

Sample path properties of permanental processes*

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Abstract

Let $X_\alpha = \{X_\alpha(t), t \in \mathcal{T}\}$, $\alpha > 0$, be an α -permanental process with kernel $u(s, t)$. We show that $X_\alpha^{1/2}$ is a subgaussian process with respect to the metric

$$\sigma(s, t) = (u(s, s) + u(t, t) - 2(u(s, t)u(t, s))^{1/2})^{1/2}.$$

This allows us to use the vast literature on sample path properties of subgaussian processes to extend these properties to α -permanental processes. Local and uniform moduli of continuity are obtained as well as the behavior of the processes at infinity. Examples are given of permanental processes with kernels that are the potential density of transient Lévy processes that are not necessarily symmetric, or with kernels of the form

$$\tilde{u}(x, y) = u(x, y) + f(y),$$

where u is the potential density of a symmetric transient Borel right process and f is an excessive function for the process.

Keywords: permanental processes; subgaussian processes; moduli of continuity of permanental processes; permanental processes at infinity.

AMS MSC 2010: 60K99; 60G15; 60G17; 60G99.

Submitted to EJP on November 3, 2017, final version accepted on May 30, 2018.

1 Introduction

An R^n valued α -permanental random variable $(X_\alpha(1), \dots, X_\alpha(n))$ is a non-negative random variable with Laplace transform

$$E \left(e^{-\sum_{i=1}^n s_i X_\alpha(i)} \right) = \frac{1}{|I + US|^\alpha}, \tag{1.1}$$

for some $n \times n$ matrix U and diagonal matrix S with positive entries s_1, \dots, s_n , and $\alpha > 0$. We refer to the matrix U as the kernel of $(X_\alpha(1), \dots, X_\alpha(n))$.

*Research of Jay Rosen was partially supported by grants from the Simons Foundation.

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An α -permanental process $X_\alpha = \{X_\alpha(t), t \in \mathcal{T}\}$ is a stochastic process that has finite dimensional distributions that are α -permanental random variables. In this paper \mathcal{T} is usually R^+ , a subset of R^+ or \mathbb{N} , the set of integers.

An α -permanental process X_α is determined by a kernel $\{u(s, t), s, t \in \mathcal{T}\}$ with the property that for all t_1, \dots, t_n in \mathcal{T} , $\{u(t_i, t_j), i, j \in [1, n]\}$ is the kernel of the α -permanental random variable $(X_\alpha(t_1), \dots, X_\alpha(t_n))$. To avoid trivialities we restrict our attention to kernels with the property that for all $\delta > 0$, $\sup_{t \leq \delta} u(t, t) > 0$.

An extensive class of examples of kernels of permanental processes is given by $U = \{u(s, t), s, t \in \mathcal{T}\}$ when U is the potential density of a transient Markov process with state space \mathcal{T} , with respect to some σ -finite measure m on \mathcal{T} and $u(s, t)$ is finite for all $s, t \in \mathcal{T}$. In this case U is the kernel of an α -permanental process X_α for all $\alpha > 0$; see [5, Theorem 3.1], and for all distinct (t_1, \dots, t_n) in \mathcal{T} , the $n \times n$ matrix $\{u(t_i, t_j)\}_{i,j=1}^n$ is invertible; (see [18, Lemma A.1]). We refer to these permanental processes as associated α -permanental processes because they are associated with the transient Markov process.

The following observation about bivariate α -permanental random variables is the key to the results in this paper: Suppose that $(X_\alpha(s), X_\alpha(t))$ is an α -permanental random variable with kernel

$$U_{s,t} = \begin{pmatrix} u(s, s) & u(s, t) \\ u(t, s) & u(t, t) \end{pmatrix}. \tag{1.2}$$

It follows from [20, p. 135] that $u(s, s)$, $u(t, t)$ and $u(s, t)u(t, s)$ are all greater than or equal to 0. Furthermore, one can see from (1.1) that $(X_\alpha(s), X_\alpha(t))$ also has the symmetric kernel

$$\tilde{U}_{s,t} = \begin{pmatrix} u(s, s) & (u(s, t)u(t, s))^{1/2} \\ (u(s, t)u(t, s))^{1/2} & u(t, t) \end{pmatrix}. \tag{1.3}$$

and

$$|\tilde{U}_{s,t}| = |U_{s,t}| = u(s, s)u(t, t) - u(s, t)u(t, s) \geq 0, \tag{1.4}$$

(see (3.73)), so that the symmetric kernel $\tilde{U}_{s,t}$ is positive definite.

Let X_α be an α -permanental process with kernel $\{u(s, t), s, t \in \mathcal{T}\}$. It follows from (1.4) and the inequality between arithmetic and geometric means that

$$u(s, s) + u(t, t) - 2(u(s, t)u(t, s))^{1/2} \geq 0. \tag{1.5}$$

We then define the function $\{\sigma(s, t), s, t \in \mathcal{T}\}$ by

$$\sigma(s, t) = \left(u(s, s) + u(t, t) - 2(u(s, t)u(t, s))^{1/2}\right)^{1/2} \geq 0. \tag{1.6}$$

We refer to $\{\sigma(s, t), s, t \in \mathcal{T}\}$ as the sigma function of X_α . Note that although we don't require that $u(s, t)$ is symmetric, $\sigma(s, t)$ obviously is symmetric. When X_α is an associated α -permanental process then $\{\sigma(s, t), s, t \in \mathcal{T}\}$ is a metric, [11, Lemma 4.2].

When $u(s, t)$ is symmetric and is a kernel that determines a 1/2-permanental process, $Y_{1/2} = \{Y_{1/2}(t), t \in \mathcal{T}\}$, then $Y_{1/2} \stackrel{law}{=} \{G^2(t)/2, t \in \mathcal{T}\}$ where $\{G(t), t \in \mathcal{T}\}$ is a mean zero Gaussian process with covariance $u(s, t)$. In this case

$$\sigma(s, t) = \left(E(G(t) - G(s))^2\right)^{1/2} = \|G(t) - G(s)\|_2. \tag{1.7}$$

In [11] and [16] we use these observations to show that many properties of Gaussian processes which are obtained using their bivariate distributions also hold for 1/2-permanental processes. In this paper we show how such properties also hold for α -permanental processes, for all $\alpha > 0$. Theorem 1.1 is the critical step in this work.

Let $\psi_2(x) = \exp(x^2) - 1$ and $\|\cdot\|_{\psi_2}$ be the Orlicz norm in the corresponding Banach space. A fundamental relationship in the study of Gaussian processes is

$$\|G(s) - G(t)\|_{\psi_2} = 2\|G(t) - G(s)\|_2. \tag{1.8}$$

We extend this result to α -permanental processes.

Theorem 1.1. *Let $X_\alpha = \{X_\alpha(t), t \in \mathcal{T}\}$ be an α -permanental process with kernel $U = \{u(s, t), s, t \in \mathcal{T}\}$. Then*

$$\|X_\alpha^{1/2}(s) - X_\alpha^{1/2}(t)\|_{\psi_2} \leq C_\alpha \sigma(s, t) := d_{C_\alpha, \sigma}(s, t), \tag{1.9}$$

for some finite constant C_α .

As we just stated, when $u(s, t)$ is symmetric and is a kernel that determines a 1/2-permanental process, the process is $\{G^2(t)/2, t \in \mathcal{T}\}$, where $\{G(t), t \in \mathcal{T}\}$ is a mean zero Gaussian process with covariance $u(s, t)$. In this case (1.9) gives

$$\| |G(s)| - |G(t)| \|_{\psi_2} \leq C' \|G(t) - G(s)\|_2. \tag{1.10}$$

for some absolute constant C' , which is just a bit weaker than (1.8) because

$$\| |G(s)| - |G(t)| \| \leq |G(s) - G(t)|.$$

Theorem 1.1 shows that $\{d_{C, \sigma}(s, t), s, t \in R^+\}$ dominates $\|X_\alpha^{1/2}(s) - X_\alpha^{1/2}(t)\|_{\psi_2}$. Therefore the proof used in [11, Theorem 3.1] to obtain results for the uniform modulus of continuity of 1/2-permanental processes extends immediately to α -permanental processes for which (1.9) holds. Similarly, the results in [16, Theorems 4.1 and 4.2] on the local modulus of continuity of 1/2-permanental processes, extend to all α -permanental processes for which (1.9) holds. These are given in Theorems 8.1 and 8.2 in the Appendix.

The reason we present these theorems in an appendix is because the results for the uniform modulus of continuity and local modulus of continuity in (8.6) and (8.8) are quite abstract. However, as we show in [16, Example 4.1], when $T = R^+$ and there exists an increasing function φ such that for all $0 \leq s, t < \infty$,

$$\sigma(s, t) \leq \varphi(|t - s|), \tag{1.11}$$

where

$$\int_0^{1/2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du < \infty, \tag{1.12}$$

then (8.6) and (8.8) give results for 1/2-permanental processes, that are the same as familiar results for Gaussian processes, although we don't get the best constants.

In order to get precise results with the best constants, we examine more closely permanental processes with kernels that allow (1.11) and (1.12) to be satisfied and obtain stronger versions of (8.6) and (8.8) by generalizing a classical inequality of Fernique. Here are some examples of results we obtain.

Theorem 1.2. *Let $X_\alpha = \{X_\alpha(t), t \in [0, 1]\}$ be an α -permanental process with kernel $u(s, t)$ and sigma function $\sigma(s, t)$ for which (1.11) and (1.12) hold for some function $\varphi(t)$ that is regularly varying at zero with positive index. Then,*

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|X_\alpha(t) - X_\alpha(0)|}{\varphi(h)(\log \log 1/h)^{1/2}} \leq 2X_\alpha^{1/2}(0) \quad a.s. \tag{1.13}$$

If $u(0, 0) = 0$, and if in addition to the conditions on φ above, $\varphi^2(h) = O(u(h, h))$, then

$$\limsup_{t \rightarrow 0} \frac{X_\alpha(t)}{u(t, t) \log \log 1/t} \leq 1 \quad a.s. \tag{1.14}$$

and

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{X_\alpha(t)}{u^*(h, h) \log \log 1/h} \leq 1 \quad a.s. \tag{1.15}$$

where

$$u^*(h, h) = \sup_{t \leq h} u(t, t). \tag{1.16}$$

Note that when $X_{1/2}$ is 1/2 times the square of Brownian motion, its kernel u satisfies $u(h, h) = h$. In this case the upper bounds in (1.14) and (1.15) are well known to be best possible. (This is also why we get $\log \log 1/t$ in the denominator of (1.14) rather than $(2 \log \log 1/t)^{1/2}$.)

The reason we get $(\log \log 1/h)^{1/2}$ in the denominator of (1.13) is because

$$X_\alpha(t) - X_\alpha(0) = \left(X_\alpha^{1/2}(t) - X_\alpha^{1/2}(0) \right) \left(X_\alpha^{1/2}(t) + X_\alpha^{1/2}(0) \right). \tag{1.17}$$

As $t \rightarrow 0$, $X_\alpha^{1/2}(t) + X_\alpha^{1/2}(0) \rightarrow 2X_\alpha^{1/2}(0)$, so that the denominator in (1.13) only has to control $\left(X_\alpha^{1/2}(t) - X_\alpha^{1/2}(0) \right)$. (See Lemma 3.9 for details.)

The next theorem gives uniform moduli of continuity.

Theorem 1.3. *Let $X_\alpha = \{X_\alpha(t), t \in [0, 1]\}$ be an α -permanental process with kernel $u(s, t)$ and sigma function $\sigma(s, t)$ for which (1.11) and (1.12) hold. Assume furthermore, that $\varphi(t)$ is regularly varying at zero with positive index. Then*

$$\limsup_{h \rightarrow 0} \sup_{\substack{|s-t| \leq h \\ s, t \in [0, 1]}} \frac{|X_\alpha(s) - X_\alpha(t)|}{\varphi(h)(\log 1/h)^{1/2}} \leq 2 \sup_{t \in [0, 1]} X_\alpha^{1/2}(t) \quad a.s. \tag{1.18}$$

We also investigate the behavior of permanental processes at infinity. The following simple limit theorem is best possible for the squares of some Gaussian processes with stationary increments.

Theorem 1.4. *Let $X_\alpha = \{X_\alpha(t), t \in R^+\}$ be an α -permanental process with kernel $u(s, t)$ and sigma function $\sigma(s, t)$ that satisfies (1.11) and (1.12). Then*

$$\limsup_{t \rightarrow \infty} \frac{X_\alpha(t)}{u^*(t, t) \log t} \leq 1 \quad a.s. \tag{1.19}$$

Under additional hypotheses we get the familiar iterated logarithm in the denominator.

Theorem 1.5. *Under the hypotheses of Theorem 1.4 assume furthermore that $u(t, t)$ is regularly varying at infinity with positive index and $\varphi^2(t) = O(u(t, t))$ as $t \rightarrow \infty$. Then*

$$\limsup_{t \rightarrow \infty} \frac{X_\alpha(t)}{u(t, t) \log \log t} \leq 1 \quad a.s. \tag{1.20}$$

We require (1.12) because we are considering the processes on R^+ so in order for them to behave well for all $0 < t < \infty$ they must be continuous. We also want to study the behavior of permanental sequences $X_\alpha = \{X_\alpha(t_n), n \in \mathbb{N}\}$ in which $\{t_n\}$ has no limit points, or in which $\lim_{n \rightarrow \infty} t_n = t_0$ but X_α is not continuous at t_0 . Processes of this sort are treated in the next theorem.

Theorem 1.6. *Let $X_\alpha = \{X_\alpha(t_n), n \in \mathbb{N}\}$ be an α -permanental sequence with kernel $u(t_j, t_k)$ and sigma function $\sigma(t_j, t_k)$.*

Assume also that $\beta^2 := \lim_{n \rightarrow \infty} \sigma^2(t_n, 0) \log n$ exists. If $\beta = \infty$,

$$\limsup_{n \rightarrow \infty} \frac{X_\alpha(t_n)}{\sigma^2(t_n, 0) \log n} \leq 1 \quad a.s. \tag{1.21}$$

If $0 < \beta < \infty$ then

$$\limsup_{n \rightarrow \infty} |X_\alpha(t_n) - X_\alpha(0)| \leq \beta^2 + 2\beta X_\alpha^{1/2}(0) \quad a.s. \tag{1.22}$$

If $\beta = 0$,

$$\limsup_{n \rightarrow \infty} \frac{|X_\alpha(t_n) - X_\alpha(0)|}{\sigma(t_n, 0)(\log n)^{1/2}} \leq 2X_\alpha^{1/2}(0) \quad a.s. \tag{1.23}$$

The reader may wonder why (1.9) is given in terms of $X_\alpha^{1/2}$ when we are really concerned with the permanental processes X_α . The reason is that for X_α we only have

$$\|X_\alpha(s) - X_\alpha(t)\|_{\psi_1} \leq C'_\alpha \sigma(s, t), \tag{1.24}$$

for some finite constant C'_α , where $\psi_1(x) = \exp(x) - 1$ and $\|\cdot\|_{\psi_1}$ is the Orlicz norm in the corresponding Banach space. Sufficient conditions for the continuity and boundedness of processes satisfying (1.24) are weaker than those for processes satisfying (1.9). See, e.g., [12, Theorem 11.4]. (It is much easier to obtain (1.24) than (1.9), we do this in [11, Lemma 3.2 and Corollary 3.1].)

We apply the results above in the following examples:

Example 1.7. FBMQ $^{\gamma, \beta}$ Let $Z_{\gamma, \beta} = \{Z_t, t \in R_+\}$ be a Lévy process with characteristic function

$$E e^{i\lambda Z_{\gamma, \beta}} = e^{-\psi_{\gamma, \beta}(\lambda)t}, \tag{1.25}$$

where

$$\psi_{\gamma, \beta}(\lambda) = |\lambda|^{\gamma+1}(1 - i\beta \operatorname{sign}(\lambda) \tan(\gamma\pi/2)), \tag{1.26}$$

for $0 < \gamma < 1$ and $|\beta| \leq 1$. Consider the transient Markov process that is $Z_{\gamma, \beta}$ killed at the first time it hits zero and let $u_{T_0; \gamma, \beta}(x, y)$ denote its zero potential. Therefore, $u_{T_0; \gamma, \beta}(x, y)$ is also the kernel of α -permanental processes for all $\alpha > 0$. These permanental processes belong to the class of processes FBMQ $^{\gamma, \beta}$. (See page 30 and [11, Section 5.1] for the explanation of this notation.)

Theorem 1.8. Let $Y_{\alpha; \gamma, \beta} = \{Y_{\alpha; \gamma, \beta}(x), x \in R^+\}$ be an α -permanental process with kernel $u_{T_0; \gamma, \beta}$. Then for all $T > 0$

$$\limsup_{h \rightarrow 0} \sup_{\substack{|s-t| \leq h \\ s, t \in [0, T]}} \frac{|Y_{\alpha; \gamma, \beta}(s) - Y_{\alpha; \gamma, \beta}(t)|}{(h^\gamma \log 1/h)^{1/2}} \leq 2\sqrt{2}((1 + |\beta|)C_{\gamma, \beta})^{1/2} \left(\sup_{t \in T} Y_{\alpha; \gamma, \beta}(t)\right)^{1/2} \tag{1.27}$$

almost surely,

$$2C_{\gamma, \beta} \leq \limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{Y_{\alpha; \gamma, \beta}(t)}{h^\gamma \log \log 1/h} \leq 2(1 + |\beta|)C_{\gamma, \beta} \quad a.s. \tag{1.28}$$

and

$$\limsup_{t \rightarrow \infty} \frac{Y_{\alpha; \gamma, \beta}(t)}{t^\gamma \log \log t} = 2C_{\gamma, \beta} \quad a.s., \tag{1.29}$$

where

$$C_{\gamma, \beta} = \frac{-\sin\left(\left(\gamma + 1\right)\frac{\pi}{2}\right)\Gamma(-\gamma)}{\pi(1 + \beta^2 \tan^2((\gamma + 1)\pi/2))} > 0. \tag{1.30}$$

Example 1.9. We also consider the transient Markov process that is $Z_{\gamma, \beta}$ killed at the end of an independent exponential time with mean ρ . Let $u_{\rho; \gamma, \beta}(x, y)$ denote its zero potential. Therefore, $u_{\rho; \gamma, \beta}(x, y)$ is also the kernel of α -permanental processes for all $\alpha > 0$. The next theorem is similar to Theorem 1.8 except that $Y_{\alpha; \gamma, \beta} \equiv 0$, whereas $Y_{\alpha; \rho, \gamma, \beta} > 0$ almost surely.

Theorem 1.10. Let $Y_{\alpha;\rho,\gamma,\beta} = \{Y_{\alpha;\rho,\gamma,\beta}(x), x \in R^+\}$ be an α -permanental process with kernel $u_{\rho,\gamma,\beta}$. Then for all $T > 0$

$$\limsup_{h \rightarrow 0} \sup_{\substack{|s-t| \leq h \\ s,t \in [0,T]}} \frac{|Y_{\alpha;\rho,\gamma,\beta}(s) - Y_{\alpha;\rho,\gamma,\beta}(t)|}{(h^\gamma \log 1/h)^{1/2}} \leq 2\sqrt{2}((1 + |\beta|)C_{\gamma,\beta})^{1/2} \left(\sup_{t \in T} Y_{\alpha;\rho,\gamma,\beta}(t) \right)^{1/2}, \tag{1.31}$$

almost surely,

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|Y_{\alpha;\rho,\gamma,\beta}(t) - Y_{\alpha;\rho,\gamma,\beta}(0)|}{(h^\gamma \log \log 1/h)^{1/2}} \leq 2\sqrt{2}((1 + |\beta|)C_{\gamma,\beta})^{1/2} Y_{\alpha;\rho,\gamma,\beta}(0) \quad a.s. \tag{1.32}$$

and

$$\limsup_{t \rightarrow \infty} \frac{Y_{\alpha;\rho,\gamma,\beta}(t)}{\log t} = D_{\rho,\gamma,\beta} \quad a.s., \tag{1.33}$$

$$D_{\rho,\gamma,\beta} = \frac{1}{\pi} \int_0^\infty \frac{\Re(\rho + \psi_{\gamma,\beta}(\lambda))}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda. \tag{1.34}$$

The potentials in Examples 1.7 and 1.9 are not symmetric when $\beta \neq 0$. Prior to this paper there have not been examples of permanental processes that do not have symmetric kernels other than this case, i.e., when the kernel is the potential of a Lévy process. The next theorem shows how we can modify a very large class of symmetric potentials so that they are no longer symmetric but are still kernels of permanental processes.

Theorem 1.11. Let S be a locally compact set with a countable base. Let $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient symmetric Borel right process with state space S and continuous strictly positive potential densities $u(x, y)$ with respect to some σ -finite measure m on S . Then for any excessive function f of X and $\alpha > 0$,

$$\tilde{u}(x, y) = u(x, y) + f(y), \quad x, y \in S, \tag{1.35}$$

is the kernel of an α -permanental process.

A function f is said to be excessive for X if $E^x(f(X_t)) \uparrow f(x)$ as $t \rightarrow 0$ for all $x \in S$. It is easy to check that for any positive measurable function h ,

$$f(x) = \int u(x, y)h(y) dm(y) = E^x \left(\int_0^\infty h(X_t) dt \right) \tag{1.36}$$

is excessive for X . Such a function f is called a potential function for X . All the potential functions considered in this paper are continuous. This is discussed at the beginning of Section 6. We describe other excessive functions, some of which are not potentials, in the next two examples.

Example 1.12. Let $\bar{B} = \{\bar{B}_t, t \in R^+\}$ be Brownian motion killed after an independent exponential time with parameter $\lambda^2/2$, or, equivalently, with mean $2/\lambda^2$. The process \bar{B} has potential densities,

$$\bar{u}(x, y) = \frac{e^{-\lambda|y-x|}}{\lambda}, \quad x, y \in R^1, \tag{1.37}$$

and f is excessive for \bar{B} if and only if f is positive and is a $\lambda^2/2$ -superharmonic function, [4, p. 659]. In particular, $f \in C^2$ is excessive for \bar{B} if and only if it is positive and $f''(x) \leq \lambda^2 f(x)$ for all $x \in R^1$. Examples of such functions are e^{rx} for $|r| \leq \lambda$ and $q + |x|^\beta$

for $\beta \geq 2$ and $q \geq q_0$, where q_0 depends on β and λ . It follows from Theorem 1.11 that for functions f that are excessive for \bar{B} ,

$$\hat{u}_f(s, t) = e^{-\lambda|s-t|} + f(t), \quad s, t \in [0, 1], \tag{1.38}$$

is the kernel of an α -permanental process on $[0, 1]$ for all $\alpha > 0$.

The function $e^{-\lambda|s-t|}$, $s, t \in R^+$, is also the covariance of a time changed Ornstein-Uhlenbeck process.

Theorem 1.13. Let $\bar{X}_{\alpha, f} = \{\bar{X}_{\alpha, f}(t), t \in [0, 1]\}$ be an α -permanental process with kernel $\hat{u}_f(s, t)$ where $f \in C^2$ and is excessive for \bar{B} . Then

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|\bar{X}_{\alpha, f}(t) - \bar{X}_{\alpha, f}(0)|}{(h \log \log 1/h)^{1/2}} \leq 2\sqrt{2\lambda} \bar{X}_{\alpha, f}^{1/2}(0) \quad a.s. \tag{1.39}$$

and, for all $T > 0$,

$$\limsup_{h \rightarrow 0} \sup_{\substack{|s-t| \leq h \\ s, t \in [0, T]}} \frac{|\bar{X}_{\alpha, f}(s) - \bar{X}_{\alpha, f}(t)|}{(h \log 1/h)^{1/2}} \leq 2\sqrt{2\lambda} \left(\sup_{t \in T} \bar{X}_{\alpha, f}(t) \right)^{1/2}, \tag{1.40}$$

almost surely.

We now obtain an upper bound for the behavior at infinity.

Theorem 1.14. Let $\hat{X}_{\alpha, f} = \{\hat{X}_{\alpha, f}(t), t \in [0, \infty)\}$ be an α -permanental process with kernel $\hat{u}_f(s, t)$ in (1.38) but for $s, t \in [0, \infty)$, where $f \in C^2$ and is excessive for \bar{B} . Then

$$\limsup_{t \rightarrow \infty} \frac{\hat{X}_{\alpha, f}(t)}{(1 + f(t)) \log t} \leq 1 \quad a.s., \tag{1.41}$$

and when $\lim_{t \rightarrow \infty} f(t) = 0$,

$$\limsup_{t \rightarrow \infty} \frac{\hat{X}_{\alpha, f}(t)}{\log t} = 1 \quad a.s. \tag{1.42}$$

In particular this holds if f is a potential for \bar{B} , with $h \in L^1_+(0, \infty)$. (See (1.36).)

Example 1.15. Let $\tilde{B} = \{\tilde{B}_t, t \in R^+\}$ be Brownian motion killed the first time it hits 0. \tilde{B} has state space $D = (0, \infty)$ and potential densities

$$\bar{u}(x, y) = 2(x \wedge y), \quad x, y > 0. \tag{1.43}$$

Theorem 1.16. For any positive concave function f on $(0, \infty)$, and $\alpha > 0$,

$$\tilde{u}_f(s, t) = s \wedge t + f(t), \quad s, t > 0. \tag{1.44}$$

is the kernel of an α -permanental processes, $\tilde{Z}_{\alpha, f} = \{\tilde{Z}_{\alpha, f}(t), t > 0\}$, and

$$1 \leq \limsup_{t \rightarrow \infty} \frac{\tilde{Z}_{\alpha, f}(t)}{t \log \log t} \leq 1 + C_0 \quad a.s., \tag{1.45}$$

where $C_0 = \lim_{t \rightarrow \infty} f(t)/t$, which is necessarily finite.

If f is a potential for \tilde{B} , with $h \in L^1_+(0, \infty)$, (see (1.36)), then $f(t) = o(t)$ and

$$\limsup_{t \rightarrow \infty} \frac{\tilde{Z}_{\alpha, f}(t)}{t \log \log t} = 1 \quad a.s. \tag{1.46}$$

The next theorem describes the behavior of $\{\tilde{Z}_{\alpha, f}(t), t > 0\}$ as $t \rightarrow 0$ in the case that $\lim_{t \rightarrow 0} f(t)/t = \infty$.

Theorem 1.17. Let $\tilde{Z}_{\alpha,f}$ be the α -permanental process defined in Theorem 1.16, for f a positive concave function on $[0, \infty)$, with the additional property that for some $\delta > 0$

$$\int_0^\delta \frac{1}{f(u)(\log 1/u)^{1/2}} du < \infty. \tag{1.47}$$

Then there exists a coupling of $\tilde{Z}_{\alpha,f}$ with a gamma random variable $\xi_{\alpha,1}$, with shape α and scale 1, such that

$$\lim_{t \rightarrow 0} \frac{\tilde{Z}_{\alpha,f}(t)}{f(t)} = \xi_{\alpha,1} \quad a.s. \tag{1.48}$$

With an additional condition on f we can describe the behavior of $\tilde{Z}_{\alpha,f}$ near $\xi_{\alpha,1}$ more precisely.

Theorem 1.18. Let $\tilde{Z}_{\alpha,f}$ be the α -permanental process defined in Theorem 1.16, for f a positive concave function on $(0, \infty)$ that is regularly varying at zero with index less than 1. Then

$$\limsup_{t \rightarrow 0} \frac{|\tilde{Z}_{\alpha,f}^{1/2}(t) - (f(t)\xi_{\alpha,1})^{1/2}|}{(t \log \log 1/t)^{1/2}} \leq 1 \quad a.s., \tag{1.49}$$

where $\xi_{\alpha,1}$ is defined in Theorem 1.17.

Other examples are given in the body of this paper.

Theorem 1.1 is the main result in this paper. It deals, simply, with pairs of permanental random variables. Let $Z_\alpha = (Z_\alpha(1), Z_\alpha(2))$ be an α -permanental random variable with kernel

$$K = \begin{pmatrix} b & \gamma \\ \gamma & a \end{pmatrix}. \tag{1.50}$$

We point out in the paragraph containing (1.2) that a, b and $|K| \geq 0$. In addition we can take $\gamma \geq 0$.¹ We point out in the paragraph containing (1.6) that $\sigma = (a + b - 2\gamma)^{1/2} \geq 0$.

The next theorem is the main ingredient in the proof of Theorem 1.1.

Theorem 1.19. Let $X_\alpha = (X_\alpha(1), X_\alpha(2))$ be an α -permanental random variable with kernel K in (1.50). Set

$$\sigma^2 = a + b - 2\gamma. \tag{1.51}$$

Then for all $\lambda \geq 1$,

$$P \left(\frac{|X_\alpha^{1/2}(1) - X_\alpha^{1/2}(2)|}{\sigma} \geq \lambda \right) \leq C_\alpha \lambda^{(4\alpha-2)\vee 0} e^{-\lambda^2}, \tag{1.52}$$

for some constant C_α , depending only on α .

Note that by inequality between arithmetic and geometric means and the fact that $|K| \geq 0$,

$$\frac{a+b}{2} \geq (ab)^{1/2} \geq \gamma. \tag{1.53}$$

Hence if $\sigma = 0$ we have equality throughout (1.53) which implies that $a = b$ and $|K| = 0$. We point out prior to the statement of Lemma 2.1 that this implies that $X_\alpha^{1/2}(1) = X_\alpha^{1/2}(2)$ almost surely. We take the quotient 0/0 in (1.52) to be 0. (See Lemma 2.1.)

¹ An α -permanental random can have more than one kernel. In particular if it has kernel (1.50) then it also has the kernel with γ replaced by $-\gamma$.

We are interested in local moduli of continuity and rate of growth of $X_\alpha^{1/2}$ and X_α . It follows from Theorem 1.19 that when $\alpha \leq 1/2$,

$$P\left(\frac{|X_\alpha^{1/2}(s) - X_\alpha^{1/2}(t)|}{\sigma(s,t)} \geq \lambda\right) \leq C_\alpha e^{-\lambda^2}, \tag{1.54}$$

for some constant C_α , depending only on α . It is well known; (see, e.g., [15, Lemma 5.1.3]), that for $\tilde{G} = \{\tilde{G}(t), t \in \mathcal{T}\}$, a mean zero Gaussian process with covariance $u(s, t)/2$,

$$P\left(\frac{|G(s) - G(t)|}{\sigma(s,t)} \geq \lambda\right) \leq e^{-\lambda^2}. \tag{1.55}$$

Comparing (1.54) and (1.55) it is clear that upper bounds for the rates of growth of Gaussian processes that are proved solely by using the Borel-Cantelli Lemma applied to increments of the process, should also hold for the square roots of permanental processes when $\alpha \leq 1/2$. This is the case even when $\alpha > 1/2$ and the exponential in (1.54) is multiplied by a power of λ .

An extensive treatment of moduli of continuity of Gaussian processes is given in [15, Chapter 7]. However the proofs use properties of Gaussian processes more sophisticated than (1.55). Earlier treatments give many of the same results and are obtained solely from (1.55). In Lemma 3.1 we modify an inequality of Fernique, (see, e.g., [8, Chapter IV.1, Lemma 1.1]), and use it to obtain the results in Theorems 1.2–1.6.

Since most of the results in this paper only give upper bounds for the asymptotic behavior of permanental processes the question arises, how good are they? We can answer this by noting that when the kernel $u(s, t)$ is symmetric, it is the covariance of a Gaussian process, and the 1/2-permanental process with kernel $u(s, t)$ is 1/2 times the square of the Gaussian process. Therefore, we know that the upper bounds are best possible if they are also lower bounds for 1/2 times the square of the corresponding Gaussian process.

However, the reader probably knows that majorants for the increments variance of a Gaussian process, as in (1.11), do not give the whole picture for sample path behavior of Gaussian processes. For that one has to consider metric entropy with respect to the right-hand side of (1.10). Then one can get precise results such as [15, Theorems 7.1.2 and 7.1.4]. The problem here is that to get reasonable estimates of the metric entropy integrals, one has to require some smoothness conditions, such as, that the Gaussian processes have stationary increments and that the two terms in (1.10) are asymptotically equivalent as $s, t \rightarrow 0$. Then we get results like [15, Theorem 7.2.14]. Since our general results include the processes considered in [15, Theorem 7.2.14] and we get the same upper bound for the limsup, we say that generally, the upper bounds in this paper are best possible.

We do give precise results for the limsup in (1.29), (1.33), (1.42) and (1.48). To get the lower bounds we use a version of Slepian’s lemma given in [18]. Slepian’s lemma is the key to finding necessary conditions for the asymptotic behavior of Gaussian processes. It applies equally well to the process and to increments of the process because increments of a Gaussian process are themselves a Gaussian process. Our version of Slepian’s lemma applies to permanental processes, but not necessarily to their increments, because we do not know whether increments of a permanental process are a permanental process. This is why the the precise results we have are for the permanental process alone. We are currently writing a paper in which we use [17, Corollary 3.2] to get equality in (1.39) for 1/2-permanental processes with kernels given in (1.38) and certain excessive functions f .

The proofs of Theorems 1.1 and 1.19 are given in Section 2. The proofs of Theorems 1.2, 1.3 and 1.13 are given in Section 3. Theorem 1.10 and the proof of the upper bounds in Theorem 1.8 are proved in Section 4. Theorems 1.4–1.6 and the proof of upper bounds in Theorems 1.14–1.18 are proved in Section 5. The proof of Theorem 1.11 is given in Section 6. The proof of the lower bounds in Theorems 1.8, 1.14 and 1.16 are given in Section 7. The extension of [11, Theorem 3.1] and [16, Theorems 4.1 and 4.2] to all α -permanental processes for which (1.9) holds is given in Section 8.

2 Proofs of Theorems 1.1 and 1.19

Let $\xi_{\alpha,v}$ denote a gamma random variable with probability density function

$$f(\alpha, v; x) = \frac{v^\alpha x^{\alpha-1} e^{-vx}}{\Gamma(\alpha)}, \quad x \geq 0 \text{ and } \alpha, v > 0, \tag{2.1}$$

and equal to 0 for $x \leq 0$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is the gamma function.

It is easy to see that $\xi_{\alpha,v} \stackrel{(law)}{=} v^{-1} \xi_{\alpha,1}$ and

$$E(e^{-s\xi_{\alpha,1}}) = \frac{1}{(1+s)^\alpha}, \tag{2.2}$$

so that $\xi_{\alpha,1}$ is an α -permanental random variable with kernel 1. We also note that

$$P(\xi_{\alpha,1}^{1/2} \geq \lambda) = P(\xi_{\alpha,1} \geq \lambda^2) = \frac{1}{\Gamma(\alpha)} \int_{\lambda^2}^\infty x^{\alpha-1} e^{-x} dx. \tag{2.3}$$

Let $\lambda \geq 1$. For $\alpha \leq 1$, (2.3) is bounded by

$$\frac{\lambda^{2(\alpha-1)}}{\Gamma(\alpha)} \int_{\lambda^2}^\infty e^{-x} dx = \frac{\lambda^{2(\alpha-1)} e^{-\lambda^2}}{\Gamma(\alpha)}. \tag{2.4}$$

For $\alpha > 1$, we see by integration by parts that (2.3) is bounded by $C_\alpha \lambda^{2(\alpha-1)} e^{-\lambda^2}$. Thus, for $\lambda \geq 1$,

$$P(\xi_{\alpha,1}^{1/2} \geq \lambda) \leq C_\alpha \lambda^{2(\alpha-1)} e^{-\lambda^2}. \tag{2.5}$$

Let $X_\alpha = (X_\alpha(1), X_\alpha(2))$ be as in Theorem 1.19. Then

$$|I + KS| = 1 + s_1 b + s_2 a + s_1 s_2 |K|. \tag{2.6}$$

If $|K| = 0$ it follows from (1.1) that

$$E\left(e^{-(s_1 X_\alpha(1) + s_2 X_\alpha(2))}\right) = \frac{1}{(1 + s_1 b + s_2 a)^\alpha}. \tag{2.7}$$

Let $(X_\alpha(1), X_\alpha(2)) \stackrel{(law)}{=} (b, a)\xi_{\alpha,1}$. Then

$$\begin{aligned} E\left(e^{-(s_1 X_\alpha(1) + s_2 X_\alpha(2))}\right) &= E\left(e^{-(s_1 b + s_2 a)\xi_{\alpha,1}}\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(s_1 b + s_2 a)x} x^{\alpha-1} e^{-x} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(1+s_1 b + s_2 a)x} x^{\alpha-1} dx \\ &= \frac{1}{(1 + s_1 b + s_2 a)^\alpha} \end{aligned} \tag{2.8}$$

Therefore, when $|K| = 0$, $(X_\alpha(1), X_\alpha(2)) \stackrel{(law)}{=} (b, a)\xi_{\alpha,1}$, and since $|K| = 0$ implies that $ab = \gamma^2$ we see that $\sigma^2 = (\sqrt{b} - \sqrt{a})^2$.

When $a \neq b$

$$\frac{|X_\alpha^{1/2}(1) - X_\alpha^{1/2}(2)|}{\sigma} \stackrel{(law)}{=} \frac{|\sqrt{b} - \sqrt{a}|\xi_{\alpha,1}}{|\sqrt{b} - \sqrt{a}|} = \xi_{\alpha,1}. \tag{2.9}$$

Thus, when $|K| = 0$ and $a \neq b$, (1.52) follows from (2.5).

When $|K| = 0$ and $a = b$, $X_\alpha(1) = X_\alpha(2)$ so the numerator on the left-hand side of (2.9) is equal to zero. Of course, σ is also equal to zero. We take $0/0 = 0$ and get (1.52) in this case also.

For general $|K|$, assume that $\gamma \leq (a \wedge b)/2$. It follows that $\sigma^2 \geq a \vee b$. Consequently

$$\begin{aligned} P\left(\frac{|X_\alpha^{1/2}(1) - X_\alpha^{1/2}(2)|}{\sigma} \geq \lambda\right) &\leq P\left(\frac{X_\alpha^{1/2}(1)}{\sigma} \geq \lambda\right) + P\left(\frac{X_\alpha^{1/2}(2)}{\sigma} \geq \lambda\right) \\ &\leq P\left(\frac{b^{1/2}\xi_{\alpha,1}^{1/2}}{\sigma} \geq \lambda\right) + P\left(\frac{a^{1/2}\xi_{\alpha,1}^{1/2}}{\sigma} \geq \lambda\right) \leq 2C_\alpha \lambda^{2(\alpha-1)} e^{-\lambda^2}, \end{aligned} \tag{2.10}$$

where we use the facts that $X_\alpha(1) \stackrel{(law)}{=} b\xi_{\alpha,1}$, $X_\alpha(2) \stackrel{(law)}{=} a\xi_{\alpha,1}$ and (2.5). Thus we get (1.52) when $\gamma \leq (a \wedge b)/2$.

We collect these observations into the following lemma:

Lemma 2.1. *If $|K| = 0$ or if $\gamma \leq (a \wedge b)/2$, Theorem 1.19 holds. In particular Theorem 1.19 holds when $\gamma = 0$.*

We continue with the proof of Theorem 1.19. From now on we assume that $\gamma > 0$ and $|K| > 0$ which implies that $\sigma > 0$. We use the probability distribution of (X_1, X_2) that is given in [13, Theorem 1.1] in terms of

$$K^{-1} = \frac{1}{|K|} \begin{pmatrix} a & -\gamma \\ -\gamma & b \end{pmatrix}. \tag{2.11}$$

Theorem 2.2. *Let $X = (X_1, X_2)$ be an α -permanental random variable with kernel K . The probability density function of X is*

$$\tilde{g}(\alpha; (x_1, x_2)) = \frac{\gamma^{1-\alpha}}{\Gamma(\alpha)\delta} \frac{\mathcal{I}_{\alpha-1}(2\gamma\sqrt{xy}/\delta)}{(xy)^{(1-\alpha)/2}} e^{-(ax/\delta+by/\delta)} \tag{2.12}$$

on $R_{+,+}^2$, and zero elsewhere, where $\delta = |K|$ and

$$\mathcal{I}_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\nu+1)n!} \left(\frac{z}{2}\right)^{2n+\nu} \tag{2.13}$$

is the modified Bessel function of the first kind.

We use the notation $E(\xi; A) = E(\xi I_A)$ for sets A .

Lemma 2.3. *Let $X = (X_1, X_2)$ be an α -permanental random variable with kernel K as in Theorem 2.2. Then when $\gamma > 0$ and $\rho \geq \sqrt{2}$,*

$$E\left(\exp\left(\frac{\rho}{\sigma}\left(X_1^{1/2} - X_2^{1/2}\right)\right); X_1 X_2 \geq (\delta/2\gamma)^2\right) \leq C_\alpha \left(\frac{\rho}{\sqrt{2}}\right)^{(4\alpha-2)\vee 0} e^{\rho^2/4}, \tag{2.14}$$

for some constant C_α depending only on α .

Proof By Theorem 2.2

$$E \left(\exp \left(\frac{\rho}{\sigma} \left(X_1^{1/2} - X_2^{1/2} \right) \right); X_1 X_2 \geq (\delta/2\gamma)^2 \right) = \frac{\gamma^{1-\alpha}}{\Gamma(\alpha)\delta} \int_0^\infty \int_0^\infty e^{\rho(x^{1/2}-y^{1/2})/\sigma} \frac{\mathcal{I}_{\alpha-1} (2\gamma\sqrt{xy}/\delta)}{(xy)^{(1-\alpha)/2}} e^{-(ax/\delta+by/\delta)} 1_{\{xy \geq (\delta/2\gamma)^2\}} dx dy. \tag{2.15}$$

We make the change of variables $x = u^2/2, y = v^2/2$ and see that (2.15)

$$= \frac{(2\gamma)^{1-\alpha}}{\Gamma(\alpha)\delta} \int_0^\infty \int_0^\infty (uv)^\alpha e^{\rho'(u-v)} \mathcal{I}_{\alpha-1} (\gamma uv/\delta) e^{-(au^2+bv^2)/2\delta} 1_{\{\gamma uv/\delta \geq 1\}} du dv \tag{2.16}$$

where $\rho' = \rho/(\sqrt{2}\sigma)$. Over the range $\gamma uv/\delta \geq 1$, we bound

$$\mathcal{I}_{\alpha-1}(\gamma uv/\delta) \leq C'_\alpha \frac{\delta^{1/2} e^{\gamma uv/\delta}}{\sqrt{2\gamma uv}} \tag{2.17}$$

for some constant C'_α , depending on α ; see [7, 8.451.5]. Therefore (2.16)

$$\leq \frac{(2\gamma)^{(1/2)-\alpha} C'_\alpha}{\Gamma(\alpha)\delta^{1/2}} \int_0^\infty \int_0^\infty (uv)^{\alpha-1/2} e^{\rho'(u-v)} e^{-(au^2-2\gamma uv+bv^2)/2\delta} 1_{\{\gamma uv/\delta \geq 1\}} du dv. \tag{2.18}$$

$$\leq \frac{(2\gamma)^{(1/2)-\alpha} C'_\alpha}{\Gamma(\alpha)\delta^{1/2}} \int_{-\infty}^\infty \int_{-\infty}^\infty |uv|^{\alpha-1/2} e^{\rho'(u-v)} e^{-(au^2-2\gamma uv+bv^2)/2\delta} du dv \tag{2.19}$$

Consider

$$\rho'(u-v) - \frac{au^2 - 2\gamma uv + bv^2}{2\delta} \tag{2.20}$$

Let $w = (u, v)$ and $z = \rho'(1, -1)$, and write this as

$$(z, w) - \frac{(w, K^{-1}w)}{2}. \tag{2.21}$$

Let $w = s + Kz$. With this substitution (2.21) is equal to

$$\frac{(z, Kz)}{2} - \frac{(s, K^{-1}s)}{2}. \tag{2.22}$$

It follows from this that if we make the change of variables $u = s_1 + K(z)_1$ and $v = s_2 + K(z)_2$, and $\alpha \geq 1/2$, (2.19) is less than or equal to

$$\frac{C''_\alpha}{2\pi|K|^{1/2}} \int_{-\infty}^\infty \int_{-\infty}^\infty (|s_1 + K(z)_1| |s_2 + K(z)_2|)^{\alpha-1/2} e^{-(s, K^{-1}s)/2} ds_1 ds_2 e^{(z, Kz)/2}, \tag{2.23}$$

in which

$$C''_\alpha = \frac{(2\pi)(2\gamma)^{(1/2)-\alpha} C'_\alpha}{\Gamma(\alpha)}. \tag{2.24}$$

(Recall that $\delta = |K|$.)

Note that (2.23)

$$\leq C''_\alpha \left(E[(|\xi_1| + |K(z)_1|) (|\xi_2| + |K(z)_2|)]^{\alpha-1/2} \right) e^{(z, Kz)/2}, \tag{2.25}$$

where (ξ_1, ξ_2) is a mean zero 2-dimensional Gaussian random variable with covariance matrix K . Furthermore, by Lemma 2.8,

$$|K(z)_1| = \frac{\rho}{\sqrt{2}} \frac{|a-\gamma|}{\sigma} \leq \frac{\rho a^{1/2}}{\sqrt{2}} \quad \text{and} \quad |K(z)_2| = \frac{\rho}{\sqrt{2}} \frac{|b-\gamma|}{\sigma} \leq \frac{\rho b^{1/2}}{\sqrt{2}}. \tag{2.26}$$

Therefore,

$$(|\xi_1| + |K(z)_1|)(|\xi_2| + |K(z)_2|) \leq (ab)^{1/2} \left(\left| \frac{\xi_1}{a^{1/2}} \right| + \frac{\rho}{\sqrt{2}} \right) \left(\left| \frac{\xi_2}{b^{1/2}} \right| + \frac{\rho}{\sqrt{2}} \right). \quad (2.27)$$

Note that $\xi_1/a^{1/2}$ and $\xi_2/b^{1/2}$ have variance 1. Using this, (2.27) and the Cauchy Schwartz Inequality, and the fact that

$$(z, Kz) = (\rho'\sigma)^2 = \frac{\rho^2}{2}, \quad (2.28)$$

we see that (2.25)

$$\leq C''_{\alpha}(ab)^{(2\alpha-1)/4} E \left(\left| \eta \right| + \frac{\rho}{\sqrt{2}} \right)^{2\alpha-1} e^{\rho^2/4}. \quad (2.29)$$

where η is a standard normal random.

Since $\alpha \geq 1/2$, this is

$$\begin{aligned} &\leq C''_{\alpha} 2^{2\alpha-1} (ab)^{(2\alpha-1)/4} \left(E(|\eta|^{2\alpha-1}) + \left(\frac{\rho}{\sqrt{2}} \right)^{2\alpha-1} \right) e^{\rho^2/4} \\ &\leq C''_{\alpha} 2^{2\alpha-1} (ab)^{(2\alpha-1)/4} E(|\eta|^{2\alpha-1}) \left(1 + \left(\frac{\rho}{\sqrt{2}} \right)^{2\alpha-1} \right) e^{\rho^2/4}. \end{aligned} \quad (2.30)$$

Note that

$$C''_{\alpha} 2^{2\alpha-1} (ab)^{(2\alpha-1)/4} = \frac{2^{\alpha+1/2} \pi C'_{\alpha}}{\Gamma(\alpha)} \left(\frac{(ab)^{1/2}}{\gamma} \right)^{(\alpha-1/2)}. \quad (2.31)$$

Therefore, if $\gamma \geq (ab)^{1/2}/\sqrt{2}$, the left-hand side of (2.15)

$$\leq \frac{2^{(5\alpha+1)/4} \pi C'_{\alpha}}{\Gamma(\alpha)} E(|\eta|^{2\alpha-1}) \left(1 + \left(\frac{\rho}{\sqrt{2}} \right)^{2\alpha-1} \right) e^{\rho^2/4}. \quad (2.32)$$

We now obtain an upper bound for the left-hand side of (2.15) when $\alpha \geq 1/2$ and $\gamma < (ab)^{1/2}/\sqrt{2}$. To begin note that left-hand side of (2.15) is bounded by the integral in (2.18), in which the integrand is restricted to the region, $\gamma uv/\delta \geq 1$. On this region

$$\frac{1}{\gamma^{\alpha-1/2}} \leq \left(\frac{uv}{\delta} \right)^{\alpha-1/2}. \quad (2.33)$$

Using this inequality we can bound the integral in (2.18) by

$$\frac{D_{\alpha}}{\delta^{\alpha-1/2} 2\pi |K|^{1/2}} \int_0^{\infty} \int_0^{\infty} (uv)^{2\alpha-1} e^{\rho'(u-v)} e^{-(au^2-2\gamma uv+bv^2)/2\delta} 1_{\{\gamma uv/\delta \geq 1\}} du dv, \quad (2.34)$$

where

$$D_{\alpha} = \frac{2\pi C'_{\alpha}}{2^{\alpha-1/2} \Gamma(\alpha)}. \quad (2.35)$$

Following the argument from (2.18)–(2.30), and in particular focusing on (2.23) we see that

$$\begin{aligned} &\frac{1}{2\pi |K|^{1/2}} \int_0^{\infty} \int_0^{\infty} (uv)^{2\alpha-1} e^{\rho'(u-v)} e^{-(au^2-2\gamma uv+bv^2)/2\delta} 1_{\{\gamma uv/\delta \geq 1\}} du dv \\ &\leq 4^{2\alpha-1} (ab)^{\alpha-1/2} E(|\eta|^{2(2\alpha-1)}) \left(1 + \left(\frac{\rho}{\sqrt{2}} \right)^{2(2\alpha-1)} \right) e^{\rho^2/4}. \end{aligned} \quad (2.36)$$

Therefore (2.34)

$$\leq \frac{4^{2\alpha-1}\pi C'_\alpha}{2^{\alpha-1/2}\Gamma(\alpha)} \left(\frac{ab}{\delta}\right)^{(\alpha-1)/2} E\left(|\eta|^{2(2\alpha-1)}\right) \left(1 + \left(\frac{\rho}{\sqrt{2}}\right)^{2(2\alpha-1)}\right) e^{\rho^2/4}. \tag{2.37}$$

Now, note that $\gamma < (ab)^{1/2}/\sqrt{2}$ implies that $\delta > (ab/2)$. Consequently, (2.37)

$$\leq \frac{4^{2\alpha-1}\pi C'_\alpha}{\Gamma(\alpha)} E\left(|\eta|^{2(2\alpha-1)}\right) \left(1 + \left(\frac{\rho}{\sqrt{2}}\right)^{2(2\alpha-1)}\right) e^{\rho^2/4}. \tag{2.38}$$

Using (2.32) and (2.38) we see that when $\alpha \geq 1/2$, $\gamma > 0$ and $\rho \geq \sqrt{2}$,

$$E\left(\exp\left(\frac{\rho}{\sigma}\left(X_1^{1/2} - X_2^{1/2}\right)\right); X_1 X_2 \geq (\delta/2\gamma)^2\right) \leq C_\alpha \left(\frac{\rho}{\sqrt{2}}\right)^{4\alpha-2} e^{\rho^2/4}, \tag{2.39}$$

for some constant C_α depending only on α .

To obtain an upper bound for the left-hand side of (2.15) when $\alpha < 1/2$ we follow the argument from (2.18)–(2.23) and use (2.28) to see that it is

$$\begin{aligned} &\leq C''_\alpha \left(E\left|\frac{1}{(\xi_1 + K(z)_1)(\xi_2 + K(z)_2)}\right|^{1/2-\alpha}\right) e^{\rho^2/4} \tag{2.40} \\ &\leq C''_\alpha \left(E\left|\frac{1}{(\xi_1 + K(z)_1)(\xi_2 + K(z)_2)}\right|^{1/2-\alpha}\right) e^{\rho^2/4} \\ &\leq C''_\alpha \left[E\left(\left(\frac{1}{|\xi_1|}\right)^{1-2\alpha}\right) E\left(\left(\frac{1}{|\xi_2|}\right)^{1-2\alpha}\right)\right]^{1/2} e^{\rho^2/4}, \end{aligned}$$

where, as we point out above, ξ_1 is mean zero normal random variable with variance a and ξ_2 is mean zero normal random variable with variance b . Let η be a standard normal random variable. The last line of (2.40) is equal to

$$\begin{aligned} &\frac{2^{(3/2)-\alpha}\gamma^{(1/2)-\alpha}\pi C''_\alpha}{\Gamma(\alpha)} \frac{1}{(ab)^{(1/4)-(\alpha/2)}} E\left(\left(\frac{1}{|\eta|}\right)^{1-2\alpha}\right) e^{\rho^2/4} \tag{2.41} \\ &\leq \frac{2^{(3/2)-\alpha}\pi C''_\alpha}{\Gamma(\alpha)} E\left(\left(\frac{1}{|\eta|}\right)^{1-2\alpha}\right) e^{\rho^2/4}, \end{aligned}$$

where we use the fact that $\gamma < \sqrt{ab}$. Using (2.39) and (2.41) we get (2.14). □

We use the next lemma in the proof of Lemma 2.5.

Lemma 2.4. For K and σ^2 in (1.50) and (1.51),

$$|K| \leq (a \wedge b)\sigma^2. \tag{2.42}$$

Proof To prove (2.42) we first show that $|K| \leq a\sigma^2$. This is

$$ab - \gamma^2 \leq a^2 + ab - 2a\gamma, \tag{2.43}$$

which is equivalent to $(a - \gamma)^2 \geq 0$. The same argument works with a replaced by b . □

Lemma 2.5. For $\gamma > 0$ and $\lambda \geq 1$,

$$P\left(\frac{|X_1^{1/2} - X_2^{1/2}|}{\sigma} \geq \lambda; X_1 X_2 \leq (\delta/2\gamma)^2\right) \leq \widehat{C}_\alpha \lambda^{2\alpha-2} e^{-\lambda^2}, \tag{2.44}$$

where \widehat{C}_α is a constant depending only on α .

Proof Let $(U, V) := 2^{1/2}(X_1^{1/2}, X_2^{1/2})$ so that the left-hand side of (2.44), without the absolute value signs, can be written as

$$P\left(U - V \geq \sqrt{2}\lambda\sigma; UV \leq \delta/\gamma\right) \leq P\left(U \geq \sqrt{2}\lambda\sigma; UV \leq \delta/\gamma\right). \tag{2.45}$$

Using (2.16) we see that

$$\frac{(2\gamma)^{1-\alpha}}{\Gamma(\alpha)\delta}(uv)^\alpha \mathcal{I}_{\alpha-1}(\gamma uv/\delta) e^{-(au^2+bv^2)/2\delta} \tag{2.46}$$

is the joint probability density function of (U, V) . To find an upper bound for (2.45) we note that by (2.13)

$$\mathcal{I}_{\alpha-1}(w) \leq C_\alpha w^{\alpha-1}, \tag{2.47}$$

for $w \leq 1$, where C_α is a constant depending on α . With this substitution (2.45) is less than or equal to

$$\frac{C_\alpha 2^{1-\alpha}}{\Gamma(\alpha)\delta^\alpha} \int_{\sqrt{2}\lambda\sigma}^\infty \int_0^\infty (uv)^{2\alpha-1} e^{-(au^2+bv^2)/2\delta} dv du. \tag{2.48}$$

We have

$$\int_0^\infty v^{2\alpha-1} e^{-bv^2/2\delta} dv = \sqrt{\frac{\pi}{2}} \left(\frac{\delta}{b}\right)^\alpha E(|Z|^{2\alpha-1}), \tag{2.49}$$

where Z is a normal random variable with mean zero and variance 1. In addition

$$\begin{aligned} \int_{\sqrt{2}\lambda\sigma}^\infty u^{2\alpha-1} e^{-au^2/2\delta} du &= \left(\frac{\delta}{a}\right)^\alpha \int_{(a/\delta)^{1/2}\sqrt{2}\lambda\sigma}^\infty s^{2\alpha-1} e^{-s^2/2} ds \\ &\leq \left(\frac{\delta}{a}\right)^\alpha \int_{\sqrt{2}\lambda}^\infty s^{2\alpha-1} e^{-s^2/2} ds, \end{aligned} \tag{2.50}$$

since $a\sigma^2/\delta \geq 1$ by Lemma 2.4. Setting $x = s^2/2$ in (2.3), it follows from (2.5) that for all $\lambda \geq 1$ and $\alpha > 0$,

$$\int_{\sqrt{2}\lambda}^\infty s^{2\alpha-1} e^{-s^2/2} ds \leq C_\alpha \lambda^{2(\alpha-1)} e^{-\lambda^2}. \tag{2.51}$$

Using (2.49)–(2.51) in (2.48) we see that (2.48) is bounded by

$$D''_\alpha \sqrt{\frac{\pi}{2}} \left(\frac{\delta}{ab}\right)^\alpha E(|Z|^{2\alpha-1}) \lambda^{2\alpha-2} e^{-\lambda^2}, \tag{2.52}$$

where D''_α is a constant depending only on α . Since $\delta/(ab) < 1$, we get (2.44). (We can include the absolute value sign by multiplying the probability on the right by 2.) \square

Proof of Theorem 1.19 When $\gamma = 0$ or $|K| = 0$ this follows from Lemma 2.1.

Now suppose that $\gamma > 0$ and $|K| > 0$. We write

$$\begin{aligned} P\left(X_1^{1/2} - X_2^{1/2} \geq \sigma\lambda\right) &\leq P\left(X_1^{1/2} - X_2^{1/2} \geq \sigma\lambda; X_1 X_2 \leq (\delta/2\gamma)^2\right) \\ &\quad + P\left(X_1^{1/2} - X_2^{1/2} \geq \sigma\lambda; X_1 X_2 \geq (\delta/2\gamma)^2\right). \end{aligned} \tag{2.53}$$

Using Lemma 2.5 we see that we have the upper bound in (1.52) for the first probability on the right-hand side of the inequality in (2.53).

By Lemma 2.3 the second term on the right-hand side of the inequality

$$\begin{aligned} &\leq e^{-\rho\lambda} E\left(\exp\left(\frac{\rho(X_1^{1/2} - X_2^{1/2})}{\sigma}\right); X_1 X_2 \geq (\delta/2\gamma)^2\right) \\ &\leq C_\alpha \left(\frac{\rho}{\sqrt{2}}\right)^{(4\alpha-2)\vee 0} e^{-\rho\lambda} e^{\rho^2/4}. \end{aligned} \tag{2.54}$$

Taking $\rho = 2\lambda$ the upper bound in (1.52) for the second probability on the right-hand side of the inequality in (2.53). \square

We need the next two lemmas in the proof of Theorem 1.1.

Lemma 2.6. *If*

$$P(|Z| \geq \lambda) \leq K e^{-\lambda^2} \tag{2.55}$$

then

$$\|Z\|_{\psi_2} \leq c^*, \tag{2.56}$$

where c^ is the value of $c > 1$ such that*

$$K^{1/c^2} \frac{c^2}{c^2 - 1} = 2. \tag{2.57}$$

Proof For $c > 1$ and $K = e^{y_0^2}$,

$$\begin{aligned} E\left(\exp\left(\frac{Z^2}{c^2}\right) - 1\right) &= -\int_0^\infty (e^{\lambda^2/c^2} - 1) dP(|Z| \geq \lambda) = \int_0^\infty P(|Z| \geq \lambda) de^{\lambda^2/c^2} \\ &\leq \int_0^{y_0} de^{\lambda^2/c^2} + K \int_{y_0}^\infty e^{-\lambda^2} de^{\lambda^2/c^2} \\ &= e^{y_0^2/c^2} - 1 + K \int_{y_0}^\infty e^{-\lambda^2} de^{\lambda^2/c^2}. \end{aligned} \tag{2.58}$$

We have

$$\begin{aligned} \int_{y_0}^\infty e^{-\lambda^2} de^{\lambda^2/c^2} &= \frac{1}{c^2(1 - 1/c^2)} \int_{y_0}^\infty e^{-\lambda^2(1-1/c^2)} 2(1 - 1/c^2)\lambda d\lambda \\ &= \frac{e^{-y_0^2(1-1/c^2)}}{c^2(1 - 1/c^2)}, \end{aligned} \tag{2.59}$$

so that (2.58)

$$\begin{aligned} &= e^{y_0^2/c^2} - 1 + e^{y_0^2} \frac{e^{-y_0^2(1-1/c^2)}}{c^2(1 - 1/c^2)} \\ &= e^{y_0^2/c^2} - 1 + \frac{e^{y_0^2/c^2}}{c^2(1 - 1/c^2)} = e^{y_0^2/c^2} \frac{c^2}{c^2 - 1} - 1, \end{aligned} \tag{2.60}$$

which gives (2.57). \square

Lemma 2.7. *If*

$$P(|Z| \geq \lambda) \leq K(\lambda^n + 1)e^{-\lambda^2}, \quad \lambda \geq 0, \tag{2.61}$$

for $n > 0$, then

$$\|Z\|_{\psi_2} \leq c^*, \tag{2.62}$$

where c^ is the value of $c > 1$ such that*

$$e^{y_0^2/c^2} + \frac{K}{c^2} \int_{y_0}^\infty \lambda(\lambda^n + 1)e^{-\lambda^2(1-1/c^2)} d\lambda = 2, \tag{2.63}$$

for y_0 , the solution of

$$K y_0^n e^{-y_0^2} = 1. \tag{2.64}$$

Proof For $c > 1$ and $Ky_0^n e^{-y_0^2} = 1$,

$$\begin{aligned} E\left(\exp\left(\frac{Z^2}{c^2}\right) - 1\right) & \tag{2.65} \\ &= -\int_0^\infty \left(e^{\lambda^2/c^2} - 1\right) dP(|Z| \geq \lambda) = \int_0^\infty P(|Z| \geq \lambda) de^{\lambda^2/c^2} \\ &= e^{y_0^2/c^2} - 1 + \frac{2K}{c^2} \int_{y_0}^\infty \lambda(\lambda^n + 1)e^{-\lambda^2(1-1/c^2)} d\lambda, \end{aligned}$$

which gives (2.63). □

Proof of Theorem 1.1 By hypothesis $(X_\alpha(s), X_\alpha(t))$ is an α -permenal random variable with kernel

$$K_{s,t} = \begin{pmatrix} u(s,s) & u(s,t) \\ u(t,s) & u(t,t) \end{pmatrix}. \tag{2.66}$$

We point out in (1.3) that $(X_\alpha(s), X_\alpha(t))$ also has the symmetric kernel

$$U_{s,t} = \begin{pmatrix} u(s,s) & (u(s,t)u(t,s))^{1/2} \\ (u(s,t)u(t,s))^{1/2} & u(t,t) \end{pmatrix}, \tag{2.67}$$

and that the function $\sigma(s,t)$ corresponding to this is as given in (1.6). Therefore, it follows from Theorem 1.19 that for $\lambda \geq 1$

$$P\left(\frac{|X_\alpha^{1/2}(s) - X_\alpha^{1/2}(t)|}{\sigma(s,t)} \geq \lambda\right) \leq C_\alpha \lambda^{(4\alpha-2)\vee 0} e^{-\lambda^2}. \tag{2.68}$$

for some absolute constant C_α that depends only on α . Theorem 1.1 follows from (2.68) and Lemmas 2.6 and 2.7. □

The next lemma is used in the proof of Lemma 2.3.

Lemma 2.8. When $|K| > 0$,

$$\frac{|a - \gamma|}{\sigma} \leq a^{1/2} \quad \text{and} \quad \frac{|b - \gamma|}{\sigma} \leq b^{1/2}. \tag{2.69}$$

Proof Since a and b are interchangeable, it suffices to prove the first inequality in (2.69). Suppose that $a - \gamma \geq 0$. Then if, in addition, $b > \gamma$,

$$\frac{a - \gamma}{\sigma} = \frac{a - \gamma}{(a + b - 2\gamma)^{1/2}} \leq (a - \gamma)^{1/2}. \tag{2.70}$$

(Note that since $\sigma > 0$, this holds when $a = \gamma$.)

Next, suppose that $a > \gamma$ and $b < \gamma$. Then, since

$$\frac{d}{d\gamma} \left(\frac{(a - \gamma)^2}{a + b - 2\gamma} \right) = 2 \frac{(\gamma - b)(a - \gamma)}{(a + b - 2\gamma)^2} > 0, \tag{2.71}$$

we see that because $\gamma \leq (ab)^{1/2}$,

$$\begin{aligned} \frac{a - \gamma}{\sigma} &= \frac{a - \gamma}{(a + b - 2\gamma)^{1/2}} \leq \frac{a - (ab)^{1/2}}{(a + b - 2(ab)^{1/2})^{1/2}} \\ &= \frac{a - (ab)^{1/2}}{a^{1/2} - b^{1/2}} = a^{1/2}. \end{aligned} \tag{2.72}$$

If $a < \gamma$, then $b > \gamma$, and

$$\frac{\gamma - a}{\sigma} = \frac{\gamma - a}{(a + b - 2\gamma)^{1/2}}. \tag{2.73}$$

Then, since by (2.71),

$$\frac{d}{d\gamma} \left(\frac{(\gamma - a)^2}{a + b - 2\gamma} \right) > 0. \tag{2.74}$$

It follows, as in (2.72), that

$$\frac{\gamma - a}{\sigma} \leq \frac{(ab)^{1/2} - a}{(a + b - 2(ab)^{1/2})^{1/2}} = \frac{(ab)^{1/2} - a}{b^{1/2} - a^{1/2}} = a^{1/2}. \tag{2.75}$$

□

3 Upper bounds for the local moduli of continuity and rate of growth of permanental processes, I

All the results in this section follow from the next lemma which is a modification of an inequality of Fernique as presented in [8, Chapter IV.1, Lemma 1.1].

Lemma 3.1. *Let $Y = \{Y(t), t \in R^+\}$ be a stochastic process. For $s, t \in [0, S]$ and $a \geq 0$, let*

$$F(a) = \sup_{s, t \in [0, S]} P \left(\frac{|Y(s) - Y(t)|}{\sigma(s, t)} \geq a \right) \tag{3.1}$$

for some positive function $\sigma(s, t)$ on $[0, S] \times [0, S]$. Assume furthermore, that there exist an increasing function φ such that for $S > 0$ and all $0 \leq s, t \leq S$,

$$\sigma(s, t) \leq \varphi(|t - s|). \tag{3.2}$$

Let n be an integer greater than 1. Then

$$\begin{aligned} P \left(\sup_{t \in [0, S]} |Y(t) - Y(0)| > a\varphi(S) + \sum_{p=1}^{\infty} \kappa(p)\varphi(S/n(p)) \right) \\ \leq n^2 F(a) + \sum_{p=1}^{\infty} n^2(p) F(\kappa(p)), \end{aligned} \tag{3.3}$$

where $n(p) = n^{2^p}$ and κ is a positive function with $\kappa \geq 1$.

To give the reader some idea of where we are heading we mention that in using (3.3) we take both a and S to depend on n in such a way that the right-hand side of (3.3) is a converging sequence in n . This enables us to use the Borel–Cantelli Lemma to get upper bounds for the limiting behavior of $|Y(t) - Y(0)|$.

Proof of Lemma 3.1 Consider [8, Chapter IV.2, Lemma 1.1]. This lemma is proved for $S = 1$. (It also assumes that Y is a Gaussian process, but only uses $F(a)$ as defined in (3.1), in which case the right-hand side of (3.1) is independent of s and t .)

Let us first assume that $S = 1$. The only other thing in [8, Chapter IV.1, Lemma 1.1] that might be confusing is the term α which is

$$\sup_{t \in [0, 1]} \sigma(0, t) \leq \varphi(1). \tag{3.4}$$

Thus we get (3.3) with $S = 1$. In particular we require that

$$\sigma(s, t) \leq \varphi(|t - s|), \quad \forall 0 \leq s, t \leq 1. \tag{3.5}$$

Now replace $\varphi(\cdot)$ by $\varphi(S\cdot)$. We have for all $0 \leq u, v \leq S$

$$\sigma(u, v) \leq \varphi(S|(u/S) - (v/S)|), \quad \forall 0 \leq (u/S), (v/S) \leq 1. \tag{3.6}$$

If we replace $\varphi(\cdot)$ with $\varphi(S\cdot)$ we get (3.3) for arbitrary S . Here we also use the fact that the right-hand side of (3.1) is defined for all $s, t \in [0, S]$. \square

Lemma 3.1 can be used to find upper bounds for the local and uniform moduli of continuity of Y and the behavior of $Y(t)$ as $t \rightarrow \infty$. This is done in [8, Theorem 1.3, Chapter IV.2] for the local and uniform moduli of continuity of Gaussian processes $G = \{G(t), t \in R^+\}$, but the same proof also gives the behavior of $G(t)$ as $t \rightarrow \infty$. We generalize [8, Theorem 1.3, Chapter IV.2] in the case of local modulus of continuity and behavior at infinity by applying Lemma 3.1 to stochastic processes with the property that

$$F(\lambda) \leq C_m (\lambda^m + 1) e^{-\lambda^2}, \tag{3.7}$$

where $m \geq 0$ and C_m is a constant.

Lemma 3.2. *Let $Y = \{Y(t), t \in R^+\}$ be a stochastic process for which (3.7) holds. Assume that there exist an increasing function φ such that for all $0 \leq s, t < \infty$,*

$$\sigma(s, t) \leq \varphi(|t - s|), \tag{3.8}$$

where

$$\int_0^{1/2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du < \infty. \tag{3.9}$$

Consider the condition

$$\lim_{\gamma \rightarrow 1} \frac{\varphi(\gamma V)}{\varphi(V)} = 1. \tag{3.10}$$

Assume that (3.10) holds uniformly in $0 < V \leq V_0$ for some $V_0 \leq 1$. For h near zero, set

$$\tau(h) = \varphi(h)(\log \log 1/h)^{1/2} + \frac{1}{\log 2} \int_0^{1/2} \frac{\varphi(hu)}{u(\log 1/u)^{1/2}} du. \tag{3.11}$$

Then

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|Y(t) - Y(0)|}{\tau(h)} \leq \sqrt{3} \quad a.s. \tag{3.12}$$

Similarly, assume that (3.10) holds uniformly in $V \geq V_0$ for some $V_0 \geq 1$. For T large, set

$$\tilde{\tau}(T) = \varphi(T)(\log \log T)^{1/2} + \frac{1}{\log 2} \int_0^{1/2} \frac{\varphi(Tu)}{u(\log 1/u)^{1/2}} du. \tag{3.13}$$

Then

$$\limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq T} \frac{|Y(t) - Y(0)|}{\tilde{\tau}(T)} \leq \sqrt{3} \quad a.s. \tag{3.14}$$

Proof We first prove (3.14). Fix $1 < \theta < 2$ and set $V = \theta^n$. Choose n_0 such that $\theta^{n_0} \geq V_0$. Consider Lemma 3.1 with $S = \theta^n$ for $n \geq n_0$. Let $\epsilon > 0$ and take

$$a = ((3 + \epsilon) \log \log \theta^n)^{1/2}. \tag{3.15}$$

Note that by (3.7) there exists a constant C such that

$$F(a) \leq 1 \wedge C \frac{(\log n)^{m/2} + 1}{n^{3+\epsilon}}. \tag{3.16}$$

Therefore

$$n^2 F(a) \leq \frac{1}{n^{1+\epsilon/2}}, \tag{3.17}$$

for all $n \geq n_0$ sufficiently large.

Consider the last term in (3.3). Take $\kappa(p) = (3 \log n(p))^{1/2}$. We have that

$$\sum_{p=1}^{\infty} n^2(p)F(\kappa(p)) \leq \tilde{C} \sum_{p=1}^{\infty} \frac{(2^p \log n)^{m/2}}{n(p)} \leq \frac{1}{n^{3/2}}, \tag{3.18}$$

for some constant \tilde{C} and all n sufficiently large. Considering (3.17) and (3.18) we see that the right-hand side of (3.3) is a converging sequence in n .

Now consider the sum in the first line of (3.3). We have

$$\begin{aligned} \sum_{p=1}^{\infty} (3 \log n(p))^{1/2} \varphi(\theta^n/n(p)) & \tag{3.19} \\ & \leq \sqrt{3} \sum_{p=1}^{\infty} (\log n(p))^{1/2} \varphi(\theta^n/n(p)) \\ & \leq \frac{\sqrt{3}}{\log 2} \int_0^{1/n} \frac{\varphi(\theta^n u)}{u(\log 1/u)^{1/2}} du. \end{aligned}$$

(This is obtained by replacing the sum by the integral with respect to p , from zero to infinity, and making the change of variables $n(p) = 1/u$). We can now use the Borel-Cantelli Lemma to get

$$\limsup_{n \rightarrow \infty} \sup_{0 \leq t \leq \theta^n} \frac{|Y(t) - Y(0)|}{\tilde{\tau}(\theta^n)} \leq \sqrt{3} \quad a.s. \tag{3.20}$$

To prove (3.14) it suffices to show that

$$\limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq T} \frac{|Y(t) - Y(0)|}{\tilde{\tau}(T)} \leq \sqrt{3} (1 + \epsilon)^2 \quad a.s., \tag{3.21}$$

for all $\epsilon > 0$.

Fix $\epsilon > 0$. We show below that we can find $1 < \theta_1 < 2$ and $n_1 < \infty$ such that

$$\frac{\tilde{\tau}(\theta_1^{n+1})}{\tilde{\tau}(\theta_1^n)} \leq 1 + \epsilon \tag{3.22}$$

for all $n \geq n_1$. Furthermore, by (3.20) for all $\omega \in \Omega$ for some set Ω with probability one, we can find $n_2 = n_2(\omega) < \infty$ such that

$$\sup_{n \geq n_2} \sup_{0 \leq t \leq \theta_1^n} \frac{|Y(t) - Y(0)|}{\tilde{\tau}(\theta_1^n)} \leq \sqrt{3}(1 + \epsilon) \quad a.s. \tag{3.23}$$

Then, for any $\theta_1^n < \rho < \theta_1^{n+1}$ with $n \geq n_1, n_2$

$$\begin{aligned} \sup_{0 \leq t \leq \rho} \frac{|Y(t) - Y(0)|}{\tilde{\tau}(\rho)} & \leq \sup_{0 \leq t \leq \theta_1^{n+1}} \frac{|Y(t) - Y(0)|}{\tilde{\tau}(\theta_1^n)} \\ & \leq \sup_{0 \leq t \leq \theta_1^{n+1}} \frac{|Y(t) - Y(0)|}{\tilde{\tau}(\theta_1^{n+1})} \frac{\tilde{\tau}(\theta_1^{n+1})}{\tilde{\tau}(\theta_1^n)} \leq \sqrt{3} (1 + \epsilon)^2, \end{aligned} \tag{3.24}$$

where the final inequality uses (3.22) and (3.23). This gives (3.21).

To obtain (3.22) let

$$\mathcal{I}(T) = \int_0^{1/2} \frac{\varphi(Tu)}{u(\log 1/u)^{1/2}} du. \tag{3.25}$$

We have

$$\tilde{\tau}(\theta_1^{n+1}) = \varphi(\theta_1^{n+1})(\log \log \theta_1^{n+1})^{1/2} + \frac{1}{\log 2} \mathcal{I}(\theta_1^{n+1}). \tag{3.26}$$

Note that

$$\log \log \theta_1^{n+1} = \log(n+1) + \log \log \theta_1 = \log \log \theta_1^n + O\left(\frac{1}{n}\right). \tag{3.27}$$

Also

$$\begin{aligned} \mathcal{I}(\theta_1^{n+1}) &= \int_0^{\theta_1^{1/2}} \frac{\varphi(\theta_1^n u)}{u(\log 1/u + \log \theta_1)^{1/2}} du \\ &\leq \int_0^{1/2} \frac{\varphi(\theta_1^n u)}{u(\log 1/u + \log \theta_1)^{1/2}} du + \int_{1/2}^{\theta_1^{1/2}} \frac{\varphi(\theta_1^n u)}{u(\log 1/u + \log \theta_1)^{1/2}} du. \end{aligned} \tag{3.28}$$

The last integral above

$$\leq \varphi(\theta_1^{n+1}/2) \int_{1/2}^{\theta_1^{1/2}} \frac{1}{u(\log 1/u)^{1/2}} du \leq 2\varphi(\theta_1^{n+1})(\log \theta_1)^{1/2}. \tag{3.29}$$

Combining (3.28) and (3.29) we have

$$\mathcal{I}(\theta_1^{n+1}) \leq \mathcal{I}(\theta_1^n) + 2\varphi(\theta_1^{n+1})(\log \theta_1)^{1/2}. \tag{3.30}$$

Using this and (3.26) and (3.27) and the fact that $\varphi(\theta_1^{n+1})/\varphi(\theta_1^n) \geq 1$ we get

$$\tilde{\tau}(\theta_1^{n+1}) \tag{3.31}$$

$$\begin{aligned} &\leq \varphi(\theta_1^{n+1}) \left(\log \log \theta_1^n + O\left(\frac{1}{n}\right) \right)^{1/2} + \frac{\mathcal{I}(\theta_1^n)}{\log 2} + \frac{2\varphi(\theta_1^{n+1})(\log \theta_1)^{1/2}}{\log 2} \\ &\leq \frac{\varphi(\theta_1^{n+1})}{\varphi(\theta_1^n)} \left(\varphi(\theta_1^n) (\log \log \theta_1^n)^{1/2} + \frac{\mathcal{I}(\theta_1^n)}{\log 2} \right. \\ &\quad \left. + \varphi(\theta_1^n) \left(\frac{2(\log \theta_1)^{1/2}}{\log 2} + O\left(\frac{1}{n^{1/2}}\right) \right) \right) \end{aligned} \tag{3.32}$$

$$= \frac{\varphi(\theta_1^{n+1})}{\varphi(\theta_1^n)} (\tilde{\tau}(\theta_1^n) + o(\tilde{\tau}(\theta_1^n))).$$

Or, equivalently,

$$\frac{\tilde{\tau}(\theta_1^{n+1})}{\tilde{\tau}(\theta_1^n)} \leq \frac{\varphi(\theta_1^{n+1})}{\varphi(\theta_1^n)} (1 + o(1_n)). \tag{3.33}$$

It follows from (3.10) when $V \geq V_0$ that we can choose $\theta_1 > 1$, sufficiently close to 1, so that

$$\frac{\varphi(\theta_1 V)}{\varphi(V)} \leq 1 + \epsilon/4 \quad \forall V \geq V_0. \tag{3.34}$$

We next choose n_1 so that $\theta_1^{n_1} \geq V_0$, and large enough so that for all $n \geq n_1$, $(1 + \epsilon/4)(1 + o(1_n)) \leq 1 + \epsilon$. (Here $(1 + o(1_n))$ is the expression in (3.33)). This completes the proof of (3.22) and hence of (3.21).

The statement in (3.12) follows similarly by taking θ less than 1. □

Remark 3.3. A proof of (3.12), the local modulus of continuity, is essentially given in [8, Chapter IV, Theorem 1.3]. But there are some differences with what is given here. Condition [8, (2.5.9)] is with regard to a different metric than $\sigma(s, t)$ but that doesn't matter since what is used in [8, Chapter IV, Theorem 1.3] is (3.7). (There is also the requirement that $\varphi(2t) \leq 2\varphi(t)$ in [8, Chapter IV, Theorem 1.3].) We don't actually prove

(3.12) here but only point out that it basically the same as the proof of (3.14). The result in (3.14) is not contained in [8]. Also contained in [8, Chapter IV, Theorem 1.3] is a uniform modulus of continuity of the type given in Theorem 3.8. We do not use it because it doesn't give the constant 1 on the right-hand side of (3.74).

Examples of the processes Y that we are studying are the processes $X_\alpha^{1/2}$ in Theorem 1.1. We see from (2.68) that (3.7) holds. Therefore, we can use Lemma 3.2.

In Theorem 3.5 we consider an important class of processes for which we can lower the upper bound in (3.12) so that it is best possible. It uses the next lemma which is an immediate consequence of (2.68).

Lemma 3.4. *Let $X_\alpha = \{X_\alpha(t), t \in [0, 1]\}$ be an α -permanental process with bounded kernel $u(s, t)$ and sigma function $\sigma(s, t)$. Then for any sequence $\{s_n, t_n\}$ in $(0, 1] \times (0, 1]$, such that $s_j \neq t_j$ for all $j \in \mathbb{N}$,*

$$\limsup_{n \rightarrow \infty} \frac{|X^{1/2}(t_n) - X^{1/2}(s_n)|}{\sigma(s_n, t_n)(\log n)^{1/2}} \leq 1 \quad a.s. \tag{3.35}$$

Theorem 3.5. *Let $X_\alpha = \{X_\alpha(t), t \in [0, 1]\}$ be an α -permanental process with kernel $u(s, t)$ and sigma function $\sigma(s, t)$ for which (3.8) and (3.9) hold for some function $\varphi(t)$ that is regularly varying at zero with positive index. Then,*

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|X_\alpha^{1/2}(t) - X_\alpha^{1/2}(0)|}{\varphi(h)(\log \log 1/h)^{1/2}} \leq 1 \quad a.s. \tag{3.36}$$

If $u(0, 0) = 0$, and if in addition to the conditions on φ above, $\varphi^2(h) = O(u(h, h))$, then

$$\limsup_{t \rightarrow 0} \frac{X_\alpha^{1/2}(t)}{(u(t, t) \log \log 1/t)^{1/2}} \leq 1 \quad a.s. \tag{3.37}$$

and

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{X_\alpha^{1/2}(t)}{(u^*(h, h) \log \log 1/h)^{1/2}} \leq 1 \quad a.s., \tag{3.38}$$

where u^* is defined in (1.16).

Proof We first prove (3.36). Let $\theta < 1$ and consider Lemma 3.1 applied to $Y_\alpha(t) = X_\alpha^{1/2}(\theta^{n+1} + t) - X_\alpha^{1/2}(\theta^{n+1})$, with $S_n = \theta^n - \theta^{n+1}$. We have

$$\begin{aligned} \sup_{t \in [0, \theta^n - \theta^{n+1}]} |Y_\alpha^{1/2}(t) - Y_\alpha^{1/2}(0)| & \tag{3.39} \\ &= \sup_{t \in [0, \theta^n - \theta^{n+1}]} |X_\alpha^{1/2}(\theta^{n+1} + t) - X_\alpha^{1/2}(\theta^{n+1})| \\ &= \sup_{t \in [\theta^{n+1}, \theta^n]} |X_\alpha^{1/2}(t) - X_\alpha^{1/2}(\theta^{n+1})|. \end{aligned}$$

Since it follows from (2.68) that (3.7) holds we see by Lemma 3.1 that

$$\begin{aligned} P \left(\sup_{t \in [\theta^{n+1}, \theta^n]} |X_\alpha^{1/2}(t) - X_\alpha^{1/2}(\theta^{n+1})| > a\varphi(S_n) + \sum_{p=1}^{\infty} \kappa(p)\varphi(S_n/n(p)) \right) \\ \leq n^2 F(a) + \sum_{p=1}^{\infty} n^2(p) F(\kappa(p)). \end{aligned} \tag{3.40}$$

If we take $a = ((3 + \epsilon) \log n)^{1/2}$ and $\kappa(p) = (3 \log n(p))^{1/2}$, as in the proof of Lemma 3.2, we see that for all n sufficiently large, the second line of (3.40) is a term of a converging

sequence. Consequently, by the calculations in (3.16)–(3.19), the event

$$\sup_{t \in [\theta^{n+1}, \theta^n]} |X_\alpha^{1/2}(t) - X_\alpha^{1/2}(\theta^{n+1})| > \left(a\varphi(S_n) + \frac{\sqrt{3}}{\log 2} \int_0^{1/n} \frac{\varphi(S_n u)}{u(\log 1/u)^{1/2}} du \right) \tag{3.41}$$

infinitely often, has probability zero.

Note that by the condition that $\varphi(t)$ is regularly varying at zero with positive index

$$\frac{\varphi(S_n u)}{\varphi(S_n)} \leq C u^\beta \quad \forall u \in [0, 1/2] \tag{3.42}$$

for some constant C and all n sufficiently large. Therefore,

$$\int_0^{1/n} \frac{\varphi(S_n u)}{u(\log 1/u)^{1/2}} du \leq C\varphi(S_n) \int_0^{1/2} \frac{u^{\beta-1}}{(\log 1/u)^{1/2}} du \leq C'\varphi(S_n) \tag{3.43}$$

for some constant C' and all n sufficiently large. This shows that the integral in (3.41) is $o((\log n)^{1/2}\varphi(S_n))$ as $n \rightarrow \infty$.

Using the regular variation hypothesis again we see that

$$\varphi(S_n) \leq 2(\theta^{-1} - 1)^\beta \varphi(\theta^{n+1}), \tag{3.44}$$

for some $\beta > 0$ and all n sufficiently large. Therefore, for any $\epsilon > 0$, by taking $1 - \theta$ sufficiently small, the right-hand side of (3.41)

$$< \epsilon \varphi(\theta^{n+1})(\log n)^{1/2}, \tag{3.45}$$

for all n sufficiently large. It follows from this that for all $\epsilon > 0$ we can choose $\theta < 1$ such that the probability that

$$\sup_{t \in [\theta^{n+1}, \theta^n]} |X_\alpha^{1/2}(t) - X_\alpha^{1/2}(\theta^{n+1})| > 2\epsilon \varphi(\theta^{n+1})(\log(n+1))^{1/2} \tag{3.46}$$

infinitely often, is zero, for all $\epsilon > 0$.

By Lemma 3.4, since $\sigma(s, t) \leq \varphi(|t - s|)$,

$$\limsup_{n \rightarrow \infty} \frac{|X_\alpha^{1/2}(\theta^{n+1}) - X_\alpha^{1/2}(0)|}{\varphi(\theta^{n+1})(\log(n+1))^{1/2}} \leq 1 \quad a.s. \tag{3.47}$$

Combining this with the statement in the sentence containing (3.46), we get

$$\limsup_{n \rightarrow \infty} \frac{\sup_{t \in [\theta^{n+1}, \theta^n]} |X_\alpha^{1/2}(t) - X_\alpha^{1/2}(0)|}{\varphi(\theta^{n+1})(\log \log \theta^{-(n+1)})^{1/2}} \leq 1 + 2\epsilon \quad a.s. \tag{3.48}$$

Since $\varphi(t)$ is asymptotic to a monotonically increasing function near zero we get,

$$\limsup_{n \rightarrow \infty} \sup_{t \in [\theta^{n+1}, \theta^n]} \frac{|X_\alpha^{1/2}(t) - X_\alpha^{1/2}(0)|}{\varphi(t)(\log \log 1/t)^{1/2}} \leq 1 + 2\epsilon \quad a.s. \tag{3.49}$$

and since this holds for any $\epsilon > 0$ we get (3.36).

By (3.46) and the fact that $\varphi(\theta^{n+1}) \leq C u^{1/2}(\theta^{n+1}, \theta^{n+1})$,

$$\limsup_{n \rightarrow \infty} \sup_{t \in [\theta^{n+1}, \theta^n]} \frac{|X_\alpha^{1/2}(t) - X_\alpha^{1/2}(\theta^{n+1})|}{(u^*(\theta^{n+1}, \theta^{n+1}) \log n)^{1/2}} \leq 2\epsilon C, \quad a.s. \tag{3.50}$$

for all $\epsilon > 0$. By Lemma 5.1

$$\limsup_{n \rightarrow \infty} \frac{X_\alpha^{1/2}(\theta^{n+1})}{u(\theta^{n+1}, \theta^{n+1})^{1/2}(\log n)^{1/2}} \leq 1 \quad a.s. \tag{3.51}$$

Hence

$$\limsup_{n \rightarrow \infty} \sup_{t \in [\theta^{n+1}, \theta^n]} \frac{X_\alpha^{1/2}(t)}{(u^*(\theta^{n+1}, \theta^{n+1}) \log n)^{1/2}} \leq 1 + 2\epsilon C \quad a.s., \tag{3.52}$$

which gives (3.38).

The proof of (3.37) is more subtle. We use the following lemma.

Lemma 3.6. *Let $X = \{X(t), t \in \mathcal{T}\}$ be an α -permanental process with kernel $u(s, t)$ and sigma function $\sigma(s, t)$ and such that $u(t, t) > 0$ for all $t \in \mathcal{T}$. Let*

$$\tilde{X}(t) = \frac{X(t)}{u(t, t)}. \tag{3.53}$$

Then $\tilde{X} = \{\tilde{X}(t), t \in \mathcal{T}\}$ is an α -permanental process with kernel

$$\tilde{u}(s, t) = \frac{u(s, t)}{(u(s, s)u(t, t))^{1/2}}. \tag{3.54}$$

Let $\tilde{\sigma}(s, t)$ be the sigma function of $(\tilde{X}_s, \tilde{X}_t)$. Then

$$\tilde{\sigma}^2(s, t) \leq \frac{\sigma^2(s, t)}{(u(s, s)u(t, t))^{1/2}}. \tag{3.55}$$

Proof To verify (3.54) note that for any t_1, \dots, t_n ,

$$E \left(e^{-\sum_{i=1}^n \lambda_i X_{t_i}} \right) = \frac{1}{|I + U\Lambda|^\alpha}, \tag{3.56}$$

where U is the $n \times n$ matrix with entries $u(t_i, t_j)$ and Λ is a diagonal matrix with entries λ_i , $1 \leq i \leq n$. Let D be is the diagonal matrix with entries $u(t_i, t_i)$, $1 \leq i \leq n$. Then

$$\begin{aligned} E \left(e^{-\sum_{i=1}^n \lambda_i \tilde{X}_{t_i}} \right) &= E \left(e^{-\sum_{i=1}^n \lambda_i X_{t_i} / u(t_i, t_i)} \right) \\ &= \frac{1}{|I + U\Lambda D^{-1}|^\alpha} = \frac{1}{|I + D^{-1/2} U D^{-1/2} \Lambda|^\alpha}, \end{aligned} \tag{3.57}$$

which gives (3.54).

Note that $\tilde{u}(s, s) = \tilde{u}(t, t) = 1$. By the inequality between geometric and arithmetic mean,

$$\begin{aligned} \tilde{\sigma}^2(s, t) &= 2 - \frac{2(u(s, t)u(t, s))^{1/2}}{(u(s, s)u(t, t))^{1/2}} \\ &= \frac{2(u(s, s)u(t, t))^{1/2} - 2(u(s, t)u(t, s))^{1/2}}{(u(s, s)u(t, t))^{1/2}} \\ &\leq \frac{u(s, s) + u(t, t) - 2(u(s, t)u(t, s))^{1/2}}{(u(s, s)u(t, t))^{1/2}}, \end{aligned} \tag{3.58}$$

which is (3.55). □

Proof of Theorem 3.5 continued To prove (3.37) we consider $\tilde{X}_\alpha(t)$ and $\tilde{\sigma}(s, t)$ as defined in (3.53) and (3.55) for $t \in (0, 1]$. Let $\theta < 1$. Then by Lemma 3.6, for all $\theta^{n+1} \leq s, t \leq \theta^n$,

$$\tilde{\sigma}^2(s, t) \leq \frac{\varphi^2(|t - s|)}{u^*(\theta^{n+1}, \theta^{n+1})} := \tilde{\varphi}_n^2(|t - s|), \tag{3.59}$$

since $u^*(\theta^{n+1}, \theta^{n+1}) = \inf_{h \in [\theta^{n+1}, \theta^n]} u^*(h, h)$. Since φ is regularly varying with index, say $\beta > 0$ we see that for all $\epsilon > 0$, there exists an n'_0 such that for all $n \geq n'_0$

$$\varphi(\theta^n - \theta^{n+1}) \leq (1 + \epsilon) \left(\frac{1 - \theta}{\theta}\right)^\beta \varphi(\theta^{n+1}). \tag{3.60}$$

Also since φ regularly varying it is asymptotic to an increasing regularly varying function with index β . To simplify the proof let us simply take φ to be increasing. By hypothesis $\varphi^2(h) \leq Cu^*(h, h)$ for all h sufficiently small. Therefore for all $h \in [\theta^{n+1}, \theta^n]$,

$$\varphi^2(\theta^{n+1}) \leq \varphi^2(h) \leq Cu^*(h, h), \tag{3.61}$$

which implies that $\varphi^2(\theta^{n+1}) \leq Cu^*(\theta^{n+1}, \theta^{n+1})$. Using (3.60) we see that for all $\epsilon' > 0$ and θ sufficiently close to 1, and $\theta^{n+1} \leq s, t \leq \theta^n$,

$$\tilde{\varphi}_n^2(|t - s|) \leq \epsilon' \tag{3.62}$$

It follows from this that the right-hand side of (3.41) applied to \tilde{X}_α with dominating function $\tilde{\varphi}_n$, is less than $\epsilon'(\log n)^{1/2}$ for all n sufficiently large. Consequently, the probability that

$$\sup_{t \in [\theta^{n+1}, \theta^n]} |\tilde{X}_\alpha^{1/2}(t) - \tilde{X}_\alpha^{1/2}(\theta^{n+1})| > \epsilon'(\log n)^{1/2} \tag{3.63}$$

infinitely often, is zero, for all $\epsilon' > 0$.

It follows from Lemma 5.1 that

$$\limsup_{n \rightarrow \infty} \frac{\tilde{X}_\alpha^{1/2}(\theta^{n+1})}{(\log n)^{1/2}} \leq 1 \quad a.s. \tag{3.64}$$

Combining (3.63) and (3.64) we get

$$\sup_{t \in [\theta^{n+1}, \theta^n]} \tilde{X}_\alpha^{1/2}(t) \leq (1 + \epsilon')(\log n)^{1/2}, \tag{3.65}$$

almost surely as $n \rightarrow \infty$. This gives

$$\limsup_{t \rightarrow 0} \frac{X_\alpha^{1/2}(t)}{(u(t, t) \log \log 1/t)^{1/2}} \leq 1 \quad a.s. \tag{3.66}$$

which is (3.37). □

The next lemma gives relationships between the sigma function of a kernel and its majoring function φ .

Lemma 3.7. *Let X be an α -permanental process with kernel $u(s, t)$ and suppose that $u(0, 0) = 0$. Then*

$$\sigma^2(0, t) = \sigma^2(t, 0) = u(t, t). \tag{3.67}$$

If, in addition, $\sigma^2(s, t)$ is a function of $|t - s|$ then, necessarily,

$$\sigma^2(s, t) = u(|t - s|, |t - s|). \tag{3.68}$$

In general, if φ satisfies (3.8)

$$\varphi^2(|t - s|) \geq (u^{1/2}(t, t) - u^{1/2}(s, s))^2. \tag{3.69}$$

Therefore, if $\lim_{t \rightarrow \infty} u(t, t) = \infty$, necessarily

$$\lim_{t \rightarrow \infty} \frac{\varphi^2(t)}{u(t, t)} \geq 1. \tag{3.70}$$

Proof Note that for any α -permanental process X

$$E[(X(s) - E(X(s)))(X(t) - E(X(t)))] = \alpha u(s, t)u(t, s), \tag{3.71}$$

[20, p. 135]. Clearly, this is equal to 0 when $s = 0$ since $u(0, 0) = 0$ implies that $X(0) = 0$. This shows that $u(t, 0)u(0, t) = 0$. This and $u(0, 0) = 0$ gives (3.67).

If, in addition, $\sigma^2(s, t)$ is a function of $|t - s|$, then by (3.67), when $t > s$,

$$\sigma^2(s, t) = \sigma^2(0, t - s) = u(t - s, t - s), \tag{3.72}$$

which gives (3.68).

The statement in (3.69) follows because

$$u(s, t)u(t, s) \leq u(s, s)u(t, t), \tag{3.73}$$

which, itself, follows from (3.71) and the Schwartz Inequality. □

The next theorem is an analogue of Theorem 3.5 for the uniform modulus of continuity.

Theorem 3.8. *Let $X_\alpha = \{X_\alpha(t), t \in [0, 1]\}$ be an α -permanental process with kernel $u(s, t)$ and sigma function $\sigma(s, t)$ for which (3.8) and (3.9) hold. Assume furthermore, that $\varphi(t)$ is regularly varying at zero with positive index. Then*

$$\limsup_{h \rightarrow 0} \sup_{\substack{|s-t| \leq h \\ s, t \in [0, 1]}} \frac{|X_\alpha^{1/2}(s) - X_\alpha^{1/2}(t)|}{\varphi(h)(\log 1/h)^{1/2}} \leq 1 \quad a.s. \tag{3.74}$$

Proof Theorem 1.19 asserts that X_α has subgaussian increments. Then (3.74) follows from [14, Theorem 4]. In this reference the theorem is proved for Gaussian processes with stationary increments but it only uses the estimate in (1.54), along with the fact that φ is greater than the L^2 estimate for the increments of the Gaussian process. The difference of the factor 2 is explained by the observation in the line prior to (1.55).

(Of course we don't have (1.54) for X_α but only (1.52). It is easy to see that the additional powers of λ do not affect the estimates used in [14, Theorem 4].)

There is another item in the proof of [14, Theorem 4] that needs explanation. In [14, Theorem 4] which deals with the Gaussian process $\{G(t), t \in [0, 1]\}$, with stationary increments

$$\sigma(h) = \left(E (G(t+h) - G(t))^2 \right)^{1/2} \tag{3.75}$$

is written as $\sigma(h) = \exp(-g(\log 1/h))$ and it is required that $1/g'(\log 1/h) = o(\log 1/h)$. Note that when $\sigma(h) = Ch^\alpha$, for some constant C , $1/g'(\log 1/h) = 1/\alpha$, much weaker than what is allowed. However it isn't necessary to require that σ is differentiable. All the estimates in the proof of [14, Theorem 4] that use the condition $1/g'(\log 1/h) = o(\log 1/h)$ follow easily from the condition that φ is regularly varying at zero with positive index. For example when φ is regularly varying at zero with index β , instead of [14, (2.16)], we have

$$\frac{1}{\varphi(t_k)} \int_0^{t_k} \frac{\varphi(u)}{u} du = \int_0^1 \frac{\varphi(t_k u)}{\varphi(t_k)} \frac{du}{u} \leq (1 + \epsilon) \int_0^1 \frac{u^\beta}{u} du = \frac{1 + \epsilon}{\beta}, \tag{3.76}$$

as $t_k \rightarrow 0$, for all $\epsilon > 0$.

It is also required that $\sigma(h)$ is concave, but wherever this is used it is easy to get the same estimates when φ is regularly varying at zero with index $\beta \leq 1$, which is always the case. □

Our interest in $\{X_\alpha^{1/2}(t) - X_\alpha^{1/2}(s); s, t \in R^+\}$ is primarily to use the results obtained to study the behavior of $\{X_\alpha(t) - X_\alpha(s); s, t \in R^+\}$. The next lemma does this.

Lemma 3.9. Assume that

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|X^{1/2}(t) - X^{1/2}(0)|}{\omega(h)} \leq 1 \quad a.s. \tag{3.77}$$

for some function ω with $\lim_{h \rightarrow 0} \omega(h) = 0$. Then

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|X(t) - X(0)|}{\omega(h)} \leq 2X^{1/2}(0) \quad a.s. \tag{3.78}$$

and if $X(0) = 0$,

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{X(t)}{\omega^2(h)} \leq 1 \quad a.s. \tag{3.79}$$

Furthermore, if

$$\limsup_{h \rightarrow 0} \sup_{\substack{|s-t| \leq h \\ s, t \in [0,1]}} \frac{|X_\alpha^{1/2}(s) - X_\alpha^{1/2}(t)|}{\rho(h)} \leq 1 \quad a.s., \tag{3.80}$$

then

$$\limsup_{h \rightarrow 0} \sup_{\substack{|s-t| \leq h \\ s, t \in [0,1]}} \frac{|X_\alpha(s) - X_\alpha(t)|}{\rho(h)} \leq 2 \sup_{t \in [0,1]} X^{1/2}(t) \quad a.s., \tag{3.81}$$

Proof The statement in (3.79) is trivial. For (3.78) we note that

$$|X(t) - X(0)| \leq |X^{1/2}(t) - X^{1/2}(0)| \left(|X^{1/2}(t) - X^{1/2}(0)| + 2X^{1/2}(0) \right). \tag{3.82}$$

Therefore, the left hand side of (3.78) is bounded by

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|X^{1/2}(t) - X^{1/2}(0)|}{\omega(h)} \left(|X^{1/2}(t) - X^{1/2}(0)| + 2X^{1/2}(0) \right). \tag{3.83}$$

However, (3.77) implies that $X(t)$ is continuous at $t = 0$, almost surely. Therefore

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} |X^{1/2}(t) - X^{1/2}(0)| = 0 \quad a.s. \tag{3.84}$$

Using this and (3.77) we get (3.78).

The result for the uniform modulus follows similarly since

$$|X(t) - X(s)| \leq |X^{1/2}(t) - X^{1/2}(s)| |X^{1/2}(t) + X^{1/2}(s)|. \tag{3.85}$$

□

Proofs of Theorems 1.2 and 1.3 The proofs follow immediately from Theorems 3.5 and 3.8 and Lemma 3.9.

4 Upper bounds for the local moduli of continuity and rate of growth of permanental processes, II

The next corollary exhibits different upper bounds for the local modulus of continuity that are larger than the ones that hold under the hypotheses of Theorem 1.2 .

Corollary 4.1. Let $X = \{X(t), t \in [0, 1]\}$ be an α -permanental process with kernel $u(s, t)$ and with sigma function $\sigma(s, t)$ for which (3.8) and (3.9) hold and for which (3.10) also holds uniformly in $V \leq V_0$ for some $V_0 < \infty$. Then

$$\limsup_{h \rightarrow 0} \sup_{0 \leq t \leq h} \frac{|X(t) - X(0)|}{\Theta(h)} \leq \frac{4\sqrt{2}}{\log 2} X^{1/2}(0) \quad a.s. \tag{4.1}$$

where

$$\Theta(h) = \int_0^{h^2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du + \varphi(h)(\log 1/h)^{1/2}. \tag{4.2}$$

The proof follows from Lemma 3.2 applied to $Y(t) = X^{1/2}(t)$, Lemma 3.9 and the next lemma.

Lemma 4.2.

$$\int_0^{1/2} \frac{\varphi(hu)}{u(\log 1/u)^{1/2}} du \leq 2\Theta(h) + o(\Theta(h)). \tag{4.3}$$

Proof We have

$$\int_0^{1/2} \frac{\varphi(hu)}{u(\log 1/u)^{1/2}} du = \int_0^h \frac{\varphi(u/2)}{u(\log 1/u - \log 1/(2h))^{1/2}} du \tag{4.4}$$

Note that

$$\int_0^{h^2} \frac{\varphi(u/2)}{u(\log 1/u - \log 1/(2h))^{1/2}} du \leq \sqrt{2} \int_0^{h^2} \frac{\varphi(u/2)}{u(\log 1/u)^{1/2}} du, \tag{4.5}$$

since $\log 1/(2h) \leq (1/2) \log 1/u$ for $u \in [0, h^2]$. Furthermore,

$$\int_{h^2}^h \frac{\varphi(u/2)}{u(\log 1/u - \log 1/(2h))^{1/2}} du \leq \varphi(h) \int_{h^2}^h \frac{1}{u(\log 1/u - \log 1/(2h))^{1/2}} du. \tag{4.6}$$

The integrand of the last integral above is the derivative of $-2(\log 1/u - \log 1/(2h))^{1/2}$. Consequently,

$$\int_{h^2}^h \frac{1}{u(\log 1/u - \log 1/(2h))^{1/2}} du = 2(\log 1/h)^{1/2} + o((\log 1/h)^{1/2}) \text{ as } h \rightarrow 0. \tag{4.7}$$

Combining these relationships we get (4.3). □

It is interesting to note that for certain functions φ , except for a multiplicative factor, we can reverse the inequality in (4.3).

Lemma 4.3. *Suppose that*

$$\limsup_{h \rightarrow 0} \frac{\varphi(h^2)}{\varphi(h)} = C. \tag{4.8}$$

Then

$$\int_0^{1/2} \frac{\varphi(hu)}{u(\log 1/u)^{1/2}} du \geq C\Theta(h) + o(\Theta(h)). \tag{4.9}$$

Proof We have

$$\begin{aligned} \int_0^{1/2} \frac{\varphi(hu)}{u(\log 1/u)^{1/2}} du &\geq \int_0^{h/2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du \\ &\geq \int_0^{h^2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du + \int_{h^2}^{h/2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \int_{h^2}^{h/2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du &\geq \varphi(h^2) \int_{h^2}^{h/2} \frac{1}{u(\log 1/u)^{1/2}} du \\ &= \varphi(h^2)((\log 1/h)^{1/2} + o(h)) \\ &\sim C\varphi(h)((\log 1/h)^{1/2} + o(h)). \end{aligned} \tag{4.11}$$

Consequently, we get (4.9). □

Example 4.4. Let $\gamma > 1/2$. Then if

$$\varphi(h) = \frac{1}{(\log 1/h)^\gamma}, \quad \Theta(h) = \frac{2^{(3/2)-\gamma} + 2\gamma - 1}{2\gamma - 1} \varphi(h)(\log 1/h)^{1/2}. \quad (4.12)$$

Let $\beta > -1$. Then if

$$\varphi(h) = \frac{(\log \log 1/h)^\beta}{(\log 1/h)^{1/2}}, \quad \Theta(h) = \left(\frac{1}{1+\beta} + o(h) \right) \varphi(h)(\log 1/h)^{1/2} \log \log 1/h. \quad (4.13)$$

Note that in both these cases $C = 1$ in (4.8).

For more examples see [15, Lemma 7.6.5 and Example 7.6.6].

We now examine the relationship between $\sigma(s, t)$ and the L_2 metric for Gaussian processes. Let

$$\rho^2(s, t) := u(s, s) + u(t, t) - (u(s, t) + u(t, s)). \quad (4.14)$$

Although we don't require that $u(s, t)$ is symmetric, when $u(s, t)$ is symmetric $\rho(s, t) = \sigma(s, t)$. In general we get the next lemma.

Lemma 4.5. Let $\sigma(s, t)$ be as defined in (1.6). Then

$$\sigma^2(s, t) = \rho^2(s, t) + (u^{1/2}(s, t) - u^{1/2}(t, s))^2. \quad (4.15)$$

In addition, when $u(s, t) \vee u(t, s) \leq u(s, s) \wedge u(t, t)$ for all $s, t \in T$,

$$\sigma(s, t) \leq \sqrt{2}\rho(s, t). \quad (4.16)$$

Proof We have

$$\begin{aligned} \sigma^2(s, t) &= \rho^2(s, t) + u(s, t) + u(t, s) - 2(u(s, t)u(t, s))^{1/2} \\ &= \rho^2(s, t) + (u^{1/2}(s, t) - u^{1/2}(t, s))^2. \end{aligned} \quad (4.17)$$

The inequality in (4.16) is given in [16, Lemma 5.5]. □

Corollary 4.6. If

$$u(s, t) = v(s, t) + h(t) \quad (4.18)$$

and v is symmetric

$$\sigma^2(s, t) \leq v(s, s) + v(t, t) - 2v(s, t) + |h(s) - h(t)|. \quad (4.19)$$

If, in addition $\inf_{s,t \in I} u(s, t) \geq \delta$, for some interval I , then for all $s, t \in I$,

$$\sigma^2(s, t) \leq v(s, s) + v(t, t) - 2v(s, t) + \frac{|h(s) - h(t)|^2}{4\delta}. \quad (4.20)$$

Proof The inequality in (4.19) follows immediately from (4.15). To obtain (4.20) note that for $a < b$

$$b^{1/2} - a^{1/2} = \int_a^b \frac{1}{2u^{1/2}} du \leq \frac{b-a}{2a^{1/2}}. \quad (4.21)$$

Consider $u(s, t)$ in (4.18) and suppose that $h(t) > h(s)$. Then by (4.21), for $s, t \in I$

$$u^{1/2}(s, t) - u^{1/2}(t, s) \leq \frac{u(s, t) - u(t, s)}{2\delta^{1/2}} \leq \frac{h(t) - h(s)}{2\delta^{1/2}}. \quad (4.22)$$

Using this and (4.15) and the fact that v is symmetric, we get (4.20). □

Remark 4.7. The inequality in (4.20) may be smaller than the one in (4.19) even when $u(s, t)$ has the form of (4.18). For example, suppose

$$u(s, t) = e^{-\lambda|t-s|} + e^{-rt} \quad r, \lambda > 0. \tag{4.23}$$

Obviously

$$|e^{-rt} - e^{-rs}| \sim r|t - s| \quad \text{as } s, t \rightarrow 0. \tag{4.24}$$

Consequently, it follows from (4.20) that $\sigma^2(s, t)$ is bounded by $\lambda|t - s|$ as $s, t \rightarrow 0$, whereas (4.19) only gives that is bounded by $(\lambda + r)|t - s|$ as $s, t \rightarrow 0$.

Proof of upper bounds in Theorem 1.8 We show in [11, Section 5] that

$$u_{T_0;\gamma,\beta}(x, y) = R(x, y)_{\gamma,\beta} + H_{\gamma,\beta}(x, y) \tag{4.25}$$

where $R_{\gamma,\beta}$ is symmetric and $H_{\gamma,\beta}$ is antisymmetric. Explicitly,

$$R_{\gamma,\beta}(x, y) = C_{\gamma,\beta} (|x|^\gamma + |y|^\gamma - |x - y|^\gamma), \tag{4.26}$$

$$H_{\gamma,\beta}(x, y) = \beta C_{\gamma,\beta} (\text{sign}(x)|x|^\gamma - \text{sign}(y)|y|^\gamma - \text{sign}(x - y)|x - y|^\gamma), \tag{4.27}$$

and

$$C_{\gamma,\beta} = \frac{-\sin((\gamma + 1)\frac{\pi}{2})\Gamma(-\gamma)}{\pi(1 + \beta^2 \tan^2((\gamma + 1)\pi/2))} > 0. \tag{4.28}$$

By Lemma 4.5 the sigma function for $Y_{\alpha;\gamma,\beta}$, which we denote by $\sigma_{T_0;\gamma,\beta}$, satisfies

$$\begin{aligned} \sigma_{T_0;\gamma,\beta}^2(x, y) &\leq u_{T_0;\gamma,\beta}(x, x) + u_{T_0;\gamma,\beta}(y, y) - (u_{T_0;\gamma,\beta}(x, y) + u_{T_0;\gamma,\beta}(x, y)) \\ &\quad + |u_{T_0;\gamma,\beta}(x, y) - u_{T_0;\gamma,\beta}(y, x)| \\ &= R_{\gamma,\beta}(x, x) + R_{\gamma,\beta}(y, y) - 2R_{\gamma,\beta}(x, y) + 2|H_{\gamma,\beta}(x, y)|, \end{aligned} \tag{4.29}$$

where, for the last equation we use the facts that $R_{\gamma,\beta}$ is symmetric, $H_{\gamma,\beta}$ is antisymmetric and $H_{\gamma,\beta}(x, x) \equiv 0$. It is easy to see that

$$|H(x, y)| \leq |\beta|C_{\gamma,\beta}|x - y|^\gamma. \tag{4.30}$$

Using this and (4.26), we get

$$\sigma_{T_0;\gamma,\beta}(x, y) \leq (2(1 + |\beta|)C_{\gamma,\beta})^{1/2}|x - y|^{\gamma/2} := \varphi(|x - y|). \tag{4.31}$$

Since φ is regularly varying at zero with positive index we see that the upper bounds in Theorem 1.8 follows from Theorems 1.2, 1.3 and 1.5. \square

We refer to the permanental process with kernel $u_{T_0;\gamma,\beta}$ as FBMQ $^{\gamma,\beta}$. We use this notation because when $\beta = 0$, $u_{T_0;\gamma,\beta}$ is the covariance of fractional Brownian motion of index γ , (i.e. FBM). We add the Q , for quadratic, to denote the square of this process, as one does in the designation of the squared Bessel processes, (BESQ).

Proof of Theorem 1.10 It follows from [11, Lemma 5.2] that

$$\begin{aligned} u_{\rho;\gamma,\beta}(x, y) &= \frac{1}{\pi} \int_0^\infty \frac{\cos \lambda(x - y)\mathcal{R}e(\rho + \psi_{\gamma,\beta}(\lambda))}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda \\ &\quad + \frac{1}{\pi} \int_0^\infty \frac{\sin \lambda(x - y)\mathcal{I}m \psi_{\gamma,\beta}(\lambda)}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda. \end{aligned} \tag{4.32}$$

Consequently, as in (4.29)

$$\begin{aligned} \sigma_{\rho;\gamma,\beta}^2(x, y) &\leq \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos \lambda(x - y))\mathcal{R}e(\rho + \psi_{\gamma,\beta}(\lambda))}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda \\ &\quad + \frac{2}{\pi} \left| \int_0^\infty \frac{\sin \lambda(x - y)\mathcal{I}m \psi_{\gamma,\beta}(\lambda)}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda \right|. \end{aligned} \tag{4.33}$$

We write

$$\begin{aligned} & \int_0^\infty \frac{(1 - \cos \lambda(x - y)) \operatorname{Re}(\rho + \psi_{\gamma,\beta}(\lambda))}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda \\ & \leq \int_0^\infty \frac{(1 - \cos \lambda(x - y)) \operatorname{Re} \psi_{\gamma,\beta}(\lambda)}{|\psi_{\gamma,\beta}(\lambda)|^2} d\lambda + 2\rho \int_0^\infty \frac{\sin^2 \frac{\lambda(x-y)}{2}}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda. \end{aligned} \tag{4.34}$$

It follows from [11, (5.39)] that the first integral to the right of the inequality sign is equal to $\pi C_{\gamma,\beta} |x - y|^\gamma$. The second integral to the right of the inequality sign is bounded by

$$\frac{2}{\rho} \int_0^1 \sin^2 \frac{\lambda\theta}{2} d\lambda + 2\rho \int_1^{1/\theta} \frac{\sin^2 \frac{\lambda\theta}{2}}{|\psi_{\gamma,\beta}(\lambda)|^2} d\lambda + 2\rho \int_{1/\theta}^\infty \frac{1}{|\psi_{\gamma,\beta}(\lambda)|^2} d\lambda, \tag{4.35}$$

where $\theta = |x - y|$. The first of these integrals is $O(\theta^2)$ as $\theta \rightarrow 0$. The third is $O(\theta^{2\gamma+1})$ as $\theta \rightarrow 0$. The second is $O(\theta^2)$ if $\gamma > 1/2$, $O(\theta^2) \log 1/\theta$ if $\gamma = 1/2$ and $O(\theta^{1+2\gamma}) \log 1/\theta$ if $\gamma < 1/2$, all as $\theta \rightarrow 0$. Thus we see that the first integral in (4.33) is bounded by $2C_{\gamma,\beta} |x - y|^\gamma$.

We now consider the last integral in (4.33)

$$\begin{aligned} & \int_0^\infty \frac{\sin \lambda(x - y) \operatorname{Im} \psi_{\gamma,\beta}(\lambda)}{|\rho + \psi_{\gamma,\beta}(\lambda)|^2} d\lambda \\ & = \int_0^\infty \frac{\sin \lambda(x - y) \operatorname{Im} \psi_{\gamma,\beta}(\lambda)}{|\psi_{\gamma,\beta}(\lambda)|^2} d\lambda + \int_0^\infty \frac{\sin \lambda(x - y) \operatorname{Im} \psi_{\gamma,\beta}(\lambda)}{|\Psi_{\gamma,\beta}(\lambda)|^2} d\lambda \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} \frac{1}{|\Psi_{\gamma,\beta}(\lambda)|^2} & = \frac{|\rho + \psi_{\gamma,\beta}(\lambda)|^2 - |\psi_{\gamma,\beta}(\lambda)|^2}{|\psi_{\gamma,\beta}(\lambda)|^2 |\rho + \psi_{\gamma,\beta}(\lambda)|^2} \\ & = \frac{(\rho + \operatorname{Re} \psi_{\gamma,\beta}(\lambda))^2 - (\operatorname{Re} \psi_{\gamma,\beta}(\lambda))^2}{|\psi_{\gamma,\beta}(\lambda)|^2 |\rho + \psi_{\gamma,\beta}(\lambda)|^2} \\ & \leq \frac{2\rho}{|\psi_{\gamma,\beta}(\lambda)| |\rho + \psi_{\gamma,\beta}(\lambda)|^2}. \end{aligned} \tag{4.37}$$

It follows from [11, (5.40)] that the first integral to the right of the equal sign in (4.36) is equal to $\pi\beta \operatorname{sign}(x - y) C_{\gamma,\beta} |x - y|^\gamma$. We now show that the second integral is little o of this. Using (4.37) we see that the last integral in (4.36) is bounded by 2ρ times

$$\int_0^\infty \frac{|\sin \lambda\theta|}{|\rho + \Psi_{\gamma,\beta}(\lambda)|^2} d\lambda \leq \frac{\theta}{2\rho^2} + \int_1^{1/\theta} \frac{\lambda\theta}{|\Psi_{\gamma,\beta}(\lambda)|^2} d\lambda + \int_{1/\theta}^\infty \frac{1}{|\Psi_{\gamma,\beta}(\lambda)|^2} d\lambda \tag{4.38}$$

It is easy to see that the first integral to the right of the equal sign in (4.38) is $O(\theta)$ as $\theta \rightarrow 0$ and the second is $O(\theta^{1+2\gamma})$ as $\theta \rightarrow 0$. Therefore, the absolute value of the second integral in (4.33) is bounded by $2|\beta| C_{\gamma,\beta} |x - y|^\gamma$.

Using the bounds for the last two integrals in (4.33) we see that

$$\sigma_{\rho;\gamma,\beta}(x, y) \leq (2(1 + |\beta|) C_{\gamma,\beta})^{1/2} |x - y|^{\gamma/2} := \varphi(|x - y|), \tag{4.39}$$

the same as in (4.31). Since φ is regularly varying at zero with positive index we see that the upper bounds in Theorem 1.8 follows from Theorems 1.2, 1.3 and 1.5. \square

Proof of Theorem 1.13 The α -permanental process $\overline{X}_{\alpha,f}$ has kernel $\widehat{u}_f(s, t)$. We see from (4.20) in Corollary 4.6 that for $s, t \in [0, \delta/\lambda]$, the sigma function of $\overline{X}_{\alpha,f}$ satisfies

$$\sigma_f^2(s, t) \leq \left(2(1 - e^{-\lambda|s-t|})\right) + \frac{|f(t) - f(s)|^2}{4(1 - \delta)} \tag{4.40}$$

for all δ sufficiently small. Using this and the fact that $f \in C^2$ implies that for $t > s$, $|f(t) - f(s)| \leq f'(t)|t - s|$, we see that

$$\sigma_f^2(s, t) \sim 2\lambda|t - s|, \quad \text{as } s, t \rightarrow 0. \tag{4.41}$$

Therefore, Theorem 1.13 follows from Theorems 1.2 and 1.3. □

5 Rate of growth of permanental processes at infinity

We begin by considering another important class of processes for which we can decrease the upper bounds that can be obtained by (3.14). First we need some preliminary results.

Let $X = \{X(t), t \in R^+\}$ be an α -permanental process with kernel $u(s, t)$ and sigma function $\sigma(s, t)$. Then by [18, Lemma 3.1], for $\lambda > 2(\alpha - 1) \vee 0$,

$$P(X(t) \geq u(t, t)\lambda) \leq \frac{2\lambda^{\alpha-1}e^{-\lambda}}{\Gamma(\alpha)}, \tag{5.1}$$

and for $\lambda \geq 2$

$$\frac{2\lambda^{\alpha-1}e^{-\lambda}}{3\Gamma(\alpha)} \leq P(X(t) \geq u(t, t)\lambda). \tag{5.2}$$

(It is interesting to note that since $E(X(t)) = \alpha u(t, t)$, if we were to consider the rate of growth of $X(t)/E(X(t))$, it would depend on α . The results we give for the rate of growth of $X(t)/u(t, t)$ do not depend on α .)

The next observation is elementary.

Lemma 5.1. *Let $X = \{X(t_n), n \in \mathbf{N}\}$ be an α -permanental sequence with kernel $u(t_i, t_j)$. Then*

$$\limsup_{n \rightarrow \infty} \frac{X(t_n)}{u(t_n, t_n) \log n} \leq 1 \quad \text{a.s.}, \tag{5.3}$$

or, equivalently,

$$\limsup_{n \rightarrow \infty} \frac{X^{1/2}(t_n)}{(u(t_n, t_n) \log n)^{1/2}} \leq 1 \quad \text{a.s.} \tag{5.4}$$

Proof The statement in (5.3) follows from (5.1) and the Borel-Cantelli □

Proof of Theorem 1.4 The proof follows from Lemma 3.1 similarly to how the results about the local modulus of continuity are obtained in Section 3. We can assume that $u^*(T_0, T_0) > 0$ for some T_0 . Redefine

$$F(a) = \sup_{s, t \geq T_0} P\left(\frac{X^{1/2}(s) - X^{1/2}(t)}{\sigma(s, t)} \geq a\right). \tag{5.5}$$

It follows from (2.68) that (3.7) holds. Let $\delta > 0$. Then, by Lemma 3.1, for all n with $n\delta \geq T_0$,

$$\begin{aligned} &P\left(\sup_{t \in [n\delta, (n+1)\delta]} |X^{1/2}(t) - X^{1/2}(n\delta)| > a\varphi(\delta) + \sum_{p=1}^{\infty} \theta(p)\varphi(\delta/n(p))\right) \\ &\leq n^2 F(a) + \sum_{p=1}^{\infty} n^2(p) F(\theta(p)). \end{aligned} \tag{5.6}$$

Taking $a = ((3 + \epsilon) \log n)^{1/2}$ and $\theta(p) = (3 \log n(p))^{1/2}$, as in (3.18), we see that for all n sufficiently large, the second line of (5.6) is a term of a converging sequence. Consequently, as in the beginning of the proof of Lemma 3.2, the event

$$\sup_{t \in [n\delta, (n+1)\delta]} |X^{1/2}(t) - X^{1/2}(n\delta)| > \left(a\varphi(\delta) + \frac{\sqrt{3}}{\log 2} \int_0^{1/n} \frac{\varphi(\delta u)}{u(\log 1/u)^{1/2}} du \right) \tag{5.7}$$

infinitely often, is zero.

For any $\epsilon > 0$ we can find a $\delta > 0$ so that

$$a\varphi(\delta) \leq \frac{\epsilon}{2} (\log n)^{1/2} \tag{5.8}$$

and

$$\int_0^{1/n} \frac{\varphi(\delta u)}{u(\log 1/u)^{1/2}} du \tag{5.9}$$

is bounded uniformly in $\delta \leq 1$. Therefore, for any $\epsilon > 0$ we can find a δ so that the right hand side of (5.7) is $\leq \epsilon (\log n)^{1/2}$ for all n sufficiently large. It follows from this that the probability that

$$\sup_{t \in [n\delta, (n+1)\delta]} |X^{1/2}(t) - X^{1/2}(n\delta)| > \epsilon (\log n)^{1/2} \tag{5.10}$$

infinitely often, is zero, for all $\epsilon > 0$. Note that

$$\begin{aligned} & \sup_{t \in [n\delta, (n+1)\delta]} \frac{X^{1/2}(t)}{(u^*(\delta n, \delta n) \log n)^{1/2}} \\ & \leq \frac{X^{1/2}(\delta n)}{(u^*(\delta n, \delta n) \log n)^{1/2}} + \sup_{t \in [n\delta, (n+1)\delta]} \frac{|X^{1/2}(t) - X^{1/2}(n\delta)|}{(u^*(\delta n, \delta n) \log n)^{1/2}}. \end{aligned} \tag{5.11}$$

It follows that

$$\limsup_{t \rightarrow \infty} \frac{X^{1/2}(t)}{(u^*(t, t) \log t)^{1/2}} = \limsup_{n \rightarrow \infty} \sup_{t \in [n\delta, (n+1)\delta]} \frac{X^{1/2}(t)}{(u^*(\delta n, \delta n) \log n)^{1/2}}. \tag{5.12}$$

Therefore, using (5.11) we see that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{X^{1/2}(t)}{(u^*(t, t) \log t)^{1/2}} & \leq \limsup_{n \rightarrow \infty} \frac{X^{1/2}(\delta n)}{(u^*(\delta n, \delta n) \log n)^{1/2}} \\ & \quad + \limsup_{n \rightarrow \infty} \sup_{t \in [n\delta, (n+1)\delta]} \frac{|X^{1/2}(t) - X^{1/2}(n\delta)|}{(u^*(\delta n, \delta n) \log n)^{1/2}}. \end{aligned} \tag{5.13}$$

Writing $\log n = \log n\delta + \log 1/\delta$ we see from (5.4) that the first term to the right of the inequality in (5.13) is less than or equal to 1 almost surely. By (5.10), the second term to the right of the inequality in (5.13) is bounded by $\epsilon/(u^*(T_0, T_0))^{1/2}$ almost surely. Since this is true for all $\epsilon > 0$ we get (1.19). \square

Proof of Theorem 1.5 Let $u(t, t)$ be regularly varying at infinity with index $\beta > 0$. Let $t_n = \theta^n$, where $\theta > 1$ so that $\tilde{S}_n = \theta^{n+1} - \theta^n$. Since

$$\varphi^2(t) \leq O(u(t, t)) \quad \text{as } t \rightarrow \infty, \tag{5.14}$$

we see that

$$\varphi^2(\tilde{S}_n) \leq Cu(\tilde{S}_n, \tilde{S}_n) \leq C(\theta - 1)^\beta u(\theta^n, \theta^n) \quad \text{as } n \rightarrow \infty, \tag{5.15}$$

for some constant C .

Let $a_n = ((3 + \epsilon) \log n)^{1/2}$. As in (5.7), the probability that

$$\sup_{t \in [\theta^n, \theta^{n+1}]} |X^{1/2}(t) - X^{1/2}(\theta^n)| > \left(a_n \varphi(\tilde{S}_n) + \frac{\sqrt{3}}{\log 2} \int_0^{1/n} \frac{\varphi(\tilde{S}_n u)}{u(\log 1/u)^{1/2}} du \right) \quad (5.16)$$

infinitely often, is zero. Note that by (5.15)

$$a_n \varphi(\tilde{S}_n) \leq C(\theta - 1)^{\beta/2} (u(\theta^n, \theta^n) \log n)^{1/2}. \quad (5.17)$$

We now show that this dominates the integral in (5.16). For all n sufficiently large, the integral in (5.16) is equal to

$$\begin{aligned} \int_0^{\tilde{S}_n/n} \frac{\varphi(u)}{u(\log 1/u + \log \tilde{S}_n)^{1/2}} du &\leq \int_0^{1/2} \frac{\varphi(u)}{u(\log 1/u)^{1/2}} du \\ &+ \int_{1/2}^1 \frac{\varphi(u)}{u(\log \tilde{S}_n)^{1/2}} du + \int_1^{\tilde{S}_n/n} \frac{\varphi(u)}{u(\log \tilde{S}_n - \log u)^{1/2}} du. \end{aligned} \quad (5.18)$$

Using (1.12) we see that the first two integrals on the right-hand side of the inequality sign in (5.18) are finite. In addition,

$$\int_1^{\tilde{S}_n/n} \frac{\varphi(u)}{u(\log \tilde{S}_n - \log u)^{1/2}} du \leq \frac{1}{(\log n)^{1/2}} \int_1^{\tilde{S}_n} \frac{\varphi(u)}{u} du. \quad (5.19)$$

Using (5.14) and (5.15) and the regular variation of $u(t, t)$, we see that (5.19) is

$$\begin{aligned} &\leq \frac{C'}{(\log n)^{1/2}} \int_1^{\tilde{S}_n} \frac{(u(x, x))^{1/2}}{x} dx \leq \frac{C''}{(\log n)^{1/2}} (u(\tilde{S}_n, \tilde{S}_n))^{1/2} \\ &\leq \frac{\tilde{C}''}{(\log n)^{1/2}} (\theta - 1)^{\beta/2} (u(\theta^n, \theta^n))^{1/2}. \end{aligned} \quad (5.20)$$

Thus we see that the right-hand side of (5.16) is asymptotic to (5.17) as $n \rightarrow \infty$. Consequently,

$$\limsup_{n \rightarrow \infty} \sup_{t \in [\theta^n, \theta^{n+1}]} \frac{|X^{1/2}(t) - X^{1/2}(\theta^n)|}{(u(\theta^n, \theta^n) \log n)^{1/2}} \leq C(\theta - 1)^{\beta/2} \quad a.s. \quad (5.21)$$

It follows from this and (5.4), with $t_n = \theta^n$, that

$$\limsup_{n \rightarrow \infty} \sup_{t \in [\theta^n, \theta^{n+1}]} \frac{X^{1/2}(t)}{(u(\theta^n, \theta^n) \log n)^{1/2}} \leq 1 + C(\theta - 1)^{\beta/2} \quad a.s. \quad (5.22)$$

Since $u(t, t)$ is regularly varying at infinity it is asymptotic to a monotonic function at infinity. Therefore,

$$\limsup_{t \rightarrow \infty} \frac{X^{1/2}(t)}{(u(t, t) \log n)^{1/2}} \leq 1 + C(\theta - 1)^{\beta/2} \quad a.s. \quad (5.23)$$

Since this holds for all $\theta > 1$ we get (1.20). □

Proof of upper bound in Theorem 1.14 Let

$$\hat{V}_{\alpha, f}(t) = \frac{\hat{X}_{\alpha, f}(t)}{\hat{u}_f(t, t)}, \quad t \geq 0. \quad (5.24)$$

By Lemma 3.6, $\hat{V}_{\alpha, f} = \{\hat{V}_{\alpha, f}(t), t \geq 0\}$ is an α -permanental process with sigma function

$$\hat{\sigma}_f^2(s, t) = 2 - \frac{2(\hat{u}_f(s, t)\hat{u}_f(t, s))^{1/2}}{(\hat{u}_f(s, s)\hat{u}_f(t, t))^{1/2}}. \quad (5.25)$$

Note that

$$\begin{aligned} \frac{\widehat{u}_f(s, t)\widehat{u}_f(t, s)}{\widehat{u}_f(s, s)\widehat{u}_f(t, t)} &= \frac{e^{-\lambda|t-s|} + f(s)}{1 + f(s)} \frac{e^{-\lambda|t-s|} + f(t)}{1 + f(t)} \\ &\geq e^{-2\lambda|t-s|}. \end{aligned} \tag{5.26}$$

Consequently

$$\widehat{\sigma}_f^2(s, t) \leq 2(1 - e^{-\lambda|t-s|}) \leq (\lambda|t - s|) \wedge 1 := \varphi^2(|t - s|). \tag{5.27}$$

Therefore, it follows from Theorem 1.4 that

$$\limsup_{t \rightarrow \infty} \frac{\widehat{V}_{\alpha, f}(t)}{\log t} \leq 1 \quad a.s. \tag{5.28}$$

This is (1.41).

For the last remark in this theorem, suppose that f is a potential for \overline{B} , with $h \in L^1_+(0, \infty)$. Then we have

$$f(t) = \int_0^\infty e^{-\lambda|t-s|} h(s) ds. \tag{5.29}$$

For any $\epsilon > 0$, choose s_0 so that $\int_{s_0}^\infty h(s) ds \leq \epsilon$. For $t \geq s_0$,

$$\begin{aligned} f(t) &= \int_0^{s_0} e^{-\lambda|t-s|} h(s) ds + \int_{s_0}^\infty h(s) ds \\ &\leq e^{-\lambda|t-s_0|} \|h\|_1 + \epsilon. \end{aligned} \tag{5.30}$$

Therefore, $\lim_{t \rightarrow \infty} f(t) \leq \epsilon$ for all $\epsilon > 0$. □

We now give some background material that may be needed to understand Example 1.15. A function f is excessive for \widetilde{B} if and only if f is positive and concave on D , which implies that f is increasing. This follows from the fact that f is excessive for \widetilde{B} if and only if f is a positive superharmonic function on D , [3, Section 4.5, Theorem 3]. That is, f is finite, lower semi-continuous and midpoint concave, which implies that f is concave, [1, Chapter I, Section 4.4, Corollary 1]. It follows from this and Theorem 1.11 that for any positive concave function f on D

$$\widetilde{u}_f(s, t) = s \wedge t + f(t), \quad s, t > 0, \tag{5.31}$$

is the kernel of an α -permanental processes, for all $\alpha > 0$. In Theorem 1.16 we denote this process by $\widetilde{Z}_{\alpha, f} = \{\widetilde{Z}_{\alpha, f}(t), t > 0\}$.

Since f is positive and increasing, we can define $f(0) = \lim_{t \downarrow 0} f(t)$. It is easy to check that this extended function f on $[0, \infty)$ is positive and concave. Any positive concave function f on $[0, \infty)$ can be written in the form

$$f(t) = \widetilde{f}(t) + C_0 t, \tag{5.32}$$

where $C_0 \geq 0$ is a constant and $\widetilde{f}(t)$ is a positive concave function that is $o(t)$ at infinity. To see this, note that $f'_r(t)$, the right hand derivative of f , is decreasing in t . Let $C_0 = \lim_{t \rightarrow \infty} f'_r(t)$. We must have $C_0 \geq 0$ since otherwise f could not remain positive. Let $\widetilde{f}(t) = f(t) - C_0 t$. This function \widetilde{f} is concave and $\widetilde{f}'_r(t) \geq 0$, which implies that \widetilde{f} is increasing. Since $\widetilde{f}(0) = f(0) \geq 0$, we see that \widetilde{f} is positive. In addition, since $\lim_{t \rightarrow \infty} \widetilde{f}'_r(t) = 0$, \widetilde{f} is $o(t)$ at infinity.

Proof of upper bounds in Theorem 1.16 To prove (1.45) it suffices to work with the α -permanental process $\{\widetilde{Z}_{\alpha, f}(t), t \geq 1\}$. This process has kernel $\widetilde{u}_f(s, t) = s \wedge t + f(t)$,

$s, t \geq 1$. Note that $\tilde{u}_f(t, t) = t + f(t)$ is increasing, and by (5.32), is regularly varying at infinity with positive index. We see from (4.19) in Lemma 4.5 that when f is concave the sigma function of $\tilde{Z}_{\alpha, f}$ satisfies

$$\begin{aligned} \sigma_f^2(s, t) &\leq |s - t| + |f(s) - f(t)| \\ &\leq (1 + f'_l(s \wedge t)) |t - s| \\ &\leq (1 + f'_l(1)) |t - s| := \varphi_f^2(|t - s|), \end{aligned} \tag{5.33}$$

where $f'_l(x)$ denotes the left-hand derivative of f at x .

Clearly, $\varphi_f^2(t) = O(\tilde{u}_f(t, t))$ as $t \rightarrow \infty$. Therefore the upper bound in (1.45) follows from Theorem 1.5 and (5.32).

To obtain (1.46) note that when

$$f(t) = \int_0^\infty (s \wedge t)h(s) ds \quad \text{and } h \in L^1_+, \tag{5.34}$$

then for all $\epsilon > 0$,

$$\begin{aligned} \frac{f(t)}{t} &= \frac{1}{t} \int_0^{\epsilon t} (s \wedge t)h(s) ds + \frac{1}{t} \int_{\epsilon t}^\infty (s \wedge t)h(s) ds \\ &\leq \epsilon \|h\|_1 + \int_{\epsilon t}^\infty h(s) ds. \end{aligned} \tag{5.35}$$

Therefore,

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t} \leq \epsilon \|h\|_1, \tag{5.36}$$

for all $\epsilon > 0$. This gives (1.46). □

Proof of Theorem 1.17 Let $\tilde{Z}_{\alpha, f} = \{\tilde{Z}_{\alpha, f}(t), t > 0\}$ be the α -permanental processes with kernel $\tilde{u}_f(s, t) = s \wedge t + f(t)$, $s, t > 0$. It follows from Remark 6.3, with the isolated point $*$ replaced by 0, that there also exists an α -permanental processes that extends $\tilde{Z}_{\alpha, f}$ to $\{\tilde{Z}_{\alpha, f}(t), t \geq 0\}$, with kernel

$$\begin{aligned} \tilde{u}_f(s, t) &= s \wedge t + f(t), \quad s, t > 0 \\ \tilde{u}_f(0, t) &= f(t), \quad t > 0 \quad \text{and} \quad \tilde{u}_f(s, 0) = \tilde{u}_f(0, 0) = 1. \end{aligned} \tag{5.37}$$

Let

$$\hat{Z}_{\alpha, f}(t) = \frac{\tilde{Z}_{\alpha, f}(t)}{\tilde{u}_f(t, t)}, \quad t \geq 0. \tag{5.38}$$

Then by Lemma 3.6, $\hat{Z}_{\alpha, f} = \{\hat{Z}_{\alpha, f}(t), t \geq 0\}$ is an α -permanental process with sigma function

$$\hat{\sigma}_f^2(s, t) = 2 - \frac{2(\tilde{u}_f(s, t)\tilde{u}_f(t, s))^{1/2}}{(\tilde{u}_f(s, s)\tilde{u}_f(t, t))^{1/2}} = 2 \left(1 - \left(\frac{\tilde{u}_f(s, t)}{\tilde{u}_f(t, t)} \right)^{1/2} \right) \tag{5.39}$$

when $0 \leq s \leq t$.

Let $\varphi_f(u) = u/f(u)$. Note that since $\varphi_f(u)/u = 1/f(u)$, (1.47) implies that for some $\delta > 0$,

$$\int_0^\delta \frac{\varphi_f(u)}{u(\log 1/u)^{1/2}} du < \infty. \tag{5.40}$$

Changing variables, $u = e^{-s^2}$, we write the integral in (5.40) as

$$2 \int_{(\log 1/\delta)^{1/2}}^\infty \varphi_f(e^{-s^2}) ds. \tag{5.41}$$

Thus, since this integral is finite and $\varphi_f(e^{-s^2})$ is decreasing, for any $s > s_0$

$$s\varphi_f(e^{-s^2}) \leq s_0\varphi_f(e^{-s^2}) + \int_{s_0}^{\infty} \varphi_f(e^{-u^2}) du \tag{5.42}$$

Therefore,

$$\lim_{s \rightarrow \infty} s\varphi_f(e^{-s^2}) \leq \int_{s_0}^{\infty} \varphi_f(e^{-u^2}) du. \tag{5.43}$$

Since this holds for all s_0 we get

$$\lim_{s \rightarrow \infty} s\varphi_f(e^{-s^2}) = 0, \tag{5.44}$$

or equivalently

$$\lim_{t \rightarrow 0} \varphi_f(t)(\log 1/t)^{1/2} = 0. \tag{5.45}$$

In particular, (5.45) implies that $\lim_{t \rightarrow 0} t/f(t) = 0$. Therefore, when $0 \leq s \leq t$, for all $\epsilon > 0$,

$$\left(\frac{\tilde{u}_f(s,t)}{\tilde{u}_f(t,t)}\right)^{1/2} = \left(\frac{s+f(t)}{t+f(t)}\right)^{1/2} \geq 1 - (1+\epsilon)\frac{t-s}{f(t)} \quad \text{as } t \rightarrow 0. \tag{5.46}$$

Therefore,

$$\hat{\sigma}_f^2(s,t) \leq (1+\epsilon)\frac{|t-s|}{f(t)} \leq (1+\epsilon)\frac{|t-s|}{f(t-s)} := \hat{\varphi}_f(t-s) \quad \text{as } s, t \rightarrow 0. \tag{5.47}$$

In preparation for using Lemma 3.2 we first note that by concavity, $f(t)/t \leq f(s)/s$ for $t \geq s$ so that $\hat{\varphi}_f(t)$ is increasing. We now show that (3.10) holds. If $\gamma > 1$, then, since f is concave, we have

$$\frac{f(\gamma V) - f(V)}{(\gamma - 1)V} \leq \frac{f(V)}{V}. \tag{5.48}$$

Consequently

$$\frac{f(\gamma V)}{f(V)} - 1 \leq \gamma - 1, \tag{5.49}$$

which gives (3.10) when $\gamma > 1$. If $\gamma < 1$,

$$\frac{f(V) - f(\gamma V)}{(1 - \gamma)V} \leq \frac{f(\gamma V)}{\gamma V}, \tag{5.50}$$

which implies that

$$\frac{f(V)}{f(\gamma V)} - 1 \leq \frac{1 - \gamma}{\gamma}, \tag{5.51}$$

which gives (3.10) when $\gamma < 1$.

We can now use Lemma 3.2 to see

$$\limsup_{t \rightarrow 0} \frac{|\hat{Z}_{\alpha,f}^{1/2}(t) - \hat{Z}_{\alpha,f}^{1/2}(0)|}{\tau(t)} \leq \sqrt{3} \quad \text{a.s.}, \tag{5.52}$$

where

$$\tau(t) = \hat{\varphi}_f(t)(\log \log 1/t)^{1/2} + \frac{1}{\log 2} \int_0^{1/2} \frac{\hat{\varphi}_f(tu)}{u(\log 1/u)^{1/2}} du. \tag{5.53}$$

By Lemma 4.2

$$\int_0^{1/2} \frac{\hat{\varphi}_f(tu)}{u(\log 1/u)^{1/2}} du \leq 2\Theta(t) + o(\Theta(t)). \tag{5.54}$$

where

$$\Theta(t) = \int_0^{t^2} \frac{\widehat{\varphi}_f(u)}{u(\log 1/u)^{1/2}} du + \widehat{\varphi}_f(t)(\log 1/t)^{1/2}. \tag{5.55}$$

By (5.40) and (5.45)

$$\lim_{t \rightarrow 0} \Theta(t) = 0, \tag{5.56}$$

which gives

$$\lim_{t \rightarrow 0} \tau(t) = 0. \tag{5.57}$$

Since $\widehat{Z}_{\alpha,f}(t) = \widetilde{Z}_{\alpha,f}(t)/(t + f(t))$, it follows from (5.52) that

$$\lim_{t \rightarrow 0} \frac{\widetilde{Z}_{\alpha,f}(t)}{f(t)} = \widehat{Z}_{\alpha,f}(0) \quad a.s. \tag{5.58}$$

The theorem now follows from the fact that an α -permanental random variable with kernel 1, such as $\widehat{Z}_{\alpha,f}(0)$, is a gamma random variable $\xi_{\alpha,1}$ with shape α and scale 1; see [18, (1.3)]. □

Proof of Theorem 1.18 If f is regularly varying at 0 with index $0 < \gamma < 1$, $\widehat{\varphi}_f$ is regularly varying at 0 with index $1 - \gamma$. By Theorem 3.5 we get

$$\limsup_{t \rightarrow 0} \frac{|\widehat{Z}_{\alpha,f}^{1/2}(t) - \widehat{Z}_{\alpha,f}^{1/2}(0)|}{\widehat{\varphi}_f(t)(\log \log 1/t)^{1/2}} \leq 1 \quad a.s. \tag{5.59}$$

Using the fact that $\widehat{Z}_{\alpha,f}(t) = \widetilde{Z}_{\alpha,f}(t)/(t + f(t))$, we get (1.49). □

Proof of Theorem 1.6 Using Theorem 1.19 on the pairs $\{t_n, 0\}$, we see that

$$\limsup_{n \rightarrow \infty} \frac{|X^{1/2}(t_n) - X^{1/2}(0)|}{\sigma(t_n, 0)(\log n)^{1/2}} \leq 1 \quad a.s. \tag{5.60}$$

When $\beta = \infty$, since $u(0, 0) < \infty$ implies that $X(0)$ is finite almost surely, we immediately get (1.21).

To consider the case when $0 < \beta < \infty$, we write

$$|X(t_n) - X(0)| \leq |X^{1/2}(t_n) - X^{1/2}(0)| \left(|X^{1/2}(t_n) - X^{1/2}(0)| + 2X^{1/2}(0) \right) \tag{5.61}$$

Using (5.60) we see that that

$$\limsup_{n \rightarrow \infty} \left(|X^{1/2}(t_n) - X^{1/2}(0)| + 2X^{1/2}(0) \right) \leq \beta + 2X^{1/2}(0) \quad a.s. \tag{5.62}$$

Combining this in (5.61) with (5.60) we get (1.22).

When $\beta = 0$, (5.62) still holds and using (5.60) we get (1.23). □

Remark 5.2. Let $X = \{X(t), t \in [0, 1]\}$ be an α -permanental sequence with kernel $u(s, t)$ and sigma function $\sigma(s, t)$. Consider the permanental sequence $Y = \{Y(n), n \in \mathbb{N}\} = \{X(1/n), n \in \mathbb{N}\}$. Obviously, the results in Theorem 1.6 hold with $X(t_n)$ replaced by $Y(n)$ and $\sigma(t_n, 0)$ replaced by $\sigma(1/n, 0)$.

The advantage of Theorem 1.6 is that it doesn't require that (1.12) holds. This is significant because (1.12) requires that

$$\lim_{u \rightarrow 0} \sigma(u, 0)(\log 1/u)^{1/2} = 0. \tag{5.63}$$

(Under the additional assumptions that $\sigma = \varphi$ and $\sigma(u, 0) = \sigma(u)$.) Therefore, the results in (3.78) would not give the results in Theorem 1.6 when $\beta > 0$. Additionally there are also some cases when $\beta = 0$ but (1.12) does not hold.

Example 5.3. Let $\{X_\alpha(0), X_\alpha(1), \dots, X_\alpha(n), \dots n \in \mathbb{N}\}$ be an α -permanental sequence determined by the kernel

$$u(0, 0) = 2, \quad u(j, 0) = 1 + f_j, \quad u(0, k) = 1 + g_k, \quad j, k = 1, \dots \quad (5.64)$$

and

$$u(j, k) = \lambda_j \delta_{j,k} + 1 + f_j g_k, \quad j, k = 1, \dots \quad (5.65)$$

where $\lambda_j \rightarrow 0$,

$$0 \leq f_j = 1 - p_j \leq 1, \quad 0 \leq g_j = 1 - q_j \leq 1, \quad p_j = o(\lambda_j^{1/2}), \quad q_j = o(\lambda_j^{1/2}). \quad (5.66)$$

It is easy to show that the inverse of $\{u(j, k)\}_{j,k=1}^m$ is an M -matrix with positive row sums, which implies that u is the kernel of an α -permanental sequences. (See [19] for details.)

We have

$$\begin{aligned} \sigma^2(j, 0) &= u(j, j) + u(0, 0) - 2(u(j, 0)u(0, j))^{1/2} \\ &= \lambda_j + 1 + f_j g_j + 2 - 2((1 + f_j)(1 + g_j))^{1/2} \\ &= \lambda_j + 4 - p_j - q_j + p_j q_j - 2((2 - p_j)(2 - q_j))^{1/2} \end{aligned} \quad (5.67)$$

which gives

$$\sigma^2(j, 0) = \lambda_j + o(\lambda_j), \quad \text{as } j \rightarrow \infty. \quad (5.68)$$

Using this, it follows from Theorem 1.6 that when $\beta = \infty$,

$$\limsup_{n \rightarrow \infty} \frac{X_\alpha(n)}{\lambda_n \log n} \leq 1 \quad a.s. \quad (5.69)$$

when $0 < \beta < \infty$,

$$\limsup_{n \rightarrow \infty} |X_\alpha(n) - X_\alpha(0)| \leq \beta^2 + 2\beta X_\alpha^{1/2}(0) \quad a.s. \quad (5.70)$$

and when $\beta = 0$,

$$\limsup_{n \rightarrow \infty} \frac{|X_\alpha(n) - X_\alpha(0)|}{(\lambda_n \log n)^{1/2}} \leq 2X^{1/2}(0) \quad a.s. \quad (5.71)$$

We show in [19] that the lim sup in (5.69) is actually equal to 1.

6 Partial rebirthing of transient Borel right processes

Let S be a locally compact set with a countable base. Let $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient Borel right process with state space S , and continuous strictly positive potential densities $u(x, y)$ with respect to some σ -finite measure m on S .

Let $\zeta = \inf\{t \mid X_t = \Delta\}$, where Δ is the cemetery state for X , and assume that $\zeta < \infty$ a.s. Let μ be a finite measure on S . We call the function

$$f(y) = \int_S u(x, y) d\mu(x) \quad (6.1)$$

a left potential for X . Since $u(x, y)$ is continuous in y uniformly in x and μ is a finite measure we see that $f(y)$ is continuous. See [6, Section 2]

The next theorem, which is interesting on its own, is also used in the proof of Theorem 1.11. Note that it does not require that u is symmetric. In this theorem we add a point $*$ to the state space S of X and modify X so that instead of going to Δ it goes to $*$. We then allow the process to return to S from $*$ with a probability $p < 1$, or to go to Δ with probability $1 - p$. Let \tilde{X} denote the modified process on the enlarged space. We see that when X "dies", \tilde{X} has a chance to be reborn, after which it continues to evolve that way X did.

Theorem 6.1. Let $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient Borel right process with state space S , as above. Then for any left potential f for X , there exists a transient Borel right process $\tilde{X} = (\Omega, \mathcal{F}_t, \tilde{X}_t, \theta_t, \tilde{P}^x)$ with state space $\tilde{S} = S \cup \{*\}$, where $*$ is an isolated point, such that \tilde{X} has potential densities

$$\begin{aligned} \tilde{u}(x, y) &= u(x, y) + f(y), & x, y \in S \\ \tilde{u}(*, y) &= f(y), & \text{and} \quad \tilde{u}(x, *) = \tilde{u}(*, *) = 1, \end{aligned} \tag{6.2}$$

with respect to the measure \tilde{m} on \tilde{S} which is equal to m on S and assigns a unit mass to $*$.

Proof We construct \tilde{X} as described prior to the statement of this theorem. Let ρ be the total mass of μ . If \tilde{X} starts in S it proceeds just like X until time ζ , at which time it goes to $*$. It stays there for an independent exponential time with parameter $1 + \rho$, $\rho > 0$, after which it returns to S with initial law $\mu/(1 + \rho)$. (This is what we mean by partial rebirthing.)

Once in S , \tilde{X} continues as we just described for \tilde{X} starting in S . Since the measure $\mu/(1 + \rho)$ has total mass $\rho/(1 + \rho)$, after each visit to $*$, \tilde{X} only has probability $\rho/(1 + \rho)$ to be reborn. With probability $1/(1 + \rho)$ the process enters a cemetery state Δ .

We now calculate the potential densities for \tilde{X} . Let g be a function on \tilde{S} with $g(*) = 0$. Then for any $x \in S$

$$\begin{aligned} E^x \left(\int_0^\infty g(\tilde{X}_t) dt \right) & \tag{6.3} \\ &= E^x \left(\int_0^\zeta g(X_t) dt \right) + \sum_{n=1}^\infty \left(\frac{\rho}{1 + \rho} \right)^{n-1} \int \frac{d\mu(z)}{1 + \rho} E^z \left(\int_0^\zeta g(X_t) dt \right). \end{aligned}$$

This is equal to

$$\begin{aligned} & \int u(x, y)g(y) dm(y) + \frac{1}{1 + \rho} \sum_{n=1}^\infty \left(\frac{\rho}{1 + \rho} \right)^{n-1} \int d\mu(z) \int u(z, y)g(y) dm(y) \\ &= \int u(x, y)g(y) dm(y) + \int f(y)g(y) dm(y), \end{aligned}$$

which gives the first line of (6.2). The first half of the second line of (6.2) follows from a similar computation, where now we no longer have the first term in the second line of (6.3). Finally, since at each visit to $*$ the process waits there an independent exponential time with parameter $1 + \rho$, and then returns to $*$ with probability $\rho/1 + \rho$, we have, for some sequence of functions $h^{(n)} \rightarrow h$.

$$E^x \left(\int_0^\infty 1_{\{\tilde{X}_t = *\}} dt \right) \frac{1}{1 + \rho} + \sum_{n=1}^\infty \left(\frac{\rho}{1 + \rho} \right)^n \frac{1}{1 + \rho} = 1. \tag{6.4}$$

The same computation holds if we start at $*$. □

The next lemma is used in the proof of Theorem 1.11.

Lemma 6.2. Assume that for each $n \in \mathbb{N}$, $u^{(n)}(s, t)$, $s, t \in S$, is the kernel of an α -permanental process. If $u^{(n)}(s, t) \rightarrow u(s, t)$ for all $s, t \in S$, then $u(s, t)$ is the kernel of an α -permanental process.

Proof By the hypothesis, for all k and $x_1, \dots, x_k \in S$, there exists an α -permanental vector $(X_\alpha^{(n)}(x_1), \dots, X_\alpha^{(n)}(x_k))$ with kernel $K_{i,j}^{(n)} = u^{(n)}(x_i, x_j)$, $1 \leq i, j \leq k$. Therefore,

by definition, for all $s_1, \dots, s_k \geq 0$,

$$E \left(e^{-\sum_{i=1}^k s_i X_\alpha^{(n)}(x_i)} \right) = \frac{1}{|I + K^{(n)}S|^\alpha}. \tag{6.5}$$

In addition, since $u^{(n)}(x_i, x_j) \rightarrow u(x_i, x_j)$, we have $|I + K^{(n)}S| \rightarrow |I + KS|$, where $K_{i,j} = u(x_i, x_j)$, $1 \leq i, j \leq k$. It follows from the extended continuity theorem [9, Theorem 5.22], that there exists a random vector $(X_\alpha(x_1), \dots, X_\alpha(x_k))$ with

$$E \left(e^{-\sum_{i=1}^k s_i X_\alpha(x_i)} \right) = \frac{1}{|I + KS|^\alpha}. \tag{6.6}$$

Since this is true for all k and all $x_1, \dots, x_k \in S$, it follows by the Kolmogorov extension theorem that $\{u(s, t), s, t \in S\}$ is the kernel of an α -permanental process. \square

Proof of Theorem 1.11 We apply Lemma 6.2 twice to prove the theorem. Consider a general excessive function f . It follows from [2, II, (2.19)] that there exists a sequence of functions $g_n \geq 0$, with both g_n and

$$Ug_n(x) = \int_S u(x, y)g_n(y) dy \tag{6.7}$$

bounded such that $f(x)$ is the increasing limit of $Ug_n(x)$. If the g_n are in L^1 then, since u is symmetric, Ug_n is a left potential as in (6.1). Therefore, by Theorem 6.1, $\{u(s, t) + Ug_n(t), s, t \in S\}$ are kernels of α -permanental processes. Consequently, by Lemma 6.2, $\{u(s, t) + f(t), s, t \in S\}$ is the kernel of α -permanental process.

If g_n is not in L^1 we proceed as follows: Let C_m be an increasing sequence of compact sets whose union is S . Then $g_n 1_{C_m} \in L^1$, so that by Theorem 6.1 $\{u(s, t) + U(g_n 1_{C_m})(t), s, t \in S\}$ is the kernel of an α -permanental process. Taking the limit as $m \rightarrow \infty$, it follows from Lemma 6.2 that $\{u(s, t) + Ug_n(t), s, t \in S\}$ is the kernel of an α -permanental process. Since $Ug_n \rightarrow f$ we can use Lemma 6.2 again to see that $\{u(s, t) + f(t), s, t \in S\}$ is the kernel of α -permanental process. \square

Remark 6.3. Theorem 1.11 shows that there exists an α -permanental process $Z_\alpha(t), t \in S$ with the kernel given in (1.35). The same proof also shows that there exists an α -permanental process $\{Z_\alpha(t), t \in S\} \cup Z_\alpha(*)$ with the kernel given in (6.2) for any function f which is excessive for X .

7 Lower bounds

We use results from [18] to obtain lower bounds for the rate of growth of permanental process or for their behavior at 0. There are several different situations that can arise depending on the kernels of the permanental processes. We give several criteria that can be used on kernels that behave differently.

Lemma 7.1. Let $X_\alpha = \{X_\alpha(t), t \in R^+\}$ be an α -permanental process with kernel $u(s, t)$ such that $u(t, t) > 0$ for all $t \in R^+$. Set

$$\tilde{u}(s, t) = \frac{u(s, t)}{(u(s, s)u(t, t))^{1/2}} \quad s, t \in R^+. \tag{7.1}$$

Let $\{t_j\}_{j=1}^\infty$ be a sequence in R^+ . Set

$$\phi^2(i, j) = 2 - (\tilde{u}(t_i, t_j) + \tilde{u}(t_j, t_i)) \quad \text{and} \quad (\phi_n^*)^2 = \inf_{\substack{1 \leq i, j \leq n \\ i \neq j}} \phi^2(i, j). \tag{7.2}$$

If

$$\sup_{\substack{1 \leq i, j \leq \infty \\ i \neq j}} \tilde{u}(t_i, t_j) + \tilde{u}(t_j, t_i) \leq \epsilon_1 \tag{7.3}$$

and

$$\sup_{\substack{1 \leq i, j \leq n \\ i \neq j}} |\tilde{u}(t_i, t_j) - \tilde{u}(t_j, t_i)| \leq \epsilon_2 (\phi_n^*)^2, \tag{7.4}$$

for ϵ_1, ϵ_2 sufficiently small, then

$$\limsup_{i \rightarrow \infty} \frac{X_\alpha(t_i)}{u(t_i, t_i) \log i} \geq 1 - 3(\epsilon_1 + \epsilon_2) \quad a.s. \tag{7.5}$$

Proof Let $\{\tilde{X}(t) = X(t)/u(t, t), t \in R^+\}$. We show in Lemma 3.6 that $\tilde{X} = \{\tilde{X}(t), t \in R^+\}$ is an α -permanental process with kernel $\tilde{u}(s, t)$. Now consider the matrix $K_n = \{\tilde{u}(t_i, t_j)\}_{i, j=1}^n$. This is the kernel of the α -permanental vector $(\tilde{X}(t_1), \dots, \tilde{X}(t_n))$. Let $\{a_{i, n}\}_{i=1}^n$ denote the diagonal elements of K_n^{-1} . By (7.3)

$$(\phi_n^*)^2 \geq 2 - \epsilon_1. \tag{7.6}$$

By (7.4) we can take $C = \epsilon_2$ in [18, (5.5)] and, since $\tilde{u}(t, t) \equiv 1$, use [18, Lemma 5.2] to get

$$a_{i, n} \leq \frac{2}{(1 - \epsilon_2)(\phi_n^*)^2} \leq \frac{2}{(1 - \epsilon_2)(2 - \epsilon_1)} \leq 1 + 2(\epsilon_1 + \epsilon_2), \tag{7.7}$$

for all $1 \leq i \leq n$ and all ϵ_1, ϵ_2 sufficiently small.

To complete the proof we use the next two lemmas.

Lemma 7.2. Let $\{\xi_{u, v}^{(i)}\}_{i=l}^n$ be independent copies of $\xi_{u, v}$. (See (2.1)). Then for all $0 < \epsilon < 1$, and $l \geq l_0 = l_0(\epsilon)$ with $(2l_0^6 / (3\Gamma(u) \log l_0)) \geq 1$,

$$P \left(\max_{l \leq i \leq n} \frac{\xi_{u, v}^{(i)}}{\log i} \geq \frac{(1 - \epsilon)}{v} \right) \geq 1 - \frac{l + 1}{n + 1}. \tag{7.8}$$

Proof We have

$$\begin{aligned} P \left(\max_{l \leq i \leq n} \frac{\xi_{u, v}^{(i)}}{\log i} > \frac{(1 - \epsilon)}{v} \right) &= 1 - P \left(\max_{l \leq i \leq n} \frac{\xi_{u, v}^{(i)}}{\log i} \leq \frac{(1 - \epsilon)}{v} \right) \\ &= 1 - \prod_{i=l}^n \left(1 - P \left(\frac{\xi_{u, v}^{(i)}}{\log i} > \frac{(1 - \epsilon)}{v} \right) \right). \end{aligned} \tag{7.9}$$

For any $i \geq l_0$,

$$P \left(\frac{\xi_{u, v}^{(i)}}{\log i} > \frac{(1 - \epsilon)}{v} \right) \geq \frac{2e^{-(1-\epsilon) \log i}}{3\Gamma(u)(1 - \epsilon) \log i} \geq \frac{1}{i}. \tag{7.10}$$

(See e.g. [18, (3.2)].) Using (7.9) and (7.10), we see that

$$P \left(\max_{l \leq i \leq n} \frac{\xi_{u, v}^{(i)}}{\log i} > \frac{(1 - \epsilon)}{v} \right) \geq 1 - \prod_{i=l}^n \left(1 - \frac{1}{i} \right) > 1 - e^{-\sum_{i=l}^n 1/i}. \quad \square$$

Lemma 7.3. Let $X = (X_1, X_2, \dots)$ be an α -permanental sequence, and for each n , let $K(n)$ be the kernel of $X = (X_1, X_2, \dots, X_n)$. If $(K(n))^{-1}$ has diagonal elements $a_{n, i} \leq a_i$, $i_0 \leq i \leq n - i_0$, for some i_0 and all n sufficiently large, then

$$\limsup_{i \rightarrow \infty} \frac{X_i}{a_i^{-1} \log i} \geq 1, \quad a.s. \tag{7.11}$$

Proof Using [18, (1.7)] and then (7.8) we see that for any $\epsilon > 0$ and $l \geq l_0 \vee i_0(\epsilon)$

$$\begin{aligned} P\left(\max_{l \leq i \leq n-i_0} \frac{X_i}{a_i^{-1} \log i} \geq 1 - \epsilon\right) &\geq P\left(\max_{l \leq i \leq n-i_0} \frac{X_i}{a_{n,i}^{-1} \log i} \geq 1 - \epsilon\right) \\ &\geq P\left(\max_{l \leq i \leq n-i_0} \frac{\xi_{\alpha,1}^{(i)}}{\log i} \geq 1 - \epsilon\right) \\ &\geq 1 - \frac{l+1}{n-i_0+1}. \end{aligned} \tag{7.12}$$

It follows from this that for any $\epsilon > 0$ and $l \geq l_0(\epsilon) \vee i_0$,

$$P\left(\sup_{l \leq i} \frac{X_i}{a_i^{-1} \log i} \geq 1 - \epsilon\right) = 1. \tag{7.13}$$

We take the limit as $l \rightarrow \infty$ and use monotone convergence to get

$$P\left(\limsup_{i \rightarrow \infty} \frac{X_i}{a_i^{-1} \log i} \geq 1 - \epsilon\right) = 1. \tag{7.14}$$

Since this holds for all $\epsilon > 0$ we obtain (7.11). □

Proof of Lemma 7.1 continued: It follows from Lemma 7.3 that

$$\limsup_{i \rightarrow \infty} \frac{\tilde{X}(t_i)}{\log i} \geq (1 - 3(\epsilon_1 + \epsilon_2)) \quad a.s. \tag{7.15}$$

This gives (7.5). □

Proof of lower bounds in Theorem 1.8 We obtain the lower bounds in (1.28) and (1.29). Following Lemma 7.1 set

$$\tilde{u}_{T_0,\gamma,\beta}(s,t) = \frac{u_{T_0,\gamma,\beta}(s,t)}{(u_{T_0,\gamma,\beta}(s,s)u_{T_0,\gamma,\beta}(t,t))^{1/2}} \quad s, t \in R^+. \tag{7.16}$$

Let $t_j = \theta^j$. Then

$$\tilde{u}_{T_0,\gamma,\beta}(\theta^i, \theta^j) + \tilde{u}_{T_0,\gamma,\beta}(\theta^j, \theta^i) = \frac{R_{\gamma,\beta}(\theta^i, \theta^j)}{(R_{\gamma,\beta}(\theta^i, \theta^i)R_{\gamma,\beta}(\theta^j, \theta^j))^{1/2}}. \tag{7.17}$$

Using (4.26) it is easy to check that for $i \neq j$, this is

$$\leq C\theta^{-\gamma/2} \quad \text{for } \theta \gg 1 \quad \text{and} \quad \leq C'\theta^{\gamma/2} \quad \text{for } \theta \ll 1, \tag{7.18}$$

for constants C and C' . Similarly

$$\tilde{u}_{T_0,\gamma,\beta}(\theta^i, \theta^j) - \tilde{u}_{T_0,\gamma,\beta}(\theta^j, \theta^i) = \frac{2|H_{\gamma,\beta}(\theta^i, \theta^j)|}{(R_{\gamma,\beta}(\theta^i, \theta^i)R_{\gamma,\beta}(\theta^j, \theta^j))^{1/2}}. \tag{7.19}$$

Using (4.27) we see that this is

$$\leq |\beta|\theta^{-\gamma/2} \quad \text{for } \theta \gg 1 \quad \text{and} \quad \leq |\beta|\theta^{\gamma/2} \quad \text{for } \theta \ll 1. \tag{7.20}$$

Therefore, (7.5) holds for $\theta \gg 1$ or $\theta \ll 1$, and since $\tilde{u}_{T_0,\gamma,\beta}(\theta^n, \theta^n) = 2C_{\alpha,\beta}\theta^{\alpha n}$ we get

$$\limsup_{n \rightarrow \infty} \frac{X(\theta^n)}{\theta^{\alpha n} \log n} \geq 2C_{\alpha,\beta}(1 - \epsilon) \quad a.s. \tag{7.21}$$

where ϵ depends on θ and goes to 0 as θ goes to 0 or ∞ , depending on whether $\theta \ll 1$ or $\theta \gg 1$. Using the facts that for $\theta \gg 1$, $\lim_{n \rightarrow \infty} \log n / \log \log \theta^n = 1$ and for $\theta \ll 1$, $\lim_{n \rightarrow \infty} \log n / \log \log \theta^{-n} = 1$ we get the lower bounds in (1.28) and (1.29). \square

Proof of lower bound in Theorem 1.14 This is an immediate application of Lemma 7.1. Consider $\{\widehat{X}_{\alpha, f}(nj), j \in \mathbb{N}\}$. It is easy to see that

$$\sup_{\substack{1 \leq j, k \leq n \\ j \neq k}} \widetilde{u}_f(nj, nk) = \sup_{\substack{1 \leq j, k \leq n \\ j \neq k}} f(nk) + e^{-\lambda n|k-j|} \tag{7.22}$$

Therefore, since $\lim_{t \rightarrow \infty} f(t) = 0$, for all $\epsilon > 0$ we can choose n such that (7.3) and (7.4) hold with ϵ_1 and ϵ_2 less than ϵ . Consequently, (1.42) follows from (7.5). \square

Proof of lower bound in Theorem 1.16 Let U be a non-singular $n \times n$ matrix. We use U^{-1} to denote the inverse, and $U^{j,k}$ to denote the elements of U^{-1} .

Let U_f be the $(n + 1) \times (n + 1)$ matrix

$$U_f = \begin{pmatrix} 1 & f(1) & \dots & f(n) \\ 1 & U_{1,1} + f(1) & \dots & U_{1,n} + f(n) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & U_{n,1} + f(1) & \dots & U_{n,n} + f(n) \end{pmatrix}. \tag{7.23}$$

One can check that

$$U_f^{-1} = \begin{pmatrix} 1 + \rho & -\sum_{i=1}^n f(i)U^{i,1} & \dots & -\sum_{i=1}^n f(i)U^{i,n} \\ -\sum_{j=1}^n U^{1,j} & U^{1,1} & \dots & U^{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{j=1}^n U^{n,j} & U^{n,1} & \dots & U^{n,n} \end{pmatrix}, \tag{7.24}$$

where

$$\rho =: \sum_{j=1}^n \sum_{i=1}^n f(i)U^{i,j}. \tag{7.25}$$

We now apply this with U replaced by $W(n) = \{s_j \wedge s_k\}_{j,k=1}^n$, where $t_i > 0, i = 1, \dots, n$, and

$$s_j = t_1 + \dots + t_j, \quad j = 1, \dots, n. \tag{7.26}$$

That is,

$$W(n) = \begin{pmatrix} s_1 & s_1 & s_1 & \dots & s_1 & s_1 \\ s_1 & s_2 & s_2 & \dots & s_2 & s_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_1 & s_2 & s_3 & \dots & s_{n-1} & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_{n-1} & s_n \end{pmatrix}. \tag{7.27}$$

It is easy to check that

$$W(n)^{-1} = \begin{pmatrix} \frac{1}{t_1} + \frac{1}{t_2} & -\frac{1}{t_2} & 0 & \dots & 0 & 0 \\ -\frac{1}{t_2} & \frac{1}{t_2} + \frac{1}{t_3} & -\frac{1}{t_3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{t_{n-1}} + \frac{1}{t_n} & -\frac{1}{t_n} \\ 0 & 0 & 0 & \dots & -\frac{1}{t_n} & \frac{1}{t_n} \end{pmatrix}. \tag{7.28}$$

Now, let

$$U_f(n + 1) = \{s_j \wedge s_k + f(k)\}_{j,k=0}^n \tag{7.29}$$

where $s_0 = 0$ and $f(0) = 1$. It follows from (7.24) that all the diagonal entries of $U_f(n+1)^{-1}$, except for the first one, are equal to diagonal entries of $W(n)^{-1}$, that is they are equal to $1/t_j + 1/t_{j+1}$, $j = 1, \dots, n-1$ and $1/t_n$.

Let $s_j = \theta^j$, for $\theta \gg 1$. Then $t_j = \theta^j - \theta^{j-1}$ and

$$U_f(n+1)^{j,j} = \frac{1}{\theta^j} \left(\frac{\theta+1}{\theta-1} \right), \quad j = 2, \dots, n-1, \tag{7.30}$$

and

$$U_f(n+1)^{n,n} = \frac{1}{\theta^n} \left(\frac{\theta}{\theta-1} \right). \tag{7.31}$$

It now follows from Lemma 7.3 that

$$\limsup_{n \rightarrow \infty} \frac{\tilde{Z}_{\alpha,f}(\theta^n)}{\theta^n \log n} \geq \frac{\theta-1}{\theta+1} \quad a.s. \tag{7.32}$$

Taking θ arbitrarily large gives the lower bound in (1.45). □

8 Appendix

In this section we simply write d for the metric $d_{C,\sigma}$ in (1.9).

Let (\mathcal{T}, d) be a separable metric or pseudo-metric space. Let $B_d(t, u)$ denote a closed ball of radius u in (\mathcal{T}, d) and μ a probability measure on \mathcal{T} we define

$$J_{\mathcal{T},d,\mu}(a) = \sup_{t \in \mathcal{T}} \int_0^a \left(\log \frac{1}{\mu(B_d(t, u))} \right)^{1/2} du. \tag{8.1}$$

The next theorem follows from [11, Theorem 3.1]. The proof of [11, Theorem 3.1] is a consequence of the fact that a 1/2-permanental process is subgaussian. Using Theorem 1.1 it extends it as follows:

Theorem 8.1. *Let $X_\alpha = \{X_\alpha(t), t \in \mathcal{T}\}$ be an α -permanental process with kernel $u(s, t)$. Assume that \mathcal{T} is separable for d with finite diameter D and that there exists a probability measure μ on \mathcal{T} such that*

$$J_{\mathcal{T},d,\mu}(D) < \infty. \tag{8.2}$$

Then there exists a version $X'_\alpha = \{X'_\alpha(t), t \in \mathcal{T}\}$ of X_α which is bounded almost surely.

If

$$\lim_{\delta \rightarrow 0} J_{\mathcal{T},d,\mu}(\delta) = 0, \tag{8.3}$$

there exists a version $X'_\alpha = \{X'_\alpha(t), t \in \mathcal{T}\}$ of X_α such that

$$\lim_{\delta \rightarrow 0} \sup_{\substack{s, t \in \mathcal{T} \\ d(s,t) \leq \delta}} |X'_\alpha(s) - X'_\alpha(t)| = 0, \quad a.s. \tag{8.4}$$

If (8.3) holds and

$$\lim_{\delta \rightarrow 0} \frac{J_{\mathcal{T},d,\mu}(\delta)}{\delta} = \infty, \tag{8.5}$$

then

$$\lim_{\delta \rightarrow 0} \sup_{\substack{s, t \in \mathcal{T} \\ d(s,t) \leq \delta}} \frac{|X'_\alpha(s) - X'_\alpha(t)|}{J_{\mathcal{T},d,\mu}(d(s, t)/2)} \leq 30 \left(\sup_{t \in \mathcal{T}} X'_\alpha(t) \right)^{1/2} \quad a.s. \tag{8.6}$$

The next theorem follows from [16, Theorems 4.2] and Theorem 1.1.

Theorem 8.2. *Under the hypotheses of Theorem 8.1 assume that (8.3) holds. For any $t_0 \in \mathcal{T}$ and $\delta > 0$, let $\mathcal{T}_\delta := \{s : d(s, t_0) \leq \delta/2\}$. Suppose $0 < \delta \leq \delta_0 < D$ which implies that $\mathcal{T}_\delta \leq \mathcal{T}_D$. Assume that for some $\beta < 1$*

$$\limsup_{k \rightarrow \infty} \frac{\mu(\mathcal{T}_{\beta^k})}{\mu(\mathcal{T}_{\beta^{k+1}})} \leq C, \tag{8.7}$$

for some constant C , and consider the probability measures $\mu_\delta(\cdot) := \mu(\cdot \cap \mathcal{T}_\delta) / \mu(\mathcal{T}_\delta)$, $0 < \delta \leq \delta_0$. Then if $X_\alpha(t_0) \neq 0$ there exists a version $X'_\alpha = \{X'_\alpha(t), t \in \mathcal{T}\}$ of X_α such that

$$\lim_{\delta \rightarrow 0} \sup_{d(s, t_0) \leq \delta/2} \frac{|X'_\alpha(s) - X'_\alpha(t_0)|}{\overline{H}_{\mathcal{T}_\delta, d, \mu_\delta}(\delta/4)} \leq C X_\alpha^{1/2}(t_0) \quad a.s. \tag{8.8}$$

where

$$\overline{H}_{\mathcal{T}_\delta, d, \mu_\delta}(\delta/4) := \delta(\log \log 1/\delta)^{1/2} + J_{\mathcal{T}_\delta, d, \mu_\delta}(\delta/4). \tag{8.9}$$

If $X_\alpha(t_0) = 0$ there exists a version $X'_\alpha = \{X'_\alpha(t), t \in \mathcal{T}\}$ of X_α such that

$$\lim_{\delta \rightarrow 0} \sup_{\hat{d}(s, t_0) \leq \delta/2} \frac{X'_\alpha(s)}{(\overline{H}_{\mathcal{T}_\delta, d, \mu_\delta}(\delta/4))^2} \leq C' \quad a.s., \tag{8.10}$$

for some constant C' .

Remark 8.3. We have pointed out on page 2, that when $\{u(s, t); s, t \in \mathcal{T}\}$ is the potential density of a transient Markov process, $\{d_{C, \sigma}(s, t); s, t \in \mathcal{T}\}$ defined in (1.6) and (1.9), is a metric on \mathcal{T} . In general, if we only assume that $\{u(s, t); s, t \in \mathcal{T}\}$ is a kernel of α -permanental processes, we don't know whether $d_{C, \sigma}$ is a metric. Actually Theorems 8.1 and 8.2 still hold if $\{d_{C, \sigma}(s, t); s, t \in \mathcal{T}\}$ is not a metric. We continue to define

$$B_{d_{C, \sigma}}(t, u) = \{s; d_{C, \sigma}(s, t) \leq u\} \tag{8.11}$$

and everything goes through. (This is the approach we took in [16] which we wrote before we knew that when $\{u(s, t); s, t \in \mathcal{T}\}$ is the potential density of a transient Markov process, $\{d_{C, \sigma}(s, t); s, t \in \mathcal{T}\}$ is a metric on \mathcal{T} . See, in particular, [16, Theorems 1.1 and 1.2] and the paragraph that precedes [16, Theorems 1.1].)

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