least N integers such that $b_{ni}X_{ni}(2) \ne 0$. Hence, by conditions (3.9)–(3.10), we can see that

$$\begin{aligned}&P\left(\left|\sum_{i=1}^n b_{ni}X_{ni}(2)\right| > \varepsilon\right)\\ &\le \sum_{1\le i_1<i_2<\ldots<i_N\le n} P\left(b_{n,i_1}X_{n,i_1}(2)\ne 0, b_{n,i_2}X_{n,i_2}(2)\ne 0,\ldots,b_{n,i_N}X_{n,i_N}(2)\ne 0\right)\\ &\le \sum_{1\le i_1<i_2<\ldots<i_N\le n} P(b_{n,i_1}X_{i_1} > n^{-q})P(b_{n,i_2}X_{i_2} > n^{-q})\cdots P(b_{n,i_N}X_{i_N} > n^{-q})\\ &\le \left(\sum_{i=1}^n P(b_{ni}X_i > n^{-q})\right)^N \le \left(C\sum_{i=1}^n P(|b_{ni}X| > n^{-q})\right)^N\\ &\le \begin{cases} C\left(n^{pq}\sum_{i=1}^n E|b_{ni}X|^p\right)^N, & \text{if } 0<p\le 1,\\ C\left(n^{2pq}\sum_{i=1}^n E|b_{ni}X|^{2p}\right)^N, & \text{if } p>1,\end{cases}\\ &\le \begin{cases} Cn^{-(\delta-pq)N}, & \text{if } 0<p\le 1,\\ C\left[n^{2pq}\left(\max_{1\le i\le n}|b_{ni}|\right)^{2p-2}\sum_{i=1}^n b_{ni}^2\right]^N, & \text{if } p>1,\end{cases}\\ &\le \begin{cases} Cn^{-(\delta-pq)N}, & \text{if } 0<p\le 1,\\ Cn^{-2(1-1/p-pq)N}(\log n)^{-N}, & \text{if } p>1.\end{cases}\end{aligned}$$

Noting that $q < \frac{1-1/p}{p}$ if $p > 1$ and $q < \delta/p$ if $0 < p \le 1$, we choose some large integer N such that $(\delta - pq)N > 1$ and $2(1 - 1/p - pq)N > 1$. Hence, $I_2 < \infty$.

Since $-\varepsilon/N \le b_{ni}X_{ni}(3) < 0$, we can see that

$$\left|\sum_{i=1}^n b_{ni}X_{ni}(3)\right| = -\sum_{i=1}^n b_{ni}X_{ni}(3) > \varepsilon$$

implies that there are at least N integers such that $b_{ni}X_{ni}(3) \neq 0$. Hence, similar to the proof of $I_2 < \infty$, we can get $I_3 < \infty$.

Finally, we will prove that $I_4 < \infty$. By Lemma A.2 and $E|X|^{2p} < \infty$, we have

$$\begin{aligned} I_4 &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|b_{ni}X_i| > \varepsilon/N) \leq C\sum_{n=1}^{\infty} nP(|X| > Cn^{1/p}) \\ &\leq C\sum_{k=1}^{\infty} k^2 P(Ck^{1/p} < |X| \leq C(k+1)^{1/p}) \\ &\leq CE|X|^{2p} < \infty. \end{aligned}$$

This completes the proof of the lemma. □

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References

[1] Deaton, A. (1985). Panel data from a time series of cross-sections. *Journal of Econometrics* **30** 109–126.
[2] Fuller, W. A. (1987). *Measurement Error Models*. Wiley, New York.
[3] Fusek, I. and Fusková, L. (1989). A combined estimator in the simple errors-in-variables model. *Statistical Papers* **30** 1–15.
[4] Carroll, R. J., Ruppert, D. and Stefanski, L. A. (1995). *Measurement Error in Nonlinear Nodels*. Chapman and Hall, New York.
[5] Hslao, C., Wang L. and Wang, Q. (1997). Estimation of nonlinear errors-in-variables models: an approximate solution. *Statistical Papers* **38** 1–25. MR1474937
[6] Amemiya, Y. and Fuller, W. A. (1984). Estimation for the multivariate errors-in-variables model with estimated error covariance matrix. *The Annals of Statistics* **12** 497–509. MR0740908
[7] Cui, H. J. (1997). Asymptotic normality of M-estimates in the EV model. *Journal of System Science and Mathematical Science* **10** 225–236. MR1469182
[8] Cui, H. J. and Chen, S. X. (2003). Empirical likelihood confidence region for parameter in the errors-in-variables models. *Journal of Multivariate Analysis* **84** 101–115. MR1965825
[9] Gleser, L. J. (1981). Estimation in a multivariate 'errors in variables' regression model: large sample results. *The Annals of Statistics* **9** 24–44. MR0600530

[10] Lai, T. L., Robbins, H. and Wei, C. Z. (1979). Strong consistency of least squares estimates in multiple regression. *Journal of Multivariate Analysis* **9** 343–362. MR0548786

[11] Liu, J. X. and Chen, X. R. (2005). Consistency of LS estimator in simple linear EV regression models. *Acta Mathematica Scientia, Series B* **25** 50–58. MR2119336

[12] Miao, Y., Wang, K. and Zhao, F. F. (2011). Some limit behaviors for the LS estimator in simple linear EV regression models. *Statistics & Probability Letters* **81** 92–102. MR2740070

[13] Chen, P. Y., Kong, N. N. and Sung, S. H. (2017). Complete convergence for weighted sums of i.i.d. random variables with applications in regression estimation and EV model. *Communications in Statistics-Theory and Methods* **46** 3599–3613. MR3589120

[14] Fazekas, I. and Kukush, A. G. (1997). Asymptotic properties of an estimator in nonlinear functional errors-in-variables models with dependent error terms. *Computers & Mathematics with Applications* **34** 23–39. MR1487730

[15] Miao, Y., Zhao, F. F., Wang, K. and Chen, Y. P. (2013). Asymptotic normality and strong consistency of LS estimators in the EV regression model with NA errors. *Statistical Papers* **54** 193–206. MR3016962

[16] Miao, Y., Wang, Y. L. and Zheng, H. J. (2015). Consistency of LS estimators in the EV regression model with martingale difference errors. *Statistics: A Journal of Theoretical and Applied Statistics* **49** 104–118. MR3304370

[17] Wang, S. C. (2015). MDP for estimators in EV regression models with α-mixing errors. *Statistics: A Journal of Theoretical and Applied Statistics* **49** 119–127. MR3304371

[18] Joag-Dev, K. and Proschan, F. (1983). Negative association of random variables with applications. *The Annals of Statistics* **11** 286–295. MR0684886

[19] Bozorgnia, A., Patterson, R. F. and Taylor, R. L. (1996). Limit theorems for dependent random variables. *World Congress Nonlinear Analysts'92* 1639–1650. MR1389197

[20] Taylor, R.L., Patterson, R. F. and Bozorgnia, A. (2002). A strong law of large numbers for arrays of rowwise negatively dependent random variables. *Stochastic Analysis and Applications* **20** 643–656. MR1900307

[21] Asadian, N., Fakoor, V. and Bozorgnia, A. (2006). Rosenthal's type inequalities for negatively orthant dependent random variables. *Journal of the Iranian Statistical Society* **5** 66–75.

[22] Zarei, H. and Jabbari, H. (2011). Complete convergence of weighted sums under negative dependence. *Statistical Papers* **52** 413–418. MR2795888

[23] Sung, S. H. (2011). On the strong convergence for weighted sums of random variables. *Statistical Papers* **52** 447–454. MR2795891

[24] Qiu, D. H., Chang, K. C., Antoninic, R. G. and Volodin, A. (2011). On the strong rates of convergence for arrays of rowwise negatively depen-

dent random variables. *Stochastic Analysis and Applications* **29** 375–385. MR2795551

[25] Wu, Q. Y. and Jiang, Y. Y. (2011). The strong consistency of M estimator in a linear model for negatively dependent random samples. *Communications in Statistics-Theory and Methods* **40** 467–491. MR2765842

[26] Shen, A. T. (2013). On the strong convergence rate for weighted sums of arrays of rowwise negatively orthant dependent random variables. *RACSAM* **107** 257–271. MR3199709

[27] Shen, A. T. (2014). On asymptotic approximation of inverse moments for a class of nonnegative random variables. *Statistics: A Journal of Theoretical and Applied Statistics* **48** 1371–1379. MR3269741

[28] Wang, X. J., Zheng, L. L., Xu, C. and Hu, S. H. (2015). Complete consistency for the estimator of nonparametric regression models based on extended negatively dependent errors. *Statistics: A Journal of Theoretical and Applied Statistics* **49** 396–407. MR3325366

[29] Wu, Q. Y. (2011). Complete convergence for weighted sums of sequences of negatively dependent random variables. *Journal of Probability and Statistics* **2011** Article ID 202015, 16 pages. MR2774947

[30] Jing, B. Y. and Liang, H. Y. (2008). Strong limit theorems for weighted sums of negatively associated random variables. *Journal of Theoretical Probability* **21** 890–909. MR2443640

[31] Zhang, G. H. (2014). Complete convergence for Sung's type weighted sums of END random variables. *Journal of Inequalities and Applications* **2014** Article ID 353, 11 pages. MR3317501

[32] Wu, Q. Y. (2006). *Probability Limit Throey for Mixing Sequences.* Science Press of China, Beijing.

[33] Shen, A. T., Zhang, Y. and Volodin, A. (2015). Applications of the Rosenthal-type inequality for negatively superadditive dependent random variables. *Metrika* **78** 295–311. MR3320899