Electron. J. Probab. **23** (2018), no. 13, 1–18.
ISSN: 1083-6489 https://doi.org/10.1214/18-EJP142

On martingale problems and Feller processes

Franziska Kühn*

Abstract

Let A be a pseudo-differential operator with negative definite symbol q. In this paper we establish a sufficient condition such that the well-posedness of the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem implies that the unique solution to the martingale problem is a Feller process. This provides a proof of a former claim by van Casteren. As an application we prove new existence and uniqueness results for Lévy-driven stochastic differential equations and stable-like processes with unbounded coefficients.

Keywords: Feller process; martingale problem; stochastic differential equation; stable-like process; unbounded coefficients.
AMS MSC 2010: Primary 60J25, Secondary 60G44; 60J75; 60H10; 60G51.
Submitted to EJP on September 4, 2017, final version accepted on January 15, 2018.
Supersedes arXiv:1706.04132.

1 Introduction

Let $(L_t)_{t\geq 0}$ be a k-dimensional Lévy process with characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ a continuous function which is at most of linear growth. It is known that there is a intimate correspondence between the Lévy-driven stochastic differential equation (SDE)

$$dX_t = \sigma(X_{t-})\, dL_t, \qquad X_0 \sim \mu, \tag{1.1}$$

and the pseudo-differential operator A with symbol $q(x,\xi) := \psi(\sigma(x)^T\xi)$, i. e.

$$Af(x) = -\int_{\mathbb{R}^d} q(x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where $\hat{f}$ denotes the Fourier transform of a smooth function f with compact support. Kurtz [6] proved that the existence of a unique weak solution to the SDE for any initial distribution μ is equivalent to the well-posedness of the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem. Recently, we have shown in [7] that a unique solution to the martingale problem – or, equivalently, to the SDE (1.1) – is a Feller process if the Lévy measure ν satisfies

$$\nu(\{y \in \mathbb{R}^k; |\sigma(x)\cdot y + x| \leq r\}) \xrightarrow{|x|\to\infty} 0 \qquad \text{for all } \ r > 0$$

*Institut für Mathematische Stochastik, Fachrichtung Mathematik, Technische Universität Dresden, Germany, E-mail: franziska.kuehn1@tu-dresden.de

which is equivalent to saying that A maps $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$, the space of continuous functions vanishing at infinity.

In this paper, we are interested in the following more general question: Consider a pseudo-differential operator A with continuous negative definite symbol q,

$$q(x,\xi) = q(x,0) - ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{y\neq 0} (1 - e^{iy\cdot\xi} + iy\cdot\xi \mathbb{1}_{(0,1)}(|y|))\,\nu(x,dy), \quad x,\xi\in\mathbb{R}^d,$$

such that the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed, i. e. for any initial distribution μ there exists a unique solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem. Under which assumptions does the well-posedness of the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem imply that the unique solution to the martingale problem is a Feller process? Since the infinitesimal generator of the solution is, when restricted to $C_c^\infty(\mathbb{R}^d)$, the pseudo-differential operator A, it is clear that A has to satisfy $Af \in C_\infty(\mathbb{R}^d)$ for all $f \in C_c^\infty(\mathbb{R}^d)$. In a paper by van Casteren [17] it was claimed that this mapping property of A already implies that the solution is a Feller process; however, this result turned out to be wrong, see [1, Example 2.27(ii)] for a counterexample. Our main result states van Casteren's claim is *correct* if the symbol q satisfies a certain growth condition; the required definitions will be explained in Section 2.

Theorem 1.1. *Let A be a pseudo-differential operator with continuous negative definite symbol q such that $q(\cdot,0)=0$ and the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed. If A maps $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$ and*

$$\lim_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} |q(x,\xi)| < \infty, \tag{G}$$

then the solution $(X_t)_{t\ge0}$ to the martingale problem is a conservative rich Feller process with symbol q, i. e. $(X_t)_{t\ge0}$ is a conservative Feller process and the infinitesimal generator $(L,\mathcal{D}(L))$ associated with $(X_t)_{t\ge0}$ satisfies $Lf = Af$ for any $f\in C_c^\infty(\mathbb{R}^d)\subseteq D(L)$.

Remark 1.2. (i). If the martingale problem is well-posed and $A(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$, then the solution is a C_b-Feller process, i. e. the associated semigroup $(T_t)_{t\ge0}$ satisfies $T_t : C_b(\mathbb{R}^d)\to C_b(\mathbb{R}^d)$ for all $t\ge0$. The growth condition (G) is needed to prove the Feller property; that is, to show that T_tf vanishes at infinity for any $f\in C_\infty(\mathbb{R}^d)$ and $t\ge0$.

(ii). There is a partial converse to Theorem 1.1: If $(X_t)_{t\ge0}$ is a Feller process and $C_c^\infty(\mathbb{R}^d)$ is a core for the generator A of $(X_t)_{t\ge0}$, then the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed, see e. g. [5, Theorem 4.10.3] or [11, Theorem 1.37] for a proof.

(iii). The mapping property $A(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ can be equivalently formulated in terms of the symbol q and its characteristics, cf. Lemma 2.1.

(iv). For the particular case that A is the pseudo-differential operator associated with the SDE (1.1), i. e. $q(x,\xi) = \psi(\sigma(x)^T\xi)$, we recover [7, Theorem 1.1]. Note that the growth condition (G) is automatically satisfied for any function σ which is at most of linear growth.

(v). If one of the conditions in Theorem 1.1 is violated – i. e. A does not map $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$ or the growth condition (G) is not satisfied – then, in general, the unique solution fails to be a Feller process. The following (counter)examples are taken from [5, Exercise 4.1.2] (see also [1, Example 2.27]) and [15, Remark 3.4].

Example I: The process

$$X_t = \frac{x}{\sqrt{1+tx^2}}$$

is the unique solution to the SDE

$$dX_t = -X_t^3\,dt, \qquad X_0 = x,$$

and, hence, also the unique solution to the martingale problem for the pseudo-differential operator A with symbol $q(x,\xi) := ix^3\xi$. As $X_t \to 1/\sqrt{2t}$ as $x \to \infty$ it follows easily that $(X_t)_{t\geq 0}$ does not satisfy the Feller property; this is not a contradiction to Theorem 1.1 since the growth condition (G) is violated.

Example II: Let $(N_t)_{t\geq 0}$ be a Poisson process. The SDE

$$dX_t = -X_{t-}\,dN_t, \qquad X_0 = x,$$

has a unique strong solution, and therefore the martingale problem for the pseudo-differential operator with symbol $q(x,\xi) = 1 - e^{-ix\xi}$ is well-posed. However, $(X_t)_{t\geq 0}$ is not a Feller process since it does not satisfy the Feller property. Indeed: Choose a function $u \in C_\infty(\mathbb{R}^d)$, $u \geq 0$, such that $u(0) = 1$. As

$$\mathbb{P}^x(X_t = 0) \geq \mathbb{P}(N_t \geq 1)$$

we have

$$\mathbb{E}^x u(X_t) \geq u(0)\mathbb{P}(N_t \geq 1) = \mathbb{P}(N_t \geq 1),$$

and so $\mathbb{E}^x u(X_t)$ does not converge to 0 as $|x| \to \infty$. This is not a contradiction to Theorem 1.1 since A does not map $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$; this follows e. g. from Lemma 2.1(i).

Although it is, in general, hard to prove the well-posedness of a martingale problem, Theorem 1.1 is very useful since it allows us to use localization techniques for martingale problems to establish new existence results for Feller processes with unbounded coefficients.

Corollary 1.3. *Let A be a pseudo-differential operator with symbol q such that $q(\cdot,0) = 0$, $A(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ and*

$$\lim_{|x|\to\infty} \sup_{|\xi|\leq |x|^{-1}} |q(x,\xi)| < \infty.$$

Assume that there exists a sequence $(q_k)_{k\in\mathbb{N}}$ of symbols such that $q_k(x,\xi) = q(x,\xi)$ for all $|x| < k$, $\xi \in \mathbb{R}^d$, and the pseudo-differential operator A_k with symbol q_k maps $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$. If the $(A_k, C_c^\infty(\mathbb{R}^d))$-martingale problem is well posed for all $k \geq 1$, then there exists conservative rich Feller process $(X_t)_{t\geq 0}$ with symbol q, and $(X_t)_{t\geq 0}$ is the unique solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem.

The paper is organized as follows. After introducing basic notation and definitions in Section 2, we prove Theorem 1.1 and Corollary 1.3. In Section 4 we present applications and examples; in particular we obtain new existence and uniqueness results for Lévy-driven stochastic differential equations and stable-like processes with unbounded coefficients.

2 Preliminaries

We consider $\mathbb{R}^d$ endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and write $B(x,r)$ for the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$; $\mathbb{R}^d_\Delta$ is the one-point compactification of $\mathbb{R}^d$. If a certain statement holds for $x \in \mathbb{R}^d$ with $|x|$ sufficiently large, we write "for $|x| \gg 1$". For a metric space (E,d) we denote by $C(E)$ the space of continuous functions $f : E \to \mathbb{R}$; $C_\infty(E)$ (resp. $C_b(E)$) is the space of continuous functions which

vanish at infinity (resp. are bounded). A function $f : [0,\infty) \to E$ is in the Skorohod space $D([0,\infty),E)$ if f is right-continuous and has left-hand limits in E. We will always consider $E = \mathbb{R}^d$ or $E = \mathbb{R}^d_\Delta$.

An E-valued Markov process $(\Omega, \mathcal{A}, \mathbb{P}^x, x \in E, X_t, t \geq 0)$ with càdlàg (right-continuous with left-hand limits) sample paths is called a *Feller process* if the associated semigroup $(T_t)_{t\geq 0}$ defined by

$$T_t f(x) := \mathbb{E}^x f(X_t), \quad x \in E, f \in \mathcal{B}_b(E) := \{f : E \to \mathbb{R}; f \text{ bounded, Borel measurable}\}$$

has the *Feller property*, i. e. $T_t f \in C_\infty(E)$ for all $f \in C_\infty(E)$, and $(T_t)_{t\geq 0}$ is *strongly continuous at* $t = 0$, i. e. $\|T_t f - f\|_\infty \xrightarrow{t\to 0} 0$ for any $f \in C_\infty(E)$. Following [13] we call a Markov process $(X_t)_{t\geq 0}$ with càdlàg sample paths a *C_b-Feller process* if $T_t(C_b(E)) \subseteq C_b(E)$ for all $t \geq 0$. An $\mathbb{R}^d_\Delta$-valued Markov process with semigroup $(T_t)_{t\geq 0}$ is *conservative* if $T_t \mathbb{1}_{\mathbb{R}^d} = \mathbb{1}_{\mathbb{R}^d}$ for all $t \geq 0$.

If the smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the generator $(L, \mathcal{D}(L))$ of a Feller process $(X_t)_{t\geq 0}$, then we speak of a *rich* Feller process. A result due to von Waldenfels and Courrège, cf. [1, Theorem 2.21], states that the generator L of an $\mathbb{R}^d$-valued rich Feller process is, when restricted to $C_c^\infty(\mathbb{R}^d)$, a pseudo-differential operator with negative definite symbol:

$$Lf(x) = -\int_{\mathbb{R}^d} e^{i\,x\cdot\xi} q(x,\xi) \hat{f}(\xi)\, d\xi, \qquad f \in C_c^\infty(\mathbb{R}^d),\, x \in \mathbb{R}^d$$

where $\hat{f}(\xi) := \mathcal{F}f(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$ denotes the Fourier transform of f and

$$q(x,\xi) = q(x,0) - ib(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{iy\cdot\xi} + iy\cdot\xi \mathbb{1}_{(0,1)}(|y|))\, \nu(x,dy). \quad (2.1)$$

We call q the *symbol* of the Feller process $(X_t)_{t\geq 0}$ and of the pseudo-differential operator; (b, Q, ν) are the *characteristics* of the symbol q. For each fixed $x \in \mathbb{R}^d$, $(b(x), Q(x), \nu(x,dy))$ is a Lévy triplet, i. e. $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d\times d}$ is a symmetric positive semidefinite matrix and $\nu(x,dy)$ a σ-finite measure on $(\mathbb{R}^d\setminus\{0\}, \mathcal{B}(\mathbb{R}^d\setminus\{0\}))$ satisfying $\int_{y\neq 0} \min\{|y|^2, 1\}\, \nu(x,dy) < \infty$. We use $q(x,D)$ to denote the pseudo-differential operator L with continuous negative definite symbol q. A family of continuous negative definite functions $(q(x,\cdot))_{x\in\mathbb{R}^d}$ is *locally bounded* if for any compact set $K \subseteq \mathbb{R}^d$ there exists $c > 0$ such that $|q(x,\xi)| \leq c(1 + |\xi|^2)$ for all $x \in K$, $\xi \in \mathbb{R}^d$. By [14, Lemma 2.1, Remark 2.2], this is equivalent to

$$\forall K \subseteq \mathbb{R}^d \text{ cpt.}: \quad \sup_{x\in K} \left(|q(x,0)| + |b(x)| + |Q(x)| + \int_{y\neq 0} (|y|^2 \wedge 1)\, \nu(x,dy) \right) < \infty. \quad (2.2)$$

If (2.2) holds for $K = \mathbb{R}^d$, we say that q has bounded coefficients. We will frequently use the following result.

Lemma 2.1. *Let L be a pseudo-differential operator with continuous negative definite symbol q and characteristics (b, Q, ν). Assume that $q(\cdot, 0) = 0$ and that q is locally bounded.*

(i). $\lim_{|x|\to\infty} Lf(x) = 0$ *for all $f \in C_c^\infty(\mathbb{R}^d)$ if, and only if,*

$$\lim_{|x|\to\infty} \nu(x, B(-x,r)) = 0 \qquad \text{for all } r > 0. \quad (2.3)$$

(ii). *If* $\lim_{|x|\to\infty} \sup_{|\xi|\leq |x|^{-1}} |\operatorname{Re} q(x,\xi)| = 0$, *then* (2.3) *holds.*

(iii). *$L(C_c^\infty(\mathbb{R}^d)) \subseteq C(\mathbb{R}^d)$ if, and only if, $x \mapsto q(x,\xi)$ is continuous for all $\xi \in \mathbb{R}^d$.*

For a proof of Lemma 2.1(i),(ii) see [1, Lemma 3.26] or [8, Theorem 1.27]; 2.1(iii) goes back to Schilling [13, Theorem 4.4], see also [10, Theorem A.1].

If the symbol q of a rich Feller process $(L_t)_{t\geq 0}$ does not depend on x, i. e. $q(x,\xi)=q(\xi)$, then $(L_t)_{t\geq 0}$ is a *Lévy process*. This is equivalent to saying that $(L_t)_{t\geq 0}$ has stationary and independent increments and càdlàg sample paths. The symbol $q=q(\xi)$ is called *characteristic exponent*. Our standard reference for Lévy processes is the monograph [12] by Sato. *Weak uniqueness* holds for the *Lévy-driven stochastic differential equation* (SDE, for short)

$$dX_t = \sigma(X_{t-})\,dL_t, \qquad X_0\sim\mu,$$

if any two weak solutions of the SDE have the same finite-dimensional distributions. We refer the reader to Situ [16] for further details.

Let $(A,\mathcal{D})$ be a linear operator with domain $\mathcal{D}\subseteq\mathcal{B}_b(E)$ and μ a probability measure on $(E,\mathcal{B}(E))$. A d-dimensional stochastic process $(X_t)_{t\geq 0}$, defined on a probability space $(\Omega,\mathcal{A},\mathbb{P}^\mu)$, with càdlàg sample paths is a *solution to the $(A,\mathcal{D})$-martingale problem with initial distribution μ*, if $X_0\sim\mu$ and

$$M_t^f := f(X_t)-f(X_0)-\int_0^t Af(X_s)\,ds, \qquad t\geq 0,$$

is a $\mathbb{P}^\mu$-martingale with respect to the canonical filtration of $(X_t)_{t\geq 0}$ for any $f\in\mathcal{D}$. By considering the measure $\mathbb{Q}^\mu$ induced by $(X_t)_{t\geq 0}$ on $D([0,\infty),E)$ we may assume without loss of generality that $\Omega=D([0,\infty),E)$ is the Skorohod space and $X_t(\omega):=\omega(t)$ the canonical process. The $(A,\mathcal{D})$-martingale problem is *well-posed* if for any initial distribution μ there exists a unique (in the sense of finite-dimensional distributions) solution to the $(A,\mathcal{D})$-martingale problem with initial distribution μ. For a comprehensive study of martingale problems see [2, Chapter 4].

3 Proof of the main results

In order to prove Theorem 1.1 we need the following statement which allows us to formulate the linear growth condition (G) in terms of the characteristics.

Lemma 3.1. *Let $(q(x,\cdot))_{x\in\mathbb{R}^d}$ be a family of continuous negative definite functions with characteristics (b,Q,ν) such that $q(\cdot,0)=0$. Then*

$$\limsup_{|x|\to\infty}\sup_{|\xi|\leq|x|^{-1}}|q(x,\xi)|<\infty \tag{G}$$

if, and only if, there exists an absolute constant $c>0$ such that each of the following conditions is satisfied for $|x|\gg 1$.

(i). $\left|b(x)+\int_{1\leq|y|<|x|/2} y\,\nu(x,dy)\right|\leq c(1+|x|)$.
(ii). $|Q(x)|+\int_{|y|\leq|x|/2}|y|^2\,\nu(x,dy)\leq c(1+|x|^2)$.
(iii). $\nu(x,\{y\in\mathbb{R}^d;|y|\geq 1\vee|x|/2\})\leq c$.

If (G) *holds and q is locally bounded, cf.* (2.2)*, then* (i)–(iii) *hold for all $x\in\mathbb{R}^d$.*

Proof. First we prove that (i)–(iii) are sufficient for (G). Because of (i) and (ii) it suffices to show that

$$p(x,\xi):=\int_{y\neq 0}(1-e^{iy\cdot\xi}+iy\cdot\xi\mathbb{1}_{(0,|x|/2)}(|y|))\,\nu(x,dy)$$

satisfies the linear growth condition (G). Using the elementary estimates

$$|1-e^{iy\cdot\xi}|\leq 2 \qquad\text{and}\qquad |1-e^{iy\cdot\xi}+iy\cdot\xi|\leq\frac{1}{2}|\xi|^2|y|^2$$

we find

$$|p(x,\xi)| \le \frac{|\xi|^2}{2} \int_{0<|y|<|x|/2} |y|^2 \, \nu(x,dy) + 2 \int_{|y|\ge|x|/2} \nu(x,dy)$$

for all $|x| \ge 1$ which implies by (ii) and (iii) that

$$\limsup_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} |p(x,\xi)| < \infty.$$

It remains to prove that (G) implies (i)-(iii). For (ii) and (iii) we use a similar idea as in [13, proof of Theorem 4.4]. It is known that the function g defined by

$$g(\eta) := \frac{1}{2} \int_{(0,\infty)} \frac{1}{(2\pi r)^{d/2}} \exp\left(-\frac{|\eta|^2}{2r} - \frac{r}{2}\right) dr, \qquad \eta \in \mathbb{R}^d,$$

has a finite second moment, i. e. $\int_{\mathbb{R}^d} |\eta|^2 g(\eta)\, d\eta < \infty$, and satisfies

$$\frac{|z|^2}{1+|z|^2} = \int_{\mathbb{R}^d} (1 - \cos(\eta \cdot z)) g(\eta)\, d\eta \tag{3.1}$$

for all $z \in \mathbb{R}^d$. As

$$\inf_{|z|\ge 1/2} \frac{|z|^2}{1+|z|^2} = \frac{1}{5} > 0$$

we obtain by applying Tonelli's theorem

$$\begin{aligned}
\frac{1}{5}\nu(x,\{y;|y|\ge|x|/2\}) &\le \int_{|y|\ge|x|/2} \frac{\left(\frac{|y|}{|x|}\right)^2}{1+\left(\frac{|y|}{|x|}\right)^2} \nu(x,dy) \\
&= \int_{|y|\ge|x|/2} \int_{\mathbb{R}^d} \left(1 - \cos\frac{\eta\cdot y}{|x|}\right) g(\eta)\, d\eta\, \nu(x,dy) \\
&\le \int_{\mathbb{R}^d} \operatorname{Re} q\left(x, \frac{\eta}{|x|}\right) d\eta.
\end{aligned}$$

Since

$$|\psi(\xi)| \le 2 \sup_{|\zeta|\le 1} |\psi(\zeta)| (1+|\xi|^2), \qquad \xi \in \mathbb{R}^d,$$

for any continuous negative definite function ψ, cf. [1, Proposition 2.17d)], we get

$$\nu(x,\{y;|y|\ge|x|/2\}) \le 10 \sup_{|\xi|\le 1} \left| q\left(x, \frac{\xi}{|x|}\right)\right| \int_{\mathbb{R}^d} (1+|\eta|^2) g(\eta)\, d\eta,$$

and this gives (iii) for $|x| \gg 1$. Next we prove (ii). First of all, we note that

$$0 \le \xi \cdot Q(x)\xi \le \operatorname{Re} q(x,\xi) \le |q(x,\xi)|$$

and therefore $|Q(x)| \le c(1+|x|^2)$ is a direct consequence of (G). On the other hand,

$$\inf_{|y|\le|x|/2} \frac{1}{|x|^2+|y|^2} \ge \frac{4}{5}\frac{1}{|x|^2}$$

implies that

$$\frac{4}{5}\frac{1}{|x|^2} \int_{|y|\le\frac{|x|}{2}} |y|^2 \, \nu(x,dy) \le \int_{|y|\le\frac{|x|}{2}} \frac{|y|^2}{|x|^2+|y|^2} \, \nu(x,dy) = \int_{|y|\le\frac{|x|}{2}} \frac{\left(\frac{|y|}{|x|}\right)^2}{1+\left(\frac{|y|}{|x|}\right)^2} \nu(x,dy).$$

Using (3.1) and applying Tonelli's theorem once more, we find

$$\int_{|y|\le|x|/2} |y|^2\,\nu(x,dy) \le \frac{5}{4}|x|^2 \int_{\mathbb{R}^d} \operatorname{Re} q\left(x,\frac{\eta}{|x|}\right) g(\eta)\,d\eta.$$

Hence,

$$\int_{|y|\le|x|/2} |y|^2\,\nu(x,dy) \le \frac{5}{4}|x|^2 \sup_{|\xi|\le 1}\left|q\left(x,\frac{\xi}{|x|}\right)\right| \int_{\mathbb{R}^d} (1+|\eta|^2) g(\eta)\,d\eta$$

and (ii) follows. Finally, as (ii) and (iii) imply that

$$\limsup_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} \left|q(x,\xi) - i\xi\cdot\left(b(x) + \int_{1\le|y|<|x|/2} y\,\nu(x,dy)\right)\right| < \infty,$$

see the first part of the proof, a straightforward application of the triangle inequality gives

$$\limsup_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} \left|i\xi\cdot\left(b(x) + \int_{1\le|y|<|x|/2} y\,\nu(x,dy)\right)\right| < \infty$$

which proves (i). □

Corollary 3.2. *Let A be a pseudo-differential operator with continuous negative definite symbol q such that $q(\cdot,0)=0$. If A maps $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$ and q satisfies the linear growth condition* (G)*, then there exists for any initial distribution μ a solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem which is conservative.*

Proof. As $A(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ and A satisfies the positive maximum principle, it follows from [2, Theorem 4.5.4] that there exists an $\mathbb{R}^d_\Delta$-valued solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem with initial distribution $\mu := \delta_x$. By considering the probability measure induced by $(X_t)_{t\ge0}$ on the Skorohod space $D([0,\infty),\mathbb{R}^d_\Delta)$, we may assume without loss of generality that $X_t(\omega) := \omega(t)$ is the canonical process on $\Omega := D([0,\infty),\mathbb{R}^d_\Delta)$. Lemma 3.1 implies that

$$\lim_{r\to\infty} \sup_{|z-x|\le 2r} \sup_{|\xi|\le r^{-1}} |q(z,\xi)| < \infty \qquad \text{for all } x\in\mathbb{R}^d,$$

and therefore [10, Corollary 3.2] shows that the solution with initial distribution δ_x does not explode in finite time with probability 1. By construction, see [2, proof of Theorem 4.5.4], the mapping $x\mapsto \mathbb{P}^x(B)$ is measurable for all $B\in\mathcal{F}_\infty^X := \sigma(X_t; t\ge0)$. If we define

$$\mathbb{P}^\mu(B) := \int_{\mathbb{R}^d} \mathbb{P}^x(B)\,\mu(dx), \qquad B\in\mathcal{F}_\infty^X$$

then $\mathbb{P}^\mu$ gives rise to a conservative solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem with initial condition μ. □

In Section 4 we will formulate Corollary 3.2 for solutions of stochastic differential equations, cf. Theorem 4.1. The next result is an important step to prove Theorem 1.1.

Lemma 3.3. *Let L be a pseudo-differential operator with continuous negative definite symbol p and characteristics (b,Q,ν) such that $p(\cdot,0)=0$ and $L(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$. Assume that $\nu(x,\{y\in\mathbb{R}^d; |y|\ge|x|/2\}) = 0$ for $|x|\gg1$ and*

$$\limsup_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} |p(x,\xi)| < \infty. \tag{G}$$

(i). *For any initial distribution μ there exists a probability measure $\mathbb{P}^\mu$ on $D([0,\infty),\mathbb{R}^d)$ such that the canonical process $(Y_t)_{t\geq 0}$ solves the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem and*

$$\mathbb{P}^\mu(B) = \int \mathbb{P}^x(B)\,\mu(dx) \qquad \text{for all } \; B \in \mathcal{F}_\infty^Y := \sigma(Y_t; t\geq 0). \tag{3.2}$$

(ii). *For any $t\geq 0$, $R>0$ and $\varepsilon>0$ there exist constants $\varrho>0$ and $\delta>0$ such that*

$$\mathbb{P}^\mu\left(\inf_{s\leq t}|Y_s| < R\right) \leq \varepsilon \tag{3.3}$$

for any initial distribution μ such that $\mu(B(0,\varrho))\leq\delta$.

(iii). *For any $t\geq 0$, $\varepsilon>0$ and any compact set $K\subseteq\mathbb{R}^d$ there exists $R>0$ such that*

$$\mu(K^c)\leq\frac{\varepsilon}{2} \implies \mathbb{P}^\mu\left(\sup_{s\leq t}|Y_s|\geq R\right)\leq\varepsilon. \tag{3.4}$$

Proof. (i) is a direct consequence of Corollary 3.2; we have to prove (ii) and (iii). To keep notation simple we show the result only in dimension $d=1$. Since L maps $C_c^\infty(\mathbb{R})$ into $C_\infty(\mathbb{R})$, the symbol p is locally bounded, cf. [1, Proposition 2.27(d)], and therefore Lemma 3.1 shows that 3.1(i)–(iii) hold for all $x\in\mathbb{R}$. Set $u(x):=1/(1+|x|^2)$, $x\in\mathbb{R}$, then

$$|u'(x)|\leq 2|x|u(x)^2 \qquad \text{and} \qquad |u''(x)|\leq 6u(x)^2 \qquad \text{for all } \; x\in\mathbb{R}. \tag{3.5}$$

Clearly, $|Lu(x)|\leq I_1+I_2$ where

$$I_1 := \left|b(x)+\int_{1\leq|y|<|x|/2} y\,\nu(x,dy)\right| |u'(x)| + \frac{1}{2}|Q(x)|\,|u''(x)|$$

$$I_2 := \left|\int_{y<|x/2} (u(x+y)-u(x)-u'(x)y)\,\nu(x,dy)\right|$$

for all $|x|\gg 1$. By Lemma 3.1 and (3.5) there exists a constant $c_1>0$ such that $I_1\leq c_1u(x)$ for all $x\in\mathbb{R}$. On the other hand, Taylor's formula shows

$$I_2\leq\frac{1}{2}\int_{|y|<|x|/2}|y|^2\,|u''(\zeta)|\,\nu(x,dy)$$

for some intermediate value $\zeta=\zeta(x,y)$ between x and $x+y$. Since $|y|<|x|/2$, we have $|\zeta|\geq|x|/2$; hence, by (3.5),

$$|u''(\zeta)|\leq 6u(\zeta)^2\leq 24u(x)^2.$$

Applying Lemma 3.1, we find that there exists a constant $c_2>0$ such that

$$I_2\leq 24u(x)^2\int_{|y|<|x|/2}|y|^2\,\nu(x,dy)\leq c_2u(x).$$

Consequently, $|Lu(x)|\leq(c_1+c_2)u(x)$ for all $|x|\gg 1$. As Lu is bounded and u is bounded away from 0 on compact sets, we can choose a constant $c_3>0$ such that

$$|Lu(x)|\leq c_3u(x) \qquad \text{for all } \; x\in\mathbb{R}. \tag{$\star$}$$

Define $\tau_R:=\inf\{t\geq 0;|Y_t|<R\}$. Using a standard truncation and stopping technique it follows that

$$\mathbb{E}^\mu u(Y_{t\wedge\tau_R})-\mathbb{E}^\mu u(Y_0)=\mathbb{E}^\mu\left(\int_{(0,t\wedge\tau_R)}Lu(Y_s)\,ds\right).$$

Hence, by ($\star$),

$$\mathbb{E}^\mu u(Y_{t\wedge\tau_R}) \le \mathbb{E}^\mu u(Y_0) + c_3\mathbb{E}^\mu\left(\int_{(0,t)} u(Y_{s\wedge\tau_R})\,ds\right).$$

An application of Gronwall's inequality shows that there exists a constant $C>0$ such that

$$\mathbb{E}^\mu u(Y_{t\wedge\tau_R}) \le e^{Ct}\mathbb{E}^\mu u(Y_0) \qquad \text{for all } \ t\ge 0.$$

By the Markov inequality, this implies that

$$\begin{aligned}\mathbb{P}^\mu\left(\inf_{s\le t}|Y_s|<R\right) \le \mathbb{P}^\mu(|Y_{t\wedge\tau_R}|\le R) \le \mathbb{P}^\mu\big(u(Y_{t\wedge\tau_R})\ge u(R)\big) &\le \frac{1}{u(R)}\mathbb{E}^\mu u(Y_{t\wedge\tau_R})\\ &\le \frac{1}{u(R)}e^{Ct}\mathbb{E}^\mu u(Y_0).\end{aligned}$$

If μ is an initial distribution such that $\mu(B(0,\varrho))\le\delta$, then $\mathbb{E}^\mu u(Y_0)\le\delta+\varrho^{-2}$. Choosing ϱ sufficiently large and $\delta>0$ sufficiently small, we get (3.3). The proof of (iii) is similar. If we set $v(x):=x^2+1$, then there exists by Lemma 3.1 a constant $c>0$ such that $|Lv(x)|\le cv(x)$ for all $x\in\mathbb{R}$. Applying Gronwall's inequality another time, we find a constant $C>0$ such that

$$\mathbb{E}^\mu v(Y_{t\wedge\sigma_R}) \le e^{Ct}\mathbb{E}^\mu v(Y_0), \qquad t\ge 0,$$

where $\sigma_R:=\inf\{t\ge 0;|Y_t|\ge R\}$ denotes the exit time from the ball $B(0,R)$. Hence, by the Markov inequality,

$$\mathbb{P}^\mu\left(\sup_{s\le t}|Y_s|\ge R\right) \le \mathbb{P}^\mu\big(v(Y_{t\wedge\sigma_R})\ge v(R)\big) \le \frac{1}{v(R)}e^{Ct}\mathbb{E}^\mu v(Y_0).$$

In particular we can choose for any compact set $K\subseteq\mathbb{R}$ and any $\varepsilon>0$ some $R>0$ such that

$$\mathbb{P}^x\left(\sup_{s\le t}|Y_s|\ge R\right)\le\frac{\varepsilon}{2} \qquad \text{for all } \ x\in K.$$

Now if μ is an initial distribution such that $\mu(K^c)\le\varepsilon/2$, then, by (3.2),

$$\begin{aligned}\mathbb{P}^\mu\left(\sup_{s\le t}|Y_s|\ge R\right) &= \int_K \mathbb{P}^x\left(\sup_{s\le t}|Y_s|\ge R\right)\mu(dx) + \int_{K^c}\mathbb{P}^x\left(\sup_{s\le t}|Y_s|\ge R\right)\mu(dx)\\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}.\end{aligned}$$

□

For the proof of Theorem 1.1 we will use the following result which follows e. g. from [4, Theorem 4.1.16, Proof of Corollary 4.6.4].

Lemma 3.4. *Let A be a pseudo-differential operator with negative definite symbol q such that $A: C_c^\infty(\mathbb{R}^d)\to C_b(\mathbb{R}^d)$. If the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed and the unique solution $(X_t)_{t\ge0}$ satisfies the compact containment condition*

$$\sup_{x\in K}\mathbb{P}^x\left(\sup_{s\le t}|X_s|\ge r\right)\xrightarrow{r\to\infty}0$$

for any compact set $K\subseteq\mathbb{R}^d$, then $x\mapsto\mathbb{E}^x f(X_t)$ is continuous for all $f\in C_b(\mathbb{R}^d)$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The well-posedness implies that the solution $(X_t)_{t\ge0}$ is a Markov process, see e. g. [2, Theorem 4.4.2], and by Corollary 3.2 the (unique) solution is conservative. In order to prove that $(X_t)_{t\ge0}$ is a Feller process, we have to show that the semigroup $T_tf(x):=\mathbb{E}^x f(X_t)$, $f\in C_\infty(\mathbb{R}^d)$, has the following properties, cf. [1, Lemma 1.4]:

(i). continuity at $t = 0$: $T_t f(x) \to f(x)$ as $t \to 0$ for any $x \in \mathbb{R}^d$ and $f \in C_\infty(\mathbb{R}^d)$.
(ii). Feller property: $T_t(C_\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ for all $t \geq 0$.

The first property is a direct consequence of the right-continuity of the sample paths and the dominated convergence theorem. Since we know that the martingale problem is well posed, it suffices to construct a solution to the martingale problem satisfying (ii). Write $\nu(x, dy) = \nu_s(x, dy) + \nu_l(x, dy)$ where

$$\nu_s(x, B) := \int_{|y|<1\vee|x|/2} \mathbb{1}_B(y)\, \nu(x, dy) \qquad \nu_l(x, B) := \int_{|y|\geq 1\vee|x|/2} \mathbb{1}_B(y)\, \nu(x, dy)$$

are the small jumps and large jumps, respectively, and denote by p the symbol with characteristics (b, Q, ν_s). By Corollary 3.2 there exists for any initial distribution μ a conservative solution to the $(p(x, D), C_c^\infty(\mathbb{R}^d))$-martingale problem, and the solution satisfies 3.3(ii) and 3.3(iii). Using the same reasoning as in [2, proof of Proposition 4.10.2] it is possible to show that we can use interlacing to construct a solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem with initial distribution $\mu = \delta_x$:

$$X_t := \sum_{k\geq 0} Y^{(k)}_{t-\tau_k} \mathbb{1}_{[\tau_k, \tau_{k+1})}(t)$$

where

- $\tau_k := \inf\{t \geq 0; N_t = k\} = \sum_{j=1}^k \sigma_j$ are the jump times of a Poisson process $(N_t)_{t\geq 0}$ with intensity $\lambda := \sup_{z\in\mathbb{R}^d} \nu_l(z, \mathbb{R}^d\backslash\{0\})$, i. e. $\sigma_j \sim \mathrm{Exp}(\lambda)$ are independent and identically distributed. Note that $\lambda < \infty$ by Lemma 3.1.
- $(Y_t^{(k,\mu_k)})_{t\geq 0} := (Y_t^{(k)})_{t\geq 0}$ is a solution to the $(p(x, D), C_c^\infty(\mathbb{R}^d))$-martingale problem with initial distribution

$$\mu_k(B) := \frac{1}{\lambda}\mathbb{E}^x\left(\int \mathbb{1}_B(z + y)\, \nu_l(z, dy) + (\lambda - \nu_l(z, \mathbb{R}^d\backslash\{0\}))\delta_z(B)\Big|_{z=Y^{(k-1)}_{\sigma_{k-1}-}} \right) \tag{3.6}$$

  for $k \geq 1$ and $\mu_0(dy) := \delta_x(dy)$. Moreover, $Y^{(k)}$ and $(\sigma_j)_{j\geq k+1}$ are independent for all $k \geq 0$.
- $\mathbb{P}^x$ is a probability measure which depends on the initial distribution $\mu = \delta_x$ of $(X_t)_{t\geq 0}$.

Note that if we define a linear operator P by

$$Pf(z) := \int f(z + y)\, \nu_l(z, dy) + (\lambda - \nu_l(z, \mathbb{R}^d\backslash\{0\}))f(z), \qquad f \in C_\infty(\mathbb{R}^d),\ z \in \mathbb{R}^d \tag{3.7}$$

then (3.4) implies that

$$\mathbb{E}^x f(Y_0^{(k)}) = \frac{1}{\lambda}\mathbb{E}^x(Pf(Y^{(k-1)}_{\sigma_{k-1}-})) \qquad \text{for all } f \in \mathcal{B}_b(\mathbb{R}^d), k \geq 1. \tag{3.8}$$

Before we proceed with the proof, let us give a remark on the construction of $(X_t)_{t\geq 0}$. The intensity of the Poisson process $(N_t)_{t\geq 0}$, which triggers the "large jumps", is $\lambda = \sup_z \lambda(z)$ where $\lambda(z) := \nu_l(z, \mathbb{R}^d\backslash\{0\})$ is the "state-space dependent intensity" of the large jumps. Roughly speaking the second term on the right-hand side of (3.6) is needed to thin out the large jumps; with probability $\lambda^{-1}\mathbb{E}^x((\lambda - \lambda(Y^{(k-1)}_{\sigma_{k-1}-}))$ there is no large jump at time σ_{k-1}, and therefore the effective jump intensity at time $t = \sigma_{k-1}$ is $\lambda(Y^{(k-1)}_{\sigma_{k-1}-})$.

We will prove that $(X_t)_{t\geq 0}$ has the Feller property. To this end, we first show that for any $t \geq 0$, $\varepsilon > 0$, $k \geq 1$ and any compact set $K \subseteq \mathbb{R}^d$ there exists $R > 0$ such that

$$\mathbb{P}^x\left(\sup_{s\leq t} |Y_s^{(j,\mu_j)}| \geq R\right) \leq \varepsilon \qquad \text{for all } x \in K, j = 0, \ldots, k; \tag{3.9}$$

we prove (3.9) by induction. Note that $\mu_j = \mu_j(x)$ depends on the initial distribution of $(X_t)_{t\geq 0}$.

Base ($k = 0$): This is a direct consequence of Lemma 3.3(ii) since $\mu_0(dy) = \delta_x(dy)$.
Inductive step ($k \to k+1$): Because of Lemma 3.3(ii) and the induction hypothesis, it suffices to show that there exists $C \subseteq \mathbb{R}^d$ compact such that $\mathbb{P}^x(Y_0^{(k+1,\mu_{k+1})} \notin C) \leq \varepsilon/2$ for all $x \in K$. Choose $m \geq 0$ sufficiently large such that $\mathbb{P}^x(\sigma_k \geq m) \leq \varepsilon' := \varepsilon/8$, and choose $R > 0$ such that (3.9) holds with $\varepsilon := \varepsilon'$, $t := m$. Then, by (3.8) and our choice of R,

$$\begin{aligned}\mathbb{P}^x(|Y_0^{(k+1)}| \geq r) &= \frac{1}{\lambda}\mathbb{E}^x\left((P\mathbb{1}_{\overline{B(0,r)}^c})(Y^{(k)}_{\sigma_k-})\right)\\ &\leq \epsilon' + \frac{1}{\lambda}\mathbb{E}^x\left(\mathbb{1}_{\{\sup_{s\leq m}|Y_s^{(k)}|\leq R\}}(P\mathbb{1}_{\overline{B(0,r)}^c})(Y^{(k)}_{\sigma_k-})\right)\end{aligned}$$

which implies for $r > R$, $x \in K$

$$\begin{aligned}&\mathbb{P}^x(|Y_0^{(k+1)}| \geq r)\\ &\leq \varepsilon' + \frac{1}{\lambda}\mathbb{E}^x\left(\mathbb{1}_{\{\sup_{s\leq m}|Y_s^{(k)}|\leq R\}}\left[\int \mathbb{1}_{B(0,r)^c}(Y^{(k)}_{\sigma_k-}+y)\,\nu_l(Y^{(k)}_{\sigma_k-},dy) + 2\lambda\mathbb{1}_{B(0,r)^c}(Y^{(k)}_{\sigma_k-})\right]\right)\\ &\leq 3\varepsilon' + \frac{1}{\lambda}\mathbb{E}^x\left(\mathbb{1}_{\{\sup_{s\leq m}|Y_s^{(k)}|\leq R\}}\int_0^m\int_{|y|\geq r-R}\nu(Y^{(k)}_{t-},dy)\,\mathbb{P}^x_{\sigma_k}(dt)\right)\\ &\leq 3\varepsilon' + \frac{1}{\lambda}\sup_{|z|\leq R}\nu(z,B(0,r-R)^c).\end{aligned}$$

The second term on the right-hand side converges to 0 as $r \to \infty$, cf. [13, Theorem 4.4] or [10, Theorem A.1], and therefore we can choose $r > 0$ sufficiently large such that $\mathbb{P}^x(|Y_0^{(k+1)}| \geq r) \leq 4\varepsilon' = \varepsilon/2$ for all $x \in K$.

This finishes the proof of (3.9). For fixed $\varepsilon > 0$ choose a number $k \geq 1$ such that $\mathbb{P}^x(N_t \geq k+1) \leq \varepsilon$. By definition of $(X_t)_{t\geq 0}$ and (3.9), we get

$$\sup_{x\in K}\mathbb{P}^x\left(\sup_{s\leq t}|X_s| \geq R\right) \leq \sup_{x\in K}\mathbb{P}^x\left(\bigcup_{j=0}^k\left\{\sup_{s\leq t}\left|Y_s^{(j,\mu_j)}\right| \geq R\right\}\right) + \varepsilon \leq (k+1)\varepsilon.$$

Thus, by Lemma 3.4, $x \mapsto T_tf(x) = \mathbb{E}^x f(X_t)$ is continuous for any $f \in C_\infty(\mathbb{R}^d)$. It remains to show that T_tf vanishes at infinity; to this end we will show that for any $r > 0$, $\varepsilon > 0$ there exists a constant $M > 0$ such that

$$\mathbb{P}^x\left(\inf_{s\leq t}|X_s| < r\right) \leq \varepsilon \qquad \text{for all } |x| \geq M. \tag{3.10}$$

It follows from Lemma 3.1 and the very definition of λ that Pf defined in (3.7) is bounded and

$$\begin{aligned}|Pf(x)| &\leq \int_{|x+y|<r}|f(x+y)|\,\nu_l(x,dy) + \int_{|x+y|\geq r}|f(x+y)|\,\nu_l(x,dy) + 2\lambda|f(x)|\\ &\leq \|f\|_\infty\nu(x,B(-x,r)) + \lambda\sup_{|z|\geq r}|f(z)| + 2\lambda|f(x)|\\ &\xrightarrow{|x|\to\infty} \lambda\sup_{|z|\geq r}|f(z)| \xrightarrow{r\to\infty} 0,\end{aligned}$$

i. e. Pf vanishes at infinity for any $f \in C_\infty(\mathbb{R}^d)$. We claim that for any $k \geq 0$, $\varepsilon > 0$, $t \geq 0$ and $r > 0$ there exists a constant $M > 0$ such that

$$\mathbb{P}^x\left(\inf_{s\leq t}|Y_s^{(j,\mu_j)}| < r\right) \leq \varepsilon \qquad \text{for all } j = 0,\dots,k, |x| \geq M. \tag{3.11}$$

We prove (3.11) by induction.

Base ($k = 0$): This follows from Lemma 3.3(ii) since $\mu_0(dy) = \delta_x(dy)$.
Inductive step ($k \to k+1$): For fixed $r > 0$ choose $\delta > 0$ and $\varrho > 0$ as in 3.3(ii). By 3.3(ii) it suffices to show that there exists $M > 0$ such that

$$\mu_{k+1}(B(0,\varrho)) \le \delta \qquad \text{for all } |x| \ge M. \tag{$\star$}$$

(Note that $\mu_{k+1} = \mu_{k+1}(x)$ depends on the initial distribution of $(X_t)_{t\ge 0}$.) Pick a cut-off function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\mathbb{1}_{B(0,\varrho)} \le \chi \le \mathbb{1}_{B(0,\varrho+1)}$, then by (3.6),

$$\mu_{k+1}(B(0,\varrho)) \le \mathbb{E}^x \chi(Y_0^{(k+1,\mu_{k+1})}) = \frac{1}{\lambda}\mathbb{E}^x\big((P\chi)(Y_{\sigma_k-}^{(k,\mu_k)})\big).$$

If $\|P\chi\|_\infty = 0$ this proves ($\star$). If $\|P\chi\|_\infty$ is strictly larger than 0, then we can choose $m \ge 1$ such that $\mathbb{P}^x(\sigma_1 \ge m) \le \delta/(2\|P\chi\|_\infty)$. Since $P\chi$ vanishes at infinity, we have $\sup_{|z|\ge R}|P\chi(z)| \le \lambda\delta/4$ for $R > 0$ sufficiently large. By the induction hypothesis, there exists $M > 0$ such that (3.11) holds with $\varepsilon := \lambda\delta/4$, $r := R$ and $t := m$. Then

$$|\mathbb{E}^x(P\chi)(Y_{s-}^{(k,\mu_k)})| \le \mathbb{P}^x\left(|Y_{s-}^{(k,\mu_k)}| < R\right)\|P\chi\|_\infty + \sup_{|z|\ge R}|P\chi(z)| \le \frac{1}{2}\lambda\delta$$

for all $s \le m$ and $|x| \ge M$, and therefore

$$\begin{aligned}\mu_{k+1}(B(0,\varrho)) = \frac{1}{\lambda}\mathbb{E}^x(P\chi)(Y_{\sigma_k-}^{(k,\mu_k)}) &\le \frac{1}{\lambda}\mathbb{E}^x\left(\int_{(0,\infty)} P\chi(Y_{s-}^{(k,\mu_k)})\,\mathbb{P}^x_{\sigma_k}(ds)\right)\\ &\le \frac{\delta}{2} + \|P\chi\|_\infty \int_{(m,\infty)} \mathbb{P}^x_{\sigma_1}(ds) \le \delta.\end{aligned}$$

This finishes the proof of (3.11). For fixed $\varepsilon > 0$ and $t \ge 0$ choose $k \ge 1$ such that $\mathbb{P}^x(N_t \ge k+1) \le \varepsilon$. Choose $M > 0$ as in (3.11), then

$$\mathbb{P}^x(|X_t| < R) \le \mathbb{P}^x\left(\bigcup_{j=0}^k \left\{\inf_{s\le t}|Y_s^{(j)}| < R\right\}\right) + \varepsilon \le 2\varepsilon \qquad \text{for all } |x| \ge M.$$

Consequently, we have shown that $(X_t)_{t\ge 0}$ is a Feller process. Since $(X_t)_{t\ge 0}$ solves the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem, we have

$$\mathbb{E}^x u(X_{t\wedge\tau_r^x}) - u(x) = \mathbb{E}^x\left(\int_{(0,t\wedge\tau_r^x)} Au(X_s)\,ds\right), \qquad u \in C_c^\infty(\mathbb{R}^d),$$

where $\tau_r^x := \inf\{t \ge 0; |X_t - x| \ge r\}$ denotes the exit time from the ball $B(x,r)$. Using that $A(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$, it is not difficult to see that the generator of $(X_t)_{t\ge 0}$ is, when restricted to $C_c^\infty(\mathbb{R}^d)$, a pseudo-differential operator with symbol q, see e. g. [7, Proof of Theorem 3.5, Step 2] for details. This means that $(X_t)_{t\ge 0}$ is a rich Feller process with symbol q. □

Proof of Corollary 1.3. By Corollary 3.2 there exists for any initial distribution μ a solution to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem, and by assumption the martingale problem for the pseudo-differential operator A_k with symbol q_k is well-posed. Therefore [3, Theorem 5.3], see also [2, Theorem 4.6.2], shows that the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed. Now the assertion follows from Theorem 1.1. □

4 Applications

In this section we apply our results to Lévy-driven stochastic differential equations (SDEs) and stable-like processes. Corollary 3.2 gives the following general existence result for weak solutions to Lévy-driven SDEs.

Theorem 4.1. *Let $(L_t)_{t\geq 0}$ be a k-dimensional Lévy process with characteristic exponent ψ and Lévy triplet (b, Q, ν). Let $\ell : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ be continuous functions which grow at most linearly. If*

$$\nu(\{y \in \mathbb{R}^k; |\sigma(x)\cdot y + x| \leq r\}) \xrightarrow{|x|\to\infty} 0 \qquad \text{for all } r > 0, \tag{4.1}$$

then the SDE

$$dX_t = \ell(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 \sim \mu \tag{4.2}$$

has for any initial distribution μ a weak solution $(X_t)_{t\geq 0}$ which is conservative.

Note that (4.1) is, in particular, satisfied if

$$\lim_{|x|\to\infty} \sup_{|\xi|\leq |x|^{-1}} |\operatorname{Re}\psi(\sigma(x)^T\xi)| = 0,$$

e. g. if σ is at most of sublinear growth, cf. Lemma 2.1(ii).

Proof. Denote by A the pseudo-differential operator with symbol

$$q(x,\xi) := -i\ell(x)\cdot\xi + \psi(\sigma(x)^T\xi).$$

Since q is locally bounded and $x \mapsto q(x,\xi)$ is continuous for all $\xi \in \mathbb{R}^d$ it follows from (4.2) that $A(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$, cf. Lemma 2.1. Because ℓ, σ are at most of linear growth, q satisfies the growth condition (G). Applying Corollary 3.2 we find that there exists a conservative solution $(X_t)_{t\geq 0}$ to the $(A, C_c^\infty(\mathbb{R}^d))$-martingale problem. By [6], $(X_t)_{t\geq 0}$ is a weak solution to the SDE (4.2). □

For $\alpha \in (0,1]$ we denote by

$$\mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^n) := \left\{ f : \mathbb{R}^d \to \mathbb{R}^n; \forall x \in \mathbb{R}^d : \sup_{|y-x|\leq 1} \frac{|f(y)-f(x)|}{|y-x|^\alpha} < \infty \right\}$$

$$\mathcal{C}^\alpha(\mathbb{R}^d,\mathbb{R}^n) := \left\{ f : \mathbb{R}^d \to \mathbb{R}^n; \sup_{x\neq y} \frac{|f(y)-f(x)|}{|y-x|^\alpha} < \infty \right\}$$

the space of (locally) Hölder continuous functions with Hölder exponent α.

Theorem 4.2. *Let $(L_t)_{t\geq 0}$ be a k-dimensional Lévy process with Lévy triplet (b, Q, ν) and characteristic exponent ψ. Suppose that there exist $\alpha, \beta \in (0,1]$ such that the Lévy-driven SDE*

$$dX_t = f(X_{t-})\,dt + g(X_{t-})\,dL_t, \qquad X_0 \sim \mu$$

has a unique weak solution for any initial distribution μ and any two bounded functions $f \in \mathcal{C}^\alpha(\mathbb{R}^d,\mathbb{R}^d)$ and $g \in \mathcal{C}^\beta(\mathbb{R}^d,\mathbb{R}^{d\times k})$ such that

$$|g(x)^T\xi| \geq c|\xi|, \qquad \xi \in \mathbb{R}^d, x \in \mathbb{R}^d$$

for some constant $c > 0$. Then the SDE

$$dX_t = \ell(X_{t-})\,dt + \sigma(X_{t-})\,dL_t, \qquad X_0 \sim \mu$$

has a unique weak solution for any $\ell \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$, $\sigma \in C^\beta_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^{d\times k})$ *which are at most of linear growth and satisfy*

$$\nu(\{y\in\mathbb{R}^k; |\sigma(x)\cdot y + x|\le r\}) \xrightarrow{|x|\to\infty} 0 \qquad \text{for all } r>0 \tag{4.3}$$

and

$$\forall n\in\mathbb{N}\ \exists c_n>0\ \forall |x|\le n, \xi\in\mathbb{R}^d:\ \ |\sigma(x)^T\xi| \ge c_n|\xi|. \tag{4.4}$$

The unique weak solution is a conservative rich Feller process with symbol

$$q(x,\xi) := -i\ell(x)\cdot\xi + \psi(\sigma(x)^T\xi), \qquad x,\xi\in\mathbb{R}^d.$$

Proof. Let $\ell\in\mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$ and $\sigma\in\mathcal{C}^\beta_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^{d\times k})$ be two functions which grow at most linearly and satisfy (4.3), (4.4). Lemma 2.1 shows that the pseudo-differential operator A with symbol q satisfies $A(C_c^\infty(\mathbb{R}^d))\subseteq C_\infty(\mathbb{R}^d)$. Moreover, since σ, ℓ are at most of linear growth, the growth condition (G) is clearly satisfied. Set

$$\ell_k(x) := \begin{cases} \ell(x), & |x|<k \\ \ell\left(k\frac{x}{|x|}\right), & |x|\ge k \end{cases} \quad\text{and}\quad \sigma_k(x) := \begin{cases} \sigma(x), & |x|<k, \\ \sigma\left(k\frac{x}{|x|}\right), & |x|\ge k. \end{cases}$$

By assumption, the SDE

$$dX_t = \ell_k(X_{t-})\,dt + \sigma_k(X_{t-})\,dL_t, \qquad X_0\sim\mu,$$

has a unique weak solution for any initial distribution μ for all $k\ge 1$. By [6] (see also [7, Lemma 3.3]) this implies that the $(A_k, C_c^\infty(\mathbb{R}^d))$-martingale problem for the pseudo-differential operator with symbol $q_k(x,\xi) := -i\ell_k(x)\cdot\xi + \psi(\sigma_k(x)^T\xi)$ is well-posed. Since σ_k is bounded, we have

$$\nu(\{y\in\mathbb{R}^k; |\sigma_k(x)\cdot y + x|\le r\}) \xrightarrow{|x|\to\infty} 0 \qquad \text{for all } r>0,$$

and therefore Lemma 2.1 shows that A_k maps $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$. Now the assertion follows from Corollary 1.3. □

Applying Theorem 4.2 we obtain the following generalization of [9, Corollary 4.7], see also [11, Theorem 5.23].

Theorem 4.3. *Let $(L_t)_{t\ge0}$ be a one-dimensional Lévy process such that its characteristic exponent ψ satisfies the following conditions:*

(i). *ψ has a holomorphic extension Ψ to*

$$U := \{z\in\mathbb{C}; |\operatorname{Im} z|<m\} \cup \{z\in\mathbb{C}\backslash\{0\}; \arg z\in(-\vartheta,\vartheta)\cup(\pi-\vartheta,\pi+\vartheta)\}$$

for some $m\ge0$ and $\vartheta\in(0,\pi/2)$.

(ii). *There exist $\alpha\in(0,2]$, $\beta\in(1,2)$ and constants $c_1,c_2>0$ such that*

$$\operatorname{Re}\Psi(z) \ge c_1|\operatorname{Re} z|^\beta \qquad \text{for all } z\in U,\ |z|\gg 1,$$

and

$$|\Psi(z)| \le c_2(|z|^\alpha \mathbb{1}_{\{|z|\le1\}} + |z|^\beta \mathbb{1}_{\{|z|>1\}}), \qquad z\in U.$$

(iii). *There exists a constant $c_3>0$ such that $|\Psi'(z)|\le c_3|z|^{\beta-1}$ for all $z\in U$, $|z|\gg1$.*

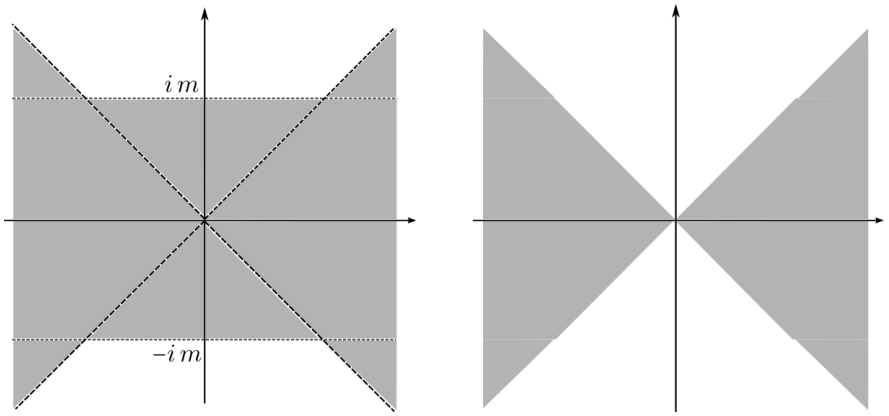

Figure 1: The domain $U = U(m, \vartheta)$ for $m > 0$ (left) and $m = 0$ (right).

Let $\ell : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ be two locally Hölder continuous functions which grow at most linearly. If

$$\nu(\{x; |\sigma(x)y + x| \le r\}) \xrightarrow{|x|\to\infty} 0 \qquad \text{for all } r > 0,$$

then the SDE

$$dX_t = \ell(X_{t-})\, dt + \sigma(X_{t-})\, dL_t, \qquad X_0 \sim \mu,$$

has a unique weak solution for any initial distribution μ. The unique solution is a conservative rich Feller process with symbol $q(x, \xi) := -i\ell(x)\xi + \psi(\sigma(x)\xi)$.

Proof. [9, Corollary 4.7] shows that the assumptions of Theorem 4.2 are satisfied, and this proves the assertion. □

Theorem 4.3 applies, for instance, to Lévy processes with the following characteristic exponents:

(i). (isotropic stable) $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}$, $\alpha \in (1, 2]$,

(ii). (relativistic stable) $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} - \varrho^\alpha$, $\xi \in \mathbb{R}$, $\varrho > 0$, $\alpha \in (1, 2)$,

(iii). (Lamperti stable) $\psi(\xi) = (|\xi|^2 + \varrho)_\alpha - (\varrho)_\alpha$, $\xi \in \mathbb{R}$, $\varrho > 0$, $\alpha \in (1/2, 1)$, where $(r)_\alpha := \Gamma(r + \alpha)/\Gamma(r)$ denotes the Pochhammer symbol,

(iv). (truncated Lévy process) $\psi(\xi) = (|\xi|^2 + \varrho^2)^{\alpha/2} \cos(\alpha \arctan(\varrho^{-1}|\xi|)) - \varrho^\alpha$, $\xi \in \mathbb{R}$, $\alpha \in (1, 2)$, $\varrho > 0$,

(v). (normal tempered stable) $\psi(\xi) = (\kappa^2 + (\xi - ib)^2)^{\alpha/2} - (\kappa^2 - b^2)^{\alpha/2}$, $\xi \in \mathbb{R}$, $\alpha \in (1, 2)$, $b > 0$, $|\kappa| > |b|$.

For further examples of Lévy processes satisfying the assumptions of Theorem 4.3 we refer to [9, 11].

We close this section with two further applications of Corollary 1.3. The first is an existence result for Feller processes with symbols of the form $p(x, \xi) = \varphi(x)q(x, \xi)$. Recall that $p(x, D)$ denotes the pseudo-differential operator with symbol p.

Theorem 4.4. *Let A be a pseudo-differential operator with symbol q such that $q(\cdot, 0) = 0$, $A(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ and*

$$\lim_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} |q(x, \xi)| < \infty.$$

Assume that for any continuous bdd. function $\sigma : \mathbb{R}^d \to (0,\infty)$ the $(\sigma(x)q(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem for the pseudo-differential operator with symbol $\sigma(x)q(x,\xi)$ is well-posed. If $\varphi : \mathbb{R}^d \to (0,\infty)$ is a continuous function such that

$$\lim_{|x|\to\infty} \sup_{|\xi|\le|x|^{-1}} \big(\varphi(x)|q(x,\xi)|\big) < \infty, \tag{4.5}$$

and

$$\varphi(x)\nu(x, B(-x,r)) \xrightarrow{|x|\to\infty} 0 \qquad \text{for all } r>0, \tag{4.6}$$

then there exists a conservative rich Feller process $(X_t)_{t\ge0}$ with symbol $p(x,\xi) := \varphi(x)q(x,\xi)$ and $(X_t)_{t\ge0}$ is the unique solution to the $(p(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem.

Theorem 4.4 is more general than [10, Theorem 4.6]. *Indeed:* If there exists a rich Feller process $(X_t)_{t\ge0}$ with symbol q and $C_c^\infty(\mathbb{R}^d)$ is a core for the infinitesimal generator of $(X_t)_{t\ge0}$, then, by [1, Theorem 4.2], there exists for any continuous bounded function $\sigma > 0$ a rich Feller process with symbol $\sigma(x)q(x,\xi)$ and core $C_c^\infty(\mathbb{R}^d)$, and therefore the $(\sigma(x)q(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed, cf. [5, Theorem 4.10.3].

Proof of Theorem 4.4. For given φ define

$$\varphi_k(x) := \varphi(x)\mathbb{1}_{B(0,k)}(x) + \varphi\left(k\frac{x}{|x|}\right)\mathbb{1}_{B(0,k)^c}(x).$$

By assumption, the $(\varphi_k(x)q(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed. Moreover, the boundedness of φ_k and the fact that $q(x,D)(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ imply that $\varphi_k(x)q(x,D)$ maps $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$. On the other hand, (4.6) gives, by Lemma 2.1, $p(x,D)(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$. Applying Corollary 1.3 proves the assertion. □

Example 4.5. Let $\varphi : \mathbb{R}^d \to (0,\infty)$ be a continuous fuction and $\alpha : \mathbb{R}^d \to (0,2]$ a locally Hölder continuous function. If there exists a constant $c > 0$ such that $\varphi(x) \le c(1+|x|^{\alpha(x)})$ for all $x \in \mathbb{R}^d$, then there exists a conservative rich Feller process $(X_t)_{t\ge0}$ with symbol

$$p(x,\xi) := \varphi(x)|\xi|^{\alpha(x)}, \qquad x,\xi\in\mathbb{R}^d,$$

and $(X_t)_{t\ge0}$ is the unique solution to the $(p(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem.

Indeed: If we set

$$\alpha_j(x) := \alpha(x)\mathbb{1}_{B(0,j)}(x) + \alpha\left(j\frac{x}{|x|}\right)\mathbb{1}_{B(0,j)^c}(x),$$

then [8, Theorem 5.2] shows that for each $j \ge 1$ there exists a rich Feller process with symbol $q_j(x,\xi) := |\xi|^{\alpha_j(x)}(x)$, and that $C_c^\infty(\mathbb{R}^d)$ is a core for the generator. By [1, Theorem 4.2], there exists for any continuous bounded function $\sigma > 0$ a rich Feller process with symbol $\sigma(x)q_j(x,\xi)$ and core $C_c^\infty(\mathbb{R}^d)$. This implies that the $(\sigma(x)q_j(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem is well posed, see e. g. [5, Theorem 4.10.3] or [8, Theorem 1.37]. Applying Theorem 4.4 we find that there exists a conservative rich Feller process with symbol $p_j(x,\xi) := \varphi(x)q_j(x,\xi)$, and that the $(p_j(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed. Now the assertion follows from Corollary 1.3.

Example 4.5 shows that Corollary 1.3 is useful to establish the existence of stable-like processes with unbounded coefficients. For relativistic stable-like processes we obtain the following general existence result.

Theorem 4.6. *Let $\alpha : \mathbb{R}^d \to (0,2]$, $m : \mathbb{R}^d \to (0,\infty)$ and $\kappa : \mathbb{R}^d \to (0,\infty)$ be locally Hölder continuous functions. If*

$$\sup_{|x|\geq 1} \frac{\kappa(x)}{|x|^2 m(x)^{2-\alpha(x)}} < \infty \tag{4.7}$$

and

$$\kappa(x)m(x)e^{-|x|m(x)/4} \xrightarrow{|x|\to\infty} 0, \tag{4.8}$$

then there exists a conservative rich Feller process $(X_t)_{t\geq 0}$ with symbol

$$q(x,\xi) := \kappa(x)\left[(|\xi|^2 + m(x)^2)^{\alpha(x)/2} - m(x)^{\alpha(x)}\right], \qquad x,\xi \in \mathbb{R}^d,$$

and $(X_t)_{t\geq 0}$ is the unique solution to the $(q(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem.

Note that κ and m do not need to be of linear growth; for instance if $\inf_x \alpha(x) > 0$, then we can choose $m(x) := e^{|x|}$ and $\kappa(x) := (1+|x|^k)$ for $k \geq 1$.

Proof of Theorem 4.6. For a function $f : \mathbb{R}^d \to \mathbb{R}$ set

$$f_i(x) := f(x)\mathbb{1}_{B(0,i)}(x) + f\left(i\frac{x}{|x|}\right)\mathbb{1}_{B(0,i)^c}(x)$$

and define

$$q_i(x,\xi) := \kappa_i(x)\left[(|\xi|^2 + m_i(x)^2)^{\alpha_i(x)/2} - m_i(x)^{\alpha_i(x)}\right].$$

Since κ_i, α_i and m_i are bounded Hölder continuous functions which are bounded away from 0, it follows from [11], see also [8], that the $(q_k(x,D), C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed. Consequently, the assertion follows from Corollary 1.3 if we can show that q satisfies (G) and that the pseudo-differential operators $q(x,D)$ and $q_i(x,D)$, $i \geq 1$, map $C_c^\infty(\mathbb{R}^d)$ into $C_\infty(\mathbb{R}^d)$. An application of Taylor's formula yields

$$\begin{aligned}\sup_{|\xi|\leq|x|^{-1}} |q(x,\xi)| &\leq \kappa(x)\big[(|x|^{-2} + m(x)^2)^{\alpha(x)/2} - (m(x)^2)^{\alpha(x)/2}\big] \\ &\leq \kappa(x)\frac{1}{|x|^2}\frac{\alpha(x)}{2}m(x)^{\alpha(x)-2},\end{aligned}$$

and by (4.7) this implies (G). It remains to prove the mapping properties of $q(x,D)$ and $q_i(x,D)$. Since $x \mapsto q_i(x,\xi)$ is continuous and

$$\sup_{|\xi|\leq|x|^{-1}} |q(x,\xi)| \leq \|\kappa_i\|_\infty \left(\inf_{|x|\leq i} m(x)\right)^{-2} \frac{1}{|x|^2} \xrightarrow{|x|\to\infty} 0,$$

it follows from Lemma 2.1 that $q_i(x,D)(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ for all $i \geq 1$. To prove $q(x,D)(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ we note that $x \mapsto q(x,\xi)$ is continuous, and therefore it suffices to show by Lemma 2.1 that

$$\lim_{|x|\to\infty} \nu(x, B(-x,r)) \xrightarrow{|x|\to\infty} 0, \qquad r > 0,$$

where $\nu(x,dy)$ is for each fixed $x \in \mathbb{R}^d$ the Lévy measure of a relativistic stable Lévy process with parameters $(\kappa(x), m(x), \alpha(x))$. It is known that $\nu(x,dy) \leq c\kappa(x)e^{-|y|m(x)/2}\,dy$ on $B(0,1)^c$, and therefore

$$\nu(x, B(-x,r)) \leq c\kappa(x)\int_{B(-x,r)} e^{-|y|m(x)/2}\,dy = c\kappa(x)\left(e^{-|x-r|m(x)/2} - e^{-|x+r|m(x)/2}\right).$$

For $|x| \gg 1$ and fixed $r > 0$ we obtain from Taylor's formula

$$\nu(x, B(-x,r)) \leq c\kappa(x)m(x)e^{-|x|m(x)/4} \xrightarrow[(4.8)]{|x|\to\infty} 0. \qquad \square$$

References

[1] Böttcher, B., Schilling, R. L., Wang, J.: Lévy-Type Processes: Construction, Approximation and Sample Path Properties. Springer Lecture Notes in Mathematics vol. **2099**, (vol. III of the "Lévy Matters" subseries). *Springer*, Cham, 2013. xvii+199pp. MR-3156646

[2] Ethier, S. N., Kurtz, T. G.: Markov processes - characterization and convergence. *Wiley*, New York, 1986. x+534pp. MR-0838085

[3] Hoh, W.: *Pseudo-Differential Operators Generating Markov Processes*. Habilitationsschrift. Universität Bielefeld, Bielefeld 1998.

[4] Jacob, N.: Pseudo Differential Operators and Markov Processes III. *Imperial College Press*, London 2005. xxvii+474pp. MR-2158336

[5] Kolokoltsov, V.: Markov Processes, Semigroups and Generators. *De Gruyter*, Berlin, 2011. xvii+430pp. MR-2780345

[6] Kurtz, T. G.: Equivalence of stochastic equations and martingale problems. In: Crisan, D. (ed.), Stochastic Analysis 2010, *Springer*, Heidelberg, 2011, pp. 113–130. MR-2789081

[7] Kühn, F.: Solutions of Lévy-driven SDEs with unbounded coefficients as Feller processes. arXiv:1610.02286. To appear in *Proc. Amer. Math. Soc.*

[8] Kühn, F.: Probability and Heat Kernel Estimates for Lévy(-Type) Processes. PhD Thesis, Technische Universität Dresden 2016. `http://nbn-resolving.de/urn:nbn:de:bsz:14-qucosa-214839`

[9] Kühn, F.: Transition probabilities of Lévy-type processes: Parametrix construction. arXiv:1702.00778.

[10] Kühn, F.: Random time changes of Feller processes. arXiv:1705.02830.

[11] Kühn, F.: Lévy-Type Processes: Moments, Construction and Heat Kernel Estimates. Springer Lecture Notes in Mathematics vol. **2187** (vol. VI of the "Lévy Matters" subseries). *Springer*, Cham, 2017. xxii+243pp. MR-3701414

[12] Sato, K.-I.: Lévy Processes and Infinitely Divisible Distributions. *Cambridge University Press*, Cambridge, 2005. xiv+521pp. MR-1739520

[13] Schilling, R. L.: Conservativeness and Extensions of Feller Semigroups. *Positivity* **2** (1998), 239–256. MR-1653474

[14] Schilling, R. L.: Growth and Hölder conditions for the sample paths of Feller processes. *Probab. Theory Related Fields* **112** (1998), 565–611. MR-1664705

[15] Schilling, R. L., Schnurr, A.: The Symbol Associated with the Solution of a Stochastic Differential Equation. *Electron. J. Probab.* **15** (2010), 1369–1393. MR-2721050

[16] Situ, R.: Theory of stochastic differential equations with jumps and applications. *Springer*, New York, 2005. xx+434pp. MR-2160585

[17] van Casteren, J. A.: On martingales and Feller semigroups. *Results Math.* **21** (1992), 274–288. MR-1157331

Acknowledgments. I would like to thank René Schilling for helpful comments and suggestions.