

Sharp minimax tests for large covariance matrices and adaptation*

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Abstract: We consider the detection problem of correlations in a p -dimensional Gaussian vector, when we observe n independent, identically distributed random vectors, for n and p large. We assume that the covariance matrix varies in some ellipsoid with parameter $\alpha > 1/2$ and total energy bounded by $L > 0$.

We propose a test procedure based on a U-statistic of order 2 which is weighted in an optimal way. The weights are the solution of an optimization problem, they are constant on each diagonal and non-null only for the T first diagonals, where $T = o(p)$. We show that this test statistic is asymptotically Gaussian distributed under the null hypothesis and also under the alternative hypothesis for matrices close to the detection boundary. We prove upper bounds for the total error probability of our test procedure, for $\alpha > 1/2$ and under the assumption $T = o(p)$ which implies that $n = o(p^{2\alpha})$. We illustrate via a numerical study the behavior of our test procedure.

Moreover, we prove lower bounds for the maximal type II error and the total error probabilities. Thus we obtain the asymptotic and the sharp asymptotically minimax separation rate $\tilde{\varphi} = (C(\alpha, L)n^2p)^{-\alpha/(4\alpha+1)}$, for $\alpha > 3/2$ and for $\alpha > 1$ together with the additional assumption $p = o(n^{4\alpha-1})$, respectively. We deduce rate asymptotic minimax results for testing the inverse of the covariance matrix.

We construct an adaptive test procedure with respect to the parameter α and show that it attains the rate $\tilde{\psi} = (n^2p/\ln \ln(n\sqrt{p}))^{-\alpha/(4\alpha+1)}$.

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1. Introduction

A large variety of applied fields collect and need to recover information from high-dimensional data. Among these we can cite communications and signal theory (functional magnetic resonance imaging, spectroscopic imaging), econometrics, climate studies, biology (gene expression micro-array) and finance (port-

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folio allocation). Testing large covariance matrix is an important problem and has recently been approached via several techniques: corrected likelihood ratio test using the theory of large random matrices, methods based on the sample covariance matrix and so on.

Let X_1, \dots, X_n , be n independent and identically distributed p -vectors following a multivariate normal distribution $\mathcal{N}_p(0, \Sigma)$, where $\Sigma = [\sigma_{ij}]_{1 \leq i, j \leq p}$ is the normalized covariance matrix, with $\sigma_{ii} = 1$, for all $i = 1$ to p . Let us denote by $X_k = (X_{k,1}, \dots, X_{k,p})^T$ for all $k = 1, \dots, n$. In this paper we also assume that the size p of the vectors grows to infinity as well as the sample size n , $p \rightarrow \infty$ and $n \rightarrow \infty$. We assume here that Σ belongs to the class $\mathcal{F}(\alpha, L)$ defined for $\alpha > 0$ by

$$\mathcal{F}(\alpha, L) = \left\{ \Sigma \geq 0 ; \frac{1}{p} \sum_{1 \leq i < j \leq p} \sigma_{ij}^2 |i - j|^{2\alpha} \leq L \text{ for all } p \text{ and } \sigma_{ii} = 1 \right. \\ \left. \text{for all } i = 1, \dots, p \right\}.$$

For any covariance matrix Σ , we recall that the Frobenius norm is computed as

$$\|\Sigma - I\|_F^2 = \text{tr}[(\Sigma - I)^2] = 2 \sum_{1 \leq i < j \leq p} \sigma_{ij}^2.$$

We consider the goodness-of-fit test, with null hypothesis

$$H_0 : \Sigma = I, \quad \text{where } I \text{ is the } p \times p \text{ identity matrix} \quad (1)$$

against the composite alternative hypothesis

$$H_1 : \Sigma \in \mathcal{F}(\alpha, L), \quad \text{such that } \frac{1}{2p} \|\Sigma - I\|_F^2 \geq \varphi^2.$$

In order to test $H_0 : \Sigma = \Sigma_0$, for some given non negative definite covariance matrix Σ_0 , we suggest rescaling the data $Z_i = \Sigma_0^{-1/2} X_i$ and then apply the same test procedure provided that $\Sigma_0^{-1/2} \Sigma \Sigma_0^{-1/2}$ belongs to $\mathcal{F}(\alpha, L)$. Let us denote by

$$Q(\alpha, L, \varphi) = \left\{ \Sigma \in \mathcal{F}(\alpha, L) ; \frac{1}{p} \sum_{1 \leq i < j \leq p} \sigma_{ij}^2 \geq \varphi^2 \right\}, \quad (2)$$

where $\varphi = \varphi_{n,p}(\alpha, L)$ is related to n and p , but also to α and L assumed fixed. The set of covariance matrices under the alternative hypothesis consists of matrices of size $p \times p$, whose elements decrease polynomially when moving away from the diagonal. This assumption is natural for covariances matrices and has been considered for estimation problems, see e.g [3], [11]. Regularization techniques, originally used for nonparametric estimation of functions, were successfully employed to the estimation of large covariance matrices. Among these works, let us mention minimax and adaptive minimax results: via banding the covariance matrix [3], thresholding the entries of the empirical covariance matrix [4], block-thresholding [10], tapering [11], ℓ_1 -estimation [12] and so on.

Unlike the estimation of the covariance matrix, there are very few works for testing in a minimax setup in the existing literature.

Several types of test statistics have been proposed in the literature in order to test the null hypothesis (1). The likelihood ratio (LR) statistic, was first designed for fixed p and $n \rightarrow +\infty$. To treat the high dimensional case when $n, p \rightarrow +\infty$, [1] proposed a correction to the LR statistic and showed its convergence in law under the null hypothesis, as soon as $p/n \rightarrow c$, for some fixed $c \in (0, 1)$. Indeed, this correction is based on the asymptotic behavior of the spectrum of the covariance matrix. Another approach is based on the largest magnitude of the off-diagonal entries of the sample correlation matrix and was introduced by [19]. Later, [8] and [22] show an original limit behavior of Gumbel type for a self-normalized version of the maximum deviation of the sample covariance matrix. We also note that a non-asymptotic sphericity test for Gaussian vectors was studied by [2]. The alternative is given by a model with rank-one and sparse additive perturbation in the variance.

Furthermore, an approach based on the quadratic form $U_n = (1/p)\text{tr}(S_n - I)^2$, where $S_n = (1/n) \sum_{i=1}^n X_i X_i^\top$ is the sample covariance matrix, was proposed by [21], to test (1). Later, [20] shows that the test of H_0 based on U_n is not consistent for large p . They introduce a corrected version of U_n and study its asymptotic behavior when $n, p \rightarrow \infty$ and $p/n \rightarrow c \in (0 + \infty)$. In order to deal with non Gaussian random vectors, and without specifying any relation between n and p , [13] proposed a U-statistic of order 2, as a new correction of the previous quadratic form. They do moment assumptions in order to show the asymptotic behavior of their U-statistic, under the null and under a fixed alternative hypothesis. Motivated by their work, [9] used the U-statistic given in [13] to test (1) from a sample of Gaussian vectors, and studied the testing problem from a minimax point of view. They consider the alternative hypothesis $H_1 : \Sigma$ such that $\|\Sigma - I\|_F \geq \varphi$ and they establish the minimax rates of order $\sqrt{p/n}$ in this case. In our setup the restriction to the ellipsoid $\mathcal{F}(\alpha, L)$ leads to different rates for testing.

In this paper, we introduce a U-statistic, which is weighted in an optimal way for our problem. This can also be seen as a regularization technique for estimating a quadratic functional, as it is often the case in minimax nonparametric test theory (see [18]). We use this test statistic to construct an asymptotically minimax test procedure. Let us stress the fact that we study the type II error probability uniformly over the set of all matrices Σ under the alternative and that induces a separation rate saying how close Σ can be to the identity matrix I and still be distinguishable from I . We describe the sharp separation rates for fixed unknown α and give an adaptive procedure free of α that allows to test at the price of a logarithmic loss in the rate.

We describe here the rate asymptotics of the error probabilities from the minimax point of view. We recall that a test procedure Δ is a measurable function with respect to the observations, taking values in $[0, 1]$. Set $\eta(\Delta) = \mathbb{E}_I(\Delta) = \mathbb{P}_I(\Delta = 1)$ its type I error probability, $\beta(\Delta, Q(\alpha, L, \varphi)) = \sup_{\Sigma \in Q(\alpha, L, \varphi)} \mathbb{E}_\Sigma(1 - \Delta) = \sup_{\Sigma \in Q(\alpha, L, \varphi)} \mathbb{P}_\Sigma(\Delta = 0)$ its maximal type II error probability over the

set $Q(\alpha, L, \varphi)$, and by

$$\gamma(\Delta, Q(\alpha, L, \varphi)) = \eta(\Delta) + \beta(\Delta, Q(\alpha, L, \varphi))$$

the total error probability of Δ . Let us denote by γ the minimax total error probability over $Q(\alpha, L, \varphi)$ which is defined by

$$\gamma = \gamma(\varphi) := \gamma(Q(\alpha, L, \varphi)) = \inf_{\Delta} \gamma(\Delta, Q(\alpha, L, \varphi))$$

where the infimum is taken over all test procedures. We want to describe the separation rate $\tilde{\varphi} = \tilde{\varphi}(n, p)$ such that, on the one hand,

$$\gamma \rightarrow 1 \quad \text{if} \quad \frac{\varphi}{\tilde{\varphi}} \rightarrow 0.$$

In this case we say that we can not distinguish between the two hypotheses. On the other hand, we exhibit an explicit test procedure Δ^* such that its total error probability tends to 0

$$\gamma(\Delta^*, Q(\alpha, L, \varphi)) \rightarrow 0 \quad \text{if} \quad \frac{\varphi}{\tilde{\varphi}} \rightarrow +\infty.$$

We say that Δ^* is asymptotically minimax consistent test and $\tilde{\varphi}$ is the asymptotically minimax separation rate.

In this paper, we find asymptotically minimax rates for testing over the class $\mathcal{F}(\alpha, L)$. The minimax consistent test procedure is based on a U-statistic of second order, weighted in an optimal way. In this, our procedure is very different from known corrected procedures based on quadratic forms of the sample covariance matrix, see e.g. [20]. This is the first time a weighted test-statistic is used for testing covariance matrices.

Moreover, our rates are sharp minimax. We show a Gaussian asymptotic behaviour of the test statistic in the neighbourhood of the separation rate. We get the following sharp asymptotic expression for the maximal type II probability error, under some assumptions relating φ , n and p ,

$$\inf_{\Delta: \eta(\Delta) \leq w} \beta(\Delta, Q(\alpha, L, \varphi)) = \Phi(z_{1-w} - n\sqrt{p}b(\varphi)) + o(1),$$

where Φ denotes the cumulative distribution function (cdf) of the standard Gaussian distribution and z_{1-w} is the $1 - w$ quantile of the standard Gaussian distribution for any $w \in (0, 1)$. We deduce that the sharp minimax total error probability is of the type

$$\gamma(\varphi) = 2\Phi(-n\sqrt{p}b(\varphi)/2) + o(1),$$

where $b^2(\varphi) = C(\alpha, L)\varphi^{4+1/\alpha}$ as $\varphi \rightarrow 0$, $C(\alpha, L)$ is explicitly given. It is usual to call the asymptotically sharp minimax rate

$$\tilde{\varphi} = (C(\alpha, L)n^2p)^{-\alpha/(4\alpha+1)},$$

corresponding to $n^2pb^2(\tilde{\varphi}) = 1$ and to the asymptotic testing constant $C(\alpha, L)$.

Analogous results were obtained by [7] in the particular case where the covariance matrix is Toeplitz, that is $\sigma_{i,j} = \sigma_{|i-j|}$ for all different i and j from 1 to p . We note a gain of a factor p in the minimax rate. The asymptotically sharp minimax rate for Toeplitz covariance matrices is

$$\tilde{\varphi}_T = (C(\alpha, L)n^2p^2)^{-\alpha/(4\alpha+1)}.$$

This additional factor p can be heuristically explained by the number of parameters $p - 1$ for a Toeplitz matrix, instead of $p(p - 1)/2$ for an arbitrary covariance matrix. For $n = 1$ the test problem for Toeplitz covariance matrices was solved in the sharp asymptotic framework, as $p \rightarrow \infty$, by [14]. Let us also recall that the adaptive rates (to α) for minimax testing are obtained for the spectral density problem by [15] by a non constructive method using the asymptotic equivalence with a Gaussian white noise model. We also give an adaptive procedure for testing without prior knowledge on α , for α belonging to a closed subset of $(1/2, +\infty)$.

Important generalizations of this problem include testing in a minimax setup of composite null hypotheses like sphericity, $H_0 : \Sigma = v^2 \cdot I$, for unknown v^2 in some compact set separated from 0, or bandedness, $H_0 : \Sigma = \Sigma_0$ such that $[\Sigma_0]_{ij} = 0$ for all $i \neq j$ with $|i - j| > K$. Our proofs rely on the Gaussian distribution of Gaussian vectors. Generalizations to non Gaussian distributions with finite moments of some order can be proposed under additional assumptions on the behaviour of higher order moments, like e.g. [13].

Section 2 introduces the test statistic and studies its asymptotic properties. Next we give upper bounds for the maximal type II error probability and for the total error probability and refine these results to sharp asymptotics under the condition that $n = o(1)p^{2\alpha}$. In Section 3 we implement our test procedure and estimate its power. In Section 4 we prove sharp asymptotic optimality and deduce the optimality of the minimax separation rates for all $\alpha > 1$ and as soon as $p = o(n^{4\alpha-1})$. In Section 5 we present the rate minimax results for testing the inverse of the covariance matrix. In Section 6 we define an adaptive test procedure and show that the price of adaptation is a loss of $(\ln \ln(n\sqrt{p}))^{\alpha/(4\alpha+1)}$ in the separation rate.

Proofs are given in Section 7 and in the Appendix.

2. Test procedure and asymptotic properties

In the minimax theory of tests developed since [17] it is well understood that optimal test statistics are estimators (suitably normalized and tuned) of the functional which defines the separation of an element in the alternative from the element of the null hypothesis. In our case this is the Frobenius norm $\|\Sigma - I\|_F^2 = \text{tr}[(\Sigma - I)^2]$.

Weighting the elements of the sample covariance matrix appeared first as hard thresholding in minimax estimation of large covariance matrices. Let us mention [3] for banding i.e. truncation of the matrix to its k first diagonals (closest to the

main diagonal), [4] for hard thresholding, then [11] where tapering was studied. It is a natural idea when coming from minimax nonparametric estimation.

However, that was never used for tests concerning large covariance matrices. In this section, we introduce a weighted U-statistic of order 2 for testing large covariance matrices, study its asymptotic properties and give asymptotic upper bounds for the minimax rates of testing.

From now on asymptotics and symbols o , O , \sim and \asymp are considered as n and p tend to infinity and as φ tends to 0. Recall that, given sequences of real numbers u and real positive numbers v , we say that they are asymptotically equivalent, $u \sim v$, if $\lim u/v = 1$. Moreover, we say that the sequences are asymptotically of the same order, $u \asymp v$, if there exist two constants $0 < c \leq C < \infty$ such that $c \leq \liminf u/v$ and $\limsup u/v \leq C$.

2.1. Test statistic and its asymptotic behaviour

Our test statistic is a weighted U-statistic of order 2. It can be also seen as a weighted functional of the sample covariance matrix. The weights w_{ij}^* are constant on each diagonal (they depend on i and j only through $i - j$), non-zero only for $|i - j| \leq T$ for some large integer T and decreasing polynomially for elements further from the main diagonal (as $|i - j|$ is increasing). More precisely, we consider the following test statistic:

$$\widehat{D}_n = \frac{1}{n(n-1)p} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq i < j \leq p} w_{ij}^* X_{k,i} X_{k,j} X_{l,i} X_{l,j} \quad (3)$$

where

$$w_{ij}^* = \frac{\lambda}{2b(\varphi)} \left(1 - \left(\frac{|i-j|}{T} \right)_+^{2\alpha} \right), \quad T = \lfloor C_T(\alpha, L) \cdot \varphi^{-\frac{1}{\alpha}} \rfloor \quad (4)$$

$$\lambda = C_\lambda(\alpha, L) \cdot \varphi^{\frac{2\alpha+1}{\alpha}}, \quad b(\varphi) = C^{1/2}(\alpha, L) \cdot \varphi^{2+\frac{1}{2\alpha}}$$

with

$$C_T(\alpha, L) = ((4\alpha + 1)L)^{\frac{1}{2\alpha}}, \quad C_\lambda(\alpha, L) = \frac{2\alpha + 1}{2\alpha} ((4\alpha + 1)L)^{-\frac{1}{2\alpha}}, \quad (5)$$

$$C(\alpha, L) = \frac{2\alpha + 1}{(4\alpha + 1)^{1+1/(2\alpha)}} L^{-\frac{1}{2\alpha}}.$$

The weights $\{w_{ij}^*\}_{i,j}$ and the parameters $T, \lambda, b^2(\varphi)$ are asymptotically equivalent (as $\varphi \rightarrow 0$ such that $p/T \asymp p\varphi^{1/\alpha} \rightarrow \infty$) to the solution of the optimization problem:

$$\frac{1}{p} \sum_{1 \leq i < j \leq p} w_{ij}^* \sigma_{ij}^{*2} = \sup_{\left\{ (w_{ij})_{i,j} : w_{ij} \geq 0; \right.} \inf_{\left. \left\{ \Sigma : \Sigma = (\sigma_{ij})_{i,j}; \right. \right\}} \frac{1}{p} \sum_{1 \leq i < j \leq p} w_{ij} \sigma_{ij}^2. \quad (6)$$

$$\left\{ \frac{1}{p} \sum_{1 \leq i < j \leq p} w_{ij}^2 = \frac{1}{2} \right\} \left\{ \Sigma \in Q(\alpha, L, \varphi) \right\}$$

Indeed our test statistic $\widehat{\mathcal{D}}_n$ will concentrate asymptotically around $\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) = (1/p) \sum_{1 \leq i < j \leq p} w_{ij} \sigma_{ij}^2$ which is 0 for $\Sigma = I$. The minimax paradigm considers the worst parameter Σ^* in the class $Q(\alpha, L, \varphi)$, that will give the smallest value $\mathbb{E}_{\Sigma^*}(\widehat{\mathcal{D}}_n(w_{ij}))$ and then finds the parameters $\{w_{ij}^*\}_{i < j}$ of the test statistic to provide the largest value $\mathbb{E}_{\Sigma^*}(\widehat{\mathcal{D}}_n(w_{ij}^*))$. Such procedure performs uniformly well over all parameters $\Sigma \in Q(\alpha, L, \varphi)$. That explains why we solve the optimization problem (6).

Note that the weights in (4) have further properties, as $\varphi \rightarrow 0$ and $p\varphi^{1/\alpha} \rightarrow \infty$:

$$w_{ij}^* \geq 0, \quad \sup_{i,j} w_{ij}^* \leq \frac{\lambda}{2b(\varphi)} \leq O(1)\varphi^{\frac{1}{2\alpha}} \asymp \frac{1}{\sqrt{T}}$$

and

$$\begin{aligned} \frac{1}{p} \sum_{1 \leq i < j \leq p} w_{ij}^{*2} &= \frac{\lambda^2}{4pb^2(\varphi)} \sum_{1 \leq i < j \leq p} \left(1 - \left(\frac{|i-j|}{T}\right)^{2\alpha}\right)_+^2 \\ &\sim \frac{\lambda^2}{4pb^2(\varphi)} \cdot pT \left(1 - \frac{2}{2\alpha+1} + \frac{1}{4\alpha+1}\right) \\ &= \frac{C_\lambda^2(\alpha, L) \cdot C_T(\alpha, L)}{4C(\alpha, L)} \cdot \frac{8\alpha^2}{(2\alpha+1)(4\alpha+1)} = \frac{1}{2}. \end{aligned}$$

The following Proposition gives the moments of $\widehat{\mathcal{D}}_n$ under the null and their bounds under the alternative hypothesis, respectively, as well as the asymptotic normality under the null hypothesis.

Proposition 1. *The test statistic $\widehat{\mathcal{D}}_n$ defined by (3) with parameters given by (4) and (5) has the following moments, under the null hypothesis:*

$$\mathbb{E}_I(\widehat{\mathcal{D}}_n) = 0, \quad \text{Var}_I(\widehat{\mathcal{D}}_n) = \frac{2}{n(n-1)p^2} \sum_{1 \leq i < j \leq p} w_{ij}^{*2} = \frac{1}{n(n-1)p}$$

and is asymptotically normal

$$n\sqrt{p} \widehat{\mathcal{D}}_n \xrightarrow{d} \mathcal{N}(0, 1).$$

Moreover, under the alternative, if we assume that $\varphi \rightarrow 0$, $p\varphi^{1/\alpha} \rightarrow \infty$ and $\alpha > 1/2$, we have, for all Σ in $Q(\alpha, L, \varphi)$:

$$\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) = \frac{1}{p} \sum_{1 \leq i < j \leq p} w_{ij}^* \sigma_{ij}^2 \geq b(\varphi) \quad \text{and} \quad \text{Var}_\Sigma(\widehat{\mathcal{D}}_n) = \frac{T_1}{n(n-1)p^2} + \frac{T_2}{np^2},$$

where

$$T_1 \leq p \cdot (1 + o(1)) + p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot O(T\sqrt{T}), \tag{7}$$

$$T_2 \leq p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot O(\sqrt{T}) + p^{3/2} \left(\mathbb{E}_\Sigma^{3/2}(\widehat{\mathcal{D}}_n) \cdot O(T^{3/4}) + \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot o(1) \right). \tag{8}$$

Note that, under the alternative, we have the additional assumption that $p\varphi^{1/\alpha} \asymp p/T \rightarrow +\infty$, when p grows to infinity.

Let us look closer at the optimization problem (6): for given $\varphi > 0$, $b(\varphi)$ is the least value that $\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)$ can take when Σ is in the alternative set of hypotheses.

Under the alternative, we shall establish the asymptotic normality under additional conditions that the underlying covariance matrix is close to the solution of (6). This will be sufficient to give upper bounds of the total error probability of Gaussian type in the next Section.

Proposition 2. *The test statistic $\widehat{\mathcal{D}}_n$ defined by (3) with parameters given by (4) and (5), such that $\varphi \rightarrow 0$, $p\varphi^{1/\alpha} \rightarrow \infty$ and under the additional assumption that $n^2pb^2(\varphi) \asymp 1$, is asymptotically normal:*

$$n\sqrt{p}(\widehat{\mathcal{D}}_n - \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)) \xrightarrow{d} \mathcal{N}(0, 1),$$

for any Σ in $Q(\alpha, L, \varphi)$ such that $\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) = O(b(\varphi))$.

2.2. Upper bounds for the error probabilities

In order to distinguish between the two hypothesis H_0 and H_1 defined previously, we propose the following test procedure

$$\Delta^* = \Delta^*(t) = \mathbf{1}(\widehat{\mathcal{D}}_n > t), \quad t > 0 \tag{9}$$

where $\widehat{\mathcal{D}}_n$ is the estimator defined in (3).

The following theorem proves that the previously defined test procedure is minimax consistent if t is conveniently chosen.

Theorem 1. *The test procedure Δ^* defined in (9) with $t > 0$ has the following properties :*

Type I error probability : if $n\sqrt{p} \cdot t \rightarrow +\infty$ then $\eta(\Delta^) \rightarrow 0$.*

Type II error probability : if $\alpha > 1/2$ and if

$$\varphi \rightarrow 0, p\varphi^{1/\alpha} \rightarrow \infty \text{ and } n^2pb^2(\varphi) \rightarrow +\infty$$

then, uniformly over t such that $t \leq c \cdot C^{1/2}(\alpha, L) \cdot \varphi^{2+\frac{1}{2\alpha}}$, for some constant c in $(0, 1)$, we have

$$\beta(\Delta^*(t), Q(\alpha, L, \varphi)) \rightarrow 0.$$

If t verifies all previous assumptions, then $\Delta^(t)$ is asymptotically minimax consistent:*

$$\gamma(\Delta^*(t), Q(\alpha, L, \varphi)) \rightarrow 0.$$

In the next Theorem we give sharp upper bounds of error probabilities of Gaussian type. The proof of this result explains the choice of the weights as solution of the optimization problem (6). Moreover, we will see that the Gaussian behavior is obtained near the separation rates.

Recall that Φ is the cumulative distribution function (cdf) of standard Gaussian random variable and, for any $w \in (0, 1)$, z_{1-w} is defined by $\Phi(z_{1-w}) = 1-w$.

Theorem 2. *The test procedure Δ^* defined in (9) with $t > 0$ has the following properties :*

Type I error probability : we have $\eta(\Delta^(t)) = 1 - \Phi(n\sqrt{p} \cdot t) + o(1)$.*

Type II error probability : if $\alpha > 1/2$ and if

$$\varphi \rightarrow 0, p\varphi^{1/\alpha} \rightarrow \infty \text{ and } n^2pb^2(\varphi) \asymp 1, \tag{10}$$

then, uniformly over t , we have

$$\beta(\Delta^*(t), Q(\alpha, L, \varphi)) \leq \Phi(n\sqrt{p} \cdot (t - b(\varphi))) + o(1).$$

In particular, for $t = t^w$ such that $n\sqrt{p} \cdot t^w = z_{1-w}$ we have $\eta(\Delta^*(t^w)) \leq w + o(1)$ and

$$\beta(\Delta^*(t^w), Q(\alpha, L, \varphi)) \leq \Phi(z_{1-w} - n\sqrt{p} \cdot b(\varphi)) + o(1).$$

Another important consequence of the previous theorem, is that the test procedure $\Delta^*(t^*)$, with $t^* = b(\varphi)/2$ has total error probability

$$\gamma(\Delta^*(t^*), Q(\alpha, L, \varphi)) \leq 2\Phi\left(-n\sqrt{p} \frac{b(\varphi)}{2}\right) + o(1).$$

3. Simulation study

We include several examples, to illustrate the numerical behavior of our test procedure. First, we estimate the type I error probabilities to test the null hypothesis $\Sigma = I$. We implement $\widehat{\mathcal{D}}_n$ defined in (3) and (4) for $\alpha = L = 1$ and $\varphi = \widehat{\varphi}$. We choose the threshold t such that $t = t_\eta$ where $t_\eta = z_{1-\eta}/(n\sqrt{p})$ and $z_{1-\eta}$ is the $(1-\eta)$ -quantile of the standard normal distribution. We show in Table 1 the estimated type I error probabilities $P_I(\widehat{\mathcal{D}}_n > t_\eta)$ which are obtained from 1000 repeated samples.

We observe that the estimated probabilities are all close to the corresponding significant levels η except for p too small compared to n .

TABLE 1
Estimated type I error probabilities $P_I(\widehat{\mathcal{D}}_n > t_\eta)$, when $\eta \in \{0.01, 0.05, 0.1\}$ and for $p \in \{10, 20, \dots, 90\}$ and $n \in \{30, 60, 90\}$

$\eta \setminus p$	10	20	30	40	50	60	70	80	90	n
0.01	0.004	0.012	0.011	0.008	0.015	0.011	0.010	0.015	0.012	30
	0.005	0.004	0.013	0.013	0.012	0.007	0.009	0.009	0.008	60
	0.002	0.005	0.006	0.008	0.008	0.011	0.007	0.007	0.006	90
0.05	0.022	0.048	0.038	0.046	0.045	0.043	0.047	0.046	0.034	30
	0.013	0.036	0.033	0.043	0.059	0.048	0.041	0.047	0.050	60
	0.008	0.030	0.040	0.044	0.044	0.039	0.039	0.041	0.048	90
0.1	0.049	0.075	0.089	0.072	0.104	0.085	0.074	0.097	0.085	30
	0.040	0.076	0.078	0.082	0.077	0.107	0.098	0.098	0.073	60
	0.019	0.061	0.058	0.074	0.085	0.078	0.094	0.095	0.089	90

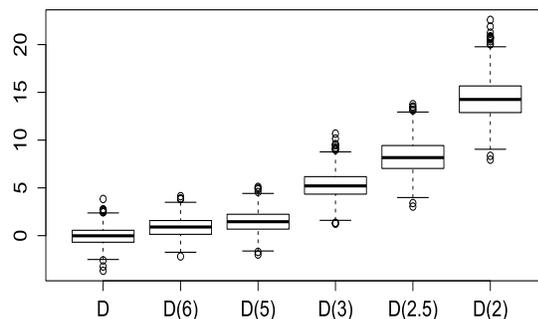


FIG 1. Boxplots of D and $D(M) = n\sqrt{p} \cdot \widehat{D}_n$, when $n = 50$ and $p = 80$, under the null hypothesis I and under the alternative hypothesis $\Sigma(M)$.

Second, we illustrate the asymptotic behavior of our test procedure under the null hypothesis and under alternative hypothesis defined by the symmetric positive matrices

$$\Sigma(M) = (\sigma_{ij}(M))_{1 \leq i, j \leq p}; \quad \sigma_{ij}(M) = \mathbf{1}_{(i=j)} + \frac{u_{ij} \cdot |i-j|^{-2}}{M} \cdot \mathbf{1}_{(i \neq j)}, \quad (11)$$

for $M \in [3/2, 9]$ and where $u_{ij} = u_{ji}$ are independently sampled from a uniform distribution on $[0, 1]$ and their role is to produce non-Toeplitz matrices. Note that this set of matrices has been considered in the literature see e.g. [10] and [11]. We implement the test statistic \widehat{D}_n for $\alpha = L = 1$, and

$$\varphi = \varphi(M) = \frac{1}{M\sqrt{p}} \cdot \left(\sum_{i < j} u_{ij}^2 \cdot |i-j|^{-4} \right)^{1/2}.$$

We denote by $D(M) = n\sqrt{p} \cdot \widehat{D}_n$ the test statistic associated to $\Sigma(M)$ and by D the one associated to I , the identity matrix.

Third we study the power of our test procedure. We test the null hypothesis $\Sigma = I$ against the alternative hypothesis constituted by the matrices $\Sigma(M)$ given in (11). We choose the threshold t of the test empirically, under the null hypothesis $\Sigma = I$, from 1000 repeated samples of size n , such that the type I error probability is close to and smaller than 0.05. We use t to estimate the type II error probability, also from 1000 repetitions and then plot the power as function of $\varphi(M)$.

Figure 2 shows that the power is an increasing function of $\varphi(M)$. Also, we can see that for a fixed value of $\varphi(M)$ the power increases with p . Indeed, our procedure benefits from large values of p , which is not a nuisance parameter here.

We also compare our test procedure to the one given in [9], which is based on a U -statistic of order 2, denoted here by CM-test. We consider different cases that are characterized by different values of n and p .

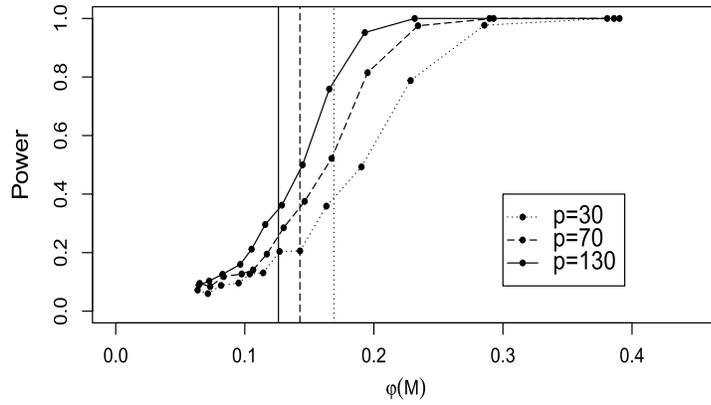


FIG 2. Power curves of the Δ -test as function of $\varphi(M)$ for $n = 30$ and $p \in \{30, 70, 130\}$

Figure 3, shows that for p smaller than, equal to or bigger than n , the Δ -test based on the weighted U-statistics \mathcal{D}_n has larger power than the CM-test based on the non weighted U-statistic. Moreover, the improvement of the Δ -test over the CM-test is larger, when the ratio p/n is bigger.

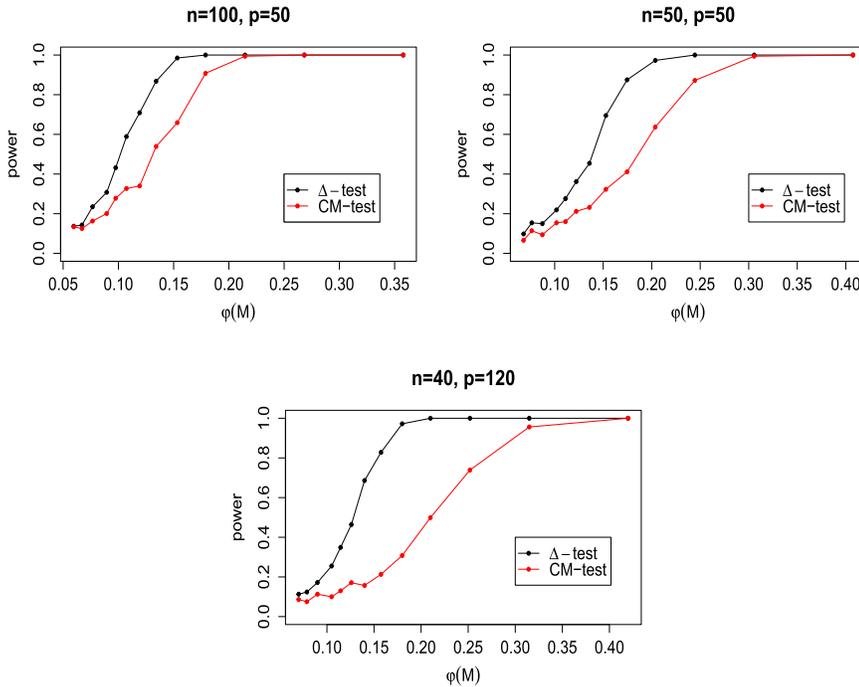


FIG 3. Power curves of the Δ -test and the CM-test as functions of $\varphi(M)$

Let us consider the tridiagonal matrices $\Sigma(\rho) = (\mathbb{1}_{\{i=j\}} + \rho\mathbb{1}_{\{|i-j|=1\}})_{1 \leq i, j \leq p}$, for $\rho \in (0, 0.35]$ under the alternative hypothesis. These are the covariance matrices of a MA(1) Gaussian processes. Moreover, the matrices $\Sigma(\rho)$ are Toeplitz, we also compare our the procedure to the one proposed in [7] for Toeplitz covariance matrices, that we denote by BZ-test. The thresholds are evaluated empirically for each procedure at type I error probability close to and smaller than 0.05. We plot the power curves of the three test procedures.

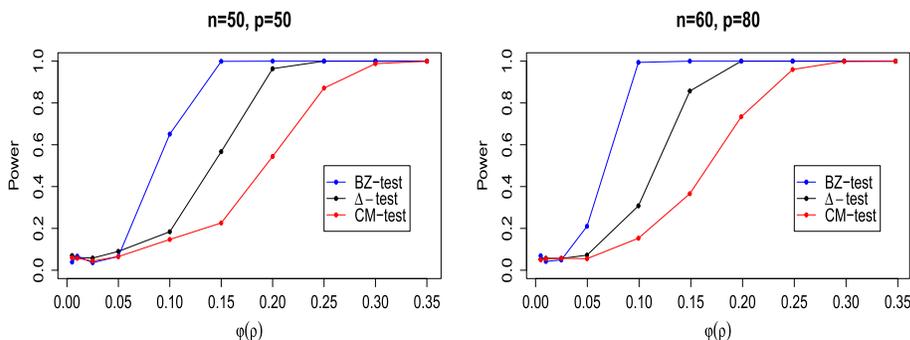


FIG 4. Power curves of the BZ-test, Δ -test and CM-test as functions of $\varphi(\rho)$, for MA(1) Gaussian processes

Figure 4 shows that, when the alternative hypothesis consists of Toeplitz matrices the BZ-test has the better performance. However if we miss the information that the matrix is Toeplitz, we see that the Δ -test is not bad and its power dominates the power of the CM-test.

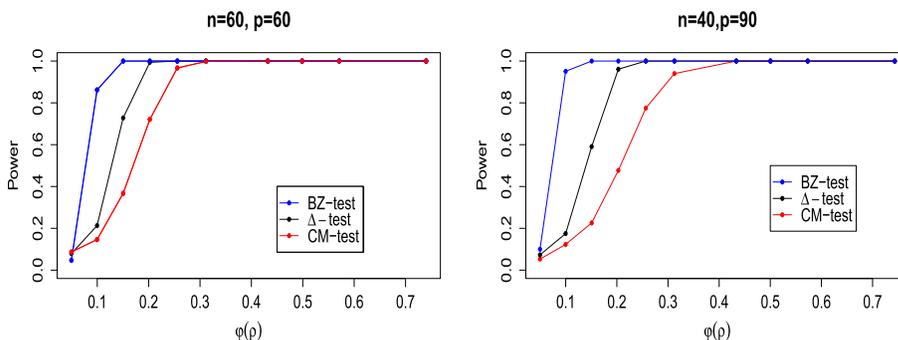


FIG 5. Power curves of the BZ-test, Δ -test and CM-test as functions of $\varphi(\rho)$, for AR(1) Gaussian processes

Finally, as a third example, we consider the covariance matrices of AR(1) Gaussian processes

$$\Sigma(\rho) = (\sigma_{ij}(\rho))_{1 \leq i, j \leq p}; \quad \sigma_{ij}(\rho) = \rho^{|i-j|}$$

for $\rho \in (0, 0.6]$, under the alternative hypothesis. In this case we take $L = 2$. Similarly to the previous cases the thresholds are estimated so that the type I error probability is fixed at 0.05, and we compute the power curves of the different test procedures.

4. Asymptotic optimality

In this section, we first state the lower bound for testing, which, in addition to the test procedure exhibited in the previous section, shows that the asymptotically minimax separation rate is

$$\tilde{\varphi} = \left(n\sqrt{p} C^{1/2}(\alpha, L) \right)^{-\frac{2\alpha}{4\alpha+1}}, \tag{12}$$

where the constant $C(\alpha, L)$ is given by (5).

Theorem 3. *Assume that, either $\alpha > 3/2$, or $\alpha > 5/8$ and $np\varphi^{6-\frac{2}{\alpha}} \rightarrow 0$. If*

$$\varphi \rightarrow 0, p\varphi^{1/\alpha} \rightarrow \infty, \text{ and } n^2pb^2(\varphi) \rightarrow 0,$$

then $\gamma = \inf_{\Delta} \gamma(\Delta, Q(\alpha, L, \varphi)) \rightarrow 1$, where the infimum is taken over all test statistics Δ .

Together with Theorem 1, the proof that $\tilde{\varphi}$ is asymptotically minimax, under our assumptions, is complete. Note that the condition $np\varphi^{6-\frac{2}{\alpha}} \rightarrow 0$ is verified when $\alpha > 3/2$ for all n and $p \rightarrow +\infty$ giving a general result in this case. When $5/8 < \alpha < 3/2$, the same condition holds for $p = o(n^{\frac{8\alpha-5}{3-2\alpha}})$. This result is proven by showing that the χ^2 distance between the null hypothesis and an averaged likelihood under the alternative (that we explicitly construct) tends to 0.

Next, the sharp lower bound for the type II error probability is of Gaussian type.

Theorem 4. *Assume that $\alpha > 1$ and if*

$$\varphi \rightarrow 0, p\varphi^{1/\alpha} \rightarrow \infty, \sqrt{p}\varphi^{2-\frac{1}{2\alpha}} \rightarrow 0 \text{ and } n^2pb^2(\varphi) \asymp 1, \tag{13}$$

$$\inf_{\Delta: \eta(\Delta) \leq w} \beta(\Delta, Q(\alpha, L, \varphi)) \geq \Phi(z_{1-w} - n\sqrt{p}b(\varphi)) + o(1),$$

where the infimum is taken over all test statistics Δ with type I error probability less than or equal to w . Moreover,

$$\gamma = \inf_{\Delta} \gamma(\Delta, Q(\alpha, L, \varphi)) \geq 2\Phi\left(-n\sqrt{p}\frac{b(\varphi)}{2}\right) + o(1).$$

Remark: Theorems 2 and 4 imply that for $\alpha > 1$, the sharp separation rate for minimax testing is $\tilde{\varphi}$, under the additional assumptions (13). Let us check when the separation rate verifies these assumptions: $p\tilde{\varphi}^{1/\alpha} \rightarrow \infty$ holds if $n = o(1)p^{2\alpha}$, and $\sqrt{p}\tilde{\varphi}^{2-\frac{1}{2\alpha}} \rightarrow 0$ holds if $p = o(n^{4\alpha-1})$.

Let us assume that $p = n^B$ for $B > 0$ and that $\varphi = \tilde{\varphi}$. In this case, $\tilde{\varphi} = (C \cdot n^{2+B})^{-\alpha/(4\alpha+1)}$ that tends to 0 with n . Upper bounds are valid as soon as $p\tilde{\varphi}^{1/\alpha} = (C^{-1}n^{2(2\alpha B-1)})^{1/(4\alpha+1)}$ tends to infinity, that is $B > 1/(2\alpha)$. In addition the lower bounds are valid if $p\tilde{\varphi}^{4-1/\alpha} = C^{-(4\alpha-1)/(4\alpha+1)}n^{-2(4\alpha-1-B)/(4\alpha+1)}$ tends to 0, that is $B < 4\alpha - 1$.

If we assume that $p = \exp(Bn)$ for some $B > 0$ and that $\varphi = \tilde{\varphi}$, then $p\tilde{\varphi}^{1/\alpha}$ always tends to infinity, so the upper bounds hold. However, the additional assumption that $p\tilde{\varphi}^{4-1/\alpha}$ tends to 0 will never hold. In this case, we only have the minimax separation rates (for $\alpha > 3/2$) but not the sharp constants. It is easy to see that for $p = (\ln(n))^B$ we have $p\tilde{\varphi}^{1/\alpha}$ tends to 0, so neither our upper bounds nor the lower bounds hold in this case.

Note that, there is a more general test procedure independent of φ , for which it is possible to derive the upper bounds as in Theorems 1 and 2. It suffices to use the test statistic \hat{D}_n with the weights w_{ij}^* replaced by the weights $w_{ij}^*(\tilde{\varphi})$ defined as in (4) and (5) for φ replaced by $\tilde{\varphi}$. For more details see section 4.2 in [5].

The proof of the lower bounds is given in Section 7. We construct a family of n large centered Gaussian vectors with covariance matrices based on $\{\sigma_{ij}^*\}_{1 \leq i, j \leq p}$ given by the optimization problem (6) and a prior measure P_π on these covariance matrices. The logarithm of the likelihood ratio associated to an arbitrary Σ with respect to I under the null hypothesis is known to drift away to infinity (see [1], who corrected this ratio to get a proper limit). However, we show that the logarithm of the Bayesian likelihood ratio with our prior measure P_π verifies

$$\log \frac{f_\pi}{f_I}(X_1, \dots, X_n) = u_n Z_n - \frac{u_n^2}{2} + \xi, \quad \text{in } P_I \text{ probability}$$

where $u_n = n\sqrt{pb}(\varphi)$, Z_n is asymptotically distributed as a standard Gaussian distribution and ξ is a random variable which converges to zero under P_I probability.

5. Testing the inverse of the covariance matrix

Let us consider the same model, but the following test problem

$$H_0 : \Sigma^{-1} = I$$

against the alternative

$$H_1 : \Sigma \in \mathcal{G}(\alpha, L, \lambda) \text{ such that } \frac{1}{2p} \|\Sigma^{-1} - I\|_F^2 \geq \psi^2,$$

where $\mathcal{G}(\alpha, L, \lambda)$ is the class of covariance matrices Σ in $\mathcal{F}(\alpha, L)$ with the additional constraint that the eigenvalues $\lambda_i(\Sigma)$ are bounded from below by some $\lambda \in (0, 1)$ for all i from 1 to p and all Σ in the set.

We prove here that previous results apply to this setup and we get the same rates, but not the sharp asymptotics. Note that, the additional hypothesis is

mild enough so that it does not change the rates for testing. Indeed, we see this case as a well-posed inverse problem. The cases of ill-posed inverse problem where the smallest eigenvalue can be allowed to tend to 0 will most certainly imply a loss in the rate and is beyond the scope of this paper.

Theorem 5. *Suppose $\alpha > 3/2$, $L > 0$ and $\lambda \in (0, 1)$. If n and p tend to infinity, such that $n = o(1)p^{2\alpha}$, then $\tilde{\varphi}$ defined in (12) is the asymptotically minimax rate for the previous test.*

Proof. Note that $\Sigma^{-1} = I$ if and only if $\Sigma = I$. Moreover, if Σ belongs to $\mathcal{G}(\alpha, L, \lambda)$ such that $\frac{1}{2p}\|\Sigma^{-1} - I\|_F^2 \geq \psi^2$, then Σ obviously belongs to $\mathcal{F}(\alpha, L)$ and is such that

$$\frac{1}{2p}\|\Sigma - I\|_F^2 \geq \frac{\lambda^2}{2p}\|\Sigma^{-1} - I\|_F^2 \geq \lambda^2\psi^2.$$

Thus we can proceed with our former test procedure, with φ replaced by $\lambda\psi$ and we obtain the upper bounds in the definition of the separation rates.

The lower bounds in the previous Section will also remain valid. Indeed, this proof is based on the construction of a subfamily $\{\Sigma_U^* : u \in \mathcal{U}\}$ on the set of alternatives. We have proven in Proposition 3, that

$$\min_i \lambda_i(\Sigma_U^*) \geq 1 - O(\varphi^{1-1/(2\alpha)}),$$

and we have $\alpha > 1/2$ and $\varphi = \lambda\psi \rightarrow 0$ as $\psi \rightarrow 0$, therefore, $1 - O(\varphi^{1-1/(2\alpha)}) \geq \lambda$ for $\psi > 0$ small enough. Thus, this family belongs to the set of alternatives we consider here, as well. Moreover, Proposition 3 proves also that

$$\|\Sigma_U^*\|_2 := \max_i \lambda_i(\Sigma_U^*) \leq 1 + O(\varphi^{1-1/(2\alpha)}) \leq \lambda_{max},$$

for some fixed λ_{max} free of α and L . Thus,

$$\frac{1}{2p}\|(\Sigma_U^*)^{-1} - I\|_F^2 \geq \frac{1}{2p \cdot \|\Sigma_U^*\|_2^2} \cdot \|\Sigma_U^* - I\|_F^2 \geq \frac{1}{\lambda_{max}^2} \frac{1}{2p} \cdot \|\Sigma_U^* - I\|_F^2.$$

Thus we proceed the same way with φ replaced by $\lambda_{max}\psi$. □

6. Adaptive testing procedure

We want to build a test procedure of H_0 in (1) which is free of the parameter α belonging to some closed interval $\mathcal{A} = [\underline{\alpha}, \bar{\alpha}] \subset (1/2, +\infty)$. The radius L plays a minor role in the procedure and we suppose that it is known (w.l.o.g we assume that $L = 1$). Such a procedure is called adaptive and it solves the test problem H_0 in (1) against a much larger set of alternative hypotheses:

$$H_1 : \Sigma \in \bigcup_{\alpha \in \mathcal{A}} Q(\alpha, L, \mathcal{C}\psi_\alpha), \tag{14}$$

where C is a large enough positive constant and

$$\psi_\alpha = \left(\frac{\rho_{n,p}}{n\sqrt{p}} \right)^{\frac{2\alpha}{4\alpha+1}}, \quad \rho_{n,p} = \sqrt{\ln \ln(n\sqrt{p})}, \quad (15)$$

depend on n and p , but also on α . In order to construct the adaptive test procedure, we define a finite regular grid over the set $\mathcal{A} = [\underline{\alpha}, \bar{\alpha}]$:

$$\mathcal{A}_N = \{\alpha_r = \underline{\alpha} + \frac{\bar{\alpha} - \underline{\alpha}}{N} \cdot r; r = 1, \dots, N\}, \text{ where } N = \lceil \ln(n\sqrt{p}) \rceil.$$

To each $r \in \{1, \dots, N\}$, we associate the weights :

$$w_{ij,r}^* = \frac{\lambda_r}{b_r} \left(1 - \left(\frac{|i-j|}{T_r} \right)^{2\alpha_r} \right)_+,$$

where the parameters λ_r, b_r and T_r are given in (4) and (5) with α replaced by α_r and φ by ψ_α . Define the adaptive test procedure, for some constant $C^* > 0$ large enough

$$\Delta_{ad}^* = \max_{r=1, \dots, N} \mathbf{1}(\widehat{\mathcal{D}}_{n,r} > C^* t_r), \quad \text{where } t_r = C_{\lambda_r} \cdot \rho_{n,p} / (n\sqrt{p}), \quad (16)$$

and where $\widehat{\mathcal{D}}_{n,r}$ is the test statistic in (3) with weights $\{w_{ij,r}\}_{i < j}$. Note that the test Δ_{ad}^* rejects the null hypothesis as soon as there exists at least on $r \in \{1, \dots, N\}$ for which $\widehat{\mathcal{D}}_{n,r} > C^* t_r$.

Theorem 6. *Assume that*

$$p \cdot \left(\frac{\rho_{n,p}}{n\sqrt{p}} \right)^{\frac{2}{4\alpha+1}} \rightarrow +\infty \quad \text{and} \quad \frac{\ln p}{n} \rightarrow 0$$

The test statistic defined in (16) with C^ large enough verifies :*

$$\gamma(\Delta_{ad}^*, \cup_{\alpha \in \mathcal{A}} Q(\alpha, L, C\psi_\alpha)) \rightarrow 0,$$

for all $C > \left(C^ + \frac{1}{C(\underline{\alpha}, \bar{\alpha})} \right)$, where ψ_α is given in (15) and $C(\underline{\alpha}, \bar{\alpha}) = \exp(-8(\bar{\alpha} - \underline{\alpha}) / (4\underline{\alpha} + 1))$.*

The proof that the adaptive procedure we propose attains the above rate is given in Section 7. By analogy to nonparametric testing of functions, we expect the loss $\rho_{n,p}$ to be optimal uniformly over the class in the alternative hypothesis (14).

7. Proofs

Proof of Theorems 1 and 2. The proof is based on the Proposition 1 and the asymptotic normality of the weighted test statistic $n\sqrt{p}\widehat{\mathcal{D}}_n$ in Proposition 2. We get for the type I error probability of Δ^*

$$\eta(\Delta) = \mathbb{P}(\widehat{\mathcal{D}}_n > t) = 1 - \Phi(n\sqrt{p} \cdot t) + o(1).$$

For the type II error probability of Δ^* , uniformly in Σ over $Q(\alpha, L, \varphi)$, we have

$$\mathbb{P}_\Sigma(\widehat{\mathcal{D}}_n \leq t) \leq \mathbb{P}_\Sigma(|\widehat{\mathcal{D}}_n - \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)| \geq \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) - t) \leq \frac{\text{Var}_\Sigma(\widehat{\mathcal{D}}_n)}{(\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) - t)^2},$$

for $t \leq c \cdot b(\varphi)$ and $0 < c < 1$. It implies that $n\sqrt{p} \cdot t \leq cn\sqrt{p}b(\varphi)$. Therefore, we distinguish the cases where $n^2pb^2(\varphi)$ tends to infinity or is bounded.

We use the fact that, under the alternative, $\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \geq b(\varphi)$. We bound from below as follows:

$$\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) - t \geq (1 - c)\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n).$$

Then, it gives

$$\mathbb{P}_\Sigma(\widehat{\mathcal{D}}_n \leq t) \leq \frac{T_1}{n(n-1)p^2(1-c)^2\mathbb{E}_\Sigma^2(\widehat{\mathcal{D}}_n)} + \frac{T_2}{np^2(1-c)^2\mathbb{E}_\Sigma^2(\widehat{\mathcal{D}}_n)} =: S_1 + S_2.$$

Let us bound from above S_1 using (7):

$$S_1 \leq \frac{1 + o(1)}{n(n-1)p(1-c)^2b^2(\varphi)} + \frac{O(T^{3/2})}{n(n-1)pb(\varphi)}.$$

We have $T^{3/2}b(\varphi) \asymp \varphi^{2-\frac{1}{\alpha}} = o(1)$, for all $\alpha > 1/2$, which proves that :

$$S_1 \leq \frac{1 + o(1)}{n(n-1)p(1-c)^2b^2(\varphi)}$$

which tends to 0 provided that $n^2pb^2(\varphi) \rightarrow +\infty$. We will see using (8) that the term S_2 tends to 0 as well:

$$\begin{aligned} S_2 &\leq \frac{O(\sqrt{T})}{npb(\varphi)} + \frac{O(T^{3/4}b^{1/2}(\varphi))}{n\sqrt{p}b(\varphi)} + \frac{o(1)}{n\sqrt{p}b(\varphi)} \\ &= o(1) \text{ for all } \alpha > 1/2, \text{ as soon as } n\sqrt{p}b(\varphi) \rightarrow +\infty. \end{aligned}$$

Now, if φ is close to the separation rate: $n^2pb^2(\varphi) \asymp 1$, we see that whenever $\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)/b(\varphi)$ tends to infinity, the bound is trivial ($S_1 + S_2 \rightarrow 0$).

The nontrivial bound is obtained when Σ under the alternative is close to the optimal matrix $\Sigma^* = (\sigma_{ij}^*)_{1 \leq i, j \leq p}$, in the sense that $\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) = O(b(\varphi))$ together with the fact that φ is close to the separation rate: $n^2pb^2(\varphi) \asymp 1$. We apply Proposition 2 to get the asymptotic normality

$$n\sqrt{p}(\widehat{\mathcal{D}}_n - \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)) \rightarrow \mathcal{N}(0, 1).$$

Thus,

$$\begin{aligned} \sup_{\Sigma \in Q(\alpha, L, \varphi)} \mathbb{P}_\Sigma(\widehat{\mathcal{D}}_n \leq t) &\leq \sup_{\Sigma \in Q(\alpha, L, \varphi)} \Phi(n\sqrt{p} \cdot (t - \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n))) + o(1) \\ &\leq \Phi(n\sqrt{p} \cdot (t - \inf_{\Sigma \in Q(\alpha, L, \varphi)} \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n))) + o(1). \end{aligned}$$

At this point, choosing optimal weights translates into

$$\begin{aligned} & \inf_{w_{ij} > 0: \sum_{i \neq j} w_{ij}^2 = 1/2} \sup_{\Sigma \in Q(\alpha, L, \varphi)} \mathbb{P}_{\Sigma}(\widehat{\mathcal{D}}_n \leq t) \\ & \leq \Phi(n\sqrt{p} \cdot (t - \sup_{w_{ij} > 0: \sum_{i \neq j} w_{ij}^2 = 1/2} \inf_{\Sigma \in Q(\alpha, L, \varphi)} \mathbb{E}_{\Sigma}(\widehat{\mathcal{D}}_n))) + o(1) \\ & \leq \Phi(n\sqrt{p} \cdot (t - b(\varphi))) + o(1), \end{aligned}$$

after solving the optimization problem in the Appendix, which ends the proof of the Theorem. \square

Proof of Theorems 3 and 4. The first step of the proof is to reduce the set of parameters to a convenient parametric family. Let $\Sigma^* = [\sigma_{ij}^*]_{1 \leq i, j \leq p}$ be the matrix which has 1 on the diagonal and off-diagonal entries σ_{ij}^* where

$$\sigma_{ij}^* = \sqrt{\lambda} \left(1 - \left(\frac{|i-j|}{T} \right)^{2\alpha} \right)_+^{\frac{1}{2}} \quad \text{for } i \neq j, \tag{17}$$

with λ and T are given by (4) and (5).

Let us define Q^* a subset of $Q(\alpha, L, \varphi)$ as follows

$$Q^* = \{ \Sigma_U^* : [\Sigma_U^*]_{ij} = I(i=j) + u_{ij} \sigma_{ij}^* I(i \neq j) \text{ for all } 1 \leq i, j \leq p, U \in \mathcal{U} \}$$

where

$$\mathcal{U} = \{ U = [u_{ij}]_{1 \leq i, j \leq p} : u_{ii} = 0, \forall i \text{ and } u_{ij} = u_{ji} = \pm I(|i-j| \leq T), \text{ for } i \neq j \}.$$

The cardinality of \mathcal{U} is $p(T-1)/2$.

Proposition 3. *For $\alpha > 1/2$, the symmetric matrix $\Sigma_U^* = [u_{ij} \sigma_{ij}^*]_{1 \leq i, j \leq p}$, with $\sigma_{ii}^* = 1$, for all i from 1 to p , and σ_{ij}^* defined in (17) is non-negative definite, for $\varphi > 0$ small enough, and for all $U \in \mathcal{U}$.*

Moreover, denote by $\lambda_{1,U}, \dots, \lambda_{p,U}$ the eigenvalues of Σ_U^ , then $|\lambda_{i,U} - 1| \leq O(1)\varphi^{1-1/(2\alpha)}$, for all i from 1 to p .*

We deduce that

$$\|\Sigma_U^*\| \leq 1 + O(\varphi^{1-\frac{1}{2\alpha}}) \text{ and } \|\Sigma_U^* - I\| \leq O(\varphi^{1-\frac{1}{2\alpha}}). \tag{18}$$

Indeed, $\|\Sigma_U^*\| = \max_{i=1, \dots, p} \lambda_{i,U} \leq 1 + O(\varphi^{1+\frac{1}{2\alpha}})$ and $\Sigma_U^* - I$ has eigenvalues $\lambda_{i,U} - 1$.

Proposition 3 shows that for all $\Sigma_U^* \in Q^*$, Σ_U^* is non-negative definite, for $\varphi > 0$ small enough.

Assume that $X_1, \dots, X_n \sim N(0, I)$ under the null hypothesis and denote by P_I the likelihood of these random variables. We assume that $X_1, \dots, X_n \sim N(0, \Sigma_U^*)$, under the alternative, and we denote P_U the associated likelihood. In addition let

$$P_{\pi} = \frac{1}{2^{p(T-1)/2}} \sum_{U \in \mathcal{U}} P_U$$

be the average likelihood over Q^* .

The problem can be reduced to the test $H_0 : X_1, \dots, X_n \sim P_I$ against the averaged distribution $H_1 : X_1, \dots, X_n \sim P_\pi$, in the sense that

$$\inf_{\Delta: \eta(\Delta) \leq w} \beta(\Delta(t), Q(\alpha, L, \varphi)) \geq \inf_{\Delta: \eta(\Delta) \leq w} \beta(\Delta(t), P_\pi) + o(1)$$

and that

$$\inf_{\Delta} \gamma(\Delta, Q(\alpha, L, \varphi)) \geq \inf_{\Delta} \gamma(\Delta, P_\pi) + o(1).$$

It is, therefore, sufficient to show that, when $u_n \asymp 1$,

$$\inf_{\Delta: \eta(\Delta) \leq w} \beta(\Delta(t), P_\pi) \geq \Phi(n\sqrt{p} \cdot (t - b(\varphi))) + o(1) \tag{19}$$

and that

$$\inf_{\Delta} \gamma(\Delta, P_\pi) \geq 2\Phi\left(-n\sqrt{p} \frac{b(\varphi)}{2}\right) + o(1). \tag{20}$$

While, for $u_n = o(1)$, we need to show that

$$\inf_{\Delta} \gamma(\Delta, P_\pi) \rightarrow 1. \tag{21}$$

In order to obtain (19) and (20), we apply results in Section 4.3.1 of [18] giving the sufficient condition that, in P_I probability:

$$L_{n,p} := \log \frac{f_\pi}{f_I}(X_1, \dots, X_n) = u_n Z_n - \frac{u_n^2}{2} + \xi, \tag{22}$$

where $u_n = n\sqrt{p}b(\varphi) \asymp 1$, $b(\varphi) = C^{\frac{1}{2}}(\alpha, L) \cdot \varphi^{2+\frac{1}{2\alpha}}$, Z_n is asymptotically distributed as a standard Gaussian distribution and ξ is a random variable which converges to zero under P_I probability. Moreover, to show (21), it suffices to show that

$$\mathbb{E}_I \left(\frac{dP_\pi}{dP_I} \right)^2 \leq 1 + o(1), \tag{23}$$

since

$$\gamma \geq 1 - \frac{1}{2} \|P_I - P_\pi\|_1 \text{ and } \|P_I - P_\pi\|_1^2 \leq \mathbb{E}_I \left(\frac{dP_\pi}{dP_I} \right)^2 - 1.$$

We first begin by showing (23), in order to finish the proof of Theorem 3. Let,

$$\begin{aligned} H_{n,p} &:= \mathbb{E}_I \left(\frac{dP_\pi}{dP_I}(X_1, \dots, X_n) \right)^2 \\ &= \mathbb{E}_I \mathbb{E}_{U,V} \left(\frac{\exp\left(-\frac{1}{2} \sum_{k=1}^n X_k^\top ((\Sigma_U)^{-1} + (\Sigma_V)^{-1} - 2I) X_k\right)}{(2\pi)^{\frac{np}{2}} \det^{\frac{n}{2}}(\Sigma_U \Sigma_V)} \right) \end{aligned} \tag{24}$$

We have

$$\begin{aligned} H_{n,p} &= \mathbb{E}_{U,V} \left(\frac{\det^{-\frac{n}{2}} \left((\Sigma_U)^{-1} + (\Sigma_V)^{-1} - I \right)}{\det^{\frac{n}{2}}(\Sigma_U \Sigma_V)} \right) \\ &= \mathbb{E}_{U,V} \left(\det^{-\frac{n}{2}} \left(\Sigma_U + \Sigma_V - \Sigma_U \Sigma_V \right) \right). \end{aligned}$$

We define $\Delta_U = \Sigma_U - I$ and note that $\Sigma_U + \Sigma_V - \Sigma_U \Sigma_V = I - \Delta_U \Delta_V$. As the matrix $\Delta_U \Delta_V$ is not necessarily symmetric, we write

$$(I - \Delta_U \Delta_V)(I - \Delta_U \Delta_V)^\top = I - M$$

where $M = M_{U,V} := \Delta_U \Delta_V + \Delta_V \Delta_U - \Delta_U \Delta_V^2 \Delta_U$ is symmetric. Moreover, we prove that for all U and $V \in \mathcal{U}$ the eigenvalues of M are in $(-1, 1)$ for all $\alpha > 1/2$ and φ small enough. Indeed, by Gershgorin's theorem, for each eigenvalue λ_M of M there exists at least one $i \in \{1, \dots, p\}$ such that

$$|\lambda_M - M_{ii}| \leq \sum_{j:j \neq i} |M_{ij}|.$$

We can show that $\sum_{j:j \neq i} |M_{ij}| = O(\varphi^{2-\frac{1}{\alpha}})$ and $|M_{ii}| \leq O(\varphi^2) + O(\varphi^{4-\frac{1}{\alpha}})$. Thus,

$$H_{n,p} = \mathbb{E}_{U,V}(\det^{-\frac{n}{4}}(I - M)) = \mathbb{E}_{U,V} \exp\left(-\frac{n}{4} \log \det(I - M)\right)$$

The Taylor expansion for the logdet of a symmetric matrix writes

$$-\frac{1}{4} \log \det(I - M) = \frac{1}{4} \text{tr}(M) + \frac{1}{8} \text{tr}(M^2) + O(\text{tr}(M^3)).$$

In more details,

$$\begin{aligned} \frac{1}{4} \text{tr}(M) &= \frac{1}{2} \text{tr}(\Delta_U \Delta_V) - \frac{1}{4} \text{tr}(\Delta_U^2 \Delta_V^2) \\ \frac{1}{8} \text{tr}(M^2) &= \frac{1}{4} \text{tr}(\Delta_U \Delta_V)^2 + \frac{1}{4} \text{tr}(\Delta_U^2 \Delta_V^2) + \frac{1}{8} \text{tr}(\Delta_U \Delta_V^2 \Delta_U)^2 \\ &\quad - \frac{1}{4} \text{tr}(\Delta_U \Delta_V^2 \Delta_U^2 \Delta_V) - \frac{1}{4} \text{tr}(\Delta_V \Delta_U^2 \Delta_V^2 \Delta_U) \end{aligned}$$

Recall that $\forall A, B \in \mathbb{R}^{p \times p}$ we have $\|AB\|_F \leq \|A\|_2 \|B\|_F$. For all $U, V \in \mathcal{U}$, we use the last inequality and the Cauchy-Schwarz inequality to get

$$\begin{aligned} \text{tr}(\Delta_U \Delta_V^2 \Delta_U^2 \Delta_V) &\leq \|\Delta_U \Delta_V\|_F \|\Delta_V \Delta_U^2 \Delta_V\|_F \\ &\leq \|\Delta_V\|_2 \|\Delta_U\|_F \|\Delta_V \Delta_U^2\|_2 \|\Delta_V\|_F \leq p \cdot \varphi^{6-\frac{2}{\alpha}}, \\ \text{tr}(\Delta_U \Delta_V^2 \Delta_U)^2 &= \|\Delta_U \Delta_V^2 \Delta_U\|_F^2 \leq \|\Delta_U \Delta_V^2\|_2^2 \|\Delta_U\|_F^2 \leq p \cdot \varphi^{8-\frac{3}{\alpha}}. \end{aligned}$$

Finally, using similar arguments we can show that

$$\text{tr}(M_{U,V}^3) = O(p\varphi^{6-\frac{2}{\alpha}}).$$

Thus,

$$-\frac{1}{4} \log \det(I - M) = \frac{1}{2} \text{tr}(\Delta_U \Delta_V) + \frac{1}{4} \text{tr}(\Delta_U \Delta_V)^2 + O(p\varphi^{6-\frac{2}{\alpha}}).$$

Now we develop the terms on the right hand side of the previous equation. We obtain

$$\text{tr}(\Delta_U \Delta_V) = \sum_{\substack{1 \leq i, j \leq p \\ 1 < |i-j| < T}} u_{ij} v_{ij} \cdot \sigma_{ij}^{*2} = 2 \sum_{\substack{1 \leq i < j \leq p \\ 1 < |i-j| < T}} u_{ij} v_{ij} \cdot \sigma_{ij}^{*2}$$

and

$$\begin{aligned}
tr(\Delta_U \Delta_V)^2 &= \sum_{\substack{1 \leq i, j, h, l \leq p \\ 1 < |i-h|, |h-j|, |i-l|, |l-j| < T}} u_{ih} v_{hj} u_{jl} v_{li} \cdot \sigma_{ih}^* \sigma_{hj}^* \sigma_{jl}^* \sigma_{li}^* \\
&= \sum_{\substack{1 \leq i, l \leq p \\ 1 < |i-l| < T}} \sigma_{ij}^{*4} + 2 \sum_{\substack{1 \leq i, j, l \leq p \\ i < j \\ 1 < |i-l|, |l-j| < T}} u_{il} u_{jl} v_{lj} v_{li} \cdot \sigma_{il}^{*2} \sigma_{lj}^{*2} \\
&+ 4 \sum_{\substack{1 \leq i, j, h, l \leq p \\ i < j, l < h \\ 1 < |i-h|, |h-j|, |i-l|, |l-j| < T}} u_{ih} u_{jl} v_{hj} v_{li} \cdot \sigma_{ih}^* \sigma_{jl}^* \sigma_{hj}^* \sigma_{li}^*.
\end{aligned}$$

Now, we can write (24) as follows:

$$\begin{aligned}
H_{n,p} &= \mathbb{E}_{U,V} \exp \left(- \frac{n \log \det(I - \Delta_U \Delta_V)}{2} \right) \\
&= \mathbb{E}_{U,V} \exp \left(n \sum_{\substack{1 \leq i < j \leq p \\ 1 < |i-j| < T}} u_{ij} v_{ij} \cdot \sigma_{ij}^{*2} + \frac{n}{2} \sum_{\substack{1 \leq i, j, l \leq p \\ i < j \\ 1 < |i-l|, |l-j| < T}} u_{il} u_{jl} v_{lj} v_{li} \cdot \sigma_{il}^{*2} \sigma_{lj}^{*2} \right. \\
&+ n \sum_{\substack{1 \leq i, j, h, l \leq p \\ i < j, l < h \\ 1 < |i-h|, |h-j|, |i-l|, |l-j| < T}} u_{ih} u_{jl} v_{hj} v_{li} \cdot \sigma_{ih}^* \sigma_{jl}^* \sigma_{hj}^* \sigma_{li}^* \left. \right) \\
&+ \frac{n}{4} \sum_{\substack{1 \leq i, l \leq p \\ 1 < |i-l| < T}} \sigma_{ij}^{*4} + O(np\varphi^{6-\frac{2}{\alpha}}).
\end{aligned}$$

We explicit the expected value with respect to the i.i.d Rademacher random variables $\{u_{ij} v_{ij}\}_{i < j}$, $\{u_{il} u_{jl} v_{lj} v_{li}\}_{i < j, l \notin \{i, j\}}$ and $\{u_{ih} u_{jl} v_{hj} v_{li}\}_{i < j, l < h}$ pairwise distinct and independent:

$$\begin{aligned}
H_{n,p} &= \prod_{\substack{i < j \\ 1 < |i-j| < T}} \cosh(n\sigma_{ij}^{*2}) \prod_{\substack{1 \leq i, j, l \leq p \\ i < j \\ 1 < |i-l|, |l-j| < T}} \cosh\left(\frac{n}{2} \sigma_{il}^{*2} \sigma_{lj}^{*2}\right) \\
&\prod_{\substack{1 \leq i, j, h, l \leq p \\ i < j, l < h \\ 1 < |i-h|, |h-j|, |i-l|, |l-j| < T}} \cosh\left(n\sigma_{ih}^* \sigma_{jl}^* \sigma_{hj}^* \sigma_{li}^*\right) \exp\left(\frac{n}{2} \sum_{\substack{1 \leq i, l \leq p \\ 1 < |i-l| < T}} \sigma_{ij}^{*4} + O(np\varphi^{6-\frac{2}{\alpha}})\right).
\end{aligned}$$

We use the inequality $\cosh(x) \leq \exp(x^2/2)$ and get

$$\begin{aligned}
H_{n,p} &\leq \exp \left\{ \frac{n^2}{2} \left(\sum_{\substack{i < j \\ 1 < |i-j| < T}} \sigma_{ij}^{*4} + \frac{1}{4} \sum_{\substack{1 \leq i, j, l \leq p \\ i < j \\ 1 < |i-l|, |l-j| < T}} \sigma_{il}^{*4} \sigma_{lj}^{*4} \right. \right. \\
&+ \left. \left. \sum_{\substack{1 \leq i, j, h, l \leq p \\ i < j, l < h \\ 1 < |i-h|, |h-j|, |i-l|, |l-j| < T}} \sigma_{ih}^{*2} \sigma_{jl}^{*2} \sigma_{hj}^{*2} \sigma_{li}^{*2} \right) \right\} \cdot \exp \left(\frac{n}{2} \sum_{\substack{1 \leq i, l \leq p \\ 1 < |i-l| < T}} \sigma_{ij}^{*4} + O(np\varphi^{6-\frac{2}{\alpha}}) \right).
\end{aligned}$$

Or, $\frac{n^2}{2} \sum_{\substack{i < j \\ 1 < |i-j| < T}} \sigma_{ij}^{*4} = n^2 pb^2(\varphi)$ and since $\varphi \rightarrow 0$ we have that

$$\begin{aligned} \frac{n^2}{8} \sum_{\substack{1 \leq i, j, l \leq p \\ i \neq j \\ 1 < |i-l|, |l-j| < T}} \sigma_{il}^{*4} \sigma_{lj}^{*4} &= \frac{n^2}{8} \sum_{\substack{i \neq l \\ 1 < |i-l| < T}} \sigma_{il}^{*4} \sum_{\substack{j \\ 1 < |l-j| < T}} \sigma_{lj}^{*4} \\ &= n^2 pb^2(\varphi) \cdot O(\lambda^2 T) = n^2 pb^2(\varphi) \cdot o(1) \end{aligned}$$

and

$$\begin{aligned} \frac{n^2}{2} \sum_{\substack{1 \leq i, j, h, l \leq p \\ i \neq j, l \neq h \\ 1 < |i-h|, |h-j|, |i-l|, |l-j| < T}} \sigma_{ih}^{*2} \sigma_{jl}^{*2} \sigma_{hj}^{*2} \sigma_{li}^{*2} \\ = n^2 O(p \lambda^4 T^3) = O(n^2 p \varphi^{4 + \frac{1}{\alpha}} \cdot \varphi^4) = n^2 pb^2(\varphi) \cdot o(1). \end{aligned}$$

Finally, $n p \varphi^{6 - \frac{2}{\alpha}} = n^2 p \varphi^{4 + \frac{1}{\alpha}} \cdot \frac{\varphi^{2 - \frac{3}{\alpha}}}{n} = o(1)$ as soon as $n^2 p \varphi^{4 + \frac{1}{\alpha}} \rightarrow 0$ and $\alpha > 3/2$ or $5/8 < \alpha < 3/2$ and $p < n^{\frac{8\alpha - 5}{-2\alpha + 3}}$.

As consequence, if $n^2 pb^2(\varphi) \rightarrow 0$ with the additional conditions on α, n and p given previously, we get

$$\mathbb{E}_I \left(\frac{dP_\pi}{dP_I} \right)^2 \leq \exp \left(n^2 pb^2(\varphi) (1 + o(1)) \right) = 1 + o(1),$$

which ends the proof of Theorem 3.

Now, we show (22) in order to finish the proof of Theorem 4. More explicitly,

$$\begin{aligned} L_{n,p} &= \log \frac{f_\pi}{f_I}(X_1, \dots, X_n) \\ &= \log \mathbb{E}_U \exp \left(-\frac{1}{2} \sum_{k=1}^n X_k^\top ((\Sigma_U^*)^{-1} - I) X_k - \frac{n}{2} \log \det(\Sigma_U^*) \right), \end{aligned} \tag{25}$$

where U is seen as a randomly chosen matrix with uniform distribution over the set \mathcal{U} . Let us denote $\Delta_U = \Sigma_U^* - I$ and recall that proposition 3 implies that $\|\Delta_U\| \leq O(1) \varphi^{1 - \frac{1}{2\alpha}} = o(1)$ for all $\alpha > 1/2$. We write the following approximations obtained by matrix Taylor expansion:

$$-\frac{1}{2} ((\Sigma_U^*)^{-1} - I) = \frac{1}{2} \sum_{l=1}^5 (-1)^{l+1} \cdot \Delta_U^l + O(1) \Delta_U^6 \tag{26}$$

$$\log \det(\Sigma_U^*) = \text{tr} \left(\sum_{l=1}^5 \frac{(-1)^{l+1}}{l} \cdot \Delta_U^l + O(1) \Delta_U^6 \right) \tag{27}$$

Note that, $\text{tr}(\Delta_U) = 0$ and that $\text{tr}(\Delta_U^2) = \|\Sigma^* - I\|_F^2 = 2 \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*2}$ does not

depend on U . Moreover,

$$\begin{aligned} \mathbb{E}_I\left(\sum_{k=1}^n X_k^\top \Delta_U^6 X_k\right) &= n \cdot \text{tr}(\Delta_U^6) \leq n \cdot \|\Delta_U\|^4 \cdot \text{tr}(\Delta_U^2) \leq O(1) \cdot n\varphi^{4-\frac{2}{\alpha}} \cdot p\varphi^2 \\ &\leq O(1) \cdot n\sqrt{p}\varphi^{2+\frac{1}{2\alpha}} \cdot \sqrt{p}\varphi^{4-\frac{5}{2\alpha}} \\ &\leq O(1) \cdot u_n \cdot \sqrt{p}\varphi^{2-\frac{1}{2\alpha}} \cdot \varphi^{2-\frac{2}{\alpha}} = o(1) \end{aligned}$$

for all $\alpha > 1$ and when $u_n = O(1)$ and $\sqrt{p}\varphi^{2-\frac{1}{2\alpha}} = O(1)$. Also, for all $\alpha > 1$

$$\text{Var}_I\left(\sum_{k=1}^n X_k^\top \Delta_U^6 X_k\right) = 2n\text{tr}(\Delta_U^{12}) \leq O(1)n\varphi^{10-\frac{5}{\alpha}} \cdot p\varphi^2 = o(1).$$

In conclusion, we use $Y_n = \mathbb{E}_I(Y_n) + O_P(\sqrt{\text{Var}(Y_n)})$ for any sequence of random variables Y_n , to get

$$\sum_{k=1}^n X_k^\top \Delta_U^6 X_k - n\text{tr}(\Delta_U^6) = o_P(1), \quad \text{in } P_I\text{-probability.}$$

We get

$$\begin{aligned} L_{n,p} &= \log \mathbb{E}_U \exp\left(\frac{1}{2} \sum_{k=1}^n X_k^\top \Delta_U X_k - \frac{1}{2} \sum_{k=1}^n X_k^\top \Delta_U^2 X_k + \frac{n}{4} \text{tr}(\Delta_U^2)\right) \quad (28) \\ &+ \frac{1}{2} \sum_{l=3}^5 (-1)^{l+1} \sum_{k=1}^n X_k^\top \Delta_U^l X_k - \frac{n}{2} \sum_{l=3}^5 \frac{(-1)^{l+1}}{l} \cdot \text{tr}(\Delta_U^l) + o_P(1). \end{aligned}$$

From $l = 3, 4$ and 5 , we treat similarly the terms

$$\begin{aligned} \sum_{k=1}^n X_k^\top \Delta_U^l X_k &= \mathbb{E}_I\left(\sum_{k=1}^n X_k^\top \Delta_U^l X_k\right) + O_P\left(\sqrt{\text{Var}_I\left(\sum_{k=1}^n X_k^\top \Delta_U^l X_k\right)}\right) \\ &= n\text{tr}(\Delta_U^l) + O_P(1) \cdot \sqrt{n\text{tr}(\Delta_U^{2l})} \quad (29) \end{aligned}$$

By (28), we have $n\text{tr}(\Delta_U^6) = o(1)$, similarly we obtain $n\text{tr}(\Delta_U^{2l}) = o(1)$ for $l = 4$ and 5 . Thus (28) becomes :

$$\begin{aligned} L_{n,p} &= \log \mathbb{E}_U \exp\left(\frac{1}{2} \sum_{k=1}^n X_k^\top \Delta_U X_k - \frac{1}{2} \sum_{k=1}^n X_k^\top \Delta_U^2 X_k + \frac{n}{4} \text{tr}(\Delta_U^2)\right) \\ &+ \frac{n}{2} \sum_{l=3}^5 (-1)^{l+1} \left(1 - \frac{1}{l}\right) \cdot \text{tr}(\Delta_U^l) + o_P(1). \quad (30) \end{aligned}$$

We have

$$\text{tr}(\Delta_U^3) = \sum_{\substack{i \neq j \neq k \\ k \neq i}} u_{ij} u_{jk} u_{ki} \sigma_{ij}^* \sigma_{jk}^* \sigma_{ki}^* = 3! \sum_{i < j < k} u_{ij} u_{jk} u_{ki} \sigma_{ij}^* \sigma_{jk}^* \sigma_{ki}^*$$

and we decompose

$$\begin{aligned} \text{tr}(\Delta_U^4) &= \sum_{\substack{i \neq j \neq k \\ i \neq l \neq k}} u_{ij} u_{jk} u_{kl} u_{li} \sigma_{ij}^* \sigma_{jk}^* \sigma_{kl}^* \sigma_{li}^* \\ &= \sum_{i \neq j} \sigma_{ij}^{*4} + 2 \sum_{i \neq j \neq k} \sigma_{ij}^{*2} \sigma_{jk}^{*2} + 4! \sum_{i < j < k < l} u_{ij} u_{jk} u_{kl} u_{li} \sigma_{ij}^* \sigma_{jk}^* \sigma_{kl}^* \sigma_{li}^*. \end{aligned}$$

Note that

$$\begin{aligned} n \sum_{i \neq j} \sigma_{ij}^{*4} &= O(np\varphi^{4+\frac{1}{\alpha}}) = O(n\sqrt{p}\varphi^{2+\frac{1}{2\alpha}} \cdot \sqrt{p}\varphi^{2+\frac{1}{2\alpha}}) = o(1) \\ 2n \sum_{i \neq j \neq k} \sigma_{ij}^{*2} \sigma_{jk}^{*2} &= O(np\lambda^2 T^2) = O(n\sqrt{p}\varphi^{2+\frac{1}{2\alpha}} \cdot \sqrt{p}\varphi^{2-\frac{1}{2\alpha}}) \\ &= O(u_n \cdot \sqrt{p}\varphi^{2-\frac{1}{2\alpha}}) = o(1), \end{aligned}$$

if $u_n \asymp 1$ and $\sqrt{p}\varphi^{2-\frac{1}{2\alpha}} \rightarrow 0$. As for the last term :

$$\begin{aligned} \text{tr}(\Delta_U^5) &= \sum_{\substack{i \neq j \neq k \\ k \neq l \neq v \\ v \neq i}} u_{ij} u_{jk} u_{kl} u_{lv} u_{vi} \sigma_{ij}^* \sigma_{jk}^* \sigma_{kl}^* \sigma_{lv}^* \sigma_{vi}^* \\ &= 5 \sum_{\substack{k \neq l \neq j \\ j \neq k}} u_{jk} u_{kl} u_{lj} \sigma_{jk}^* \sigma_{kl}^* \sigma_{lj}^* \left(\sum_i \sigma_{ij}^{*2} \right) \\ &+ 5 \sum_{\substack{i \neq j \neq k \\ k \neq i}} u_{ij} u_{jk} u_{ki} \sigma_{ij}^* \sigma_{jk}^* \sigma_{ki}^* (\sigma_{ij}^{*2}) \\ &+ 5! \sum_{i < j < k < l < v} u_{ij} u_{jk} u_{kl} u_{lv} u_{vi} \sigma_{ij}^* \sigma_{jk}^* \sigma_{kl}^* \sigma_{lv}^* \sigma_{vi}^*. \end{aligned}$$

The first two terms in the decomposition of $\text{tr}(\Delta_U^5)$ group with $\text{tr}(\Delta_U^3)$ and $\sigma_{ij}^{*2} \searrow 0$ for all $1 < |i-j| < T$,

$$\begin{aligned} 0 < \sigma_{ij}^{*2} &\leq \sum_{j:1 < |j-i| < T} \sigma_{ij}^{*2} = \sum_{j:|j-i| < T} \lambda \left(1 - \left(\frac{|i-j|}{T} \right)^{2\alpha} \right)_+ \\ &= O(\lambda \cdot T) = O(\varphi^2) = o(1), \end{aligned}$$

as $\varphi \rightarrow 0$, therefore we ignore these terms in further calculations.

Let us denote by $W_{ij} = \sum_{k=1}^n X_{k,i} X_{k,j}$, then $\sum_{k=1}^n X_k^\top \Delta_U X_k = \sum_{1 \leq i \neq j \leq p} u_{ij} \sigma_{ij}^* W_{ij}$

$$\begin{aligned} \sum_{k=1}^n X_k^\top \Delta_U^2 X_k &= \sum_{1 \leq i, j \leq p} [\Delta_U^2]_{ij} W_{ij} \\ &= \sum_{1 \leq i \neq j \leq p} \sum_{h \notin \{i, j\}} u_{ih} u_{hj} \sigma_{ih}^* \sigma_{hj}^* W_{ij} + \sum_{i=1}^p \sum_{h \neq i} \sigma_{ih}^{*2} W_{ii} \end{aligned}$$

and $\frac{n}{4} \text{tr}(\Delta_U^2) = \frac{n}{4} \sum_{1 \leq i \neq h \leq p} \sigma_{ih}^{*2}$. Then, from (30) we get

$$\begin{aligned}
 L_{n,p} &= \log \mathbb{E}_U \exp \left(\frac{1}{2} \sum_{1 \leq i \neq j \leq p} u_{ij} \sigma_{ij}^* W_{ij} - \frac{1}{2} \sum_{1 \leq i \neq h \leq p} \sigma_{ih}^{*2} \left(W_{ii} - \frac{n}{2} \right) \right. \\
 &\quad \left. - \frac{1}{2} \sum_{\substack{1 \leq j \neq h \neq i \leq p \\ i \neq j}} u_{ih} u_{hj} \sigma_{ih}^* \sigma_{hj}^* W_{ij} + \frac{n}{2} \sum_{l=3}^5 (-1)^{l+1} \cdot \frac{l-1}{l} \cdot \text{tr}(\Delta_U^l) \right) + o_P(1) \\
 &= \log \mathbb{E}_U \exp \left(\sum_{1 \leq i < j \leq p} u_{ij} \sigma_{ij}^* W_{ij} - \sum_{\substack{1 \leq i \neq j \neq h \neq i \leq p \\ i < j}} u_{ih} u_{hj} \sigma_{ih}^* \sigma_{hj}^* W_{ij} \right. \\
 &\quad \left. + \frac{n}{2} \sum_{l=3}^5 (-1)^{l+1} \cdot \frac{(l-1)}{l} \cdot l! \sum_{k_1 < k_2 < \dots < k_l} u_{k_1 k_2} \dots u_{k_l k_1} \sigma_{k_1 k_2}^* \dots \sigma_{k_l k_1}^* \right) \\
 &\quad - \frac{1}{2} \sum_{1 \leq i \neq h \leq p} \sigma_{ih}^{*2} \left(W_{ii} - \frac{n}{2} \right) + o_P(1). \tag{31}
 \end{aligned}$$

Now, we explicit the expected value with respect to the i.i.d Rademacher random variables $u_{ij}, u_{ih} u_{hj}, u_{k_1 k_2} u_{k_2 k_3} u_{k_3 k_1}, \dots$ for all $i < j, h, k_1 < k_2 < \dots < k_l$ pairwise distinct. Indeed, products of independent Rademacher random variables are still Rademacher and independent. Thus,

$$\begin{aligned}
 L_{n,p} &= \sum_{1 \leq i < j \leq p} \log \cosh(\sigma_{ij}^* W_{ij}) + \sum_{\substack{1 \leq i < j \leq p \\ h \notin \{i, j\}}} \log \cosh(\sigma_{ih}^* \sigma_{hj}^* W_{ij}) \\
 &\quad + \sum_{l=3}^5 \sum_{k_1 < \dots < k_l} \log \cosh \left(\frac{n(-1)^{l+1}}{2} \cdot (l-1) \cdot (l-1)! \cdot \sigma_{k_1 k_2}^* \dots \sigma_{k_l k_1}^* \right) \\
 &\quad - \frac{1}{2} \sum_{1 \leq i \neq h \leq p} \sigma_{ih}^{*2} \left(W_{ii} - \frac{n}{2} \right) + o_P(1). \tag{32}
 \end{aligned}$$

We shall use repeatedly the Taylor expansion of $\log \cosh(u) = u^2/2 - (u^4/12)(1 + o(1))$ as $u \rightarrow 0$. Indeed, $\mathbb{E}_I(W_{ij}) = 0$ and $\mathbb{E}_I(|\sigma_{ij}^* W_{ij}|^2) \leq O(1) \cdot \lambda n = O(1) \cdot n^{-\frac{1}{(4\alpha+1)}} p^{-\frac{2\alpha+1}{(4\alpha+1)}} = o(1)$, giving that $|\sigma_{ij}^* W_{ij}| = o_P(1)$. Thus

$$\log \cosh(\sigma_{ij}^* W_{ij}) = \frac{1}{2} (\sigma_{ij}^* W_{ij})^2 - \frac{1}{12} (\sigma_{ij}^* W_{ij})^4 (1 + o_P(1)). \tag{33}$$

Similarly, using the first order Taylor expansion, we get

$$\log \cosh(\sigma_{ih}^* \sigma_{hj}^* W_{ij}) = \frac{1}{2} (\sigma_{ih}^* \sigma_{hj}^* W_{ij})^2 (1 + o_P(1))$$

and for $l = 3, 4$ and 5 ,

$$\begin{aligned}
 &\log \cosh \left(\frac{n(-1)^{l+1}}{2} \cdot (l-1) \cdot (l-1)! \cdot \sigma_{k_1 k_2}^* \dots \sigma_{k_l k_1}^* \right) \\
 &= \frac{n^2 \left((l-1) \cdot (l-1)! \right)^2}{8} \cdot \sigma_{k_1 k_2}^{*2} \dots \sigma_{k_l k_1}^{*2} (1 + o(1))
 \end{aligned}$$

Recall now that $\sigma_{ij}^* = 0$, for all i, j such that $|i - j| \geq T$ and $\sigma_{ij}^{*2} \leq \lambda = O(1)\varphi^{2+\frac{1}{\alpha}}$. Then,

$$\begin{aligned} \mathbb{E}\left(\sum_{\substack{1 \leq i < j \leq p \\ h \notin \{i, j\}}} \sigma_{ih}^{*2} \sigma_{hj}^{*2} W_{ij}^2\right) &= O(np\lambda^2 T^2) = O(np\varphi^4) \\ &= O(1) \cdot n\sqrt{p}\varphi^{2+\frac{1}{2\alpha}} \cdot \sqrt{p}\varphi^{2-\frac{1}{2\alpha}} = o(1), \end{aligned}$$

as soon as $u_n \asymp 1$ and $\sqrt{p}\varphi^{2-\frac{1}{2\alpha}} \rightarrow 0$. In conclusion, as the convergence in $\mathbb{L}_1(P_I)$ implies convergence in P_I probability, we get

$$\sum_{\substack{1 \leq i < j \leq p \\ h \notin \{i, j\}}} \log \cosh(\sigma_{ih}^* \sigma_{hj}^* W_{ij}) \rightarrow 0 \text{ in } P_I \text{ probability.} \tag{34}$$

Moreover, for $l = 3, 4$ and 5 ,

$$\begin{aligned} n^2 \sum_{k_1 < \dots < k_l} \frac{\left((l-1) \cdot (l-1)!\right)^2}{8} \cdot \sigma_{k_1 k_2}^{*2} \dots \sigma_{k_l k_1}^{*2} \\ = O(n^2 p \lambda^l T^{l-1}) = O(n^2 p \varphi^{2l+\frac{1}{\alpha}}) = o(1). \end{aligned} \tag{35}$$

Using (34) and (35), (32) gives

$$\begin{aligned} L_{n,p} &= \frac{1}{2} \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*2} W_{ij}^2 - \frac{1}{12} \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*4} W_{ij}^4 (1 + o_P(1)) \\ &\quad - \frac{1}{2} \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*2} \left(W_{ii} - \frac{n}{2}\right) + o_P(1) \end{aligned} \tag{36}$$

we further decompose as follows :

$$\begin{aligned} &\frac{1}{2} \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*2} W_{ij}^2 \\ &= \frac{1}{2} \sum_{k=1}^n \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*2} X_{k,i}^2 X_{k,j}^2 + \frac{1}{2} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*2} X_{k,i} X_{k,j} X_{l,i} X_{l,j} \end{aligned}$$

With our definition: $\sigma_{ij}^{*2} = 2 \cdot w_{ij}^* \cdot b(\varphi)$, we can write

$$\frac{1}{2} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*2} X_{k,i} X_{k,j} X_{l,i} X_{l,j} = n\sqrt{p}b(\varphi) \cdot (n-1)\sqrt{p} \cdot \widehat{D}_n$$

and we put $Z_n = (n-1)\sqrt{p} \cdot \widehat{D}_n$ which has asymptotically standard Gaussian under P_I probability, by Proposition 1.

By Proposition 4 given in the Appendix, we have $\mathbb{E}(W_{ij}^4) = 3n^2(1 + o(1))$, then

$$\frac{1}{12} \cdot \mathbb{E}_I \left(\sum_{1 \leq i < j \leq p} \sigma_{ij}^{*4} W_{ij}^4 \right) = \frac{1}{12} \cdot 3n^2 \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*4} (1 + o(1)) = \frac{u_n^2}{2} (1 + o(1)).$$

Moreover,

$$\begin{aligned} \text{Var}_I\left(\sum_{i < j} \sigma_{ij}^{*4} W_{ij}^4\right) &= \sum_{i < j} \sigma_{ij}^{*8} \text{Var}_I(W_{ij}^4) + \sum_{1 \leq i < j \neq j' \leq p} \sum \sigma_{ij}^{*4} \sigma_{ij'}^{*4} \text{Cov}_I(W_{ij}^4, W_{ij'}^4) \\ &= O(n^4 p \lambda^4 T) + O(n^3 p \lambda^4 T^2) \\ &= O(n^4 p \varphi^{8 + \frac{3}{\alpha}}) + O(n^3 p \varphi^{8 + \frac{2}{\alpha}}) = o(1). \end{aligned}$$

We deduce that,

$$\frac{1}{12} \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*4} W_{ij}^4 = \frac{u_n^2}{2} + o_P(1)$$

Remaining terms in (36) can be grouped as follows:

$$\frac{1}{4} \sum_{k=1}^n \sum_{1 \leq i \neq j \leq p} \sigma_{ij}^{*2} (X_{k,i}^2 - 1)(X_{k,j}^2 - 1) = o_P(1)$$

since the random variable in the previous display is centered and

$$\begin{aligned} &E_I\left(\sum_{k=1}^n \sum_{1 \leq i \neq j \leq p} \sigma_{ij}^{*2} (X_{k,i}^2 - 1)(X_{k,j}^2 - 1)\right)^2 \\ &= \sum_{k=1}^n \sum_{1 \leq i \neq j \leq p} \sigma_{ij}^{*4} \cdot E_I(X_{k,i}^2 - 1)^2 E_I(X_{k,j}^2 - 1)^2 \\ &= 4n \sum_{1 \leq i < j \leq p} \sigma_{ij}^{*4} = O(npb^2(\varphi)) = o(1), \end{aligned}$$

which concludes the proof of (22). □

Proof of Theorem 6. The type I error probability tends to 0 as a consequence of the Berry-Essen type inequality in Lemma 1 in the Appendix applied to the degenerate U-statistic $\widehat{\mathcal{D}}_{n,p}$. We have that, for some $\varepsilon \in (0, 1/2)$ and any $t > 0$:

$$\left| \mathbb{P}_I(\widehat{\mathcal{D}}_{n,r} \leq t) - \Phi(n\sqrt{p} \cdot t) \right| \leq 16\varepsilon^{1/2} \exp\left(-\frac{n^2 p t^2}{4}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{pT_r}\right)$$

for all $1 \leq r \leq N$. We use the relation $1 - \Phi(u) \leq (1/u) \exp(-u^2/2)$ for all $u \in \mathbb{R}$, to deduce that

$$\mathbb{P}_I(\widehat{\mathcal{D}}_{n,r} > x) \leq \left(\frac{1}{n\sqrt{p} \cdot x} + 16\varepsilon^{1/2}\right) \exp\left(-\frac{n^2 p x^2}{4}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{pT_r}\right)$$

We use this previous result to show that the type I error probability tends to 0. See that for all $r \in \{1, \dots, N\}$, $C_{\lambda_r} \geq c(\underline{\alpha}, \bar{\alpha})$, where $c(\underline{\alpha}, \bar{\alpha}) = (2\underline{\alpha} + 1)/(2\bar{\alpha}(4\bar{\alpha} + 1)^{\frac{1}{2\bar{\alpha}}})$. Thus since $n\sqrt{p} \cdot t_r = C_{\lambda_r} \sqrt{\ln \ln(n\sqrt{p})}$, we obtain that $n\sqrt{p} \cdot t_r \geq c(\underline{\alpha}, \bar{\alpha}) \sqrt{\ln \ln(n\sqrt{p})} =: t$ for all $r \in \{1, \dots, N\}$. Recall that $N =$

$\lceil \ln(n\sqrt{p}) \rceil$, therefore

$$\begin{aligned} \mathbb{P}_I(\Delta_{ad}^* = 1) &= \mathbb{P}_I(\exists r \in \{1, \dots, N\}; \widehat{\mathcal{D}}_{n,r} > \mathcal{C}^* t_r) \leq \sum_{r=1}^N \mathbb{P}_I(\widehat{\mathcal{D}}_{n,r} > \mathcal{C}^* t_r) \\ &\leq \sum_{r=1}^N \left\{ \left(\frac{1}{n\sqrt{p} \cdot \mathcal{C}^* t_r} + 16\varepsilon^{1/2} \right) \exp\left(-\frac{n^2 p \cdot \mathcal{C}^{*2} t_r^2}{4}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{pT_r}\right) \right\} \\ &\leq N \left(\frac{1}{n\sqrt{p} \cdot c(\underline{\alpha}, \bar{\alpha}) \mathcal{C}^* t} + 16\varepsilon^{1/2} \right) \exp\left(-\frac{n^2 p \cdot c^2(\underline{\alpha}, \bar{\alpha}) \mathcal{C}^{*2} t^2}{4}\right) + O\left(\frac{N}{n}\right) \\ &+ O\left(\frac{1}{p}\right) \sum_{r=1}^N \frac{1}{T_r} \\ &\leq \left(\frac{1}{c(\underline{\alpha}, \bar{\alpha}) \mathcal{C}^* \sqrt{\ln \ln(n\sqrt{p})}} + 16\varepsilon^{1/2} \right) (\ln(n\sqrt{p}))^{1-(c(\underline{\alpha}, \bar{\alpha}) \mathcal{C}^*/2)^2} \\ &+ \frac{O(\ln(n\sqrt{p}))}{n} + \frac{O(1)}{p} \sum_{r=1}^N \frac{1}{T_r}. \end{aligned}$$

See that $1/T_r = (\psi_{\alpha_r})^{\frac{1}{\alpha_r}} \leq (\rho_{n,p}/(n\sqrt{p}))^{2/(4\bar{\alpha}+1)}$ and therefore:

$$\frac{1}{p} \sum_{r=1}^N \frac{1}{T_r} \leq \frac{\lceil \ln(n\sqrt{p}) \rceil}{p} \left(\frac{\rho_{n,p}}{n\sqrt{p}} \right)^{\frac{2}{4\bar{\alpha}+1}} = o(1).$$

Moreover if $(\ln p)/n = o(1)$, then $\ln(n\sqrt{p})/n = o(1)$, and if $\mathcal{C}^* \geq 2/c(\underline{\alpha}, \bar{\alpha})$ we obtain

$$\mathbb{P}_I(\Delta_{ad}^* = 1) = o(1).$$

Now, we move to the type II error probability. Let us consider $\Sigma \in \mathcal{F}(\alpha, L)$ such that $(1/2p)\|\Sigma - I\|_F^2 = (1/p) \sum_{i < j} \sigma_{ij}^2 \geq (\mathcal{C}\psi_\alpha)^2$ for some $\alpha \in \mathcal{A}$. We defined α_{r_0} as the smallest point on the grid such that $\alpha < \alpha_{r_0}$. We denote by $\widehat{\mathcal{D}}_{n,r_0}$, t_{r_0} , λ_{r_0} , b_{r_0} and T_{r_0} the test statistic, the threshold and the parameters depending on α_{r_0} . Also we define $C_{T_{r_0}}$, $C_{\lambda_{r_0}}$ and $C_{b_{r_0}}$ the constants defined in (5) for α_{r_0} instead of α and $L = 1$. We have $C_{b_r} < 1$ and $C_{T_r} > 1$, for all $r \in \{1, \dots, N\}$. The type II error probability is bounded from above as follows, $\forall \alpha \in [\underline{\alpha}, \bar{\alpha}]$ and $\forall \Sigma \in Q(\alpha, L, \mathcal{C}\psi_\alpha)$:

$$\begin{aligned} \mathbb{P}_\Sigma(\Delta_{ad}^* = 0) &= \mathbb{P}_\Sigma(\forall 1 \leq r \leq N, \widehat{\mathcal{D}}_{n,r} \leq \mathcal{C}^* t_r) \leq \mathbb{P}_\Sigma(\widehat{\mathcal{D}}_{n,r_0} \leq \mathcal{C}^* t_{r_0}) \\ &\leq \mathbb{P}_\Sigma(\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_{n,r_0}) - \widehat{\mathcal{D}}_{n,r_0} \geq \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_{n,r_0}) - \mathcal{C}^* t_{r_0}) \end{aligned}$$

First we have

$$\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_{n,r_0}) = \frac{1}{p} \sum_{i < j} w_{ij,r_0}^* \sigma_{ij}^2 = \frac{1}{p} \cdot \frac{\lambda_{r_0}}{b_{r_0}} \sum_{i < j} \left(1 - \left(\frac{|i-j|}{T_{r_0}} \right)^{2\alpha_{r_0}} \right)_+ \sigma_{ij}^2$$

$$\begin{aligned}
 &\geq \frac{1}{p} \cdot \frac{\lambda_{r_0}}{b_{r_0}} \left(\sum_{i < j} \sigma_{ij}^2 - \sum_{\substack{i < j \\ |i-j| > T_{r_0}}} \sigma_{ij}^2 - \sum_{\substack{i < j \\ |i-j| < T_{r_0}}} \frac{|i-j|^{2\alpha_{r_0}}}{T_{r_0}^{2\alpha_{r_0}}} \cdot \sigma_{ij}^2 \right) \\
 &\geq \frac{\lambda_{r_0}}{b_{r_0}} \left(\mathcal{C}^2 \cdot \psi_\alpha^2 - \sum_{i < j} \frac{|i-j|^{2\alpha}}{T_{r_0}^{2\alpha}} \cdot \sigma_{ij}^2 \right) \geq \frac{\lambda_{r_0}}{b_{r_0}} \left(\mathcal{C}^2 \cdot \psi_\alpha^2 - L \cdot T_{r_0}^{-2\alpha} \right) \\
 &\geq C_{\lambda_{r_0}} \cdot (C_{b_{r_0}})^{-\frac{1}{2}} (\psi_{\alpha_{r_0}})^{\frac{1}{2\alpha_{r_0}}} \left(\mathcal{C}^2 \cdot \psi_\alpha^2 - (C_{T_{r_0}})^{-2\alpha} \cdot (\psi_{\alpha_{r_0}})^{\frac{4\alpha}{2\alpha_{r_0}}} \right) \\
 &\geq C_{\lambda_{r_0}} \left(\mathcal{C} \cdot (\psi_{\alpha_{r_0}})^{\frac{1}{2\alpha_{r_0}}} \cdot \psi_\alpha^2 - (C_{T_{r_0}})^{-2\alpha} \cdot (\psi_{\alpha_{r_0}})^{\frac{4\alpha+1}{2\alpha_{r_0}}} \right) \\
 &=: (E_1 - E_2).
 \end{aligned}$$

Now we show that, since $\alpha < \alpha_{r_0}$ we have,

$$\begin{aligned}
 E_1 \cdot t_{r_0}^{-1} &= \mathcal{C} \cdot (\psi_{\alpha_{r_0}})^{\frac{1}{2\alpha_{r_0}}} \cdot (\psi_\alpha)^2 \cdot ((n\sqrt{p})/\rho_{n,p}) \\
 &= ((n\sqrt{p})/\rho_{n,p})^{\frac{4(\alpha_{r_0}-\alpha)}{(4\alpha_{r_0}+1)(4\alpha+1)}} \geq \mathcal{C}
 \end{aligned}$$

Moreover, use that $0 > \alpha - \alpha_{r_0} \geq -(\bar{\alpha} - \underline{\alpha})/\ln(n\sqrt{p})$, to obtain

$$\begin{aligned}
 t_{r_0} \cdot E_2^{-1} &= (\rho_{n,p}/(n\sqrt{p})) \cdot (C_{T_{r_0}})^{2\alpha} \cdot (\psi_{\alpha_{r_0}})^{-\frac{4\alpha+1}{2\alpha_{r_0}}} \geq ((n\sqrt{p})/\rho_{n,p})^{\frac{4(\alpha-\alpha_{r_0})}{(4\alpha_{r_0}+1)}} \\
 &= \exp \left\{ \frac{4(\alpha - \alpha_{r_0})}{4\alpha_{r_0} + 1} \cdot \ln((n\sqrt{p})/\rho_{n,p}) \right\} \\
 &\geq \exp \left\{ -\frac{4(\bar{\alpha} - \underline{\alpha})}{4\underline{\alpha} + 1} (1 + o(1)) \right\} \geq C(\underline{\alpha}, \bar{\alpha}).
 \end{aligned}$$

We deduce that,

$$\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_{n,r_0}) \geq \left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} \right) \cdot t_{r_0}.$$

Let us denote by \mathcal{T}_1 and \mathcal{T}_2 the right-hand side terms in (7) and (8), respectively.

Then by Markov inequality, for $\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} - \mathcal{C}^* > 0$, we get

$$\begin{aligned}
 \mathbb{P}_\Sigma(\Delta_{ad}^* = 0) &\leq \mathbb{P}_\Sigma \left(|\widehat{\mathcal{D}}_{n,r_0} - \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_{n,r_0})| \geq \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_{n,r_0}) - \mathcal{C}^* t_{r_0} \right) \\
 &\leq \frac{\text{Var}_\Sigma(\widehat{\mathcal{D}}_{n,r_0})}{\left(\mathbb{E}_\Sigma(\widehat{\mathcal{D}}_{n,r_0}) - \mathcal{C}^* t_{r_0} \right)^2} \leq \frac{\left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} \right)^2 \text{Var}_\Sigma(\widehat{\mathcal{D}}_{n,r_0})}{\left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} - \mathcal{C}^* \right)^2 \mathbb{E}_\Sigma^2(\widehat{\mathcal{D}}_{n,r_0})} \\
 &\leq \frac{\left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} \right)^2 \cdot (\mathcal{T}_1 + (n-1)\mathcal{T}_2)}{n(n-1)p^2 \left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} - \mathcal{C}^* \right)^2 \mathbb{E}_\Sigma^2(\widehat{\mathcal{D}}_{n,r_0})} := F_1 + F_2.
 \end{aligned}$$

We use (7) to show that F_1 tends to zero.

$$\begin{aligned}
 F_1 &:= \frac{\left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})}\right)^2 \cdot \mathcal{T}_1}{n(n-1)p^2 \left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} - \mathcal{C}^*\right)^2 \mathbb{E}_{\Sigma}^2(\widehat{\mathcal{D}}_{n,r_0})} \\
 &\leq \frac{1 + o(1)}{n(n-1)p \left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} - \mathcal{C}^*\right)^2 t_{r_0}^2} + \frac{O(T_{r_0}^{\frac{3}{2}} \cdot t_{r_0})}{n(n-1)pt_{r_0}^2} = O(\rho_{n,p}^{-2}) = o(1),
 \end{aligned}$$

since $T_{r_0}^{\frac{3}{2}} \cdot t_{r_0} = O\left((\rho_{n,p}/n\sqrt{p})^{-\frac{3}{4\alpha_{r_0}+1}+1}\right) = o(1)$ for $\alpha_{r_0} > 1/2$. Similarly we use (8) to show that

$$F_2 := \frac{\left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})}\right)^2 \cdot \mathcal{T}_2}{np^2 \left(\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} - \mathcal{C}^*\right)^2 \mathbb{E}_{\Sigma}^2(\widehat{\mathcal{D}}_{n,r_0})} = o(1).$$

Thus we get, for $\mathcal{C} - \frac{1}{C(\underline{\alpha}, \bar{\alpha})} - \mathcal{C}^* > 0$,

$$\sup_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \sup_{\substack{\Sigma \in \mathcal{F}(\alpha, L); \\ \frac{1}{2p} \|\Sigma - I\|_F^2 \geq \mathcal{C}^2 \psi_{\alpha}^2}} \mathbb{P}_{\Sigma}(\Delta_{ad}^* = 0) = o(1). \quad \square$$

Appendix

Proof of Proposition 1. We recall that under the null hypothesis the coordinates of the vector X_k are independent, so using this fact we have:

$$\begin{aligned}
 \text{Var}_I(\widehat{\mathcal{D}}_n) &= \frac{2}{n^2(n-1)^2 p^2} \text{Var}\left(\sum_{\substack{i=1 \\ i < j}}^p \sum_{j=1}^p \sum_{\substack{l=1 \\ k \neq l}}^n \sum_{k=1}^n w_{ij}^* X_{k,i} X_{k,j} X_{l,i} X_{l,j}\right) \\
 &= \frac{2}{n(n-1)p^2} \sum_{\substack{i=1 \\ i < j}}^p \sum_{j=1}^p w_{ij}^{*2} \mathbb{E}^4(X_{1,i}^2) = \frac{2}{n(n-1)p^2} \sum_{\substack{i=1 \\ i < j}}^p \sum_{j=1}^p w_{ij}^{*2} \\
 &= \frac{1}{n(n-1)p}
 \end{aligned}$$

For $\Sigma \in Q(\alpha, L, \varphi)$,

$$\begin{aligned}
 \mathbb{E}_{\Sigma}(\widehat{\mathcal{D}}_n) &= \frac{1}{n(n-1)p} \sum_{\substack{i=1 \\ i < j}}^p \sum_{j=1}^p \sum_{\substack{l=1 \\ k \neq l}}^n \sum_{k=1}^n w_{ij}^* \mathbb{E}(X_{k,i} X_{k,j} X_{l,i} X_{l,j}) \\
 &= \frac{1}{p} \sum_{\substack{i=1 \\ i < j}}^p \sum_{j=1}^p w_{ij}^* \mathbb{E}(X_{1,i} X_{1,j}) \mathbb{E}(X_{2,i} X_{2,j}) = \frac{1}{p} \sum_{\substack{i=1 \\ i < j}}^p \sum_{j=1}^p w_{ij}^* \sigma_{ij}^2
 \end{aligned}$$

and this is $p(1+o(1))$ since $\sup_{i,j} w_{ij}^2 \asymp (1/T) \rightarrow 0$. When the indices are in A_2 , we have three indices out of four which are equal. We assume $i = i'$, therefore it is sufficient to check that,

$$\begin{aligned} T_{1,2} &= 2 \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ j \neq j'}}^p w_{ij}^* w_{ij'}^* (\sigma_{jj'} + \sigma_{ij} \sigma_{ij'})^2 \\ &\leq 4 \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ j \neq j'}}^p w_{ij}^* w_{ij'}^* \sigma_{jj'}^2 + 4 \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ j \neq j'}}^p w_{ij}^* w_{ij'}^* \sigma_{ij}^2 \sigma_{ij'}^2 \end{aligned}$$

Now let us bound from above the first term of $T_{1,2}$,

$$\begin{aligned} T_{1,2,1} &:= \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ j \neq j'}}^p w_{ij}^* w_{ij'}^* \sigma_{jj'}^2 \\ &\leq \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ |j-j'| < T}}^p w_{ij}^* w_{ij'}^* \sigma_{jj'}^2 + \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ |j-j'| \geq T}}^p \frac{|j-j'|^{2\alpha}}{T^{2\alpha}} \sigma_{jj'}^2 \sum_{i=1}^p w_{ij}^* w_{ij'}^* \quad (40) \end{aligned}$$

Again we will treat each term of $T_{1,2,1}$ separately. We recall that the weights w_{ij}^* verify the following properties

$$(w_{ij}^* \geq w_{i'j'}^* \quad \text{for } |i-j| \leq |i'-j'|) \quad \text{and} \quad \sum_{i=1}^p w_{ij}^* \asymp \sqrt{T}.$$

In the rest of the proof we denote by $k_0(\alpha, L), k_1(\alpha, L), \dots$ different constants that dependent only on α and/or on L .

$$\begin{aligned} T_{1,2,1,1} &:= \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ |j-j'| < T}}^p w_{ij}^* w_{ij'}^* \sigma_{jj'}^2 \\ &= \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ |j-j'| \leq |i-j| < T}}^p w_{ij}^* w_{ij'}^* \sigma_{jj'}^2 + \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ |i-j| < |j-j'| < T}}^p w_{ij}^* w_{ij'}^* \sigma_{jj'}^2 \\ &\leq \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ |j-j'| < |i-j| < T}}^p w_{jj'}^* \sigma_{jj'}^2 \sum_{i=1}^p w_{ij}^* + \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ |i-j| < |j-j'| < T}}^p \frac{w_{ij}^* w_{ij'}^* |j-j'|^{2\alpha}}{|i-j|^{2\alpha}} \sigma_{jj'}^2 \end{aligned}$$

We have for $\alpha > 1/2$,

$$\begin{aligned}
 T_{1,2,1,1} &\leq k_0(\alpha, L)\sqrt{T} \cdot p \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \\
 &\quad + \left(\sup_{i,j} w_{ij}^*\right)^2 \sum_{j=1}^p \sum_{\substack{j'=1 \\ j \neq j'}}^p |j-j'|^{2\alpha} \sigma_{jj'}^2 \left(\sum_{i=1}^p \frac{1}{|i-j|^{2\alpha}}\right) \\
 &\leq k_0(\alpha, L) \cdot \sqrt{T} \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) + k_1(\alpha, L) \cdot L \cdot p \cdot \left(\sup_{i,j} w_{ij}^*\right)^2 \\
 &\leq p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) O(\sqrt{T}) + o(p). \tag{41}
 \end{aligned}$$

For the second term in (40), where $|j-j'| \geq T$, we use the following bound:

$$\sum_{\substack{i=1 \\ i \neq j, j'}}^p w_{ij}^* w_{ij'}^* \leq \sum_{\substack{i=1 \\ i \neq j, j'}}^p (w_{ij}^*)^2 \leq \frac{1}{2},$$

then we prove that,

$$T_{1,2,1,2} := \sum_{\substack{j=1 \\ j \neq i \\ |j-j'| \geq T}}^p \sum_{\substack{j'=1 \\ j' \neq i}}^p \frac{|j-j'|^{2\alpha}}{T^{2\alpha}} \sigma_{jj'}^2 \sum_{i=1}^p w_{ij}^* w_{ij'}^* \leq \frac{L \cdot p}{T^{2\alpha}} = O\left(\frac{p}{2T^{2\alpha}}\right) = o(p). \tag{42}$$

Note that $\sup_{i,j} \sigma_{ij} \leq 1$. The second term of $T_{1,2}$, is bounded as follows:

$$\begin{aligned}
 T_{1,2,2} &:= \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i \\ j \neq j'}}^p \sum_{\substack{j'=1 \\ j' \neq i \\ j' \neq j}}^p w_{ij}^* w_{ij'}^* \sigma_{ij}^2 \sigma_{ij'}^2 = \sum_{i=1}^p \left(\sum_{\substack{j=1 \\ j \neq i}}^p w_{ij}^* \sigma_{ij}^2\right) \left(\sum_{\substack{j'=1 \\ j' \neq i}}^p w_{ij'}^* \sigma_{ij'}^2\right) \\
 &\leq \left(\sup_{i,j} w_{ij}^*\right) \sup_i \left(\sum_{\substack{j=1 \\ 1 \leq |j-i| < T}}^p \sigma_{ij}^2\right) \left(\sum_{i=1}^p \sum_{\substack{j'=1 \\ j' \neq i}}^p w_{ij'}^* \sigma_{ij'}^2\right) \\
 &\leq 2L \cdot \left(\sup_{i,j} w_{ij}^*\right) \cdot T \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \leq p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot O(\sqrt{T}) \tag{43}
 \end{aligned}$$

As a consequence of (41) to (43),

$$T_{1,2} \leq p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot O(\sqrt{T}) + o(p) \tag{44}$$

The last case, where (i, j, i', j') vary in A_3 , the indices are pairwise distinct,

$$\begin{aligned}
 T_{1,3} &= \sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* (\sigma_{ii'} \sigma_{jj'} + \sigma_{ij'} \sigma_{i'j})^2 \\
 &\leq 2 \sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* \sigma_{ii'}^2 \sigma_{jj'}^2 + 2 \sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* \sigma_{ij'}^2 \sigma_{i'j}^2
 \end{aligned}$$

As the two previous terms have the same upper bound, let us deal with the first one say $T_{1,3,1}$. We should distinguish two cases, the first when $|i - i'| < T$ and the second when $|i - i'| \geq T$. We begin by the first case, which in turn will be decomposed into three terms. First,

$$\begin{aligned}
T_{1,3,1,1} &:= \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-j| \geq |i-i'|, |i'-j'| \geq |i-i'|}} w_{ij}^* w_{i'j'}^* \sigma_{ii'}^2 \sigma_{jj'}^2 \leq \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-j| \geq |i-i'|, |i'-j'| \geq |i-i'|}} w_{ii'}^{*2} \sigma_{ii'}^2 \sigma_{jj'}^2 \\
&\leq (\sup_{ij} w_{ij}^*) \sum_{1 \leq i, i' \leq p} w_{ii'}^* \sigma_{ii'}^2 \sum_{\substack{1 \leq j, j' \leq p \\ 1 < |i-j|, |i'-j'| < T}} \sigma_{jj'}^2 \\
&\leq (\sup_{ij} w_{ij}^*) \cdot T^2 \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)
\end{aligned} \tag{45}$$

Then,

$$\begin{aligned}
T_{1,3,1,2} &:= \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-j| < |i-i'| < T, |i'-j'| \geq |i-i'|}} w_{ij}^* w_{i'j'}^* \sigma_{ii'}^2 \sigma_{jj'}^2 \leq \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-j| < |i-i'| < T, |i'-j'| \geq |j-j'|}} w_{ij}^* w_{i'j'}^* \sigma_{ii'}^2 \sigma_{jj'}^2 \\
&\leq (\sup_{ij} w_{ij}^*) T^2 \cdot p \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \leq k_2(\alpha, L) \cdot T \sqrt{T} \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)
\end{aligned} \tag{46}$$

Finally, using Cauchy-Schwarz inequality, we have,

$$\begin{aligned}
T_{1,3,1,3} &:= \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-j| < |i-i'| < T, |i'-j'| < |i-i'| < T}} w_{ij}^* w_{i'j'}^* \sigma_{ii'}^2 \sigma_{jj'}^2 \\
&= \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-j| < |i-i'| < T, |i'-j'| < |j-j'| < T}} w_{ij}^* w_{i'j'}^* \cdot \frac{|i-i'|^{2\alpha}}{|i-j|^\alpha |i'-j'|^\alpha} \cdot \sigma_{ii'}^2 \sigma_{jj'}^2 \\
&\leq (\sup_{ij} w_{ij}^*)^2 \sum_{i=1}^p \sum_{\substack{i'=1 \\ i' \neq i}}^p |i-i'|^{2\alpha} \sigma_{ii'}^2 \sum_{\substack{1 \leq j, j' \leq p \\ 1 \leq |i-j|, |i'-j'| < T}} \frac{\sigma_{jj'}^2}{|i-j|^\alpha |i'-j'|^\alpha} \\
&\leq k_3(\alpha, L) \cdot T^{-1} \cdot 2pL \cdot \max\{1, T^{-2\alpha+2}\} = o(p)
\end{aligned} \tag{47}$$

for all $\alpha > \frac{1}{2}$. Now we suppose that we have $|i - i'| > T$, then,

$$\begin{aligned}
T_{1,3,2} &:= \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-i'| > T}} w_{ij}^* w_{i'j'}^* \sigma_{ii'}^2 \sigma_{jj'}^2 = \sum_{\substack{(i,j,i',j') \in A_3 \\ |i-i'| > T}} w_{ij}^* w_{i'j'}^* \frac{|i-i'|^{2\alpha}}{T^{2\alpha}} \sigma_{ii'}^2 \sigma_{jj'}^2 \\
&\leq \frac{(\sup_{ij} w_{ij}^*)^2}{T^{2\alpha}} \sum_{1 \leq i, i' \leq p} |i-i'|^{2\alpha} \sigma_{ii'}^2 \sum_{\substack{1 \leq j, j' \leq p \\ 1 \leq |i-j|, |i'-j'| < T}} \sigma_{jj'}^2 \\
&\leq \frac{(\sup_{ij} w_{ij}^*)^2}{T^{2\alpha}} \cdot 2pL \cdot T^2 \leq \frac{k_4(\alpha, L) \cdot p}{T^{2\alpha-1}} = o(p) \quad \text{for } \alpha > \frac{1}{2}.
\end{aligned} \tag{48}$$

Finally we obtain, from (45) to (48) :

$$T_{1,3} \leq p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot O(T\sqrt{T}) + o(p). \tag{49}$$

Put together (39), (44) and (49) to obtain (7). Let us give an upper bound for the second term of (38),

$$\begin{aligned} T_2 &= 4\mathbb{E}_\Sigma\left\{\sum_{i=1}^p \sum_{j=1}^p w_{ij}^* (X_{k,i}X_{k,j} - \sigma_{ij})\sigma_{ij}\right\}^2 \\ &= \sum_{i=1}^p \sum_{j=1}^p \sum_{i' < j'}^p \sum_{j'=1}^p w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} \mathbb{E}_\Sigma(X_{1,i}X_{1,j} - \sigma_{ij})(X_{1,i'}X_{1,j'} - \sigma_{i'j'}) \\ &= \sum_{i=1}^p \sum_{j=1}^p \sum_{i' \neq j}^p \sum_{j' \neq j'}^p w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} (\sigma_{i'i}^* \sigma_{j'j'}^* + \sigma_{i'j}^* \sigma_{ij}^*) \end{aligned}$$

Proceeding similarly, we shall distinguish three kind of terms. Let us begin by the case when the indices belong to A_1 ,

$$\begin{aligned} T_{2,1} &= 2 \sum_{i=1}^p \sum_{\substack{j=1 \\ i \neq j}}^p w_{ij}^{*2} \sigma_{ij}^2 \mathbb{E}_\Sigma[(X_{1,i}X_{1,j} - \sigma_{ij})^2] = 2 \sum_{i=1}^p \sum_{\substack{j=1 \\ i \neq j}}^p w_{ij}^{*2} \sigma_{ij}^2 (1 + \sigma_{ij}^2) \\ &\leq 4(\sup_{i,j} w_{ij}^*) \sum_{i=1}^p \sum_{\substack{j=1 \\ i \neq j}}^p w_{ij}^* \sigma_{ij}^2 = 8(\sup_{i,j} w_{ij}^*) \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) = o(1) \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n). \end{aligned} \tag{50}$$

Next, when $(i, j, i', j') \in A_2$,

$$\begin{aligned} T_{2,2} &= 4 \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i}}^p w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} (\sigma_{jj'} + \sigma_{ij} \sigma_{i'j'}) \\ &= 4 \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i}}^p w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} \sigma_{jj'} + 4 \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i}}^p w_{ij}^* w_{i'j'}^* \sigma_{ij}^2 \sigma_{i'j'}^2 \end{aligned}$$

We bound from each term of $T_{2,2}$ separately. Using Cauchy-Schwarz inequality two times we obtain,

$$\begin{aligned} T_{2,2,1} &:= \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \sum_{\substack{j'=1 \\ j' \neq i}}^p w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} \sigma_{jj'} \\ &\leq \sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p w_{ij}^* \sigma_{ij} \left(\sum_{\substack{j'=1 \\ j' \neq i}}^p w_{i'j'}^{*2} \sigma_{i'j'}^2 \right)^{1/2} \left(\sum_{\substack{j'=1 \\ j' \neq i}}^p \sigma_{jj'}^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned} &\leq \left(\sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p w_{ij}^{*2} \sigma_{ij}^2 \right)^{1/2} \left(\sum_{i=1}^p \sum_{\substack{j=1 \\ j \neq i}}^p \left(\sum_{\substack{j'=1 \\ j' \neq i}}^p w_{ij'}^{*2} \sigma_{ij'}^2 \right) \left(\sum_{\substack{j'=1 \\ j' \neq i}}^p \sigma_{jj'}^2 \right) \right)^{1/2} \\ &\leq \left(\sup_{i,j} w_{ij}^* \right) \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot O(T) = O(\sqrt{T}) \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n). \end{aligned}$$

The second term in $T_{2,2}$ is $T_{1,2,2}$ and therefore,

$$T_{2,2} = O(\sqrt{T}) \cdot p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n). \tag{51}$$

Finally, when $(i, j, i', j') \in A_3$, we have to bound from above

$$T_{2,3} = \sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} \sigma_{ii'}^* \sigma_{jj'}^* + \sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} \sigma_{ij'}^* \sigma_{i'j}^*.$$

These last two terms, in $T_{2,3}$ are treated similarly, so let us deal with :

$$\begin{aligned} T_{2,3,1} &:= \sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* \sigma_{ij} \sigma_{i'j'} \sigma_{ii'}^* \sigma_{jj'}^* \\ &\leq \sum_j \sum_{i'} \left(\sum_i w_{ij}^* \sigma_{ii'}^2 \right)^{1/2} \left(\sum_i w_{ij}^* \sigma_{ij}^2 \right)^{1/2} \left(\sum_{j'} w_{i'j'}^* \sigma_{i'j'}^2 \right)^{1/2} \left(\sum_{j'} w_{i'j'}^* \sigma_{jj'}^2 \right)^{1/2} \\ &\leq \left(\sum_j \sum_{i'} \left(\sum_i w_{ij}^* \sigma_{ij}^2 \right) \left(\sum_{j'} w_{i'j'}^* \sigma_{i'j'}^2 \right) \right)^{1/2} \left(\sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* \sigma_{jj'}^2 \sigma_{ii'}^2 \right)^{1/2} \\ &\leq p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot \left(\sum_{(i,j,i',j') \in A_3} w_{ij}^* w_{i'j'}^* \sigma_{jj'}^2 \sigma_{ii'}^2 \right)^{1/2} \end{aligned}$$

Using the upper bound of $T_{1,3}$ obtained previously, we have

$$T_{2,3} \leq p\sqrt{p} \cdot \left(\mathbb{E}_\Sigma^{3/2}(\widehat{\mathcal{D}}_n) \cdot O(T^{3/4}) + \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot o(1) \right) \tag{52}$$

Put together (50), (51) and (52) to get (8).

The asymptotic normality under the null hypothesis is obvious. □

Proof of Proposition 2. We use the decomposition (37) in the proof of the Proposition 1 and we treat each term separately. Recall that, by our assumptions, $n\sqrt{p} \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) = O(1)$. Use (8) to get

$$\begin{aligned} &\text{Var}_\Sigma \left(\frac{2}{\sqrt{p}} \sum_{l=1}^n \sum_{1 \leq i < j \leq p} w_{ij}^* (X_{l,i} X_{l,j} - \sigma_{ij}) \sigma_{ij} \right) \\ &\leq \frac{n}{p} \left(p^{3/2} \left(o(1) \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) + O(T^{3/4}) \mathbb{E}_\Sigma^{3/2}(\widehat{\mathcal{D}}_n) \right) + p \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) O(\sqrt{T}) \right) \\ &= o(1)n\sqrt{p} \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) + (n\sqrt{p} \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n))^{3/2} \cdot \frac{O(T^{3/4})}{n^{1/2} p^{1/4}} + n\sqrt{p} \cdot \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n) \cdot o(1) \end{aligned} \tag{53}$$

This tends to 0, since $T^3/n^2p = (n^2pb^2(\varphi))^{-1} \cdot \varphi^{4-2/\alpha} = o(1)$, which is true for all $\alpha > 1/2$.

It follows that, for proving the asymptotic normality, it is sufficient to prove the asymptotic normality of

$$n\sqrt{p} \cdot \frac{1}{n(n-1)p} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq i < j \leq p} w_{ij}^*(X_{k,i}X_{k,j} - \sigma_{ij})(X_{l,i}X_{l,j} - \sigma_{ij}).$$

We study V_n which is a centered, 1-degenerate U-statistic, with symmetric kernel $H_n(X_1, X_2)$ defined as follows

$$\begin{aligned} V_n &= \sum_{1 \leq k \neq l \leq n} H_n(X_k, X_l), \\ H_n(X_1, X_2) &= \frac{1}{n\sqrt{p}} \sum_{1 \leq i < j \leq p} w_{ij}^*(X_{k,i}X_{k,j} - \sigma_{ij})(X_{l,i}X_{l,j} - \sigma_{ij}). \end{aligned}$$

We apply Theorem 1 of [16]. Therefore we check that $\mathbb{E}_\Sigma(H_n^2(X_1, X_2)) < +\infty$ and that

$$\frac{\mathbb{E}_\Sigma(G_n^2(X_1, X_2)) + n^{-1}\mathbb{E}_\Sigma(H_n^4(X_1, X_2))}{\mathbb{E}_\Sigma^2(H_n^2(X_1, X_2))} \rightarrow 0,$$

where, for $x, y \in \mathbb{R}^p$,

$$\begin{aligned} G_n(x, y) &= \mathbb{E}_\Sigma(H_n(X_1, x)H_n(X_1, y)) \\ &= \frac{1}{n^2p} \sum_{1 \leq i < j \leq p} \sum_{1 \leq i' < j' \leq p} w_{ij}^*w_{i'j'}^*(x_i x_j - \sigma_{ij})(y_i y_j - \sigma_{ij})(\sigma_{ii'}\sigma_{jj'} + \sigma_{i'j}\sigma_{ij'}). \end{aligned}$$

Since $n\sqrt{p} \cdot \mathbb{E}_\Sigma(\widehat{D}_n) = O(1)$, and from the inequality (7), we have

$$\mathbb{E}_\Sigma(H_n^2(X_1, X_2)) = \frac{1}{2n^2}(1 + o(1)).$$

In order to prove that $\mathbb{E}_\Sigma(G_n^2(X_1, X_2))/\mathbb{E}_\Sigma^2(H_n^2(X_1, X_2)) = o(1)$, it is sufficient to show that

$$\begin{aligned} &\mathbb{E}_\Sigma \left(\sum_{1 \leq i < j \leq p} \sum_{1 \leq i' < j' \leq p} w_{ij}^*w_{i'j'}^*(X_{1,i}X_{1,j} - \sigma_{ij})(X_{2,i'}X_{2,j'} - \sigma_{i'j'}) \right. \\ &\qquad \qquad \qquad \left. \cdot (\sigma_{ii'}\sigma_{jj'} + \sigma_{i'j}\sigma_{ij'}) \right)^2 \\ &= o(p^2). \end{aligned}$$

In fact,

$$\begin{aligned} &\mathbb{E}_\Sigma \left(\sum_{1 \leq i < j \leq p} \sum_{1 \leq i' < j' \leq p} w_{ij}^*w_{i'j'}^*(X_{1,i}X_{1,j} - \sigma_{ij})(X_{2,i'}X_{2,j'} - \sigma_{i'j'}) \right. \\ &\qquad \qquad \qquad \left. \cdot (\sigma_{ii'}\sigma_{jj'} + \sigma_{i'j}\sigma_{ij'}) \right)^2 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{1 \leq i_1 < j_1 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} \sum_{1 \leq i_2 < j_2 \leq p} \sum_{1 \leq i'_2 < j'_2 \leq p} w_{i_1 j_1}^* w_{i'_1 j'_1}^* w_{i_2 j_2}^* w_{i'_2 j'_2}^* \\
 &\quad \cdot (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i'_1 j_1} \sigma_{i_1 j'_1}) (\sigma_{i_2 i'_2} \sigma_{j_2 j'_2} + \sigma_{i'_2 j_2} \sigma_{i_2 j'_2}) \\
 &\quad \cdot \mathbb{E}[(X_{1, i_1} X_{1, j_1} - \sigma_{i_1 j_1})(X_{1, i_2} X_{1, j_2} - \sigma_{i_2 j_2})] \\
 &\quad \cdot \mathbb{E}[(X_{2, i'_1} X_{2, j'_1} - \sigma_{i'_1 j'_1})(X_{2, i'_2} X_{2, j'_2} - \sigma_{i'_2 j'_2})] \\
 &= \sum_{1 \leq i_1 < j_1 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} \sum_{1 \leq i_2 < j_2 \leq p} \sum_{1 \leq i'_2 < j'_2 \leq p} w_{i_1 j_1}^* w_{i'_1 j'_1}^* w_{i_2 j_2}^* w_{i'_2 j'_2}^* \\
 &\quad \cdot (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i'_1 j_1} \sigma_{i_1 j'_1}) (\sigma_{i_2 i'_2} \sigma_{j_2 j'_2} + \sigma_{i'_2 j_2} \sigma_{i_2 j'_2}) \\
 &\quad \cdot (\sigma_{i_1 i_2} \sigma_{j_2 j_1} + \sigma_{i_1 j_2} \sigma_{i_2 j_1}) (\sigma_{i'_1 i'_2} \sigma_{j'_2 j'_1} + \sigma_{i'_1 j'_2} \sigma_{i'_2 j'_1}) \\
 &\hspace{15em} (54)
 \end{aligned}$$

To bound from above (54), we shall distinguish four cases. The first one is when all couples of indices are equal,

$$\begin{aligned}
 \mathcal{G}_1 &:= \sum_{1 \leq i_1 < j_1 \leq p} w_{i_1 j_1}^{*4} (1 + \sigma_{i_1 j_1}^2)^4 \\
 &\leq (\sup_{i_1, j_1} w_{i_1 j_1}^{*2}) \cdot (\sup_{i_1, j_1} (1 + \sigma_{i_1 j_1}^2)^4) \cdot \sum_{1 \leq i_1 < j_1 \leq p} w_{i_1 j_1}^{*2} \\
 &\leq 8 \cdot (\sup_{i_1, j_1} w_{i_1 j_1}^{*2}) \cdot p = o(p) = o(p^2).
 \end{aligned}$$

The second one is when we have two different pairs of couples of indices, which can be obtained by two different combinations of the couples of indices. When we have equal pairs of couples of indices, as for example $(i_1, j_1) = (i_2, j_2)$, $(i'_1, j'_1) = (i'_2, j'_2)$ and $(i_1, j_1) \neq (i'_1, j'_1)$, we get

$$\begin{aligned}
 \mathcal{G}_{2,1} &:= \sum_{1 \leq i_1 < j_1 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} w_{i_1 j_1}^{*2} w_{i'_1 j'_1}^{*2} (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i'_1 j_1} \sigma_{i_1 j'_1})^2 \\
 &\quad \cdot (1 + \sigma_{i_1 j_1}^2) (1 + \sigma_{i'_1 j'_1}^2) \\
 &\leq (\sup_{i_1, j_1} w_{i_1 j_1}^{*2}) \cdot (\sup_{i_1, j_1} (1 + \sigma_{i_1 j_1}^2)^2) \cdot \sum_{1 \leq i_1 < j_1 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} w_{i_1 j_1}^* w_{i'_1 j'_1}^* \\
 &\quad \cdot (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i'_1 j_1} \sigma_{i_1 j'_1})^2 \\
 &\leq 4 \cdot (\sup_{i_1, j_1} w_{i_1 j_1}^{*2}) \cdot n^2 p \cdot \mathbb{E}_\Sigma(H_n^2(X_1, X_2)) = 4 \cdot (\sup_{i_1, j_1} w_{i_1 j_1}^{*2}) \cdot p = o(p^2).
 \end{aligned}$$

When we have three couples of indices equal, for example $(i_1, j_1) = (i_2, j_2) = (i'_2, j'_2)$ and $(i_1, j_1) \neq (i'_1, j'_1)$, we get

$$\begin{aligned}
 \mathcal{G}_{2,2} &:= \sum_{1 \leq i_1 < j_1 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} w_{i_1 j_1}^{*3} w_{i'_1 j'_1}^* (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i'_1 j_1} \sigma_{i_1 j'_1})^2 \\
 &\quad \cdot (1 + \sigma_{i_1 j_1}^2) (1 + \sigma_{i'_1 j'_1}^2) \\
 &\leq 4 \cdot (\sup_{i_1, j_1} w_{i_1 j_1}^{*2}) \cdot n^2 p \cdot \mathbb{E}_\Sigma(H_n^2(X_1, X_2)) = o(p^2).
 \end{aligned}$$

For the third case, there are three different couples of pairs of indices, for example, $(i_1, j_1) = (i'_2, j'_2)$ and $(i_1, j_1) \neq (i'_1, j'_1) \neq (i_2, j_2)$. Using Cauchy-Schwarz inequality we obtain,

$$\begin{aligned}
\mathcal{G}_3 &:= \sum_{1 \leq i_1 < j_1 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} \sum_{1 \leq i_2 < j_2 \leq p} w_{i_1 j_1}^* w_{i'_1 j'_1}^* w_{i_2 j_2}^{*2} (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i'_1 j_1} \sigma_{i_1 j'_1}) \\
&\quad \cdot (\sigma_{i_1 i_2} \sigma_{j_2 j_1} + \sigma_{i_1 j_2} \sigma_{i_2 j_1}) (\sigma_{i'_1 i_2} \sigma_{j_2 j'_1} + \sigma_{i'_1 j_2} \sigma_{i_2 j'_1}) (1 + \sigma_{i_2, j_2}^2) \\
&\leq \sum_{1 \leq i'_1 < j'_1 \leq p} \sum_{1 \leq i_2 < j_2 \leq p} w_{i'_1 j'_1}^* w_{i_2 j_2}^{*2} (\sigma_{i'_1 i_2} \sigma_{j_2 j'_1} + \sigma_{i'_1 j_2} \sigma_{i_2 j'_1}) (1 + \sigma_{i_2, j_2}^2) \\
&\quad \cdot \left(\sum_{1 \leq i_1 < j_1 \leq p} w_{i_1 j_1}^* (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i'_1 j_1} \sigma_{i_1 j'_1})^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{1 \leq i_1 < j_1 \leq p} w_{i_1 j_1}^* (\sigma_{i_1 i_2} \sigma_{j_2 j_1} + \sigma_{i_1 j_2} \sigma_{i_2 j_1})^2 \right)^{1/2} \\
&\leq \sum_{1 \leq i_2 < j_2 \leq p} w_{i_2 j_2}^{*2} (1 + \sigma_{i_2, j_2}^2) \left(\sum_{1 \leq i'_1 < j'_1 \leq p} w_{i'_1 j'_1}^* (\sigma_{i'_1 i_2} \sigma_{j_2 j'_1} + \sigma_{i'_1 j_2} \sigma_{i_2 j'_1})^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{1 \leq i'_1 < j'_1 \leq p} \sum_{1 \leq i_1 < j_1 \leq p} w_{i'_1 j'_1}^* w_{i_1 j_1}^* (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i_1 j'_1} \sigma_{i'_1 j_1})^2 \right)^{1/2} \\
&\quad \cdot \left(\sum_{1 \leq i_1 < j_1 \leq p} w_{i_1 j_1}^* (\sigma_{i_1 i_2} \sigma_{j_2 j_1} + \sigma_{i_1 j_2} \sigma_{i_2 j_1})^2 \right)^{1/2}.
\end{aligned}$$

Moreover, we recognize in these bounds

$$\sum_{i'_1 < j'_1} \sum_{i_1 < j_1} w_{i'_1 j'_1}^* w_{i_1 j_1}^* (\sigma_{i_1 i'_1} \sigma_{j_1 j'_1} + \sigma_{i_1 j'_1} \sigma_{i'_1 j_1})^2 = n^2 p \cdot \mathbb{E}_\Sigma(H_n^2(X_1, X_2))$$

which is $O(p)$. Thus,

$$\begin{aligned}
\mathcal{G}_3 &\leq \sup_{i_2, j_2} (1 + \sigma_{i_2, j_2}^2) \left(\sum_{1 \leq i_2 < j_2 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} w_{i_2 j_2}^{*2} w_{i'_1 j'_1}^* (\sigma_{i'_1 i_2} \sigma_{j_2 j'_1} + \sigma_{i'_1 j_2} \sigma_{i_2 j'_1})^2 \right)^{1/2} \\
&\cdot \left(n^2 p \mathbb{E}_\Sigma(H_n^2(X_1, X_2)) \sum_{1 \leq i_2 < j_2 \leq p} \sum_{1 \leq i_1 < j_1 \leq p} w_{i_2 j_2}^{*2} w_{i_1 j_1}^* (\sigma_{i_1 i_2} \sigma_{j_2 j_1} + \sigma_{i_1 j_2} \sigma_{i_2 j_1})^2 \right)^{1/2} \\
&\leq 2 (\sup_{i_1, j_1} w_{i_1 j_1}^*) \cdot n^3 p^{3/2} \cdot \mathbb{E}_\Sigma^{3/2}(H_n^2(X_1, X_2)) \leq (\sup_{i_1, j_1} w_{i_1 j_1}^*) \cdot p^{3/2} \\
&= o(p^{3/2}) = o(p^2).
\end{aligned}$$

Now we will treat the last case, when the pairs of indices are pairwise distinct, in this case, we have 16 terms to handle. As all terms are treated the same way, let us deal with:

$$\begin{aligned}
\mathcal{G}_4 &:= \sum_{1 \leq i_1 < j_1 \leq p} \sum_{1 \leq i'_1 < j'_1 \leq p} \sum_{1 \leq i_2 < j_2 \leq p} \sum_{1 \leq i'_2 < j'_2 \leq p} w_{i_1 j_1}^* w_{i'_1 j'_1}^* w_{i_2 j_2}^* w_{i'_2 j'_2}^* \\
&\quad \cdot \sigma_{i_1 i'_1} \sigma_{i_2 i'_2} \sigma_{j_2 j'_2} \sigma_{j_1 j'_1} \sigma_{i_1 i_2} \sigma_{j_2 j_1} \sigma_{i'_1 i'_2} \sigma_{j'_2 j'_1}
\end{aligned}$$

In order to find an upper bound for \mathcal{G}_4 , we decompose the previous sums, into several sums, similarly to the upper bound of (49). That is $(i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in J_1 \cup J_2 \cup \dots \cup J_{16}$, where J_1, \dots, J_{16} , form a partition of the set

$$J := \{(i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in \{1, \dots, p\}^8\}.$$

Let us define,

$$J_1 := \{(i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in J; 1 < |i_1 - i'_1|, |i_1 - i_2|, |i_2 - i'_2|, |i'_1 - i'_2| < T\},$$

$$J_2 := \{(i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in J; 1 < |i_1 - i'_1|, |i_1 - i_2|, |i_2 - i'_2| < T, \text{ and } |i'_1 - i'_2| > T\},$$

and so on, for all $J_r, r = 3, \dots, 16$. To bound from above the sum over J_1 , we partition again $J_1, J_1 = J_{1,1} \cup \dots \cup J_{1,16}$ such that,

$$J_{1,1} := \{(i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in J_1; |i_1 - i'_1| \leq |i_1 - j_1|, |i'_1 - i'_2| \leq |i'_1 - j'_1|, |i_1 - i_2| \leq |i_2 - j_2| \text{ and } |i_2 - i'_2| \leq |i'_2 - j'_2|\},$$

and so on, until we get the partition of J_1 .

$$\begin{aligned} \mathcal{G}_{4,1} &:= \sum_{1 \leq i_1 < j_1 \leq p} \sum_{\substack{1 \leq i'_1 < j'_1 \leq p \\ (i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in J_{1,1}}} \sum_{1 \leq i_2 < j_2 \leq p} \sum_{1 \leq i'_2 < j'_2 \leq p} w_{i_1 j_1}^* w_{i'_1 j'_1}^* w_{i_2 j_2}^* w_{i'_2 j'_2}^* \\ &\quad \cdot \sigma_{i_1 i'_1} \sigma_{j_1 j'_1} \sigma_{i_2 i'_2} \sigma_{j_2 j'_2} \sigma_{i_1 i_2} \sigma_{j_2 j_1} \sigma_{i'_1 i'_2} \sigma_{j'_2 j'_1} \\ &\leq \sum_{1 \leq i_1, i'_1 \leq p} \sum_{1 \leq i_2, i'_2 \leq p} w_{i_1 i'_1}^* w_{i_1 i_2}^* w_{i'_1 i'_2}^* w_{i_2 i'_2}^* \sigma_{i_1 i'_1} \sigma_{i_2 i'_2} \sigma_{i_1 i_2} \sigma_{i'_1 i'_2} \\ &\quad \cdot \sum_{\substack{1 \leq j_1, j'_1 \leq p \\ 1 \leq j_2, j'_2 \leq p \\ 1 < |i_1 - j_1|, |i'_1 - j'_1|, |i_2 - j_2|, |i'_2 - j'_2| < T}} \sigma_{j_2 j'_2} \sigma_{j_1 j'_1} \sigma_{j_2 j_1} \sigma_{j'_2 j'_1} \\ &\leq T^4 (\sup_{i_1, j_1} w_{i_1 j_1}^*)^2 \sum_{1 \leq i_1, i'_1 \leq p} \sum_{1 \leq i_2, i'_2 \leq p} (w_{i_1 i'_1}^* w_{i_1 i_2}^* w_{i'_1 i'_2}^* w_{i_2 i'_2}^*)^{\frac{1}{2}} \sigma_{i_1 i'_1} \sigma_{i_2 i'_2} \sigma_{i_1 i_2} \sigma_{i'_1 i'_2} \\ &\leq T^4 \cdot (\sup_{i_1, j_1} w_{i_1 j_1}^*)^2 \left(\sum_{1 \leq i_1, i'_1 \leq p} \sum_{1 \leq i_2, i'_2 \leq p} w_{i_1 i'_1}^* w_{i_2 i'_2}^* \sigma_{i_1 i'_1}^2 \sigma_{i_2 i'_2}^2 \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\sum_{1 \leq i_1, i'_1 \leq p} \sum_{1 \leq i_2, i'_2 \leq p} w_{i_1 i_2}^* w_{i'_1 i'_2}^* \sigma_{i_1 i_2}^2 \sigma_{i'_1 i'_2}^2 \right)^{\frac{1}{2}} \\ &\leq T^4 \cdot (\sup_{i_1, j_1} w_{i_1 j_1}^*)^2 \cdot p^2 \cdot \mathbb{E}_{\Sigma}^2(\widehat{D}_n) \end{aligned}$$

Again, by our assumption that $n^2 p \cdot \mathbb{E}_{\Sigma}^2(\widehat{D}_n) = O(1)$, we can see that :

$$\mathcal{G}_{4,1} \leq \kappa_0(\alpha, L) \cdot T^3 \cdot p^2 \cdot \mathbb{E}_{\Sigma}^2(\widehat{D}_n) = p^2 \cdot O\left(\frac{T^3}{n^2 p}\right) = p^2 \cdot o(1)$$

where, from now on, $\kappa_0(\alpha, L), \kappa_1(\alpha, L), \dots$, denote constants that depend on α and L . Now, we define $J_{1,2} := \{(i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in \{1, \dots, p\}^8\}$, such that $|i - i'| \leq |i - j|$, $|i' - i'_1| \leq |i' - j'|$, $|i - i_1| \leq |i_1 - j_1|$ and $|i_1 - i'_1| > |i'_1 - j'_1|$, thus we have,

$$\begin{aligned} \mathcal{G}_{4,2} &:= \sum_{(i_1, j_1, i'_1, j'_1, i_2, j_2, i'_2, j'_2) \in J_{1,2}} w_{i_1 j_1}^* w_{i'_1 j'_1}^* w_{i_2 j_2}^* w_{i'_2 j'_2}^* \sigma_{i_1 i'_1} \sigma_{i_2 i'_2} \sigma_{j_1 j'_1} \sigma_{j_2 j'_2} \\ &\quad \cdot \sigma_{i_1 i_2} \sigma_{j_2 j_1} \sigma_{i'_1 i'_2} \sigma_{j'_2 j'_1} \\ &\leq (\sup_{i_1, j_1} w_{i_1 j_1}^*)^{\frac{5}{2}} \sum_{1 \leq i_1, i'_1 \leq p} \sum_{1 \leq i_2, i'_2 \leq p} (w_{i_1 i'_1}^* w_{i_1 i_2}^* w_{i'_1 i'_2}^*)^{\frac{1}{2}} |i_2 - i'_2|^\alpha \sigma_{i_1 i'_1} \sigma_{i_2 i'_2} \sigma_{i_1 i_2} \sigma_{i'_1 i'_2} \\ &\quad \cdot \sum_{\substack{1 \leq j_1, j'_1 \leq p \\ 1 \leq j_2, j'_2 \leq p \\ 1 < |i_1 - j_1|, |i'_1 - j'_1|, |i_2 - j_2|, |i'_2 - j'_2| < T}} \frac{1}{|i'_2 - j'_2|^\alpha} \cdot \sigma_{j_2 j'_2} \sigma_{j_1 j'_1} \sigma_{j_2 j_1} \sigma_{j'_2 j'_1} \\ &\leq (\sup_{i_1, j_1} w_{i_1 j_1}^*)^{5/2} \left(\sum_{1 \leq i_1, i'_1 \leq p} \sum_{1 \leq i_2, i'_2 \leq p} w_{i_1 i'_1}^* |i_2 - i'_2|^{2\alpha} \sigma_{i_1 i'_1}^2 \sigma_{i_2 i'_2}^2 \right)^{1/2} \\ &\quad \cdot \left(\sum_{1 \leq i_1, i'_1 \leq p} \sum_{1 \leq i_2, i'_2 \leq p} w_{i_1 i_2}^* w_{i'_1 i'_2}^* \sigma_{i_1 i_2}^2 \sigma_{i'_1 i'_2}^2 \right)^{1/2} \cdot T^3 \cdot \max\{1, T^{-\alpha+1}\} \\ &\leq \sqrt{2L} \cdot (\sup_{i_1, j_1} w_{i_1 j_1}^*)^{5/2} \cdot T^3 \cdot \max\{1, T^{-\alpha+1}\} \cdot p^2 \cdot \mathbb{E}_\Sigma^{3/2}(\widehat{D}_n) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{G}_{4,2} &\leq \kappa_1(\alpha, L) \cdot \max\{T^{7/4}, T^{11/4-\alpha}\} \cdot \mathbb{E}_\Sigma^{3/2}(\widehat{D}_n) \\ &\leq \kappa_1(\alpha, L) \cdot \max\{T^{7/4}, T^{11/4-\alpha}\} \cdot O\left(\frac{1}{n^{3/2} p^{3/4}}\right) \\ &= o(1) \quad \text{since } T^3/n^2 p \rightarrow 0 \end{aligned} \tag{55}$$

Using similar arguments, we can prove that all remaining terms tend to zero. In consequence,

$$\frac{\mathbb{E}_\Sigma(G_n^2(X_1, X_2))}{\mathbb{E}_\Sigma^2(H_n^2(X_1, X_2))} \rightarrow 0.$$

Now let us prove that, $\mathbb{E}_\Sigma(H_n^4(X_1, X_2))/\mathbb{E}_\Sigma^2(H_n^2(X_1, X_2)) = o(n)$,

$$\begin{aligned} \mathbb{E}_\Sigma(H_n^4(X_1, X_2)) &= \frac{1}{n^4 p^2} \sum_{i_1 < j_1} \sum_{i_2 < j_2} \sum_{i_3 < j_3} \sum_{i_4 < j_4} w_{i_1 j_1}^* w_{i_2 j_2}^* w_{i_3 j_3}^* w_{i_4 j_4}^* \\ &\quad \cdot \mathbb{E}_\Sigma^2[(X_{1, i_1} X_{1, j_1} - \sigma_{i_1 j_1})(X_{1, i_2} X_{1, j_2} - \sigma_{i_2 j_2})(X_{1, i_3} X_{1, j_3} - \sigma_{i_3 j_3})(X_{1, i_4} X_{1, j_4} - \sigma_{i_4 j_4})] \end{aligned}$$

The above squared expected value is a sum of a large number of terms that are all treated similarly. Let us consider examples of terms containing squared terms and products of terms, respectively. For $\alpha > 1/2$,

$$\mathcal{H}_1 := \sum_{i_1 < j_1} \sum_{i_2 < j_2} \sum_{i_3 < j_3} \sum_{i_4 < j_4} w_{i_1 j_1}^* w_{i_2 j_2}^* w_{i_3 j_3}^* w_{i_4 j_4}^* \sigma_{i_1 i_2}^2 \sigma_{j_1 j_2}^2 \sigma_{i_3 i_4}^2 \sigma_{j_3 j_4}^2$$

$$\begin{aligned} &\leq 4(\sup_{i,j} w_{ij}^*)^4 \sum_{i_1=1}^p \sum_{i_2=1}^p |i_1 - i_2|^{2\alpha} \sigma_{i_1 i_2}^2 \sum_{\substack{j_1=1 \\ |i_1-j_1|<T}}^p \sup_{j_2} \sigma_{j_1 j_2}^2 \sum_{\substack{j_2=1 \\ |i_2-j_2|<T}}^p \frac{1}{|j_1 - j_2|^{2\alpha}} \\ &\quad \cdot \sum_{i_3=1}^p \sum_{i_4=1}^p |i_3 - i_4|^{2\alpha} \sigma_{i_3 i_4}^2 \sum_{\substack{j_3=1 \\ |i_3-j_3|<T}}^p \sup_{j_4} \sigma_{j_3 j_4}^2 \sum_{\substack{j_4=1 \\ |i_4-j_4|<T}}^p \frac{1}{|j_3 - j_4|^{2\alpha}} \\ &\leq 16L^2 \cdot (2\alpha - 1)^{-2} \cdot (\sup_{i,j} w_{ij}^*)^4 \cdot p^2 T^2 \leq \kappa_2(\alpha, L) \cdot p^2 \end{aligned}$$

The terms containing no squared values are treated as, e.g.,

$$\mathcal{H}_2 := \sum_{i_1 < j_1} \sum_{i_2 < j_2} \sum_{i_3 < j_3} \sum_{i_4 < j_4} w_{i_1 j_1}^* w_{i_2 j_2}^* w_{i_3 j_3}^* w_{i_4 j_4}^* \sigma_{i_1 i_2} \sigma_{j_1 j_2} \sigma_{i_3 i_4} \sigma_{j_3 j_4} \cdot \sigma_{i_1 i_3} \sigma_{j_1 j_3} \sigma_{i_2 i_4} \sigma_{j_2 j_4}$$

We can see that \mathcal{H}_2 coincides with $\mathcal{G}_{4,2}$. Then we can deduce that ,

$$\frac{\mathbb{E}_\Sigma(H_n^4(X_1, X_2))}{\mathbb{E}_\Sigma^2(H_n^2(X_1, X_2))} = O(1) = o(n).$$

Finally we can apply [16], and we obtain:

$$V_n = \frac{1}{n\sqrt{p}} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq i < j \leq p} w_{ij}^*(X_{k,i} X_{k,j} - \sigma_{ij})(X_{l,i} X_{l,j} - \sigma_{ij}) \xrightarrow{\mathcal{L}} N(0, 1). \tag{56}$$

Combining (53) and (56), we have by Slutsky theorem that:

$$n\sqrt{p} \cdot (\widehat{\mathcal{D}}_n - \mathbb{E}_\Sigma(\widehat{\mathcal{D}}_n)) \xrightarrow{\mathcal{L}} N(0, 1). \quad \square$$

Proof of Proposition 3. Let us check the case where $u_{ij} = 1$ for all i, j such that $|i - j| \leq T$ and the generalization to all U in \mathcal{U} will be obvious. Using Gershgorin's Theorem we get that each eigenvalue of $\Sigma_U^* = [u_{ij} \sigma_{ij}^*]_{1 \leq i, j \leq p}$ lies

in one of the disks centered in $\sigma_{ii} = 1$ and radius $R_i = \sum_{\substack{j=1 \\ j \neq i}}^p |u_{ij} \sigma_{ij}^*| = \sum_{\substack{j=1 \\ j \neq i}}^p \sigma_{ij}^*$.

We have,

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq i}}^p \sigma_{ij}^* &= \sqrt{\lambda} \sum_{\substack{j=1 \\ j \neq i}}^p \left(1 - \left(\frac{|i-j|}{T} \right)^{2\alpha} \right)_+^{\frac{1}{2}} \leq 2\sqrt{\lambda} \sum_{k=1}^T \left(1 - \left(\frac{k}{T} \right)^{2\alpha} \right)^{\frac{1}{2}} \\ &\leq 2\sqrt{\lambda} \left(\sum_{k=1}^T \left(1 - \left(\frac{k}{T} \right)^{2\alpha} \right) \right)^{\frac{1}{2}} T^{\frac{1}{2}} = O(1) T \sqrt{\lambda} \\ &\leq O(1) \varphi^{1-\frac{1}{2\alpha}} \rightarrow 0 \text{ provided that } \alpha > 1/2. \end{aligned}$$

We deduce that the smallest eigenvalue is bounded from below by

$$\min_{i=1, \dots, p} \lambda_{i,U} \geq \min_i \left\{ \sigma_{ii}^* - \sum_{\substack{j=1 \\ j \neq i}}^p \sigma_{ij}^* \right\} = 1 - \max_i \sum_{\substack{j=1 \\ j \neq i}}^p \sigma_{ij}^* \geq 1 - O(1) \varphi^{1-\frac{1}{2\alpha}}$$

which is strictly positive for $\varphi > 0$ small enough. □

Proposition 4. For all $1 \leq i < j \leq p$, W_{ij} is a centered random variable with variance, $\text{Var}_I(W_{ij}) = n$. Moreover, for $1 \leq i < j \neq j' \leq p$, we have

$$\begin{aligned} \mathbb{E}_I(W_{ij}^4) &= 3n^2 + 6n, & \mathbb{E}_I(W_{ij}^2 W_{ij'}^2) &= n^2 + 2n, \\ \mathbb{E}_I(W_{ij}^4 W_{ij'}^4) &= 9(n^4 + 12n^3 + 44n^2 + 48n). \end{aligned}$$

Also we have that $\mathbb{E}_I(W_{ij}^8) = 105n^4(1 + o(1))$. Note that if we have $i \neq i'$ and $j \neq j'$, then W_{ij}^d and $W_{i'j'}^d$ are not correlated for d finite integer. Moreover, for all $1 \leq i \leq p$, the random variables W_{ii} are such that,

$$\mathbb{E}(W_{ii}) = n, \quad \mathbb{E}_I(W_{ii}^2) = n^2 + 2n, \quad \mathbb{E}_I(W_{ii}^4) = n^4 + 12n^3 + 44n^2 + 48n.$$

Proof of Proposition 4. To show the results we use lemma 3 and some technical computation of [9].

$$\begin{aligned} \text{Var}_I(W_{ij}) &= \mathbb{E}_I(W_{ij}^2) = \mathbb{E}_I(X_{\cdot i}^\top X_{\cdot j})^2 = \mathbb{E}_I(\text{tr}(X_{\cdot i} X_{\cdot i}^\top X_{\cdot j} X_{\cdot j}^\top)) = \text{tr}(I_n^2) = n. \\ \mathbb{E}_I(W_{ij}^4) &= \mathbb{E}_I(X_{\cdot i}^\top X_{\cdot j})^4 = 3\text{tr}^2(I_n^2) + 6\text{tr}(I_n^4) = 3n^2 + 6n \\ \mathbb{E}_I(W_{ij}^2 W_{ij'}^2) &= \mathbb{E}_I((X_{\cdot i}^\top X_{\cdot j})^2 (X_{\cdot i'}^\top X_{\cdot j'})^2) = \text{tr}^2(I_n^2) + 2\text{tr}(I_n^4) = n^2 + 2n \\ \mathbb{E}_I(W_{ij}^4 W_{ij'}^4) &= \mathbb{E}_I((X_{\cdot i}^\top X_{\cdot j})^4 (X_{\cdot i'}^\top X_{\cdot j'})^4) = \mathbb{E}_I\left(\mathbb{E}_I\left((X_{\cdot i}^\top X_{\cdot j})^4 (X_{\cdot i'}^\top X_{\cdot j'})^4 \mid X_{\cdot i}\right)\right). \end{aligned}$$

Or $\mathbb{E}_I\left((X_{\cdot i}^\top X_{\cdot j})^4 (X_{\cdot i'}^\top X_{\cdot j'})^4 \mid X_{\cdot i}\right) = g(X_{\cdot i})$, where

$$\begin{aligned} g(x_{\cdot i}) &= \mathbb{E}_I\left((x_{\cdot i}^\top X_{\cdot j})^4 (x_{\cdot i'}^\top X_{\cdot j'})^4\right) = \mathbb{E}_I\left((x_{\cdot i}^\top X_{\cdot j})^4\right) \mathbb{E}_I\left((x_{\cdot i'}^\top X_{\cdot j'})^4\right) \\ \mathbb{E}_I\left((x_{\cdot i}^\top X_{\cdot j})^4\right) &= \mathbb{E}_I\left(\sum_{k=1}^p x_{k,i} X_{k,j}\right)^4 \\ &= \sum_{k=1}^p x_{k,i}^4 \mathbb{E}_I(X_{k,j}^4) + 3 \sum_{k_1 \neq k_2} x_{k_1,i}^2 x_{k_2,i}^2 \mathbb{E}_I(X_{k_1,j}^2) \mathbb{E}_I(X_{k_2,j}^2) \\ &= 3 \left(\sum_{k=1}^p x_{k,i}^2\right)^2 = 3(x_{\cdot i}^\top x_{\cdot i})^2 \end{aligned}$$

then we obtain that

$$\mathbb{E}_I(W_{ij}^4 W_{ij'}^4) = 9 \mathbb{E}_I(X_{\cdot i}^\top X_{\cdot i})^4 = 9(n^4 + 12n^3 + 44n^2 + 48n).$$

Also we have that

$$\begin{aligned} \mathbb{E}_I(W_{ij}^8) &= \mathbb{E}_I\left(\sum_{k=1}^n X_{k,i} X_{k,j}\right)^8 \\ &= \sum_{k=1}^n \mathbb{E}_I^2(X_{k,i}^8) + C_8^6 \cdot \sum_{k_1 \neq k_2} \mathbb{E}_I^2(X_{k_1,i}^6) \cdot \mathbb{E}_I^2(X_{k_2,i}^2) \end{aligned}$$

$$\begin{aligned}
 &+ \frac{C_8^4}{2!} \cdot \sum_{k_1 \neq k_2} \mathbb{E}_I^2(X_{k_1,i}^4) \cdot \mathbb{E}_I^2(X_{k_2,i}^4) \\
 &+ \frac{C_8^4 \cdot C_4^2}{2!} \cdot \sum_{k_1 \neq k_2 \neq k_3} \mathbb{E}_I^2(X_{k_1,i}^4) \cdot \mathbb{E}_I^2(X_{k_2,i}^2) \cdot \mathbb{E}_I^2(X_{k_3,i}^2) \\
 &+ \frac{C_8^2 \cdot C_6^2 \cdot C_4^2}{4!} \sum_{k_1 \neq k_2 \neq k_3 \neq k_4} \mathbb{E}_I^2(X_{k_1,i}^2) \cdot \mathbb{E}_I^2(X_{k_2,i}^2) \cdot \mathbb{E}_I^2(X_{k_3,i}^2) \cdot \mathbb{E}_I^2(X_{k_4,i}^2) \\
 &= 105^2 n + (28 \times 15^2 + 35 \times 9)n(n-1) + (210 \times 9)n(n-1)(n-2) \\
 &+ 105n(n-1)(n-2)(n-3) \\
 &= 105n^4 + 1220n^3 + 2100n^2 + 7560n
 \end{aligned}$$

We use similar arguments to calculate the moments of W_{ii} . □

Lemma 1. *Let $0 < \varepsilon < 1/2$, for any $t > 0$ we have that,*

$$\left| \mathbb{P}_I(\widehat{\mathcal{D}}_{n,r} \leq t) - \Phi(n\sqrt{p} \cdot t) \right| \leq 16\varepsilon^{1/2} \exp\left(-\frac{n^2 p t^2}{4}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{pT_r}\right)$$

for all $1 \leq r \leq N$.

Proof of Lemma 1. For each $r \in \{1, \dots, N\}$, $\widehat{\mathcal{D}}_{n,r}$ is a degenerated U-statistic of order 2, and can be written as follows:

$$\widehat{\mathcal{D}}_{n,r} = \sum_{1 \leq k \neq l \leq n} K(X_k, X_l),$$

where

$$K(X_k, X_l) = \frac{1}{n(n-1)p} \sum_{1 \leq i < j \leq p} w_{i,j,r}^* X_{k,i} X_{k,j} X_{l,i} X_{l,j}.$$

Define,

$$Z_k = \frac{1}{\sqrt{\text{Var}_I(\widehat{\mathcal{D}}_{n,r})}} \sum_{l=1}^{k-1} K(X_k, X_l) \quad \text{and} \quad V_n^2 = \sum_{k=2}^n \mathbb{E}_I(Z_k^2 / \mathcal{F}_{k-1})$$

where \mathcal{F}_k is the σ -field generated by the random variables $\{X_1, \dots, X_k\}$. Moreover, fix $0 < \delta \leq 1$, and define

$$J_n = \sum_{k=2}^n \mathbb{E}_I(Z_k)^{2+2\delta} + \mathbb{E}_I|V_n^2 - 1|^{1+\delta}.$$

Then by Theorem 3 of [6] we get that, there exists a positive constant k depending only on δ such that for any $0 < \varepsilon < 1/2$ and any real t ,

$$\left| \mathbb{P}_I(\widehat{\mathcal{D}}_{n,r} \leq t) - \Phi\left(\frac{x}{\sqrt{\text{Var}_I(\widehat{\mathcal{D}}_{n,r})}}\right) \right| \leq 16\varepsilon^{1/2} \exp\left(-\frac{t^2}{4\text{Var}_I(\widehat{\mathcal{D}}_{n,r})}\right) + \frac{k}{\varepsilon^{1+\delta}} \cdot J_n.$$

Now, we give upper bounds for $\sum_{k=2}^n \mathbb{E}_I(Z_k)^{2+2\delta}$ and $\mathbb{E}_I|V_n^2 - 1|^{1+\delta}$ for $\delta = 1$ and get,

$$\begin{aligned} \sum_{k=2}^n \mathbb{E}_I(Z_k)^4 &= \frac{1}{n^2(n-1)^2p^2} \sum_{k=2}^n \mathbb{E}_I \left(\sum_{l=1}^{k-1} \sum_{1 \leq i < j \leq p} w_{ij,r}^* X_{k,i} X_{k,j} X_{l,i} X_{l,j} \right)^4 \\ &= \frac{1}{n^2(n-1)^2p^2} \sum_{k=2}^n \left\{ (k-1) \left(3^4 \sum_{1 \leq i < j \leq p} w_{ij,r}^{*4} + 3 \sum_{1 \leq i < j \leq p} \sum_{1 \leq i' < j' \leq p} w_{ij,r}^{*2} w_{i'j',r}^{*2} \right) \right. \\ &\quad \left. + 3(k-1)(k-2) \left(3^2 \sum_{1 \leq i < j \leq p} w_{ij,r}^{*4} + \sum_{1 \leq i < j \leq p} \sum_{1 \leq i' < j' \leq p} w_{ij,r}^{*2} w_{i'j',r}^{*2} \right) \right\} \\ &\leq \frac{1}{n^2(n-1)^2p^2} \left\{ \frac{n(n-1)}{2} \left(\frac{81}{2} \cdot p(\sup_{i,j} w_{ij,r}^{*2}) + \frac{3p^2}{4} \right) \right. \\ &\quad \left. + \frac{(n-1)n(2n-1)}{6} \left(\frac{9}{2} \cdot p(\sup_{i,j} w_{ij,r}^{*2}) + \frac{p^2}{4} \right) \right\} \\ &= O\left(\frac{1}{n}\right). \end{aligned}$$

Similarly we can show that $\mathbb{E}_I(V_n^2 - 1)^2 = O\left(\frac{1}{n}\right) + O\left(\frac{1}{pT_r}\right)$. Thus we obtain the desired result. \square

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