

Reflected Brownian motion: selection, approximation and linearization

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Abstract

We construct a family of SDEs with smooth coefficients whose solutions select a reflected Brownian flow as well as a corresponding stochastic damped transport process (W_t) , the limiting pair gives a probabilistic representation for solutions of the heat equations on differential 1-forms with the absolute boundary conditions. The transport process evolves pathwise by the Ricci curvature in the interior, by the shape operator on the boundary where it is driven by the boundary local time, and with its normal part erased at the end of the excursions to the boundary of the reflected Brownian motion. On the half line, this construction selects the Skorohod solution (and its derivative with respect to initial points), not the Tanaka solution; on the half space it agrees with the construction of N. Ikeda and S. Watanabe [29] by Poisson point processes. The construction leads also to an approximation for the boundary local time, in the topology of uniform convergence but not in the semi-martingale topology, indicating the difficulty in proving convergence of solutions of a family of random ODE's to the solution of a stochastic equation driven by the local time and with jumps. In addition, we obtain a differentiation formula for the heat semi-group with Neumann boundary condition and prove also that (W_t) is the weak derivative of a family of reflected Brownian motions with respect to the initial point.

Keywords: Brownian motion; reflection; local time; boundary; stochastic flow; heat equation.

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1 Introduction

A. Throughout this paper $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is an underlying filtered probability space satisfying the usual assumptions, and M is a smooth connected d -dimensional Riemannian manifold with boundary ∂M . We denote by M° the interior of M and by ν the unit, inward pointing, normal vector on the boundary. The boundary is a smooth manifold, but not necessarily connected. Let \wedge denote the space of differential forms of all degrees

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and let d denote the exterior differential and d^* its dual operator on $L^2(\wedge)$. Furthermore, let $\Delta = -d^*d$ denote the Laplacian on real valued functions and $\Delta^1 = -(d^*d + dd^*)$ the restriction of the Hodge-Laplace-Beltrami operator to differential 1-forms. We study the heat equation on differential 1-forms, on a manifold with boundary.

The solutions to the initial value problem for heat equations on scalar functions and that on differential 1-forms are naturally related. For manifolds without a boundary, the differential of the former with an initial value f , assumed to be reasonably regular, solves the latter with the initial value df . This is an easy consequence of the commuting relation $d\Delta = \Delta^1d$, between d and the Laplacians. If (x_t) is a Brownian motion (BM) with initial value x_0 and W_t the damped stochastic parallel transport along x_\cdot , then

$$e^{\frac{1}{2}\Delta t} f(x_0) = \mathbf{E}f(x_t), \quad e^{\frac{1}{2}\Delta^1 t}(df) = \mathbf{E}[df(W_t(\cdot))].$$

The damped stochastic parallel transport (W_t) solves the parallel translation equation, damped by $-\frac{1}{2}\text{Ric}^\sharp$, where $\text{Ric}_x^\sharp : T_xM \rightarrow T_xM$ is given by $\text{Ric}_x^\sharp(v) = \text{Ric}_x(v, -)$ for Ric_x the Ricci curvature and $v \in T_xM$. The involvement of the Ricci curvature follows from the Weitzenböck formula: $\Delta^1\phi = \text{trace}\nabla^2\phi - \phi(\text{Ric}^\sharp)$ and the fact that the commutator between d and $\text{trace}\nabla^2$, where ∇ is the Levi-Civita connection, is the interior product of the differential 1-form with Ric^\sharp .

A simple application of Itô's formula transfers the derivative on the initial function to W_t and leads to the formula $d(e^{\frac{1}{2}\Delta t} f) = \mathbf{E}[f(x_t)M_t]$ where M_t is a martingale, a stochastic integral which is a path integration of the damped parallel translation. This procedure captures the essence of Malliavin calculus and the smoothing property of the heat semi-group.

If the manifold has a boundary, we impose the Neumann boundary condition on the scalar heat equation, which models heat conduction in a perfect insulator, and consider:

$$\begin{cases} \frac{\partial u(t, \cdot)}{\partial t} = \frac{1}{2}\Delta u(t, \cdot), & \text{on } M^\circ, t > 0 \\ \frac{\partial u(t, \cdot)}{\partial \nu} \Big|_{\partial M} = 0, & u(0, \cdot) = f. \end{cases} \quad (1.1)$$

Then its solution has the probabilistic representation $u_t(x_0) = \mathbf{E}[f(Y_t)]$, where (Y_t) is a reflected Brownian motion (RBM) with initial value x_0 . A RBM on M is a solution to a stochastic differential equation (SDE) of the form:

$$dY_t = \chi_{M^\circ}(Y_t) "dx_t" + \nu \chi_{\partial M}(Y_t) dL_t,$$

where (L_t) is the boundary local time of the RBM. The differential du , with respect to the space variable, solves also the heat equation on differential forms. However the induced boundary condition, $df(\nu) = 0$, is not sufficient for the well-posedness of the heat equation on differential form. We assume the absolute boundary condition and consider the equations:

$$\begin{cases} \frac{\partial \phi(t, \cdot)}{\partial t} = \frac{1}{2}\Delta^1\phi(t, \cdot) & \text{on } M^\circ, t > 0 \\ \phi(0, \cdot) = \phi_0, \\ \phi(t, \nu)|_{\partial M} = 0, \quad d\phi(t, \nu)|_{\partial M} = 0, \end{cases} \quad (1.2)$$

where $d\phi$ denotes exterior differential of ϕ , a differential two form. In local coordinates, $d\phi(v_1, v_2) = (\nabla_{v_1}\phi)(v_2) - (\nabla_{v_2}\phi)(v_1)$. Since $d^2 = 0$, the absolute boundary condition is satisfied by $du(t, \cdot)$. Suppose the Dirichlet boundary condition $\phi(\nu) = 0$ holds, then the absolute boundary condition is equivalent to $\nabla_\nu\phi - \phi(\nabla\nu) = 0$ on the tangent spaces on the boundary.

A reflected Brownian motion, Y_t , is obtained together with its local time L_t at the boundary, and its transport equation involves ‘differentiating’ the local time with respect to the initial value of the reflected Brownian motion. The local time vanishes when Y_t is in the interior of the manifold, so does not respond well to differentiating. The equation for the damped parallel transport along the sample paths of a reflected Brownian motion is also obtained by damping the parallel translation equation by the Ricci curvature, before the RMB touches the boundary for the first time, after which both the shape operator and the boundary local time are involved.

Our questions are (a) How would we simulate a RBM and its damped stochastic parallel transports? (b) Could we select a RBM so it leads naturally to a probabilistic representation for the solution to the heat equation on differential 1-forms with the absolute boundary conditions? (3) Which properties of the Brownian flow on manifolds without a boundary can be passed to manifolds with a boundary? For example, an immediate consequence of the construction is that a differentiation formula for heat semigroups which does not involve the differential of the initial value, are valid for heat equation with the Neumann boundary condition. See [10, 21, 31] for compact and non-compact manifolds without a boundary. In this article we prove that there exists a family of stochastic differential equations, whose solution do not visit the boundary, such that the following convergence hold.

1. Their solution flows converge to a reflected Brownian motions;
2. Their damped (resp. undamped) stochastic parallel transports converge to the respected damped (resp. undamped stochastic parallel transport) along the limiting reflected Brownian motions.

The existence of a family of SDE’s satisfying the first convergence is well known, especially for RBM defined on an Euclidean domain. The idea is to build a hard barrier on the boundary by adding large drifts pointing away from the boundary.

B. We explain the construction mentioned earlier in the context of known results, beginning by quoting the following theorem from the book [29] by N. Ikeda and S. Watanabe.

Theorem 1.1 ([29, Thm 7.2]). *Let $D = \mathbf{R}_d^+ = \{(x_1, \dots, x^d), x^d \geq 0\}$. Let $\{b^i, \sigma_k^i, i = 1, \dots, d, k = 1, \dots, m\}$ be real valued functions on D and δ a real valued function on ∂D , all are bounded Lipschitz continuous. Suppose that $\sum_{k=1}^m (\sigma_k^d)^2 \geq C$ and $\delta \geq C$ where C is a positive number. Then there exists a pair of adapted stochastic processes (Y_t, L_t) such that for $1 \leq i \leq d - 1$, their components satisfy the equations*

$$\begin{aligned}
 dY_t^i &= \sum_{k=1}^m \sigma_k^i(Y_t) \chi_{D^0}(Y_t) dB_t^k + b^i(Y_t) \chi_{D^0}(Y_t) dt, & Y_0^i &= y_0^i, \\
 dY_t^d &= \sum_{k=1}^m \sigma_k^d(Y_t) \chi_{D^0}(Y_t) dB_t^k + b^d(Y_t) \chi_{D^0}(Y_t) dt, + \delta(Y_t) dL_t, & Y_0^d &= y_0^d,
 \end{aligned}
 \tag{1.3}$$

and the following conditions: (a) $L_0 = 0$, (b) L_t is a continuous and increasing process, (3) $\int_0^t \chi_{\{Y_s \in \partial D\}} dL_s = L_t$, for all $t \geq 0$, a.s.. Furthermore uniqueness in law holds.

If $\delta \equiv 1$, the solution of the resulting non-sticky stochastic differential equation is a RBM. By working in local charts, this implies that RBM’s exist on any smooth manifolds with a boundary. By this theorem, other reflected stochastic processes, with not necessarily normal reflection, can also be constructed. All reasonable constructions should lead to the same essential quantity: the local time of the Brownian motion on the boundary. This philosophy follows from the uniqueness to the associated sub-martingale problem.

To solve the heat equation on differential 1-forms with the absolute boundary condition, N. Ikeda and S. Watanabe sought a tangent space valued process satisfying $\phi(t, v) = \mathbf{E}\phi(0, W_t(v))$. They constructed the solution for the half space, by the method of orthonormal frames, and remarked that, by working in charts, this leads trivially to the existence of a solution for general manifolds [29].

An ansatz for (W_t) is the solution to the random (covariant) ordinary differential equation along the sample paths of a given RBM: $\frac{D}{dt}W_t = -\frac{1}{2}\text{Ric}_{Y_t}^\#(W_t)$. Here D denotes covariant differentiation of $(W_t(\omega))$ along a sample path $Y_t(\omega)$. We use $W_t(\omega)$ to stand both for a tangent vector $W_t(\omega) \in T_{Y_t(\omega)}M$ with initial value $W_0(\omega)$ and the linear map $W_t(\omega) : T_{Y_0}M \rightarrow T_{Y_t(\omega)}M$. The latter translates a tangent vector at Y_0 ‘parallelly’ to a tangent vector in the tangent space at $Y_t(\omega)$.

The above representation holds for a manifold without a boundary, and is essentially true for the half space. As we will explain, it is also necessary to consider the shape of the boundary and add the shape operator $\mathcal{S} : TM \rightarrow TM$:

$$DW_t = -\frac{1}{2}\text{Ric}_{Y_t}^\#(W_t) \chi_{\{M^o\}}(Y_t) dt - \mathcal{S}_{Y_t}(W_t) dL_t. \tag{1.4}$$

Here \mathcal{S} denotes the shape operator of the boundary. If $y \in \partial M$ and $w \in T_yM$ then $\mathcal{S}_y = (-\nabla_w\nu)^T$ which is the tangential part of $-\nabla_w\nu$. For the half space, the shape operator vanishes identically, so the boundary local time is not involved. For a general manifold, the shape operator compensates with the variation of ϕ in the normal direction: if $\phi(\nu) \equiv 0$ then $\nabla_w\phi(\nu) = \phi(\mathcal{S}_y(w))$, see (C.5) for detail. There remains a term of the form $\int_0^t \nabla\phi_{T-s}(\nu(Y_s), W_s^\nu)ds$ where W_s^ν is the ‘normal part’ of W_s . To solve the absolute boundary problem one simply assumes that the normal part of W_s vanishes.

The damped stochastic parallel translation process exists and has a number of constructions, the basic idea is as follows. Denote by R_t the distance of Y_t to the boundary. We erase the normal part of the solution to (1.4), at the end of each excursion of the process Y_t into the interior M^o , of size larger or equal to ϵ , and obtain a process W_t^ϵ . As ϵ approaches zero, W_t^ϵ converges to a process W_t with the required properties. The Poisson point process of (R_t) determines the frequency of the projection, the ‘local time’ or the boundary time of the RBM is the local time of R_t at 0 and is essentially the δ -measure on the boundary.

We construct a nice family of SDEs of the form $dY_t^a = “dx_t” + A^a(Y_t^a)dt$ whose solutions (Y_t^a) are smooth and stay in the interior for all time, in particular they have no boundary times. We seek the additional properties: (1) $L_t^a := \int_0^t |A^a(Y_s^a)|ds$ approximates $\int_0^t \chi_{\partial M}(Y_s)ds$; (2) as $a \rightarrow 0$, the solutions to the equations

$$DW_t^a = -\frac{1}{2}\text{Ric}_{Y_t^a}^\#(W_t^a)dt + \nabla_{W_t^a}A_t^a dt \tag{1.5}$$

converge to (W_t) . Observe that dL_t^a and dL_t have mutually singular supports and it is clear that not every choice of “ dx_t ” yields an approximation. The solutions (Y_t^a) together with their damped parallel translations (W_t^a) select a RBM (Y_t) together with a damped stochastic parallel transport $W_t = \lim_{\epsilon \rightarrow 0} W_t^\epsilon$. Note that, although (Y_t^a) and (W_t^a) are sample continuous, the selected stochastic process (Y_t, W_t) has jumps on the boundary. In addition we observe that (W_t^a) can also be obtained by an appropriate variation in the initial values of a stochastic flow (Y_t^a) .

The construction is motivated by the following well known facts for manifold without boundaries: (1) damped parallel translations can be obtained from perturbation to a Brownian system with respect to its initial data and (2) a formula for the derivative of the heat semigroup follows naturally from differentiating a Brownian flow with respect to its initial value. More precisely if $(dF_t)(v)$ is the derivative of a Brownian flow $(F_t(x))$

with respect to its initial data x in the direction $v \in T_x M$, then $W_t(v)$ is the conditional expectation of $(dF_t)(v)$ with respect to the filtration of the Brownian flow. From our construction it would be trivial to see the formula $du_t(Y_0) = \frac{1}{t} \mathbf{E}[f(Y_t)M_t]$ holds for manifolds with a boundary, where M_t is a suitable local martingale.

C. Historical Remarks. Heat equations on differential forms were already considered 60 years ago in [14, P. E. Conner], where the authors related them to the existence problem for harmonic differential forms, and were studied by P. Malliavin [34] and H. Airault [2] who used perturbations to a boundary operator. The space of harmonic differential forms are related to the de Rham cohomology groups which are the quotients of closed smooth differential forms by exact differential forms. For compact manifolds, de Rham cohomology groups are typically obtained by an L^2 Hodge decomposition theorem and are topological invariants. Furthermore the alternative sum of the dimensions of the de Rham cohomology groups is the Euler characteristic. On manifolds with a boundary, a vanishing theorem on the first de Rham cohomology was obtained by A. Méritet [35]; a probabilistic proof for the Gauss-Bonnet-Chern Theorem which concerns a formula relating the Euler characteristics to the Gauss curvature, was obtained by I. Shigekawa, N. Ueki, and S. Watanabe [37].

The study of reflected Brownian motions, in conjunction with heat equation on functions, goes back to the late 50's and early 60's, see [15, F. Dario], [26, N. Ikeda], [38, A. V. Skorohod], and [27, N. Ikeda, T. Ueno, H. Tanaka and K. Satô]. A weak solution for smooth domains is given in [39, D. W. Stroock and S. R. S. Varadhan], see also the book [9, A. Bensoussan and J. L. Lions]. On convex Euclidean domains this was studied by H. Tanaka [40]. A comprehensive study can be found in [33, P. L. Lions and A. S. Sznitman], see also [42, S. R. S. Varadhan and R. J. Williams]. In terms of Dirichlet forms, see [13, Z.-Q. Chen, P.J. Fitzsimmons, and R. Song], and [12, Burdzy-Chen-Jones].

The study of stochastic damped parallel transports on manifolds with boundary began with the upper half plane, precluding which reflected Brownian motions was constructed by the canonical horizontal SDE on the orthonormal frame bundle (OM) with drift given by the horizontal lift of the 'reflecting' vector field, see e.g. [26, N. Ikeda], [44, Watanabe] and [28, N. Ikeda and S. Watanabe]. The same SDE on OM defines also the stochastic parallel transport process $//_t(Y)$ which is a sample continuous stochastic process, this is quite unlike the damped stochastic parallel translations whose construction is also more difficult. The former reduces to solving a 'reflected' stochastic differential equation while the latter is a stochastic process with seemingly arbitrarily induced jumps. Our construction shows that these jumps are not as arbitrary as they seem to be.

Since a manifold with boundary can be transformed to the upper half plane by local charts, c.f. [28, N. Ikeda and the S. Watanabe], this implies the local existence of damped parallel transport on a general manifold, where the Dirichlet Neumann problems were studied in [1, 2, H. Airault] using multiplicative functionals. H. Airault's approach was followed up in [25, E. Hsu] to give a neat treatment for the reflected SDE on the orthonormal frame bundle. In Appendix C, we extend and explain Ikeda-Watanabe's construction [29] for damped parallel transport, the formulation involves an additional curvature term, the shape operator of the normal vector ν .

For manifolds without a boundary, the concept of stochastic parallel translation goes back to K. Itô [30], J. Eells and D. Elworthy [17], P. Malliavin [34], H. Airault [2], and N. Ikeda and S. Watanabe [28]. The damped parallel translations along a Brownian motion (x_t) are constructed using stochastic parallel translation. Together with (x_t) it is an $\mathbb{L}(TM, TM)$ -valued diffusion process along (x_t) with Markov generator $\frac{1}{2}\Delta^1$. It was well understood that $\mathbf{E}[df(W_t)]$, where $f \in BC^2(M)$, solves the heat equation on differential 1-forms. See e.g. [18, D. Elworthy], [10, J.-M. Bismut], [28, N. Ikeda and S. Watanabe], and [36, P.-A. Meyer]. Furthermore if P_t is the heat semi-group, the process $dP_{T-t}f(W_t)$

is a local martingale. See [31] for a systematic study of probabilistic representations of heat semigroups on differential forms using both the damped parallel translation and the derivative flow of a Brownian system. Furthermore, (W_t) can be obtained from the derivative process of a Brownian flow by conditioning, from which information on the derivative of the heat semigroup on functions can also be obtained. The filtering method was used by M. Arnaudon, B. Driver, K. D. Elworthy, Y. LeJan, Xue-Mei Li, A. Thalmaier and Feng-Yu Wang [31, 20, 19, 22, 41, 5], and in many other related works. This plays also a role in the study of the regularity of finely harmonic maps, c.f. [8, M. Arnaudon, Xue-Mei Li and A. Thalmaier]. The stochastic Jacobi field point of view was also developed in [6, M. Arnaudon and A. Thalmaier], leading to estimates for the derivatives of harmonic maps between manifolds and to Liouville-type theorems.

2 Outline and main theorems

Let ρ denote the Riemannian distance in M and $p \geq 1$. If $\mathbf{E} \left[\sup_{0 \leq t \leq T} \rho(Y_t^a, Y_t)^p \right] \rightarrow 0$ we say that the family of processes (Y_t^a) converges to (Y_t) in $\mathcal{S}_p([0, T])$. We may also consider the convergence of a stochastic process after embedding the state space in an Euclidean space. Contrary to the convergence in the topology of uniform convergence (UCP), the $\mathcal{S}_p([0, T])$ convergence in terms of the Euclidean distance does depend on the embedding. However if M is compact, convergence in $\mathcal{S}_p([0, T])$ for the Riemannian distance is equivalent to convergence in $\mathcal{S}_p([0, T])$ for the Euclidean distance of an Euclidean space into which the manifold is embedded. It is also equivalent to the UCP convergence on the manifold. The notation, concerning the convergence of stochastic processes on manifolds, is introduced in Appendix B where we also summarise the relevant estimates for stochastic integrals in the \mathcal{S}_p and \mathcal{H}_p norm, as well as relating different notions of convergences.

The paper is organised as follows. We first construct, in §3, a family of SDEs whose solution flows $(Y_t^a, a > 0)$ exist and has a reflected Brownian flow as their limit. Away from the boundary, $(Y_t^a, a > 0)$ are Brownian motions with drift A^a , where A^a is the gradient of $\ln \tanh\left(\frac{R}{a}\right)$ and where $R(x)$ is the distance of x to the boundary. This construction selects a special reflected Brownian motion and their damped parallel translations.

Theorem 3.3. (a) For each $a > 0$ and initial value $Y_0 \in M^0$, (Y_t^a) remains in M^0 ; (b) As $a \rightarrow 0$, (Y_t^a) converges to the RBM process (Y_t) in UCP.

For $M = \mathbf{R}_+$, the solution to the Skorohod problem is selected while the solution given by Tanaka's formula is not, see Appendix A for detail.

The convergence of their stochastic parallel translations, $\|_s(Y_t^a)$, is given in §4, while the more difficult convergence theorem for their damped stochastic parallel translations (W_t^a) is stated in §5, together with a detailed description for the limiting process W_t and a differentiation formula for the heat semi-group with the Neumann boundary condition. The proof for the main theorem in §5 is given in §7. In §6 we study a variation, $(Y_t^a(u), u \in [0, 1])$, of (Y_t^a) and discuss the convergence of their derivatives with respect to u to the damped stochastic parallel translation process W_t . In the Appendix C we provide a proof for the existence of the stochastic damped parallel translation of Ikeda and Watanabe, whose proof uses Poisson point processes and was explicit only for the half plane. A formula for the normal part, W_t^a , of W_t , are given in section 5.

The main idea behind the construction is that $R(Y_t^a)$ involves one single real valued Brownian motion for all parameters a , and hence pathwise analysis is possible. The construction involves a tubular neighbourhood of the boundary in which the product metric is used for the estimation. By the construction, the convergence of $\{Y_t^a, a > 0\}$ in

the ‘tangential directions’ of the product tubular neighbourhood is trivial. Outside of the tubular neighbourhood the drift $A^a(x)$ vanishes, while inside it converges to zero exponentially fast as $a \rightarrow 0$ for every $x \in M^o$.

We also show that there exists a family of stochastic processes $\{L_t^a, a > 0\}$ which converges to the local time L_t in $\mathcal{S}_p([0, T])$. The convergence is not in the $\mathcal{H}_p([0, T])$ topology. The former convergence is a key for the convergence of the damped stochastic parallel translations, while the lack of the convergence in the latter topology makes it difficult to follow the standard methods for proving the convergence of solutions of SDEs with a parameter.

Corollary 3.5 and Lemma 7.8 *Let M be compact. Then for any $p > 1$ and $T > 0$,*

$$\lim_{a \rightarrow 0} \|L_t^a - L_t\|_{\mathcal{S}_p([0, T])} = 0.$$

In spite of the convergence, dL_t and dL_t^a are mutually singular measures. As a approaches zero, the total variation norms of their differences converge to a non-zero measure: $|d(L_t - L_t^a)| \rightarrow 2|dL_t|$. We also observe that $|L_t^a|_{\mathcal{H}_p([0, T])} = |L_t^a|_{\mathcal{S}_p[0, T]}$, $|L_t|_{\mathcal{H}_p[0, T]} = |L_t|_{\mathcal{S}_p[0, T]}$ and

$$|L_t - L_t^a|_{\mathcal{H}_p([0, T])} = |L_t|_{\mathcal{H}_p([0, T])} + |L_t^a|_{\mathcal{H}_p([0, T])}.$$

Below we state our main theorem, whose proof is the content of Section 7. A preliminary and easier result on the stochastic parallel transports is given in §4.

If $v \in T_y M$, where $y \in \partial M$, we denote v^T and v^ν respectively its tangential and normal component. Thus $(W_t^{a, T})$ is the tangential part of W_t^a , the latter solves (1.5), and

$$W_t^a = W_t^{a, T} + f_a(t)\nu(Y_t^a)$$

where $f_a(t) = \langle W_t^a, \nu(Y_t^a) \rangle$.

Lemma 7.11, Lemma 7.12, Theorem 5.6 and Corollary 5.7. Let M be a compact Riemannian manifold and let (W_t) be the solution to (1.4). Let $p \in [1, \infty)$. Then

1. $\lim_{a \rightarrow 0} W_t^{a, T} = W_t^T$, in the topology of uniform convergence in probability.
2. Write $W_t = W_t^T + f(t)\nu_{Y_t}$. As $a \rightarrow 0$,

$$\mathbf{E} \left[\int_0^t |f_a(s) - f(s)|^p ds \right] \rightarrow 0.$$

3. For any C^2 differential 1-form ϕ such that $\phi(\nu)$ vanishes on the boundary,

$$\lim_{a \rightarrow 0} \sup_{s \leq t} \mathbf{E} |\phi(W_s^a) - \phi(W_s)|^p = 0.$$

The family of sample continuous stochastic processes $\{(W_t^a), a > 0\}$ cannot converge to the stochastic process (W_t) with jumps, in the topology of uniform convergence in probability, nor in the \mathcal{S}_p norm, nor in the Skorohod topology. The uniform distance between a continuous path and a path with jumps is at least half of the size of the largest jump.

In Lemma 7.9 and Corollary 7.10 we see the simple ideas which are key to the proof. We show that there exists a real continuous process $c_a(u)$ such that

$$df_a(t) = -c_a(t)f_a(t) dt + \text{other interacting terms},$$

in which $-\int_0^t c_a(u) f_a(u) du$ is the only term that contributes to creating jumps (as a approaches 0). More precisely, for $t \notin \mathcal{R}(\omega)$, let

$$\alpha_t = \sup\{s : Y_s \in \partial M, s \in [0, t]\}.$$

Then for all $s, t \in [0, S]$ with $s < t$,

$$\begin{aligned} \lim_{a \rightarrow 0} e^{-\int_s^t c_a(u) du} &= 1 \quad \text{if } s > \alpha_t \\ \lim_{a \rightarrow 0} e^{-\int_s^t c_a(u) du} &= 0 \quad \text{if } s < \alpha_t. \end{aligned}$$

Let us follow Appendix C and explain the jumps in the limiting process W_t . Denote by $R(x) = \rho(\cdot, \partial)$ the distance function to the boundary and set $R_t = \rho(Y_t, \partial M)$. Viewed by the distance process R_t , the RBM reaches the boundary at a stopping time ζ where it makes excursions into the interior. By an equivalent change of probability measures, these excursions are similar to those made by a real valued RBM. The complement to the set of boundary points of R_t are disjoint open intervals. Let $\mathcal{R}(\omega)$ denote the set of the ‘right most points’ of the excursions of the distance function R_t and $\mathcal{R}_\epsilon(\omega)$ its subset coming from excursions of length at least ϵ .

The stochastic process $W_t \in L(T_{Y_0}M; T_{Y_t}M)$ is the unique càdlàg process along the RBM (Y_t) satisfying the following equation during an excursion to the interior:

$$\begin{aligned} W_t(v) &= v - \frac{1}{2} \int_0^t \int_s^t \parallel_s^{-1} \text{Ric}_{Y_{s-}}^\sharp(W_{s-}(v)) ds - \int_0^t \int_s^t \parallel_s^{-1} \mathcal{S}_{Y_{s-}}(W_{s-}(v)) dL_s \\ &\quad - \sum_{s \in \mathcal{R}(\omega) \cap [0, t]} \langle W_{s-}(v), \nu(Y_s) \rangle \parallel_{s,t} \nu(Y_s), \end{aligned} \tag{2.1}$$

where $\parallel_{s,t} = \parallel_{s,t}(Y_\cdot)$ denotes the parallel transport process along (Y_\cdot) from $T_{Y_s}M$ to $T_{Y_t}M$. We abbreviate $\parallel_{0,t}$ to \parallel_t . If we remove only the normal part of W_t on excursion intervals whose lengths are equal to or exceed size ϵ , the above procedure makes sense and the resulting processes (W_t^ϵ) has a limit. The above equation is understood in this limiting sense. More precisely, let W_t^ϵ denote the solution to

$$\begin{aligned} DW_t^\epsilon &= -\frac{1}{2} \text{Ric}^\sharp(W_t^\epsilon) dt - \mathcal{S}(W_t^\epsilon) dL_t - \chi_{\{t \in \mathcal{R}_\epsilon(\omega)\}} \langle W_{t-}^\epsilon, \nu_{Y_t} \rangle \nu_{Y_t}, \\ W_0^\epsilon &= \text{Id}_{T_{Y_0}M}. \end{aligned} \tag{2.2}$$

There exists an adapted right continuous stochastic process W_t such that $\lim_{\epsilon \rightarrow 0} W_t^\epsilon = W_t$ in UCP, and in \mathcal{S}_p for compact manifolds M , where $p \geq 1$. Furthermore for any $\alpha \in \mathcal{T}$, the set of stochastic processes taking their values in the set of bounded differential 1-forms above Y_t and vanishing outside some compact set, c.f. (C.6), we have

$$\lim_{\epsilon \rightarrow 0} \left(\int_0^{\cdot \wedge \tau_D} \alpha_s(DW_s^\epsilon) \right) \stackrel{\mathcal{S}_2}{\rightrightarrows} \left(\int_0^{\cdot \wedge \tau_D} \alpha_s(DW_s) \right).$$

This construction agrees with the derivative flow of the Skorohod reflected Brownian motion on the half line, extending stochastic damped parallel translation of Ikeda and Watanabe to general manifolds with boundary. This was further elaborated in Section 6 where we prove damped parallel translation is a weak derivative of the reflected Brownian flow. See also [25, E. Hsu] and [43, F. Wang] for two other constructions for damped parallel translations. In Proposition 5.1 we explain that removing the normal part of the damped parallel translation at the beginning of excursions leads to the same object in the limit, using the fact that the beginnings of excursions are left limits of the ends of excursions.

In *Theorem 5.3* we observe that (W_t) has the local martingale property when composed with the differential of a solution to the heat equation with the Neumann boundary condition, c.f. [29, N. Ikeda and S. Watanabe]. A stochastic representation for the semigroup on differential one forms with absolute boundary conditions follows.

Theorem 5.2 *Suppose that the tubular neighbourhood of the boundary has positive radius, and that the curvatures $\text{Ric}^\#$ and S are bounded from below. If $\phi(t, \cdot)$ is a solution to the heat equation with absolute boundary conditions, then $\phi(t, \cdot) = \mathbf{E}\phi(0, W_t(\cdot))$.*

Finally we consider the problem of obtaining (W_t^a) by varying the initial value in a family of stochastic flows. We have previously obtained a family of processes $(Y_t^a, 0 < a \leq 1)$ which converges as $a \rightarrow 0$ to a RBM in L^p . For a fixed we consider a variation $(Y_t^a(u), u \in [0, 1])$ with the property that $Y_0^a(u) = \gamma(u)$ where $\gamma(u)$ is a C^1 curve and $\partial_u Y_t^a(u) = W_t^a(u)\dot{\gamma}(u)$, see [4, M. Arnaudon, K.A. Coulibaly and A. Thalmaier] for the construction.

Proposition 6.1, Theorem 6.2 and Theorem 6.3 *If M is compact, then the family of two parameter stochastic processes $\{(Y_t^a(u))_{t \in [0, T], u \in [0, 1]}, a \in (0, 1]\}$, indexed by a , is tight in the weak topology. Let $\{a_k\}$ be a sequence of positive numbers converging to 0 and such that $(Y_t^{a_k}(u))_{t \in [0, T], u \in [0, 1]}$ converges weakly to $(Y_t(u))_{t \in [0, T], u \in [0, 1]}$. Then the following statements hold:*

- (1) for every $u \in [0, 1]$, $Y_t(u)$ is a RBM on M with initial value $\gamma(u)$;
- (2) for every $p \in [1, \infty)$ there exists a number $C'(p, T)$ s.t. for all $0 \leq u_1, u_2 \leq 1$,

$$\sup_{0 \leq t \leq T} \mathbf{E} [\rho^p(Y_t(u_1), Y_t(u_2))] \leq C'(p, T) \|\dot{\gamma}\|_\infty |u_1 - u_2|^p. \tag{2.3}$$

- (3) For all $f \in C^2(M)$ with $df(\nu) = 0$,

$$\mathbf{E} \left[f(Y_t(u_2)) - f(Y_t(u_1)) - \int_{u_1}^{u_2} df(W_t(u)\dot{\gamma}(u)) du \right] = 0.$$

The last identity implies that if $\dot{\gamma}(u) = v$, then $\frac{\partial}{\partial u} Y_t(\gamma(u)) = W_t(u)(v)$. Thus the damped parallel translation can be interpreted as a derivative of the RBM with respect to the initial point, making connection with the study of K. Burdzy [11], S. Anders [3], and J.-D. Deuschel and L. Zambotti [16] for Euclidean domains. It would be interesting to prove that (W_t) can be constructed from the derivative of the flow in strong sense.

3 A reflected Brownian flow

Let ρ denote the Riemannian distance function on M and R the distance function to the boundary, so $R(x) = \inf\{\rho(x, y) : y \in \partial M\}$. Let ν denote the inward-pointing unit normal vector field on the boundary and \exp the exponential map. By inward-pointing we mean that the sign of ν is chosen so that $\exp(t\nu)$ belongs to M^0 for sufficiently small t .

If $\{\sigma_1, \dots, \sigma_m\}$ is a family of vector fields spanning the tangent space at each point, we associate to it a bundle map $\sigma : M \times \mathbf{R}^m \rightarrow TM$ given by $\sigma(x)(e) = \sum_{k=1}^m \sigma_k(x)\langle e_k, e \rangle$ where $\{e_i\}$ is an orthonormal basis of \mathbf{R}^m . Let σ_0 be a smooth vector field. Let (B_t) be an \mathbf{R}^m -valued Brownian motion. A Brownian system is a stochastic differential equations (SDE) of the form

$$dx_t = \sigma(x_t) \circ dB_t + \sigma_0(x_t)dt$$

with infinitesimal generator $\frac{1}{2}\Delta$ where \circ denotes Stratonovich integration. Such equations always exist and are not unique, for example if σ_i are the gradient vector fields obtained by an isometric embedding of M into \mathbf{R}^m , the SDE is a gradient Brownian system.

A RBM is a solution to the Skorohod problem, i.e. a sample continuous strong Markov process satisfying the following properties: (1) its Markov generator is $\frac{1}{2}\Delta$ when restricted to $C_K^2(M^0)$; (2) it spends almost all the time in the interior of the manifold (with respect to Lebesgue measure on time set); and (3) its drift on the boundary is co-linear with ν . More precisely,

Definition 3.1. A stochastic flow is a Riemannian RBM if it solves the Skorohod problem

$$dY_t = \sigma(Y_t) \circ dB_t + \sigma_0(Y_t)dt + A(Y_t)dL_t, \tag{3.1}$$

where A is a smooth vector field extending the inward normal vector field and L_t is a non-decreasing process satisfying

$$\int_0^t \chi_{(\partial M)^c}(Y_s)dL_s = 0.$$

We say that L_t is the local time of the RBM at ∂M , or the boundary time of the RBM. Since dL_t is supported on the boundary of M , Y_t behaves exactly like a Brownian motion in the interior.

The aim of the section is to construct a family of Brownian systems on M with large drift A^a pushing away from the boundary, approximating a reflected Brownian motion in the topology of uniform convergence in probability. Furthermore the following properties are desired: the convergence is ‘uniform’ and the limiting process is continuous with respect to the initial data.

By the tubular neighbourhood theorem, there exists a continuous function $\delta : \partial M \rightarrow (0, \infty)$ such that for $E_0 = \{(x, t) : x \in \partial M, 0 \leq t < \delta(x)\}$, the map

$$\begin{aligned} \Phi : E_0 &\rightarrow F_0 := \Phi(E_0) \subset M \\ (x, t) &\mapsto \exp_x(t\nu_x) \end{aligned} \tag{3.2}$$

is a diffeomorphism such that $\Psi(y) = (\pi(y), R(y))$ on F_0 where $\pi(y)$ is the boundary point given by $\rho(y, \pi(y)) = R(y)$ and $\Psi = \Phi^{-1}$. In other words, on F_0 , $R(\Phi(x, t)) = t$ and the distance function R is smooth. For $0 \leq c \leq 1$, define

$$E_c = \{(x, t) : 0 \leq t < (1 - c)\delta(x)\}, \quad F_c = \Phi(E_c).$$

Since ∂M is a Riemannian manifold of its own right, we may represent $\Delta_{\partial M} = \sum_{j=2}^m \bar{\sigma}_j$ as the sum of squares of vector fields, for example by taking $\{\bar{\sigma}_j, j = 2, \dots, m\}$ to be a gradient system on ∂M .

In the tubular neighbourhood around a relatively compact set U of the boundary, the width $\delta(x)$ can be taken to be a positive constant $3\delta_0$. Let R be a real valued 1-Lipschitz smooth function on M which on $F_{1/3}$ agrees with the distance function to the boundary and such that $R \geq \delta_0$ on $F_{1/3}^c$. This can be obtained by modifying the distance function to the boundary.

Proposition 3.2. Let $r \geq 1$ and $c \in (0, 1)$. Let $\{\bar{\sigma}_0, \bar{\sigma}_j, j = 2, \dots, m\}$ be a family of C^r vector fields on ∂M with the property that

$$\frac{1}{2} \sum_{j=2}^m L_{\bar{\sigma}_j} L_{\bar{\sigma}_j} + L_{\bar{\sigma}_0} = \frac{1}{2} \Delta_{\partial M}.$$

Suppose that E_0 has strictly positive radius, i.e. there exists a positive number δ_0 such that $\inf_{\partial M} \delta \geq 3\delta_0$. Then there exist a finite number of C^r vector fields $\{\sigma_j, j = 0, \dots, N\}$ such that

- (1) $\frac{1}{2} \sum_{j=1}^N L_{\sigma_j} L_{\sigma_j} + L_{\sigma_0} = \frac{1}{2} \Delta$ on M ,
- (2) $\sigma_1 = \nabla R$ on $F_{2/3}$,
- (3) For $2 \leq j \leq m$, σ_j extends $\bar{\sigma}_j$,
- (4) For all $p \in \Phi(\partial M \times [0, \delta_0])$, for all $j \geq 2$: $\langle \sigma_1, \sigma_j \rangle_p = 0$.

In the sequel we denote $A = \sigma_1$.

Proof. We first assume that there exist a family of vector fields $\sigma_j, 1 \leq j \leq m$ defined in the tubular neighbourhood F_0 . Denote $\sigma : M \times \mathbf{R}^m \rightarrow TM$ the corresponding bundle map, as indicated earlier. Let us extend the construction to M and then return to the local construction.

Let $\tilde{\sigma}_i, m+1 \leq i \leq m+m'$ be a family of vector fields in M such that

$$\sum_{i=m+1}^{m+m'} L_{\tilde{\sigma}_i} L_{\tilde{\sigma}_i} = \Delta.$$

Let $\tilde{\sigma}$ be the corresponding bundle map from $M \times \mathbf{R}^{m+m'} \rightarrow TM$ and let $N = m+m'$. We take a real valued function $\beta \in C^\infty(M; [0, 1])$ with the property that $\sqrt{\beta}$ and $\sqrt{1-\beta}$ are smooth (this is implied by the other assumptions), $\beta|_{F_{2/3}} = 1$, and $\beta|_{M \setminus F_{1/3}} = 0$. Let us define a new bundle map $\hat{\sigma} : \mathbf{R}^m \times \mathbf{R}^{m'} \rightarrow TM$ as following:

$$\hat{\sigma}(e_1, e_2) = \sqrt{\beta} \sigma(e_1) + \sqrt{1-\beta} \tilde{\sigma}(e_2),$$

and prove that it is a surjection. Let

$$(\hat{\sigma})^*(v) = (\sqrt{\beta} \sigma^*(v), \sqrt{1-\beta} \tilde{\sigma}^*(v)).$$

Then $(\hat{\sigma})^* : T_x M \rightarrow \mathbf{R}^N$ is the right inverse to $\hat{\sigma}$. Indeed

$$\hat{\sigma} \hat{\sigma}^* = \beta \sigma \sigma^* + (1-\beta) \tilde{\sigma} \tilde{\sigma}^* = id,$$

and $\hat{\sigma}$ induces the Laplace-Beltrami operator. It is the desired map.

Let us now construct σ on F_0 . If $y \in m$, let $\pi(y)$ denote the projection of y to M , γ_y the geodesic from $\pi(y)$ to y , and $\parallel(\gamma_y)$ the parallel transport along γ_y which is a linear map from $T_{\pi(y)} M$ to $T_y M$. For $j \neq 1$, along a geodesic normal to the boundary, we may extend $\bar{\sigma}_j$ by parallel transport along the geodesic in the normal vector direction:

$$\sigma_j(y) = \parallel(\gamma_y) \bar{\sigma}_j(\pi(y)).$$

In other words it is constant along the geodesic. It is clear that $\langle \sigma_j(x), \sigma_1(x) \rangle = 0$ for all $j > 1$ and $x \in F_0$.

From the assumption, $\frac{1}{2} \sum_{k=2}^m \nabla_{\bar{\sigma}_k} \bar{\sigma}_k + \bar{\sigma}_0 = 0$ on ∂M and $\sum_{j=2}^m \bar{\sigma}_j(x) \bar{\sigma}_j^b(x) = Id_{T_x \partial M}$. Here $\bar{\sigma}_j^b(x)$ denotes the 1-form $\langle \sigma_j(x), \cdot \rangle$. Let us prove that for all $x \in F_0$,

$$\sum_{j=1}^m \sigma_j(x) \sigma_j^b(x) = Id_{T_x M}$$

where $\sigma_j^b(x)$ is defined in a similar way. Since the vectors $\{\sigma_k(x)\}$ generate $T_x M$, it is sufficient to prove that for all $k = 1, \dots, m$,

$$\left(\sum_{j=1}^m \sigma_j(x) \sigma_j^b(x) \right) (\sigma_k(x)) = \sigma_k(x).$$

For $k = 1$:

$$\left(\sum_{j=1}^m \sigma_j(x) \sigma_j^b(x) \right) (\sigma_1(x)) = \sigma_1(x) \langle \sigma_1(x), \sigma_1(x) \rangle = \sigma_1(x).$$

For $k \geq 2$:

$$\begin{aligned} \left(\sum_{j=1}^m \sigma_j(x) \sigma_j^b(x) \right) (\sigma_k(x)) &= \sum_{j=2}^m \sigma_j(x) \langle \sigma_j(x), \sigma_k(x) \rangle \\ &= \sum_{j=2}^m \langle (\gamma_x) \bar{\sigma}_j(\pi(x)) \langle \bar{\sigma}_j(\pi(x)), \bar{\sigma}_k(\pi(x)) \rangle \rangle \\ &= \langle (\gamma_x) \bar{\sigma}_k(\pi(x)) \rangle = \sigma_k(x). \end{aligned}$$

To complete the construction, we take the drift vector with the following property:

$$\sigma_0(x) = -\frac{1}{2} \sum_{j=1}^m \nabla_{\sigma_j(x)} \sigma_j = -\frac{1}{2} \sum_{j=2}^m \nabla_{\sigma_j(x)} \sigma_j, \quad x \in F_0.$$

This completes the proof. □

The vector field $A = \sigma_1$, constructed in Proposition 3.2 extends the unit inward normal vector field, defined on ∂M , and coincides with ∇R on $F_{2/3}$. Off the cut locus of the R , ∇R exists almost everywhere. For the Skorohod problem, we will only need the information of A on $F := F_{2/3}$ and in particular we do not need to worry the effect of the cut locus.

Next we take a family of additional drift vector fields converging to 0 in the interior of M and to the local time on the boundary. We divide the manifold M into three regions: inner tubular neighbourhood, the middle region and the outer region. The inner region, a subset of E_0 with the product metric is quasi isometric to its image, i.e. there is a constant $C > 0$ such that for all $x, y \in \partial M$, for all $s, t \in [0, \delta_0]$, denoting $\bar{\rho}$ the distance in ∂M ,

$$\frac{1}{C} (\bar{\rho}(x, y) + |s - r|) \leq \rho(\Phi(x, s), \Phi(y, r)) \leq C (\bar{\rho}(x, y) + |s - r|).$$

Outside of the tubular neighbourhood the drifts will be chosen to be uniformly bounded and to converge to zero uniformly. If x_n is a sequence of points in the outer region with limit x_0 , we need to assume that the solution with initial value x_n converges in some sense. In the first region we have convergence in probability and in the second we will need a control on the rate of convergence that induces the property of a flow.

By the convergence of the manifold valued stochastic process (Y_t^a) to (Y_t) we mean that $H(Y_t^a)$ converges to $H(Y_t)$ where $H : M \rightarrow \mathbf{R}^k$ is an embedding, with the same notion of convergence. We recall that (Y_t^a) converges to Y_t in UCP implies that for any $\epsilon > 0$, the explosion times ξ^a of Y_t^a and its exit times from relatively compact sets, for sufficiently small a , are bounded below by the corresponding ones for $Y_t = Y_t^0$ minus ϵ (in other words $\liminf_{a \rightarrow 0} \xi^a \geq \xi^0$). Also, if we assume sufficient growth control on the curvatures and the shape of the tubular neighbourhood, the convergence will be in \mathcal{S}_p . We only discuss this aspect for a compact manifold.

Let $R : M \rightarrow \mathbf{R}$ be a smooth function such that $R|_F$ is the distance to the boundary, and $R|_{F^c} \geq \tilde{\delta}$ for some number $\tilde{\delta}$, and $R(x)$ is a constant on the complement of F^c .

Theorem 3.3. *Let a be a positive number. Suppose that σ_k and A satisfy the properties stated in Proposition 3.2. Let Y_t^a and Y_t denote respectively the maximal solution to the*

equations, with the same initial value x ,

$$dY_t^a = \sum_{k=1}^m \sigma_k(Y_t^a) \circ dB_t^k + \sigma_0(Y_t^a)dt + \nabla \ln \left(\tanh \left(\frac{R(Y_t^a)}{a} \right) \right) dt, \tag{3.3}$$

$$dY_t = \sum_{k=1}^m \sigma_k(Y_t) \circ dB_t^k + \sigma_0(Y_t)dt + A(Y_t)dL_t. \tag{3.4}$$

Suppose that $\rho(Y_0^a, Y_0)$ converges to 0 in probability.

1. Then $\lim_{a \rightarrow 0} Y^a = Y$, in the topology of uniform convergence in probability.
2. If M is compact, then the SDEs do not explode and for all $p \in [1, \infty)$ and for all $T > 0$, Y^a converges to Y in $\mathcal{S}_p([0, T])$, i.e.

$$\lim_{a \rightarrow 0} \mathbf{E} \sup_{0 \leq s \leq T} \rho(Y_s^a, Y_s)^p = 0.$$

Proof. Since UCP convergence is local and is implied by local convergence in \mathcal{S}_p , (1) is a consequence of (2). See Corollary B.4. So we assume that M is compact and choose a constant $\delta_0 > 0$ such that the function δ is bounded below by $3\delta_0$. Then we replace δ by $3\delta_0$ in the definition of E_0, F_0, E_c, F_c .

We define $h^a(x) = \ln \left(\tanh \left(\frac{R(x)}{a} \right) \right)$ and

$$A^a(x) = \nabla h^a(x) = \frac{2\nabla R(x)}{a \sinh \left(\frac{2R(x)}{a} \right)}.$$

This is an approximation for a vector field that vanishes on M^0 and exerts an ‘infinity’ force in the direction of $\nabla R = A$ on the boundary.

Let $R_t^a = R(Y_t^a)$. Then

$$R_t^a = R_0^a + \sum_k \int_0^t \langle dR, \sigma_k(Y_s^a) \rangle dB_s^k + \frac{1}{2} \int_0^t \Delta R(Y_s^a) ds + \int_0^t \frac{2}{a \sinh \left(\frac{2R_s^a}{a} \right)} ds.$$

Let us denote by β_t^a the stochastic term:

$$R_t^a = R_0^a + \beta_t^a + \frac{1}{2} \int_0^t \Delta R(Y_s^a) ds + \int_0^t \frac{2}{a \sinh \left(\frac{2R_s^a}{a} \right)} ds.$$

For $Y_t^a \in F_{2/3}$ the tubular neighbourhood of ∂M , we have by Proposition 3.2 (2) that $d\beta_t^a = dB_t^1$ is independent of a and of Y_t^a , which will be crucial for the sequel:

$$dR_t^a = dB_t^1 + \frac{1}{2} \Delta R(Y_t^a) dt + \frac{2}{a \sinh \left(\frac{2R_t^a}{a} \right)} dt. \tag{3.5}$$

Since we assumed that M is compact, $|\Delta R|$ is bounded, so the drift is essentially $\frac{2}{a \sinh \left(\frac{2R_t^a}{a} \right)}$ and R_t^a never touches the boundary and the equation is well defined.

Recall that π is the map that sends a point $x \in M$ to the nearest point on ∂M , it is defined on F_0 . The tubular neighbourhood map $\Psi : F_0 \rightarrow E_0$ splits into two parts, $\Psi(x) = (\pi(x), R(x))$. Since Ψ is a diffeomorphism onto its image, on $\{Y_t \in F_0\}$, the processes Y_t^a converges to Y_t in the Riemannian metric on M if and only if $\Psi(Y_t^a)$ converges to $\Psi(Y_t)$ in the product metric of $\partial M \times [0, \delta]$.

On any subset of M not intersecting the tubular neighbourhood that is distance $c\delta$ from ∂M for some $c < 1$, the functions $|\nabla A^a|$ are uniformly bounded in a and converge to

zero as $a \rightarrow 0$. The local time does not charge any real time if Y_t is not on the boundary. For a C^3 embedding $\Phi : M \rightarrow \mathbf{R}$,

$$\begin{aligned} \Phi(Y_t^a) - \Phi(Y_t) &= \Phi(Y_0^a) - \Phi(Y_0) + \int_0^t \langle (\sigma^* \nabla \Phi)(Y_s^a) - (\sigma^* \nabla \Phi)(Y_s), dB_s \rangle \\ &\quad + \frac{1}{2} \int_0^t (\Delta \Phi(Y_s^a) - \Delta \Phi(Y_s)) ds. \end{aligned}$$

By standard estimates, if $Y_0^a \rightarrow Y_0$ in probability, the processes $\Phi(Y_t^a)$ started outside the closed tubular set $F_{\frac{1}{3}}$ and stopped at the first entrance time of $F_{\frac{2}{3}}$ converge to $\Phi(Y_t)$ in UCP. In particular this holds for isometric embeddings and since the intrinsic Riemannian distance is controlled by the extrinsic distance function, we see that the stochastic process $\rho(Y_t^a, Y_t)$ converges in UCP.

Splitting in a proper way the times, for the UCP topology it is enough to prove that the processes (Y_t^a) started inside the open set $F_{\frac{2}{3}}$ and stopped at exiting $F_{\frac{1}{3}}$ converge to (Y_t) whenever $Y_0^a \rightarrow Y_0$.

So we assume that Y_0^a and Y_0 belong to $F_{2/3}$ and $\rho(Y_0^a, Y_0)$ converges to 0 in probability. We let

$$\tau = \inf\{t \geq 0, R(Y_t) = 2\delta_0\}, \quad \tau^a = \inf\{t \geq 0, R(Y_t^a) = 2\delta_0\}.$$

We first prove that for all $T > 0$,

$$\forall T > 0, \quad \lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{t \leq \tau^a \wedge \tau \wedge T} \rho^2(Y_t^a, Y_t) \right] = 0. \tag{3.6}$$

Notice if (3.6) holds, $\sup_{s \leq t} R(Y_s) < 2\delta_0$ implies that $\sup_{s \leq t} R(Y_s^a) < 2\delta_0$ for sufficiently small a , consequently,

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{t \leq \tau \wedge T} \rho^2(Y_t^a, Y_t) \right] = 0. \tag{3.7}$$

This in turn shows that

$$\liminf_{a \rightarrow \infty} \tau^a \wedge T \geq \tau \wedge T \tag{3.8}$$

and the convergence of Y_t^a to Y_t in the UCP topology follows.

Let $R_t^a = R(Y_t^a)$ and $R_t = R(Y_t)$. Denote $\bar{\rho}$ the Riemannian distance on ∂M . Using the tubular neighbourhood map, proving (3.6) is equivalent to proving the following limits:

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{t \leq \tau^a \wedge \tau \wedge T} (R_t^a - R_t)^2 \right] = 0, \quad \lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{t \leq \tau^a \wedge \tau \wedge T} (\bar{\rho})^2(\pi(Y_t^a), \pi(Y_t)) \right] = 0. \tag{3.9}$$

If $t \leq \tau^a \wedge \tau$, from (3.5) and (3.1), we obtain the equation

$$R_t^a - R_t = R_0^a - R_0 + \int_0^t \frac{ds}{a \sinh\left(\frac{2R_s^a}{a}\right)} - L_t + \frac{1}{2} \int_0^t (\Delta R(Y_s^a) - \Delta R(Y_s)) ds, \tag{3.10}$$

where there is no martingale part. Let $\varepsilon > 0$, and let

$$\tilde{L}_t^a = \int_0^t \frac{ds}{a \sinh\left(\frac{2R_s^a}{a}\right)} - L_t.$$

We apply the Itô-Tanaka formula to the convex function $\max(y, \varepsilon)$ and obtain:

$$\begin{aligned} \varepsilon \vee |R_t^a - R_t| &= |R_0^a - R_0| \vee \varepsilon + \int_0^t \chi_{\{R_s^a - R_s > \varepsilon\}} d\tilde{L}_s^a - \int_0^t \chi_{\{R_s^a - R_s < -\varepsilon\}} d\tilde{L}_s^a \\ &\quad + \frac{1}{2} \int_0^t \chi_{\{|R_s^a - R_s| > \varepsilon\}} (\Delta R(Y_s^a) - \Delta R(Y_s)) ds. \end{aligned}$$

It is vital to remark that $L_s > 0$ if and only if $R_s^a - R_s = R_s^a$. Also $R_s^a - R_s < -\varepsilon$, if only if $R_s \neq 0$, and so $-d\tilde{L}_s^a$ is a negative measure. We may ignore the third term on the right hand side of the identity. For each $\alpha > 0$ and $\varepsilon > 0$, there exists a number $a(\varepsilon, \alpha) > 0$ such that $\frac{1}{a \sinh(\frac{2\varepsilon}{a})} < \alpha$, for all $a \leq a(\varepsilon, \alpha)$ and for all $r \geq \varepsilon$. Hence,

$$\begin{aligned} & \int_0^t \chi_{\{R_s^a - R_s > \varepsilon, R_s = 0\}} d\tilde{L}_s^a \\ & \leq \int_0^t \frac{1}{a \sinh\left(\frac{2(R_s^a - R_s)}{a}\right)} \chi_{\{R_s^a - R_s > \varepsilon, R_s = 0\}} ds - L_t \leq \alpha t, \\ & \int_0^t \chi_{\{R_s^a - R_s > \varepsilon, R_s \neq 0\}} d\tilde{L}_s^a = \int_0^t \frac{1}{a \sinh\left(\frac{2R_s^a}{a}\right)} \chi_{\{R_s^a - R_s > \varepsilon, R_s \neq 0\}} ds \\ & \leq \int_0^t \frac{1}{a \sinh\left(\frac{2\varepsilon}{a}\right)} ds \leq \alpha t. \end{aligned}$$

It follows that

$$\varepsilon \vee |R_t^a - R_t| \leq |R_0^a - R_0| + \varepsilon + 2\alpha t + \frac{1}{2} \|\nabla \Delta R\|_{L^\infty(F_0)} \int_0^t \sup_{r \leq s} \rho(Y_r^a, Y_r) ds. \tag{3.11}$$

So

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau \wedge \tau^a \wedge T} (R_t^a - R_t)^2 \right] & \leq 8\mathbf{E} [(R_0^a - R_0)^2] + 8\varepsilon^2 + 16\alpha^2 t^2 + \\ & 8T (\|\nabla \Delta R\|_{L^\infty(F_0)})^2 \int_0^T \mathbf{E} \left[\sup_{s \leq \tau \wedge \tau^a \wedge t} \rho^2(Y_s^a, Y_s) \right] dt. \end{aligned} \tag{3.12}$$

Before continuing with the estimate above, we estimate $\bar{\rho}(\pi(Y_t^a), \pi(Y_t))$. The distance function $\bar{\rho}$ is not smooth on $\partial M \times \partial M$. So we will consider an isometric embedding $\iota : \partial M \rightarrow \mathbf{R}^{m'}$ (in fact since ∂M is compact any embedding would do) and instead of proving the second limit in (3.9) we will prove that

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{t \leq \tau^a \wedge \tau \wedge T} (\iota(\pi(Y_t^a)) - \iota(\pi(Y_t)))^2 \right] = 0 \tag{3.13}$$

We extend ι to F_0 to obtain $\tilde{\iota}(y) = (\iota \circ \pi)(y)$, then

$$\begin{aligned} \tilde{\iota}(Y_t^a) - \tilde{\iota}(Y_t) & = \tilde{\iota}(Y_0^a) - \tilde{\iota}(Y_0) + \int_0^t \langle \sigma^* \nabla \tilde{\iota}(Y_s^a) - \sigma^* \nabla \tilde{\iota}(Y_s), dB_s \rangle \\ & + \frac{1}{2} \int_0^t (\Delta \tilde{\iota}(Y_s^a) - \Delta \tilde{\iota}(Y_s)) ds + \int_0^t d\tilde{\iota}(A^a(y_s^a)) ds - \int_0^t d\tilde{\iota}(A(y_s)) dL_s. \end{aligned} \tag{3.14}$$

Since $d\pi(A) = 0$ and $d\pi(A^a) = 0$, the last two terms vanish. By standard calculation,

$$\begin{aligned} & \mathbf{E} \left[\sup_{t \leq T \wedge \tau \wedge \tau^a} \|\tilde{\iota}(Y_t^a) - \tilde{\iota}(Y_t)\|^2 \right] \\ & \leq 4\mathbf{E} [\|\tilde{\iota}(Y_0^a) - \tilde{\iota}(Y_0)\|^2] + 16 \|\nabla \sigma^* \nabla \tilde{\iota}\|_{L^\infty(F_0)}^2 \int_0^T \mathbf{E} \left[\sup_{s \leq t \wedge \tau \wedge \tau^a} \rho^2(Y_s^a, Y_s) \right] dt \\ & + 2 \|\nabla \Delta \tilde{\iota}\|_{L^\infty(F_0)} \int_0^T \mathbf{E} \left[\sup_{s \leq \tau \wedge \tau^a \wedge t} \rho^2(Y_s^a, Y_s) \right] dt. \end{aligned} \tag{3.15}$$

Since ∂M is compact, F_0 is compact. The quantities $\nabla \sigma^*$ and $\nabla \tilde{\iota} = \nabla \pi(\nabla \iota)$ are bounded. Similarly $\|\nabla \Delta \tilde{\iota}\|_{L^\infty(F_0)}$ is finite. For $x \in F_0$, set

$$H(x) = (\tilde{\iota}(x), R(x)) \in \mathbf{R}^{m'+1}. \tag{3.16}$$

Let $C_H > 0$ be a constant such that for all $x, x' \in F_0$,

$$\frac{1}{C_H} \|H(x) - H(x')\| \leq \rho(x, x') \leq C_H \|H(x) - H(x')\|. \tag{3.17}$$

Define

$$C = \left(16 \|\nabla \sigma^* \nabla \bar{i}\|_{L^\infty(F_0)}^2 + 2 \|\nabla \Delta H\|_{L^\infty(F_0)} \right) (C_H)^2. \tag{3.18}$$

From (3.15), using Gronwall lemma we obtain that if $a < a(\varepsilon)$,

$$\mathbf{E} \left[\sup_{s \leq T \wedge \tau \wedge \tau^a} \|H(Y_s^a) - H(Y_s)\|^2 \right] \leq 4 \left(\|H(Y_0^a) - H(Y_0)\|^2 + \varepsilon^2 + \alpha^2 T^2 \right) e^{CT}. \tag{3.19}$$

Since ε and α can be chosen as small as we like and C is independent of ε , α , a , $Y_0^a \rightarrow Y_0$ and H is bounded, we obtain that

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{s \leq T \wedge \tau \wedge \tau^a} \|H(Y_s^a) - H(Y_s)\|^2 \right] = 0. \tag{3.20}$$

Together with (3.12), we see that

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{s \leq T \wedge \tau \wedge \tau^a} \rho^2(Y_s^a, Y_s) \right] = 0. \tag{3.21}$$

This implies that $\tau^a \wedge \tau \rightarrow \tau$ almost surely, and since the distance is bounded,

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{s \leq T \wedge \tau} \rho^2(Y_s^a, Y_s) \right] = 0. \tag{3.22}$$

This completes the proof for the convergence of Y^a to Y in UCP, and also in \mathcal{S}_p for compact manifold M . \square

It would be interesting to use the method in [32] to study whether there exists a global smooth solution flow to the SDEs. It is also worth noting that if $\tau^U(Y)$ (resp. $\tau^U(Y^a)$) is the exit time of Y (resp. Y^a) from a relatively compact open set U , then

$$\liminf_{a \rightarrow 0} \tau^U(Y^a) \geq \tau^U(Y).$$

Corollary 3.4. *Let S_1, S_2 be stopping times such that $S_1 < S_2$ and let a_0 be a positive constant. Suppose that $Y_t^a \in F_0$ for any $a \in (0, a_0]$ and $t \in [S_1, S_2]$. Then on the interval $[S_1, S_2]$, $\lim_{a \rightarrow 0} \pi(Y^a) = \pi(Y)$. The convergence is in the semi-martingale topology. If moreover M is compact and S_2 is bounded, then the convergence holds in \mathcal{H}_p for all $p \in [1, \infty)$.*

Proof. Since the drifts A and A^a belong to the kernel of the differential $T\pi$, we obtain with Itô formula the following equations:

$$\begin{aligned} d(\pi(Y_t^a)) &= \sum_{k=1}^m T\pi \sigma_k(Y_t^a) \circ dB_t^k + T\pi \circ \sigma_0(Y_t^a) dt \\ d(\pi(Y_t)) &= \sum_{k=1}^m T\pi \circ \sigma_k(Y_t) \circ dB_t^k + T\pi \circ \sigma_0(Y_t^a) dt. \end{aligned}$$

Since $\frac{1}{2} \sum_{k=1}^m \nabla_{\sigma_k} \sigma_k + \sigma_0 = 0$, we only need to be concerned with the following term from the Itô correction: $\frac{1}{2} \sum_{k=1}^m \nabla T\pi(\cdot)(\sigma_k, \sigma_k)$. By Theorem 3.3, both $T\pi \circ \sigma_k(Y_t^a)$ and $\nabla T\pi(Y_t^a)(\sigma_k, \sigma_k)$ converge in the UCP topology, and they are locally uniformly bounded. The limits are respectively $T\pi \sigma_k(Y_t)$ and $\nabla T\pi(Y_t)(\sigma_k, \sigma_k)$. By Theorem 2 in [23, M. Emery], see [7, M. Arnaudon and A. Thalmaier] for the manifold case, $\pi(Y^a)$ converges to $\pi(Y)$ in the semi-martingale topology. \square

Define $L_t^a = \int_0^t \frac{ds}{a \sinh\left(\frac{2R_s^a}{a}\right)}$. Then in the tubular neighbourhood,

$$A^a(Y_t^a) = \nabla R(Y_t^a) \frac{d}{dt} L_t^a.$$

Corollary 3.5. *Suppose that M is compact. Then for all $p \geq 1$ and $T > 0$,*

$$\lim_{a \rightarrow 0} \mathbf{E} \left(\sup_{s \leq T} |L_s^a - L_s|^p \right) = 0.$$

Moreover, letting $L^0 = L$, for all $\lambda > 0$, there exists $C(T, \lambda)$ such that for all $a \in [0, 1]$,

$$\mathbf{E} \left[e^{\lambda L_T^a} \right] \leq C(T, \lambda). \tag{3.23}$$

Proof. Firstly we take Y_0^a, Y_0 in $F_{2/3}$, the $\frac{2}{3}$ tubular neighbourhood of the boundary. Let $\tau = \inf\{R(Y_t) = 2\delta_0\}$ and $\tau^a = \inf\{R(Y_t^a) = 2\delta_0\}$ be respectively the first exit times of Y and Y^a from $\{x : R(x) < 2\delta_0\} \subset F_{\frac{2}{3}}$. On $\{t < \tau^a \wedge \tau\}$ we have (3.10):

$$L_t^a - L_t = -R_0^a + R_0 + R_t^a - R_t - \frac{1}{2} \int_0^t (\Delta R(Y_s^a) - \Delta R(Y_s)) ds. \tag{3.24}$$

By the convergence of Y^a to Y in $\mathcal{S}_p([0, T])$,

$$\mathbf{E} \sup_{t < \tau^a \wedge \tau} |L_t^a - L_t|^p < \infty.$$

Outside of the $2/3$ tubular neighbourhood $F_{\frac{2}{3}}$, $\frac{1}{a \sinh\left(\frac{2R(x)}{a}\right)}$ converges to 0 uniformly in x and L_t vanishes. Note that $\lim_{a \rightarrow 0} \frac{1}{a \sinh\left(\frac{2r}{a}\right)} = 0$ for any $r > 0$. The required convergence result follows.

To prove (3.23) we write for $a \in [0, 1]$

$$L_t^a = R_t^a - R_0^a + \int_0^t \alpha_s^a dZ_s^a + \int_0^t \beta_s^a ds$$

where for all a , Z_t^a is a real valued Brownian motion, and $|R_t^a - R_0^a|$, α_t^a and β_t^a are uniformly bounded independently of a . The result immediately follows. \square

4 Convergence of the parallel transports

Let (Y_t) and (Y_t^a) be respectively the solutions of (3.4) and (3.3). The parallel transport along (Y_t) and (Y_t^a) are respectively the solution to the canonical horizontal stochastic differential equations on the orthonormal frame bundle with drift the horizontal lift of the drift vector fields A and A^a respectively.

Denote by \parallel_t^a the parallel transport along Y_t^a , \parallel_t the parallel transport along Y_t . Recall that $\sigma_y^* : T_y M \rightarrow \mathbf{R}^m$ is the right inverse to σ_y . Take $v^a \in T_{Y_0^a} M$ and $v \in T_{Y_0} M$ with the property that $\lim_{a \rightarrow 0} \sigma^*(Y_t^a)(v^a) = \sigma^*(Y_t^a)(v)$. Let U be a continuous vector field. Then

$$\langle \parallel_t^a v^a, U(Y_t^a) \rangle = \langle \sigma^*(Y_t^a)(\parallel_t^a v^a), \sigma^*(Y_t^a)(U(Y_t^a)) \rangle.$$

Since $Y_t^a \rightarrow Y_t$ as $a \rightarrow 0$, so does $\sigma^*(Y_t^a)U(Y_t^a)$ to $\sigma^*(Y_t)U(Y_t)$. We prove below that

$$\sigma^*(Y_t^a)(\parallel_t^a v^a) \rightarrow \sigma^*(Y_t)(\parallel_t v).$$

Proposition 4.1. *Let $v^a \in T_{Y_0^a} M$ and $v \in T_{Y_0} M$. Suppose that $\sigma_{Y_0^a}^* v^a$ converges to $\sigma_{Y_0}^* v$ in probability as $a \rightarrow 0$. Then $\lim_{a \rightarrow 0} \sigma^*(Y_t^a)(\parallel_t^a v^a) = \sigma^*(Y_t)(\parallel_t v)$, with convergence in the semi-martingale topology. Also $\lim_{a \rightarrow 0} \parallel_t^a v^a \stackrel{UCP}{=} \parallel_t v$.*

If M is compact, the convergences hold respectively in $\mathcal{H}_p([0, T])$ and $\mathcal{S}_p([0, T])$, for all $T > 0$ and for all $p \geq 1$.

Proof. Since $\nabla_{\circ dY_t^a}(\|_t^a v^a) = 0$ by the definition, the stochastic differential of $\|_t^a v^a$ satisfies the following equation:

$$d(\sigma^*(Y_t^a)\|_t^a v^a) = \nabla_{\circ dY_t^a} \sigma^*(Y_t^a)\|_t^a v^a = \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)} \sigma^*)\|_t^a v^a \circ dB_t^j$$

where we used the fact that $\nabla_{\nu} \sigma^* = 0$. Upon converting the Stratonovich integral on the right hand side we see that

$$\begin{aligned} d(\sigma^*(Y_t^a)\|_t^a v^a) &= \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)} \sigma^*)\|_t^a v^a dB_t^j + \frac{1}{2} \sum_{j=2}^m d(\nabla_{\sigma_j(Y_t^a)} \sigma^*)\|_t^a v^a dB_t^j \\ &= \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)} \sigma^*)\|_t^a v^a dB_t^j + \frac{1}{2} \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)} \nabla_{\sigma_j(Y_t^a)} \sigma^*)\|_t^a v^a dt. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} d(\sigma^*(Y_t^a)\|_t^a v^a) &= \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)} \sigma^*) \sigma(Y_t^a) (\sigma^*(Y_t^a)\|_t^a v^a) dB_t^j \\ &\quad + \frac{1}{2} \sum_{j=2}^m (\nabla_{\sigma_j(Y_t^a)} \nabla_{\sigma_j(Y_t^a)} \sigma^*) \sigma(Y_t^a) (\sigma^*(Y_t^a)\|_t^a v^a) dt. \end{aligned}$$

Since the coefficients of the SDE converges as $a \rightarrow 0$ uniformly in probability, we get that $\sigma^*(Y_t^a)\|_t^a v^a$ converges to $\sigma^*(Y_t)\|_t v$ in semi-martingale topology, see Theorem 2 in [23, M. Emery].

Finally, since the linear maps $\sigma(Y_t^a) : \mathbf{R}^m \rightarrow T_{Y_t^a} M$ converge to $\sigma(Y_t) : \mathbf{R}^m \rightarrow T_{Y_t} M$ in UCP topology and $\|_t^a = \sigma(Y_t^a)\sigma^*(Y_t^a)\|_t^a$, we see that $\|_t^a$ converges to $\|_t$ in the same topology. \square

5 Convergence of the damped parallel translations

Let (Y_t) be the reflected Brownian motion and (Y_t^a) the approximate reflected Brownian motions, constructed by (3.4) and (3.3) respectively. Let $A^a = \nabla \ln \tanh(\frac{R}{a})$. Denote by (W_t^a) the damped parallel translations (Y_t^a) , solving the equation

$$\frac{DW_t^a}{dt} = -\frac{1}{2} \text{Ric}^\#(W_t^a) + \nabla_{W_t^a} A^a, \quad W_0^a = \text{Id}. \tag{5.1}$$

Let (W_t) the the damped parallel translation along (Y_t) . We take the version constructed by Theorem C.3, so (W_t) is an adapted right continuous stochastic process such that $\lim_{\varepsilon \rightarrow 0} W_t^\varepsilon = W_t$ in the topology of uniform convergence in probability (UCP) where (W_t^ε) are solutions to the equations (2.2). Formally,

$$DW_t = -\frac{1}{2} \text{Ric}^\#(W_t) dt - \mathcal{S}(W_t) dL_t - \chi_{\{t \in \mathcal{R}(\omega)\}} \langle W_{t-}, \nu_{Y_t} \rangle \nu_{Y_t}, \quad W_0 = \text{Id}_{T_{Y_0} M}.$$

Our aim is to prove that W_t^a converges to W_t . It is fairly easy to see the convergence when Y_t^a and Y_t are in M^0 . When they are in a neighbourhood of ∂M , we use the pathwise construction for W . Let $(\varepsilon_n)_{n \geq 0}$ be a sequence of positive numbers converging to 0. As soon as a continuous version of (Y_t) and parallel translations along (Y_t) are chosen, each W^{ε_n} is constructed pathwise. Moreover W^{ε_n} converges to W locally in \mathcal{S}_2 and there exists a subsequence of ε_{n_k} such that $W^{\varepsilon_{n_k}}(\omega)$ converges locally uniformly for almost surely all ω .

Denote $\mathfrak{L}(\omega)$ the set of times Y_t spend on the boundary. Let F_0 be a tubular neighbourhood of ∂M . On $\{Y_t \in F_0\}$, we write

$$W_t = W_t^T + f(t) \nu_{Y_t}, \tag{5.2}$$

where $f(t)$ is its component along ν_{Y_t} and W_t^T its orthogonal complement.

The proposition below is a local result. We prove the following two ways of removing the normal part from the damped parallel translation are equivalent. (1) During an excursion (l_α, r_α) , evolve (W_t) with the continuous damped parallel translation equation, then remove the normal part at the touching down time r_α ; (2) at the beginning of every excursion remove the normal part of W_t and then evolve (W_t) with the continuous damped parallel translation equation during an excursion. This equivalence is due to the fact that every beginning of excursion is the right limit of ends of excursions and every end of excursion is the left limit of beginning of excursions. Notice the integral with respect to the local time is well explained by the approximation by W_t^ε and later by the approximation by W_t^a , but is absent of the description here.

Proposition 5.1. *Let S_1, S_2 be stopping times and $t \in [S_1(\omega), S_2(\omega)]$. Let*

$$\zeta = \inf\{t > 0 : Y_t \in \partial M\}.$$

We assume the following conditions.

1. The Ricci curvature and the shape operator are bounded on E_δ .
2. $Y_t(\omega) \in F_0$ whenever $t \in [S_1(\omega), S_2(\omega)]$.

Write $W_t = W_t^T + f(t) \nu_{Y_t}$. Let

$$\alpha_t(\omega) = \sup\{s \leq t, Y_s(\omega) \in \partial M\} = \sup([S_1(\omega), t \wedge S_2(\omega)] \cap \mathfrak{L}(\omega))$$

and let

$$r_t = \langle W_{S_1}, \nu_{Y_{S_1}} \rangle - \frac{1}{2} \int_{S_1}^t \text{Ric}(W_s, \nu_{Y_s}) ds + \int_{S_1}^t \langle W_s, D\nu_{Y_s} \rangle. \tag{5.3}$$

Then $f(t, \omega)$ is a right continuous real-valued process vanishing on $[S_1(\omega), S_2(\omega)] \cap \mathfrak{L}(\omega)$, such that

$$f(t) = \begin{cases} r_t & \text{if } t < \zeta \\ r_t - r_{\alpha_t} & \text{if } t \geq \zeta, \end{cases} \tag{5.4}$$

Furthermore,

$$\begin{aligned} DW_t^T = & -\frac{1}{2} \text{Ric}^\sharp(W_t)^T dt - \mathcal{S}(W_t^T) dL_t - \langle W_t, D\nu_{Y_t} \rangle \nu_{Y_t} \\ & - \langle W_t, \nu_{Y_t} \rangle D\nu_{Y_t} - \frac{1}{2} \sum_{k=2}^m \langle W_t, \nabla_{\sigma_k(Y_t)} \nu(Y_t) \rangle \nabla_{\sigma_k(Y_t)} \nu dt. \end{aligned} \tag{5.5}$$

Conversely, if a right-continuous $L(T_{Y_0}M, T_{Y_t}M)$ -valued process W_t' satisfies (5.2-5.5), then it satisfies (C.7).

As a local result, this can be reduced to the half plane model, the latter was dealt with in [29, N. Ikeda and S. Watanabe]. Our global description and the proof we give below will be used for our approximation result (Theorem 5.6 and Corollary 5.7).

Proof. Denote $f_t = f(t) = \langle W_t, \nu_{Y_t} \rangle$. The formulas below in the proof are interpreted and obtained as following: we first prove the corresponding identity for W_t^ε and then take $\varepsilon \rightarrow 0$. Firstly we compute the stochastic differential of f_t :

$$df_t = -\frac{1}{2} \langle \text{Ric}^\sharp(W_t), \nu_{Y_t} \rangle dt + \langle W_t, D\nu_{Y_t} \rangle - \chi_{\{t \in \mathfrak{L}(\omega)\}} \langle W_{t-}, \nu_{Y_t} \rangle,$$

for which we used the fact that $\mathcal{S}(W_t^T) = \mathcal{S}(W_t)$ is orthogonal to ν_{Y_t} . Then from $W_t^T = W_t - f_t \nu_{Y_t}$,

$$DW_t^T = DW_t - df_t \nu_{Y_t} - f_t D\nu_{Y_t} - \frac{1}{2} D[f, \nu_{Y_t}]_t$$

where the covariant square bracket $D[f, \nu_{Y_t}]_t$ is the martingale bracket including the jump part. The jump part of the bracket disappears since ν_{Y_t} is a sample continuous process. Thus

$$DW_t^T = -\frac{1}{2}(\text{Ric}^\sharp(W_t))^T dt - \mathcal{S}(W_t^T) dL_t - \langle W_t, D\nu_{Y_t} \rangle \nu_{Y_t} - f_t D\nu_{Y_t} - \frac{1}{2} D\langle f, \nu_{Y_t} \rangle_t,$$

where $D\langle f, \nu_{Y_t} \rangle_t$ is the continuous part of the martingale bracket. The martingale part of ν_{Y_t} is $\sum_{k=2}^m \nabla_{\sigma_k(Y_t)} \nu d B_t^k$; while the martingale part of $\langle W_t, D\nu_{Y_t} \rangle$ is

$$\sum_{k=2}^m \langle W_t, \nabla_{\sigma_k(Y_t)} \nu(Y_t) \rangle d B_t^k.$$

This means that

$$D\langle f, \nu_{Y_t} \rangle_t = \sum_{k=2}^m \langle W_t, \nabla_{\sigma_k(Y_t)} \nu(Y_t) \rangle \nabla_{\sigma_k(Y_t)} \nu dt,$$

concluding (5.5).

For $t < \zeta$, (5.4) clearly holds. We prove it holds also for $t > \zeta$. If $t \in \mathcal{R}(\omega) \equiv \{r_\alpha(\omega)\}$, $f_t = 0$ by the definition. This agrees with (5.4): $\alpha_t = t$ and $r_t - r_{\alpha_t} = 0$.

On $[S_1, S_2]$ the process R_t is equivalent in law to a reflected Brownian motion see Lemma C.1. So for every $t \in \mathcal{L}(\omega) \setminus \mathcal{R}(\omega)$, there exists an increasing sequence $(t_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{R}(\omega)$ converging to t . For all $n \in \mathbb{N}$ we have $f(t_n) = 0$ and

$$f(t) = f(t_n) + \int_{t_n}^t df(s) = 0 + \int_{t_n}^t \langle DW_s, \nu_{Y_s} \rangle + \int_{t_n}^t \langle W_s, D\nu_{Y_s} \rangle.$$

This formula makes sense by choosing a continuous version of the integral $\int_{S_1}^\cdot \langle W_s, D\nu_{Y_s} \rangle$ and by remarking that (W_t) is the pathwise solution to equation (C.7).

So we have

$$\begin{aligned} f(t)^2 &= \int_{t_n}^t 2f(s) df(s) + \int_{t_n}^t df(s) df(s) \\ &= \int_{t_n}^t 2f(s) (\langle DW_s, \nu_{Y_s} \rangle + \langle W_s, D\nu_{Y_s} \rangle) + \int_{t_n}^t \text{trace} \langle W_s, \nabla \cdot \nu \rangle \langle W_s, \nabla \cdot \nu \rangle \\ &\quad + \sum_{s \in]t_n, t] \cap \mathcal{R}(\omega)} \langle W_s, \nu_{Y_s} \rangle^2 \\ &= \int_{t_n}^t 2f(s) \left(\left\langle -\frac{1}{2} \text{Ric}^\sharp(W_s) ds - \mathcal{S}(W_s) dL_s, \nu_{Y_s} \right\rangle + \langle W_s, D\nu_{Y_s} \rangle \right) \\ &\quad + \int_{t_n}^t \text{trace} \langle W_s, \nabla \cdot \nu \rangle \langle W_s, \nabla \cdot \nu \rangle - \sum_{s \in]t_n, t] \cap \mathcal{R}(\omega)} f(s)^2. \end{aligned}$$

Notice that the last term combines the jump term from $\langle DW_s, \nu_{Y_s} \rangle$ and from $\langle W_s, \nu_{Y_s} \rangle^2$. It is the sum:

$$-2 \sum_{s \in]t_n, t] \cap \mathcal{R}(\omega)} f(s) \langle W_s, \nu_{Y_s} \rangle + \sum_{s \in]t_n, t] \cap \mathcal{R}(\omega)} \langle W_s, \nu_{Y_s} \rangle^2.$$

Since the jumps are all non-positive and $\langle \mathcal{S}(W_s), \nu_{Y_s} \rangle = 0$, we get

$$f(t)^2 \leq \int_{t_n}^t 2f(s) \left(-\frac{1}{2} \text{Ric}^\#(W_s \nu_{Y_s}) ds + \langle W_s, D\nu_{Y_s} \rangle \right) + \int_{t_n}^t \text{trace} \langle W_s, \nabla \cdot \nu \rangle \langle W_s, \nabla \cdot \nu \rangle.$$

But W_s is pathwise bounded in compact intervals, and

$$\int_u^t f(s) \langle W_s, D\nu_{Y_s} \rangle, \quad \int_u^t \text{trace} \langle W_s, \nabla \cdot \nu \rangle \langle W_s, \nabla \cdot \nu \rangle$$

are continuous in u . So the right hand side converges to 0 as $n \rightarrow \infty$.

This implies that $f(t) = 0$ for all $t \in \mathcal{L}(\omega)$. In particular for all $t > \xi$, $f(\alpha_t) = 0$ and the second equality of (5.4) is valid.

Conversely let W'_t be a right-continuous process satisfying the conditions of Proposition 5.1. Clearly W'_t satisfies (C.7) when $Y_t \in M^0$. On the other hand $f(t)$ vanishes on left hand sides of excursion, it is right continuous, and all right hand times of excursions are limits of decreasing sequences of left hand times of excursions, again by Lemma C.1. So it also vanishes on \mathcal{R} , and consequently $W'_t = W_t$. \square

We can now state the representation theorem for the heat equation on differential 1-forms, c.f. (1.2), with the absolute boundary conditions $\phi(\nu) = 0$ and $d\phi(\nu) = 0$.

Theorem 5.2. *Suppose that the tubular neighbourhood of the boundary has positive radius, and that the curvatures $\text{Ric}^\#$ and \mathcal{S} are bounded from below respectively on M^0 and on ∂M . If ϕ_t is a solution to the heat equation on differential 1-forms, (1.2), with the absolute boundary conditions, then for any $v \in T_{Y_0}M$, $\phi_t(v) = \mathbf{E}[\phi(W_t(v))]$.*

Proof. It is clear that the reflected Brownian motion (Y_t) is globally defined. Let ψ be a C^2 differential 1-form. Since $\psi_x(w)$ is linear in $w \in T_xM$ and $\frac{1}{2} \sum_{k=1}^m \nabla_{\sigma_k} \sigma_k + \sigma_0 = 0$, we see that

$$\begin{aligned} \psi(W_t) = & \psi(W_0) + \sum_{k=1}^m \int_0^t (\nabla_{\sigma_k(Y_s)} \psi)(W_{s-}) dB_s^k + \frac{1}{2} \int_0^t \Delta^1 \psi(W_{s-}) ds \\ & + \int_0^t \nabla_{\nu(Y_{s-})} \psi(W_{s-}) dL_s - \int_0^t \psi(\mathcal{S}(W_{s-})) dL_s \\ & - \sum_{s \in \mathcal{R}(\omega) \cap [0,t]} (\langle W_s, \nu_{Y_s} \rangle - \langle W_{s-}, \nu_{Y_s} \rangle) \psi(\nu_{Y_s}). \end{aligned}$$

We used Weitzenböck formula $\Delta^1 \psi = \text{trace} \nabla^2 \psi - \psi(\text{Ric}^\#)$. By Palais's formula for two vector fields ν and V :

$$d\psi(\nu, V) = L_V(\psi(\nu)) - L_\nu(\psi(V)) - \psi([\nu, V]) = (\nabla_\nu \psi)(V) - (\nabla_V \psi)(\nu). \tag{5.6}$$

Hence we may commute the directions in $\nabla_{\nu(Y_{s-})} \psi(W_{s-})$.

Suppose that ψ satisfies the additional condition: $\psi(\nu) = 0$ and $d\psi(\nu, \cdot) = 0$ on the boundary. Since Y_s is continuous, $\psi(\nu_{Y_{s-}}) = 0$ at the ends of an excursion, the last line vanishes. For any vector w in the tangent space of the boundary, $L_w(\psi(\nu)) = 0$ and so $(\nabla_w \psi)(\nu) - \psi(\mathcal{S}(w)) = 0$. Since $\nu_{Y_s} \langle W_{s-}, \nu_{Y_s} \rangle$ vanishes on the boundary and L_s increases only on the boundary,

$$\int_0^t (\nabla_\nu \psi)(\nu_{Y_s} \langle W_{s-}, \nu_{Y_s} \rangle) dL_s = 0.$$

Together with the earlier argument we see the sum of the terms in the second line vanishes:

$$\begin{aligned} & \int_0^t \nabla_{\nu(Y_{s-})} \psi(W_{s-}) dL_s - \int_0^t \psi(\mathcal{S}(W_{s-})) dL_s \\ &= \int_0^t (\nabla_{\nu(Y_{s-})} \psi(W_{s-}^T) + \nabla_{\nu(Y_{s-})} \psi(\langle W_{s-}, \nu(Y_{s-}) \rangle \nu(Y_{s-})) - \psi(\mathcal{S}(W_{s-}))) dL_s = 0. \end{aligned}$$

The last identity follows from the fact that $\nabla_{\nu} \nu$ vanishes. The above argument should be interpreted in the following way: we first replace W_t by W_t^ε everywhere for ε fixed and let $\varepsilon \rightarrow 0$ as in Theorem C.3.

If $\phi(t, \cdot)$ is the solution to the heat equation on 1-forms with absolute boundary conditions and initial value ϕ , on a neighbourhood of the boundary,

$$\phi(W_t) = \phi(t, W_0) + \sum_{k=1}^m \int_0^t (\nabla_{\sigma_k(Y_s)} \phi(t-s, W_{s-})) dB_s^k. \tag{5.7}$$

Note that ϕ is bounded and $\mathbf{E} \sup_{s \leq t} |W_s|^2$ is finite, c.f. Lemma 7.6, we take expectations of both sides of (5.7) to obtain $\phi(t, \nu) = \mathbf{E}[\phi(W_t(\nu))]$. □

Let $T > 0$. If $F(t, x)$ is a real valued function on $[0, T] \times M$, we denote by $dF(t, x)$ its differential in the second variable and $\nabla F(t, x)$ the corresponding gradient. In the next theorem we do not need to assume that ∂M has a tubular neighbourhood of positive radius.

Theorem 5.3. *Let (W_t) be the solution of (C.7). If $F : [0, T] \times M \rightarrow \mathbf{R}$ is a $C^{1,2}$ function such that $F(t, Y_t)$ is a continuous local martingale (or equivalently F solves (5.9) below), then $dF(t, Y_t)(W_t)$ is also a local martingale.*

Remark 5.4. The statements in Theorem 5.3, also in Corollary 5.5 and Theorem 5.2, are valid with W_t replaced by W_t^ε . But they are more powerful (and more intrinsic) with W_t , for the reason that $|W_t|$ is expected to be smaller than $|W_t^\varepsilon|$.

Proof. It is clear that, on $\{Y_t \in M^0\}$, $d(\langle \nabla F(t, Y_t), W_t \rangle)$ is the differential of a local martingale, hence we only need to prove the result on $\{Y_t \in F_0\}$. We write the Itô formula for $F(t, Y_t)$, the Itô differential $d(F(t, Y_t))$ satisfies the following identity:

$$\begin{aligned} d(F(t, Y_t)) &= \langle \nabla F(t, Y_t), \sigma(Y_t) dB_t \rangle \\ &+ \left(\partial_t + \frac{1}{2} \Delta \right) F(t, Y_t) dt + \langle dF(t, Y_t), \nu_{Y_t} \rangle dL_t. \end{aligned} \tag{5.8}$$

By the local martingale property of $F(t, Y_t)$ the last two terms vanishes and

$$\begin{aligned} \left(\partial_t + \frac{1}{2} \Delta \right) F(t, y) &= 0, \quad (t, y) \in [0, T] \times M^0, \\ \nu_y \in \ker dF(t, y), \quad &(t, y) \in [0, T] \times \partial M. \end{aligned} \tag{5.9}$$

Since W_t has finite variation on the set $\{Y_t \notin \partial M\}$ there is no covariation term between $dF(t, Y_t)$ and W_t . Writing an Itô formula for $\langle dF(t, Y_t), W_t \rangle$ yields

$$\begin{aligned} d\langle \nabla F(t, Y_t), W_t \rangle &= \nabla dF(t, Y_t)(\sigma(Y_t) dB_t, W_t) + \nabla dF(t, Y_t)(\nu_{Y_t}, W_t) dL_t \\ &+ \left(\partial_t + \frac{1}{2} \text{trace } \nabla^2 \right) dF(t, Y_t)(W_t) dt \\ &- \frac{1}{2} \left\langle \nabla F(t, Y_t), \text{Ric}^\#(W_t) \right\rangle dt - \langle \nabla F(t, Y_t) \mathcal{S}(W_t) \rangle dL_t. \end{aligned}$$

where in the last term we used (5.9). We note that $\Delta^1 = \text{trace } \nabla^2 - \frac{1}{2} \text{Ric}^\sharp$ and $\Delta^1 d = d\Delta$. This together with (5.9), $(\partial_t + \frac{1}{2} \Delta^1) dF(t, y) = 0$, yields

$$d\langle \nabla F(t, Y_t), W_t \rangle = \nabla dF(t, Y_t)(\sigma(Y_t)dB_t, W_t) + \nabla dF(t, Y_t)(\nu_{Y_t}, W_{t-}) dL_t - \langle \nabla F(t, Y_t), \mathcal{S}(W_t) \rangle dL_t.$$

Now for $y \in \partial M$ and $w \in T_y M$, since $\nu(y) \in \ker dF(t, y) \in T_y \partial M$ we have

$$-\langle \nabla F(t, y), \mathcal{S}(w) \rangle = \langle \nabla F(t, y), \nabla_w \nu \rangle = -\langle \nabla_w dF(t, y), \nu_y \rangle = -\nabla dF(t, y)(\nu_y, w).$$

For the second equality we used the fact that $\nu(y) \in \ker dF(t, y)$. Putting all the calculations together we finally get

$$d\langle dF(t, Y_t), W_t \rangle = \nabla dF(t, Y_t)(\sigma(Y_t)dB_t, W_t),$$

which proves that $\langle \nabla F(t, Y_t), W_t \rangle$ is a continuous local martingale. □

Applying this theorem to $F(t, y) = \mathbf{E}[f(Y_{T-t}(y))]$ where $(Y_t(y))$ is reflected Brownian motion started at $y \in M$, f is a smooth function on M with $df(\nu) = 0$ on the boundary, (under this condition F is $C^{1,2}$, see e.g. [43, F.-Y. Wang]), we immediately get the following Bismut type formula:

Corollary 5.5. *Assume that M is compact. Let $f : M \rightarrow \mathbf{R}$ be a smooth bounded function with $df(\nu) = 0$ on the boundary and let $T > 0$. Let $y \in M$, $v \in T_y M$ and (Y_t) a reflected Brownian motion started at $y \in M$, constructed as in Theorem 3.3. If u_t solves the heat equation (1.1) with Neumann boundary conditions and $u_0 = f$ then*

$$du_T(v) = \frac{1}{T} \mathbf{E} \left[f(Y_T) \int_0^T \langle W_s(v), \sigma(Y_s)dB_s \rangle \right].$$

For the analogous formula for manifold without boundary, see [31, Li] and [21, K. D. Elworthy and X.-M. Li]. Such results are also obtained in [45, L. Zambotti] and [24, T. Funaki and K. Ishitani].

Let T and a be positive numbers. Recall that the damped parallel translation along a sample continuous stochastic process (Y_t^a) is the solution to the stochastic covariant differential equation with initial value $W_0^a = \text{Id}_{T_{Y_0^a} M}$,

$$DW_t^a = \left(\nabla_{W_t^a} A^a - \frac{1}{2} \text{Ric}^\sharp(W_t^a) \right) dt. \tag{5.10}$$

The following Theorem will be proved in Section 7

Theorem 5.6. *Let M be a compact Riemannian manifold. Let*

$$A^a(x) = \nabla \ln \tanh \left(\frac{R(x)}{a} \right).$$

Let $(Y_t^a, t \in [0, T])$ and $(Y_t, t \in [0, T])$ be the stochastic processes defined in Theorem 3.3. Let W_t^a denote the damped parallel translation along Y_t^a . Then for all $p \in [1, \infty)$ and for any C^2 differential 1-form ϕ vanishing on the normal bundle $\nu(\partial M)$,

$$\limsup_{a \rightarrow 0} \mathbf{E} \sup_{s \leq t} [|\phi(W_s^a) - \phi(W_s)|^p] = 0.$$

For a non-compact manifold, we have the following result, see Appendix B.

Corollary 5.7. *Let M be a Riemannian manifold, not necessarily compact. Then for any C^2 differential 1-form ϕ such that $\phi(\nu) = 0$ in ∂M , $\phi(W^a)$ converges to $\phi(W)$ in UCP topology.*

6 Damped parallel translation as a derivative flow

In this section we assume that M is compact and prove that the damped stochastic parallel translation W_t along the sample paths of a reflected Brownian motion is the weak derivative of the stochastic flow $\{Y_t^a(u)\}$, which we define below.

For $a > 0$ let us denote by Ψ^a the solution flow to equation (3.3),

$$dY_t = \sum_{k=1}^m \sigma_k(Y_t) \circ dB_t^k + \sigma_0(Y_t)dt + A^a(Y_t) dt,$$

where $A^a(x) = \nabla \ln \tanh \left(\frac{R(x)}{a} \right)$.

Let $Y_t^a = \Psi^a(x_0)$ denote its solution with the initial value x_0 where x_0 is any point in the interior of M . Suppose that $\gamma : [0, 1] \rightarrow M^o$ is a C^1 curve with $\gamma(0) = x_0$. We build a family of Brownian flows, with drift A^a and initial value $\gamma(u)$, such that its derivative with respect to u is locally uniformly bounded for a.s. ω . For $u \in (0, 1]$, let $Y_t^a(u)$ denote the solution to the following Itô equation:

$$\begin{aligned} dY_t^a(u) &= //_{0,u}(Y_t^a(\cdot))dY_t^a(0) + A^a(Y_t^a(u))dt, \\ Y_0^a(u) &= \gamma(u) \end{aligned} \tag{6.1}$$

where $//_{0,u}(Y_t^a(\cdot))$ denotes parallel translation along the C^1 path $u \mapsto Y_t^a(u)$. Notice that the first differential on the right hand side is an Itô differential in a fixed vector space and the second one is a Stratonovich differential in a manifold.

Recall that the Itô differentials $dY_t^a(u)$ in (6.1) are defined by

$$dY_t^a(u) = //_{0,t}d \left(\int_0^t //_{0,s}^{-1} \circ dY_s^a(u) \right)_t \tag{6.2}$$

where $//_{0,t}$ is parallel transport along $t \mapsto Y_t^a(u)$, and they are, formally, tangent vectors.

More precisely, putting (3.3) in Itô form $dY_t^a = \sum_{k=1}^m \sigma_k(Y_t^a)dB_t^k + A^a(Y_t^a) dt$ we have

$$\begin{cases} dY_t^a(u) &= \sum_{k=1}^m //_{0,u}(Y_t^a(\cdot)) (\sigma_k(Y_t^a(0))) dB_t^k + A^a(Y_t^a(u)) dt \\ Y_0^a(u) &= \gamma(u). \end{cases} \tag{6.3}$$

The existence of a solution should follow from an iteration method, a proof for it is given in [4, M. Arnaudon, K. A. Coulibaly and A. Thalmaier], where an approximation procedure with iterated parallel couplings is used to obtain a Cauchy sequence in H_2 . The advantage is that at each step and each value of u we have a diffusion with the same generator $\frac{1}{2}\Delta + A^a$, and as the mesh goes to 0 all problems with cut locus disappear. The solution curves $u \mapsto Y_t^a(u)$ are almost surely differentiable and that their derivatives $\partial_u Y_t^a(u)$ are locally uniformly bounded for almost surely all ω and

$$\partial_u Y_t^a(u) = W_t^{a,u}(\dot{\gamma}(u)), \tag{6.4}$$

where $W_t^{a,u}$ is the damped parallel translation along $Y^a(u)$. This is, to our knowledge, the only known construction for $\partial_u Y_t^a(u)$ a.s. locally uniformly bounded. Our aim is to obtain a similar property for reflected Brownian motion. For this we will let $a \rightarrow 0$ in (6.4) and obtain a limiting identity in a weak sense. However we believe that our construction indeed yields (6.4) for $a = 0$ in a strong sense.

Proposition 6.1. *Let M be a smooth compact manifold with a boundary. The family*

$$\{(Y_t^a(u))_{0 \leq t \leq T, 0 \leq u \leq 1, a \in (0, 1]}\}$$

of two parameter stochastic processes is tight.

Proof. We will use the Kolmogorov criterion. For t_1, t_2, u_1, u_2 satisfying $0 \leq t_1 < t_2 \leq T$ and $0 \leq u_1 < u_2 \leq 1$ and $p \geq 1$,

$$\begin{aligned} & \mathbf{E} [\rho^p (Y_{t_1}^a(u_1), Y_{t_2}^a(u_2))] \\ & \leq 2^{p-1} (\mathbf{E} [\rho^p (Y_{t_1}^a(u_1), Y_{t_2}^a(u_1))] + \mathbf{E} [\rho^p (Y_{t_2}^a(u_1), Y_{t_2}^a(u_2))]) \\ & \leq 2^{p-1} \left(\mathbf{E} [\rho^p (Y_{t_1}^a(u_1), Y_{t_2}^a(u_1))] + \mathbf{E} \left[\left(\int_{u_1}^{u_2} |W_{t_2}^a(u)| \cdot |\dot{\gamma}(u)| du \right)^p \right] \right) \\ & \leq 2^{p-1} \left(\mathbf{E} [\rho^p (Y_{t_1}^a(u_1), Y_{t_2}^a(u_1))] + (u_2 - u_1)^{p-1} \|\dot{\gamma}\|_\infty \int_{u_1}^{u_2} \mathbf{E} [|W_{t_2}^a(u)|^p] du \right) \\ & \leq 2^{p-1} \left(\mathbf{E} [\rho^p (Y_{t_1}^a(u_1), Y_{t_2}^a(u_1))] + (u_2 - u_1)^p \|\dot{\gamma}\|_\infty \sup_{u \in [0,1], t \in [0,T]} \mathbf{E} [|W_t^a(u)|^p] \right) \\ & \leq 2^{p-1} (\mathbf{E} [\rho^p (Y_{t_1}^a(u_1), Y_{t_2}^a(u_1))] + C'(p, T)(u_2 - u_1)^p \|\dot{\gamma}\|_\infty), \end{aligned}$$

where $C'(p, T)$ is a constant. We used an estimate on $|W_t^{a,u}|$ given in (7.13) below. Here and several time in the sequel, we use the equality in law of the processes $(Y^a(u), W^a(u))$, for each fixed u , and (Y^a, W^a) . The latter process was constructed in Sections 3 and 5.

For the first term on the right hand side we again use the fact that u_1 is fixed and use estimates for Y^a , from Theorem 3.3. Since M is compact we can replace the distance $\rho(x, y)$ on M by the equivalent distance $\|H(x) - H(y)\|$ where $H : M \rightarrow \mathbf{R}^d$ is an embedding. We can also assume that $H = (\iota, R)$ is an extension of the construction in (3.16) around the boundary. In particular we can assume that the image of ∂M by H is included in $\{R = 0\}$. Then we easily check that

- the drift of $\tilde{\nu}(Y_t^a(u_1))$ is bounded,
- the drift of $(R(Y_t^a(u_1)) - R(Y_{t_1}^a(u_1)))^4$ is bounded on $\{R(Y_t^a(u_1)) \geq \delta_0\}$,
- the drift of $(R(Y_t^a(u_1)) - R(Y_{t_1}^a(u_1)))^4$ is negative on

$$\{R(Y_t^a(u_1)) \leq \delta_0\} \cap \{R(Y_t^a(u_1)) \leq \{R(Y_{t_1}^a(u_1))\}\},$$

- the drift of $(R(Y_t^a(u_1)) - R(Y_{t_1}^a(u_1)))^4$ is positive on

$$\{R(Y_t^a(u_1)) \leq \delta_0\} \cap \{R(Y_t^a(u_1)) \geq \{R(Y_{t_1}^a(u_1))\}\}$$

and bounded above by $b(R(Y_t^a(u_1)) - R(Y_{t_1}^a(u_1)))^2$ where $b > 0$ is independent of a (this is a consequence of (3.5)).

This implies, by a standard calculation, that for some constant $C' > 0$,

$$\mathbf{E} \left[\|H(Y_{t_2}^a(u_1)) - H(Y_{t_1}^a(u_1))\|^4 \right] \leq C' |t_2 - t_1|^2. \tag{6.5}$$

Finally, for some positive constant C ,

$$\mathbf{E} [\rho^4 (Y_{t_1}^a(u_1), Y_{t_2}^a(u_2))] \leq 8C |t_2 - t_1|^2 + 8C'(4, T)(u_2 - u_1)^4 \|\dot{\gamma}\|_\infty. \tag{6.6}$$

This concludes the required tightness. □

With this result at hand we construct our limiting process.

Theorem 6.2. *Let M be a smooth compact manifold with a boundary. There is a two parameter continuous process $(Y_t(u))_{0 \leq t \leq T, 0 \leq u \leq 1}$ with the following properties:*

- (1) for every $u \in [0, 1]$, $Y_t(u)$ is a reflected Brownian motion on M started at $\gamma(u)$;

(2) for every $p \in [1, \infty)$ there exists a number $C'(p, T)$ s.t. for all $0 \leq u_1 < u_2 \leq 1$,

$$\sup_{0 \leq t \leq T} \mathbf{E} [\rho^p(Y_t(u_1), Y_t(u_2))] \leq C'(p, T) \|\dot{\gamma}\|_\infty (u_2 - u_1)^p. \tag{6.7}$$

Proof. By Proposition 6.1, there exists a sequence $a_k \rightarrow 0$ such that the two parameter family of stochastic processes $Y_t^{a_k}(\cdot)$ converges in law whose limit we denote by $Y_t(\cdot)$.

Let us fix $u \in [0, 1]$. Since the convergence considered is in the weak topology, we are allowed to use another construction of $Y_t^a(u)$, namely part (2) in Theorem 3.3, in which the convergence is stronger. The limit is reflected Brownian motion started at $\gamma(u)$. This yields (1).

Let us then take $t_1 = t_2 = t$ in the computation for tightness in Proposition 6.1. Then take $k \rightarrow \infty$ to obtain (2). □

For each $u \in [0, 1]$ fixed, the stochastic processes $Y_t^{a_k}(u)$ converges in law to $Y_t(u)$. So the damped parallel translations $W_t^{a_k}(u)$, as stochastic processes on $[0, T]$, converge in law to $W_t(u)$ in the following sense: if ϕ is a C^2 differential 1-form such that $\phi|_{\partial M} = 0$, then $\phi(W_t^{a_k}(u))$ converges in law to $\phi(W_t(u))$. This is due to the fact that W^a is a functional of Y^a , W is a functional of Y , so we can apply Corollary 5.7.

Unfortunately this argument does not allow us to prove the convergence of $W_t^{a_k}(u)$ converges to $W_t(u)$, which would yield $\partial_u Y_t(u) = W_t(u)(\dot{\gamma}(u))$. However the following theorem asserts this equality in a weak sense.

Theorem 6.3. *Let M be a smooth compact manifold with a boundary. For all $f \in C^2(M)$ satisfying $\langle \nabla f, \nu \rangle = 0$ on the boundary, then*

$$\mathbf{E} \left[f(Y_t(u_2)) - f(Y_t(u_1)) - \int_{u_1}^{u_2} \langle df(Y_t(u)), W_t(u)(\dot{\gamma}(u)) \rangle du \right] = 0. \tag{6.8}$$

Proof. By (6.4),

$$f(Y_t^{a_k}(u_2)) - f(Y_t^{a_k}(u_1)) - \int_{u_1}^{u_2} \langle df(Y_t^{a_k}(u)), W_t^{a_k}(u)\dot{\gamma}(u) \rangle du = 0.$$

Since all the terms are integrable we can take the expectation, So

$$\mathbf{E} [f(Y_t^{a_k}(u_2))] - \mathbf{E} [f(Y_t^{a_k}(u_1))] - \int_{u_1}^{u_2} \mathbf{E} [\langle df(Y_t^{a_k}(u)), W_t^{a_k}(u)\dot{\gamma}(u) \rangle] du = 0.$$

Now by Theorem 3.3 and Corollary 5.7 and dominated convergence theorem:

$$\mathbf{E} [f(Y_t^{a_k}(u_2))] - \mathbf{E} [f(Y_t^{a_k}(u_1))] - \int_{u_1}^{u_2} \mathbf{E} [\langle df(Y_t^{a_k}(u)), W_t(u)\dot{\gamma}(u) \rangle] du = 0.$$

Finally we use Fubini-Tonelli Theorem to obtain (6.8). □

7 Proof of Theorem 5.6

We first reduce the proof of Theorem 5.6 to the class of C^2 differential 1-forms ϕ vanishing in a neighbourhood of the boundary, this is the content of Section 7.1. We then prove the convergence of the tangential part of the parallel transport in the topology of UCP, followed by the convergence of its normal part f_t^a . By the latter we mean that for all smooth $\phi : M \rightarrow \mathbf{R}_+$ vanishing in a neighbourhood of ∂M , $\phi(Y_t^a)f_a(t) \rightarrow \phi(Y_t)f(t)$ in the UCP topology. See Sections 7.4 and 7.5.

We describe briefly the strategy and the main difficulties. Thanks to the convergence of parallel transports established in Proposition 4.1 we only need to prove that $w_t^a \rightarrow w_t$ where $w_t^a = (\//_t^a)^{-1}W_t^a$ and $w_t = (\//_t)^{-1}W_t$, more precisely that, writing

$$w_t^a = w_t^{a,T} + f_a(t)n_t^a \tag{7.1}$$

with $n_t^a = (\| \cdot \|_t^a)^{-1} \nu(Y_t^a)$ and $w_t^{a,T}$ orthogonal to n_t^a , $w_t^{a,T} \rightarrow w_t^T$ and for any C^2 map $\phi : M \rightarrow \mathbf{R}$ vanishing in a neighbourhood of ∂M , $\phi(Y_t^a) f_a(t) n_t^a \rightarrow \phi(Y_t) f(t) n_t$.

Firstly, the integral equation for $(f_a, w^{a,T})$ has the following form

$$\begin{pmatrix} f_a(t) \\ w_t^{a,T} \end{pmatrix} = \begin{pmatrix} f_a(0) \\ w_0^{a,T} \end{pmatrix} + \int_0^t dM_s^a \begin{pmatrix} f_a(s) \\ w_s^{a,T} \end{pmatrix} + \begin{pmatrix} f_a(0) e^{-\tilde{C}_a(t)} \\ 0 \end{pmatrix}, \tag{7.2}$$

where (M_t^a) is a matrix valued process for the following form

$$M_t^a = \begin{pmatrix} 0 & \tilde{u}_t^a \\ v_t^{a,N} & v_t^{a,T} \end{pmatrix} \tag{7.3}$$

whose components are to be specified later. Also,

$$\begin{pmatrix} f(t) \\ w_t^T \end{pmatrix} = \begin{pmatrix} f(0) \\ w_0^T \end{pmatrix} + \int_0^t dM_s \begin{pmatrix} f(s) \\ w_s^T \end{pmatrix} + \begin{pmatrix} f(0) e^{-\tilde{C}(t)} \\ 0 \end{pmatrix} \tag{7.4}$$

where M_t is of the following form:

$$M_t = \begin{pmatrix} 0 & \tilde{u}_t \\ v_t^N & v_t^T \end{pmatrix}. \tag{7.5}$$

If (V_t) is a vector valued stochastic process, denote

$$\|V\|_{\mathcal{S}_p([0,T])} = \mathbf{E} \left(\sup_{0 \leq s \leq T} \mathbf{E} |V_s|^p \right).$$

We will see that the components of M^a converge to the corresponding components of M in $\mathcal{H}_p([0, T])$ for all $p \geq 1$, $T > 0$, with the exception \tilde{u}^a which contains $e^{-\tilde{C}_a(t)}$ and v_t^T which contains local time of the distance to boundary. The main difficulty is the convergence of $f_a(0) e^{-\tilde{C}_a(t)}$ to $f(0) e^{-\tilde{C}(t)}$. The convergence is only in $L^p(dt \times \mathbb{P})$, see Corollary 7.10. Also the convergence of L_t^a to L_t : it is in $\mathcal{S}_p([0, T])$ but not in $\mathcal{H}_p([0, T])$ and will require several integrations by parts. We note in the last term in equation (7.2), the tangential and the normal part decouples. The matrix (7.3) is in the lower triangular form. It is therefore possible to split the proof into the convergence of $w^{a,T}$ to w^T and the convergence of $f_a(t)$ to $f(t)$. This procedure is essential for our proof to work.

7.1 Localisation

Lemma 7.1. *Let S_1 and S_2 be stopping times such that for sufficiently small a , $Y_t \in E_0$ and $Y_t^a \in E_0$ on $\{\omega : S_1(\omega) \leq t \leq S_2(\omega)\}$.*

1. *If Theorem 5.6 holds for the class of C^2 differential 1-forms ϕ with $\phi(\nu)$ vanishing in a neighbourhood of the boundary, then it holds for all C^2 1-form ϕ such that $\phi(\nu) = 0$ on ∂M .*
2. *If $\phi(W^a)$ converges to $\phi(W)$ in the UCP topology and M is compact, then*

$$\lim_{a \rightarrow 0} \mathbf{E} \sup_{s \leq t} |\phi(W^a) - \phi(W)|^p \rightarrow 0.$$

3. *For the statement in Theorem 5.6, it is sufficient to prove that for $t \in [S_1, S_2]$, $\phi(W^a)$ converges to $\phi(W)$ in UCP.*

Without loss of generality we will assume that $S_1 = 0$ and let $S_2 = S$.

Proof. Let ϕ be a C^2 differential 1-form such that $\phi(\nu) = 0$ on ∂M . Then there exists a family of C^2 differential 1-forms ϕ^ε such that $\langle \phi^\varepsilon, \nu \rangle = 0$ in a neighbourhood of ∂M and $\sup_{x \in M} \|\phi^\varepsilon(x) - \phi(x)\| < \varepsilon$: choose for instance $\phi^\varepsilon(u) = \phi(u) - \langle u, \nu \rangle f^\varepsilon(\pi(u)) \phi(\nu)$ where f^ε is a smooth function on M satisfying

- $f^\varepsilon = 0$ on $\{R \geq \alpha\}$,
- $f^\varepsilon = 1$ on $\{R \leq \alpha/2\}$,
- $0 \leq f \leq 1$ on M ,

and $\alpha \in (0, \delta_0)$ is chosen in such a way that $|\phi(\nu)| < \varepsilon$ on $\{R < \alpha\}$.

By the assumption, $\phi^\varepsilon(W^a) \rightarrow \phi^\varepsilon(W)$ in $\mathcal{S}_p([0, T])$. On the other hand

$$\begin{aligned} |\phi(W_t^a) - \phi(W_t)| &\leq |\phi(W_t^a) - \phi^\varepsilon(W_t^a)| + |\phi^\varepsilon(W_t^a) - \phi^\varepsilon(W_t)| + |\phi^\varepsilon(W_t) - \phi(W_t)| \\ &\leq \varepsilon|W_t^a| + |\phi^\varepsilon(W_t^a) - \phi^\varepsilon(W_t)| + \varepsilon|W_t|. \end{aligned}$$

If a is sufficiently small then $\|\phi^\varepsilon(W^a) - \phi^\varepsilon(W)\|_{\mathcal{S}_p([0, T])} < \varepsilon$ by Theorem 5.6. Using lemma 7.5 we get

$$\|\phi(W^a) - \phi(W)\|_{\mathcal{S}_p([0, T])} < \varepsilon(1 + 2C)$$

for a sufficiently small. Taking $\varepsilon \rightarrow 0$ we obtain $\|\phi(W^a) - \phi(W)\|_{\mathcal{S}_p([0, T])} \rightarrow 0$.

(2) By Lemma 7.8 below, $\int_0^t \frac{2}{a \sinh(\frac{2R(Y_s^a)}{a})} ds$ converges to L_t in \mathcal{S}_p for all $p \in [1, \infty)$.

Let $t > 0$. Since M is compact, by corollary B.5 and lemma 7.5, if $\phi(W^a)$ converges to $\phi(W)$ in UCP topology,

$$\lim_{a \rightarrow 0} \mathbf{E} \left(\sup_{s \leq t} |\phi(W_s^a) - \phi(W_s)|^p \right) = 0.$$

(3) This is due to the fact that Y_t^a converges to Y_t in UCP topology and inside M^0 the coefficients for W_t^a converge smoothly and uniformly to the coefficients of the equation for W_t . \square

7.2 Preliminary computations

Let $a > 0$. The damped parallel translation along Y_t^a satisfies the following equations:

$$DW_t^a = \nabla_{W_t^a} A^a dt - \frac{1}{2} \text{Ric}_{Y_t^a}^\sharp(W_t^a) dt.$$

In the tubular neighbourhood F_0 on which the approximating SDEs were constructed, take $y \in M^0$ and $w \in T_y M$. Then

$$\begin{aligned} \nabla_w A^a &= \nabla_w \nabla \ln \tanh \left(\frac{R}{a} \right) = \nabla_w \left(\frac{2 \nabla R}{a \sinh \left(\frac{2R}{a} \right)} \right) \\ &= - \frac{4 \cosh \left(\frac{2R}{a} \right)}{a^2 \sinh^2 \left(\frac{2R}{a} \right)} \langle w, \nu_y \rangle \nu_y - \frac{2}{a \sinh \left(\frac{2R}{a} \right)} \mathcal{S}(w), \end{aligned}$$

Let us define $R_t^a = R(Y_t^a)$, $R_t = R(Y_t)$,

$$c_a(t) = \frac{4 \cosh \left(\frac{2R_t^a}{a} \right)}{a^2 \sinh^2 \left(\frac{2R_t^a}{a} \right)}, \tag{7.6}$$

and

$$f_a(t) = \langle w_t^a, n_t^a \rangle = \langle W_t^a, \nu(Y_t^a) \rangle. \tag{7.7}$$

We also denote $W_t^{a,T}$ the tangential part of W_t^a :

$$W_t^{a,T} = W_t^a - f_a(t) \nu(Y_t^a). \tag{7.8}$$

Definition 7.2. Let

$$L_t^a = \int_0^t \frac{2}{a \sinh \left(\frac{2R(Y_s^a)}{a} \right)} ds. \tag{7.9}$$

Below $//_t^{-1}$ is shorthand for $//_t^{-1}(Y^a)$. For $x \in M$, denote $\|\nabla\nu(x)\|^2 = \sum_k \|\nabla_{\sigma_k}\nu\|_x^2$. The latter is the Hilbert-Schmidt norm of the linear operator $\nabla\nu : T_xM \rightarrow T_xM$. In the following formulas we should consider the integrals are in Itô form. Hence the equation for $W_t^{a,T}$ should be interpreted as for $//_t^{-1}W_t^{a,T}$.

Lemma 7.3. *In the tubular neighbourhood, the following formulae hold.*

$$DW_t^a = -\frac{1}{2}\text{Ric}^\sharp(W_t^a) dt - c_a(t)f_a(t)\nu(Y_t^a) dt - \mathcal{S}(W_t^a) dL_t^a,$$

$$d//_t^{-1}\nu(Y_t^a)dt = //_t^{-1} \sum_k \nabla_{\sigma_k}\nu(Y_t^a)dB_t^k + \frac{1}{2}//_t^{-1} \text{trace } \nabla^2\nu(Y_t^a)dt.$$

Finally the stochastic differential of the tangential part of W_t^a has the following tangential and normal decomposition

$$DW_t^{a,T} = -\frac{1}{2} \left(\text{Ric}^\sharp(W_t^{a,T})\right)^T dt - \frac{1}{2}f_a(t) \left(\text{Ric}^\sharp(\nu(Y_t^a))\right)^T dt - \mathcal{S}(W_t^a) dL_t^a$$

$$- f_a(t)\nabla_{\sigma_k}\nu(Y_t^a)dB_t^k - \frac{1}{2}f_a(t) \text{trace } \nabla^2\nu(Y_t^a)dt$$

$$- \frac{1}{2}\langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle \nabla_{\sigma_k}\nu(Y_t^a) dt$$

$$- \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle \nu(Y_t^a)dB_t^k - \langle W_t^{a,T}, \text{trace } \nabla^2\nu(Y_t^a)\rangle \nu(Y_t^a)dt$$

$$+ f_a(t)\|\nabla\nu(Y_t^a)\|^2\nu(Y_t^a)dt,$$

$$df_a(t) = -c_a(t)f_a(t)dt - \frac{1}{2}\text{Ric}(W_t^a, \nu(Y_t^a))dt + \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k}\nu(Y_t^a)\rangle dB_t^k$$

$$+ \frac{1}{2}\langle W_t^{a,T}, \text{trace } \nabla^2\nu(Y_t^a)\rangle dt - \frac{1}{2}f_a(t)\|\nabla\nu(Y_t^a)\|^2.$$

Proof. The first formula is clear, the second is straight forward after applying Itô's formula to the equation for (Y_t^a) :

$$((//_t^a)^{-1}D\nu(Y_t^a)) = ((//_t^a)^{-1} \left(\langle \nabla\nu, \circ dY_t^a \rangle + \frac{1}{2} \text{trace } \nabla^2\nu(Y_t^a) dt \right)).$$

Since $\langle \nabla\nu, A^a \rangle = 0$, this yields

$$((//_t^a)^{-1}D\nu(Y_t^a)) = ((//_t^a)^{-1} \left(\sum_k \langle \nabla\nu, \sigma_k(Y_t^a) \rangle dB_t^k + \frac{1}{2} \text{trace } \nabla^2\nu(Y_t^a) dt \right))$$

$$+ ((//_t^a)^{-1} \nabla_{A^a(Y_t^a)}\nu)dt.$$

Note that $\nu = \nabla R$. If γ_t is the geodesic from x to $\pi(x)$ then $\dot{\gamma}(t) = \nabla R(\gamma(t))$ and hence $\nabla_{\nu}\nu = 0$ and the second formula follows. We note also that

$$\langle W_t^a, \circ D(\nu(Y_t^a)) \rangle = \sum_k \langle W_t^a, \nabla_{\sigma_k}\nu(Y_t^a) \rangle dB_t^k + \langle W_t^a, \frac{1}{2} \text{trace } \nabla^2\nu(Y_t^a) \rangle dt$$

$$= \sum_k \langle W_t^a, \nabla_{\sigma_k}\nu(Y_t^a) \rangle dB_t^k - \frac{1}{2}f_t\|\nabla\nu(Y_t^a)\|^2$$

$$+ \frac{1}{2}\langle W_t^{a,T}, \text{trace } \nabla^2\nu(Y_t^a) \rangle dt.$$

Note that the left hand side is in Stratonovich form and the right hand side in Itô form. We work on the third equation:

$$\begin{aligned} df_t &= \langle DW_t^a, \nu(Y_t^a) \rangle + \langle W_t^a, D(\nu(Y_t^a)) \rangle \\ &= -c_a(t)f_a(t)dt - \frac{1}{2}\text{Ric}(W_t^a, \nu(Y_t^a))dt + \sum_k \langle W_t^a, \nabla_{\sigma_k} \nu(Y_t^a) \rangle dB_t^k \\ &\quad + \frac{1}{2} \langle W_t^a, \text{trace } \nabla^2 \nu(Y_t^a) \rangle dt. \end{aligned}$$

All stochastic integrals in the above formula are in Itô form. The required identity follows from the observation below:

$$\left\langle \sum_k \nabla^2 \nu(\sigma_k, \sigma_k), \nu \right\rangle = - \sum_k \langle \nabla_{\sigma_k} \nu, \nabla_{\sigma_k} \nu \rangle = -\|\nabla \nu\|^2.$$

Next we compute the tangential part of the damped parallel translation.

$$\begin{aligned} DW_t^{a,T} &= DW_t^a - D(f_t \nu(Y_t^a)) \\ &= -\frac{1}{2} \text{Ric}^\sharp(W_t^a) dt - c_a(t)f_a(t)\nu(Y_t^a) dt - \mathcal{S}(W_t^a) dL_t^a - D(f_t \nu(Y_t^a)). \end{aligned}$$

For the normal part of the damped parallel transport, we use product rule

$$\begin{aligned} D(f_t \nu(Y_t^a)) &= \nu(Y_t^a) df_t + f_t D(\nu(Y_t^a)) + df_t D(\nu(Y_t^a)) \\ &= \nu(Y_t^a) \left(-c_a(t)f_a(t)dt - \frac{1}{2} \text{Ric}(W_t^a, \nu(Y_t^a))dt \right) \\ &\quad + \nu(Y_t^a) \left(\sum_k \langle W_t^a, \nabla_{\sigma_k} \nu(Y_t^a) \rangle dB_t^k + \frac{1}{2} \langle W_t^a, \text{trace } \nabla^2 \nu(Y_t^a) \rangle dt. \right) \\ &\quad + f_t \left(\sum_k \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k + \frac{1}{2} \|\nu\|^{-1} \text{trace } \nabla^2 \nu(Y_t^a) dt \right) \\ &\quad + \sum_k \langle W_t^a, \nabla_{\sigma_k(Y_t^a)} \nu \rangle \nabla_{\sigma_k(Y_t^a)} \nu dt. \end{aligned}$$

Since $\langle \nabla \cdot \nu, \nu \rangle$ vanishes, $\langle \nabla_{W_t^a} A^a, \nu(Y_t^a) \rangle dt = \mathcal{S}(W_t^a) dL_t^a$. Finally, we bring the above formula back to the equation for $W_t^{a,T}$ and observe that the cancellation of the term involving $f_a(t)$.

$$\begin{aligned} DW_t^{a,T} &= -\frac{1}{2} \text{Ric}^\sharp(W_t^a) dt - \mathcal{S}(W_t^a) dL_t^a + \frac{1}{2} \text{Ric}(W_t^a, \nu(Y_t^a)) \nu(Y_t^a) dt \\ &\quad - \sum_k \langle W_t^a, \nabla_{\sigma_k} \nu(Y_t^a) \rangle \nu(Y_t^a) dB_t^k + \frac{1}{2} f_t \|\nabla \nu\|^2 \nu(Y_t^a) \\ &\quad - \frac{1}{2} \langle W_t^{a,T}, \text{trace } \nabla^2 \nu(Y_t^a) \rangle \nu(Y_t^a) dt - f_t \sum_k \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k \\ &\quad - \frac{1}{2} f_t \text{trace } \nabla^2 \nu(Y_t^a) dt - \sum_k \langle W_t^a, \nabla_{\sigma_k(Y_t^a)} \nu \rangle \nabla_{\sigma_k(Y_t^a)} \nu dt. \end{aligned}$$

Following this up and observing that

$$-\frac{1}{2} \text{Ric}^\sharp(W_t^a) + \frac{1}{2} \text{Ric}(W_t^a, \nu(Y_t^a)) \nu(Y_t^a) = -\frac{1}{2} (\text{Ric}^\sharp(W_t^a))^T,$$

we see

$$\begin{aligned} DW_t^{a,T} &= -\frac{1}{2}(\text{Ric}^\#(W_t^a))^T dt - \mathcal{S}(W_t^a) dL_t^a - \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k} \nu(Y_t^a) \rangle dB_t^k \\ &\quad + \frac{1}{2} f_t \|\nabla \nu\|^2 - \frac{1}{2} \langle W_t^{a,T}, \text{trace } \nabla^2 \nu(Y_t^a) \rangle dt - f_t \sum_k \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k \\ &\quad - f_t \text{trace } \nabla^2 \nu(Y_t^a) dt - \sum_k \langle W_t^a, \nabla_{\sigma_k(Y_t^a)} \nu \rangle \nabla_{\sigma_k(Y_t^a)} \nu dt. \end{aligned}$$

This completes the proof. □

Lemma 7.4. *Let $n_t^a = (\//_t^a)^{-1} \nu(Y_t^a)$. Then $\lim_{a \rightarrow 0} n^a = n$, in the topology of semi-martingales.*

Proof. By the definition, (7.15), the stochastic differential $D\nu(Y_t^a)$ is essentially n_t^a :

$$dn_t^a = (\//_t^a)^{-1} \left(\langle \nabla \nu, \sigma_k(Y_t^a) \rangle dB_t^k + \frac{1}{2} \text{trace } \nabla^2 \nu(Y_t^a) dt \right). \tag{7.10}$$

By the same computation,

$$dn_t = (\//_t)^{-1} \left(\langle \nabla \nu, \sigma_k(Y_t) \rangle dB_t^k + \frac{1}{2} \text{trace } \nabla^2 \nu(Y_t) dt \right). \tag{7.11}$$

We recall that $Y^a \rightarrow Y$ and $\//^a(Y^a) \rightarrow \//(Y)$. Let

$$\Delta_{\partial M} \nu := \text{trace } \nabla^2 \nu,$$

where the trace is taken in the vector space orthogonal to ν . It follows that

$$\nabla \nu(Y^a) \rightarrow \nabla \nu(Y), \quad \sigma(Y^a) \rightarrow \sigma(Y), \quad \Delta_{\partial M} \nu(Y^a) \rightarrow \Delta_{\partial M} \nu(Y),$$

all in the topology of UCP. This implies that n^a converges to n in the topology of semi-martingales. □

Let us define

$$\underline{\text{Ric}}(x) = \inf_{v \in T_x M, |v|=1} \{\text{Ric}(v, v)\}, \quad x \in M \quad \text{and} \quad \underline{\text{Ric}} = \inf_{x \in M} \underline{\text{Ric}}(x).$$

We also define

$$\underline{\mathcal{S}}(x) = \inf_{v \in T_x M, |v|=1, \langle v, \nabla R \rangle = 0} \{\mathcal{S}(v, v)\}, \quad x \in M \quad \text{and} \quad \underline{\mathcal{S}} = \inf_{x \in M} \underline{\mathcal{S}}(x).$$

Lemma 7.5. *For any $t > 0$ and $a > 0$,*

$$|W_t^a|^2 \leq |W_0^a|^2 e^{-\int_0^t \underline{\text{Ric}}(Y_s^a) ds - 2 \int_0^t \underline{\mathcal{S}}(Y_s^a) dL_s^a}. \tag{7.12}$$

Suppose that $\underline{\text{Ric}}$ and $\underline{\mathcal{S}}$ are bounded. Then $\sup_a \|W^a\|_{\mathcal{S}_p([0,T])}$ is finite. Furthermore

$$\mathbf{E} \left[\sup_{t \leq T} |W_t^a|^p \right] \leq |W_0^a|^p e^{-p \underline{\text{Ric}} T} C(T, -p \underline{\mathcal{S}}) \tag{7.13}$$

where $C(T, \lambda)$ is defined in (3.23)

Proof. We begin with W^a . Firstly,

$$|W_t^a|^2 = |W_0^a|^2 - \int_0^t \langle \text{Ric}^\sharp(W_s^a), W_s^a \rangle ds - 2 \int_0^t \langle \mathcal{S}(W_s^a), W_s^a \rangle dL_s^a - \int_0^t c_s^a \langle W_s^a, \nu_{Y_s^a} \rangle^2 ds$$

where c_s^a is the scalar normal part of W_t^a , see (7.6). It is easy to see that $c_s^a > 0$. So

$$\begin{aligned} |W_t^a|^2 &\leq |W_0^a|^2 - \int_0^t \text{Ric}(W_s^a, W_s^a) ds - 2 \int_0^t \langle \mathcal{S}(W_s^a), W_s^a \rangle dL_s^a \\ &\leq |W_0^a|^2 - \int_0^t |W_s^a|^2 \underline{\text{Ric}}(Y_s^a) ds - 2 \int_0^t |W_s^a|^2 \underline{\mathcal{S}}(Y_s^a) dL_s^a. \end{aligned}$$

This implies (7.12) and (7.13) immediately follows. □

Lemma 7.6. *We also have*

$$\|W\|_{\mathcal{S}_p([0,T])}^p \leq |W_0|^p e^{-pT \underline{\text{Ric}}} C(T, -p \underline{\mathcal{S}}) \tag{7.14}$$

Proof. Since W^ε converges to W , a similar computation holds for W^ε , the conclusion for (W_t) follows. □

7.3 The local time

Let us recall the notation

$$n_t^a = (\|t^a\|)^{-1} \nu(Y_t^a), \text{ric}_t^a = (\|t^a\|)^{-1} \text{Ric}^\sharp(\|t^a\|(\cdot)), s_t^a = (\|t^a\|)^{-1} \mathcal{S}(\|t^a\|(\cdot)). \tag{7.15}$$

Denote $w_t^a = \|t^a\|^{-1} (Y^a) W_t^a$. Then

$$dw_t^a = -c_a(t) f_a(t) n_t^a dt - s_t^a(w_t^a) dL_t^a - \frac{1}{2} \text{ric}_t^a(w_t^a) dt.$$

Recall c_a is a real valued stochastic process defined in (7.6). Let us define a new stochastic process

$$\tilde{c}_a(t) = c_a(t) + \frac{1}{2} \|\nabla \nu(Y_t^a)\|_{\text{H.S.}}^2 + \frac{1}{2} \langle \text{ric}_t^a(n_t^a), n_t^a \rangle, \tag{7.16}$$

and also the stochastic processes

$$C_a(t) = \int_0^t c_a(s) ds, \quad \tilde{C}_a(s, t) = \int_s^t \tilde{c}_a(s) ds. \tag{7.17}$$

In the tubular neighbourhood, the following holds.

Lemma 7.7. *Let S be a bounded stopping time such that $Y_t \in F_0$ on $\{t < S\}$. Let $0 \leq s < t \leq S$. Then*

$$f_a(t) = f_a(0) e^{-\int_0^t \tilde{c}_a(r) dr} + \int_0^t e^{-\int_s^t \tilde{c}_a(r) dr} d\tilde{r}_s^a, \tag{7.18}$$

where

$$\begin{aligned} \tilde{r}_t^a &= \int_0^t \langle w_s^{a,T}, \nabla_{\sigma_k(Y_s^a)} \nu \rangle dB_s^k - \frac{1}{2} \int_0^t \langle \text{ric}_s^a(w_s^{a,T}), n_s^a \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \langle w_s^{a,T}, (\|s^a\|)^{-1} \text{trace } \nabla^2 \nu(Y_s^a) \rangle ds. \end{aligned} \tag{7.19}$$

Proof. By Lemma 7.3, the function $f_a(t)$ is a solution to the following equation,

$$df_t = -c_a(t)f_a(t)dt - \frac{1}{2}\text{Ric}(W_t^a, \nu(Y_t^a))dt + \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k} \nu(Y_t^a) \rangle dB_t^k + \frac{1}{2} \langle W_t^{a,T}, \text{trace } \nabla^2 \nu(Y_t^a) \rangle dt - \frac{1}{2} f_a(t) \|\nabla \nu(Y_t^a)\|^2.$$

Split the W_t^a terms into its tangential and normal parts:

$$df_t = -c_a(t)f_a(t)dt - \frac{1}{2}f_a(t)\text{Ric}(\nu(Y_t^a), \nu(Y_t^a))dt - \frac{1}{2}f_a(t)\|\nabla \nu\|^2 dt - \frac{1}{2}\text{Ric}(W_t^{a,T}, \nu(Y_t^a))dt + \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k(Y_t^a)} \nu \rangle dB_t^k + \frac{1}{2} \langle W_t^{a,T}, \text{trace } \nabla^2 \nu(Y_t^a) \rangle dt.$$

The required identity follows from the variation of constant method. □

Lemma 7.8. *Let S be a stopping time as given in Lemma 7.7. Define for $t \in [0, S]$,*

$$L_t^a = \int_0^t \frac{2 ds}{a \sinh\left(\frac{2R_s^a}{a}\right)}. \tag{7.20}$$

Let $p, q \in [1, \infty]$ and r defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let Z_t^a and Z_t be continuous real semi-martingales defined on $[0, S]$. Then

$$\left\| \int_0^\cdot Z_s^a dL_s^a - \int_0^\cdot Z_s dL_s \right\|_{\mathcal{F}_r} \leq \|Z^a - Z\|_{\mathcal{F}_p} \|L_S^a\|_q + \|L^a - L\|_{\mathcal{F}_p} (\|Z\|_{\mathcal{F}_q} + \|Z\|_{\mathcal{H}_q}). \tag{7.21}$$

Proof. We have for $t \in [0, S]$

$$\begin{aligned} & \int_0^t Z_s^a dL_s^a - \int_0^t Z_s dL_s \\ &= \int_0^t (Z_s^a - Z_s) dL_s^a + \int_0^t Z_s d(L_s^a - L_s) \\ &= \int_0^t (Z_s^a - Z_s) dL_s^a + \int_0^t (L_s^a - L_s) dZ_s + Z_t(L_t^a - L_t). \end{aligned}$$

Since L^a is nondecreasing we have

$$\|L^a\|_{\mathcal{H}_q([0,S])}^q = \mathbf{E} \left(\sup_{s \leq S} |L_s^a|^q \right) = \|L^a\|_{\mathcal{F}_q([0,S])}^q,$$

so we get by (B.2),

$$\left\| \int_0^\cdot (Z_s^a - Z_s) dL_s^a \right\|_{\mathcal{H}_r([0,S])} \leq \|Z^a - Z\|_{\mathcal{F}_p([0,S])} \|L^a\|_{\mathcal{F}_q([0,S])}^q.$$

Similar estimates holds for the last two terms on the right hand side of the identity. This concludes the proof. □

Let S be a stopping time such that $Y_t \in F_0$ on $\{t < S\}$.

Lemma 7.9. Let $\alpha_t = \sup_{s \leq t} \{s \leq t : Y_s \in \partial M\}$. Suppose that $t \notin \mathcal{R}(\omega)$. For all $s, t \in [0, S]$ satisfying $s < t$,

$$\begin{aligned} \lim_{a \rightarrow 0} e^{-\int_s^t c_a(u) du} &= 1, \quad \text{if } s > \alpha_t \\ \lim_{a \rightarrow 0} e^{-\int_s^t c_a(u) du} &= 0, \quad \text{if } s < \alpha_t. \end{aligned} \tag{7.22}$$

The convergence is in probability. As a consequence, for all $p \geq 1$,

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\int_0^S \left| e^{-C_a(t)} - \chi_{\{s > \alpha_t\}} \right|^p dt \right] = 0; \tag{7.23}$$

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\int_0^S \left(\int_0^t \left| e^{-\int_s^t c_a(u) du} - \chi_{\{s > \alpha_t\}} \right|^p ds \right) dt \right] = 0. \tag{7.24}$$

Proof. From (7.22) it is easy to get (7.23) and (7.24) using the fact that $e^{-\int_s^t c_a(u) du}$ and $\chi_{\{s > \alpha_t\}}$ are uniformly bounded and Fubini-Tonelli theorem.

So let us prove (7.22). Write

$$\int_s^t c_a(u) du = \int_s^t \frac{2 \cosh\left(\frac{2R_u^a}{a}\right)}{a \sinh\left(\frac{2R_u^a}{a}\right)} \times \frac{2}{a \sinh\left(\frac{2R_u^a}{a}\right)} du.$$

If $s > \alpha_t$ then there exists $\varepsilon(\omega) > 0$ such that for $u \in [s, t]$, $R_u > \varepsilon(\omega)$. Since $R^a \rightarrow R$ in UCP topology, $\sup_{u \in [s, t]} c_a(u)$ converges to 0 in probability, and this implies that $e^{-\int_s^t c_a(u) du} \rightarrow 1$.

If $s < \alpha_t$ then $L_t - L_s > 0$. Indeed, this would be true if R_t was a reflected Brownian motion. But by Girsanov transform we obtain that the law of R_t is equivalent to the one of a reflected Brownian motion (Lemma C.1). So this is true.

Now we have

$$\begin{aligned} \int_s^t \frac{2 \cosh\left(\frac{2R_u^a}{a}\right)}{a \sinh\left(\frac{2R_u^a}{a}\right)} \times \frac{2}{a \sinh\left(\frac{2R_u^a}{a}\right)} du &> \int_s^t \frac{2}{a} \times \frac{2}{a \sinh\left(\frac{2R_u^a}{a}\right)} du \\ &= \frac{2}{a} (L_t^a - L_s^a). \end{aligned}$$

Since $R^a \rightarrow R$ in UCP topology we have that $L^a \rightarrow L$ in UCP topology. So for all $a_0 > 0$ the \liminf of the right hand side is larger than $\frac{2}{a_0} (L_t - L_s)$. This yields

$$\limsup_{a \rightarrow 0} e^{-\int_s^t c_a(u) du} < e^{-\frac{2}{a_0} (L_t - L_s)}$$

in probability. Letting $a_0 \rightarrow 0$ we get

$$\lim_{a \rightarrow 0} e^{-\int_s^t c_a(u) du} = 0 \quad \text{in probability.} \quad \square$$

From this result we get the following

Corollary 7.10. Define

$$\tilde{c}(t) = -\frac{1}{2} \|\nabla \nu(Y_t)\|_{\text{H.S.}}^2 - \frac{1}{2} \langle \text{ric}_t(n_t), n_t \rangle \tag{7.25}$$

where ric_t is defined in (C.9). For $s, t \in [0, S]$ satisfying $s < t$ we define

$$\tilde{C}(s, t) = \begin{cases} \int_s^t \tilde{c}(s) ds, & \text{if } s > \alpha_t \\ +\infty, & \text{if } s \leq \alpha_t \end{cases} \tag{7.26}$$

Then the following convergence holds in probability for $\tilde{C}_a(s, t)$ defined in (7.17):

$$\lim_{a \rightarrow 0} e^{-\tilde{C}_a(s, t)} = e^{-\tilde{C}(s, t)}. \tag{7.27}$$

Consequently, for all $p \geq 1$,

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\int_0^S \left| e^{-\tilde{C}_a(t)} - e^{-\tilde{C}(t)} \right|^p dt \right] = 0, \tag{7.28}$$

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\int_0^S \left(\int_0^t \left| e^{-\tilde{C}_a(s, t)} - e^{-\tilde{C}(s, t)} \right|^p ds \right) dt \right] = 0. \tag{7.29}$$

With these notations Equation (5.4) rewrites as

$$f(t) = f(0)e^{-\tilde{C}(t)} + \int_0^t e^{-\tilde{C}(s, t)} d\tilde{r}_s \tag{7.30}$$

where

$$\begin{aligned} \tilde{r}_t &= \int_0^t \langle w_s^T, \nabla_{\sigma_k(Y_s)} \nu \rangle dB_s^k - \frac{1}{2} \int_0^t \langle \text{ric}_s(w_s^T), n_s \rangle ds \\ &+ \frac{1}{2} \int_0^t \langle w_s^T, //s^{-1} \text{trace } \nabla^2 \nu(Y_s) \rangle ds. \end{aligned} \tag{7.31}$$

Proof. The convergences are obvious. For equation (7.31), we see if $t < \zeta$, $e^{-\tilde{C}(s, t)} \rightarrow 0$ for any $s \geq 0$. Hence

$$f(t) = f(0) + \int_0^t d\tilde{r}_s = f(0) + \tilde{r}_t.$$

If $t \geq \zeta$, then

$$\int_0^t e^{-\tilde{C}(s, t)} d\tilde{r}_s = \int_{\alpha(t)}^t e^{-\tilde{C}(s, t)} d\tilde{r}_s.$$

Consequently

$$f(t) = \int_{\alpha(t)}^t d\tilde{r}_s = \tilde{r}(t) - \tilde{r}(\alpha(t)). \quad \square$$

The new expression (7.30) for $f(t)$ is the same form as the equation for $f_a(t)$:

$$\begin{aligned} f_a(t) &= f_a(0)e^{-\int_0^t \tilde{c}_a(r) dr} + \int_0^t e^{-\int_s^t \tilde{c}_a(r) dr} d\tilde{r}_s^a \\ &= f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t e^{-\tilde{C}_a(s, t)} d\tilde{r}_s^a. \end{aligned}$$

We observe also that $c_a(t)$ does not converge to a finite stochastic process, hence we only expect that $f_a(t)$ converges to $f(t)$ in a weak sense. Especially it is only for a set of t of full measure that $f_a(t) \rightarrow f(t)$. This will be made precise in part 7.5

7.4 Convergence of the tangential parts

We will see that tangential parts of W_t^a converges in UCP topology, as for normal parts we have to exclude the boundary times. But both of them converge in $L^p([0, T] \times \Omega)$, this will be proved at the very end of the proof. Let us begin with the first convergence.

Lemma 7.11. *As $a \rightarrow 0$, $W^{a,T} \rightarrow W^T$ in UCP topology.*

Proof. Since $\| \cdot \|_t^a \rightarrow \| \cdot \|_t$ in the UCP topology it is sufficient to prove that $w^{a,T} \rightarrow w^T$ in the UCP topology. We recall from Lemma 7.3, the term involving $c_a(t)$ cancels and we have

$$\begin{aligned}
 DW_t^{a,T} = & -\frac{1}{2} \left(\text{Ric}^\#(W_t^{a,T}) \right)^T dt - \frac{1}{2} f_a(t) \left(\text{Ric}^\#(\nu(Y_t^a)) \right)^T dt - \mathcal{S}(W_t^a) dL_t^a \\
 & - f_a(t) \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k - \frac{1}{2} f_t \text{trace } \nabla^2 \nu(Y_t^a) dt \\
 & - \frac{1}{2} \langle W_t^{a,T}, \nabla_{\sigma_k} \nu(Y_t^a) \rangle \nabla_{\sigma_k} \nu(Y_t^a) dt \\
 & - \sum_k \langle W_t^{a,T}, \nabla_{\sigma_k} \nu(Y_t^a) \rangle \nu(Y_t^a) dB_t^k - \langle W_t^{a,T}, \text{trace } \nabla^2 \nu(Y_t^a) \rangle \nu(Y_t^a) dt \\
 & + f_t \| \nabla \nu(Y_t^a) \|^2 \nu(Y_t^a) dt.
 \end{aligned}$$

Hence

$$\begin{aligned}
 dw_t^{a,T} = & -\frac{1}{2} \text{ric}_t^a(w_t^{a,T}) dt - \frac{1}{2} f_a(t) \left(\text{Ric}^\#(\nu(Y_t^a)) \right)^T dt - s_t^a(w_t^{a,T}) dL_t^a \\
 & - f_a(t) (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k - \frac{1}{2} f_a(t) (\| \cdot \|_t^a)^{-1} \text{trace } \nabla^2 \nu(Y_t^a) dt \\
 & - \frac{1}{2} \langle w_t^{a,T}, (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) \rangle (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) dt \\
 & - \sum_k \langle w_t^{a,T}, (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) \rangle n_t^a dB_t^k \\
 & - \langle w_t^{a,T}, \text{trace} (\| \cdot \|_t^a)^{-1} \nabla^2 \nu(Y_t^a) \rangle n_t^a dt \\
 & + f_a(t) \| \nabla \nu(Y_t^a) \|^2 n_t^a dt.
 \end{aligned} \tag{7.32}$$

We define the processes $v_t^{a,T}, v_t^{a,\bar{n}}$:

$$\begin{aligned}
 v_u^{a,T}(\cdot) = & -\frac{1}{2} \int_0^u \text{ric}_t^a(\cdot) dt - \int_0^u s_t^a(\cdot) dL_t^a \\
 & - \frac{1}{2} \int_0^u \langle \cdot, (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) \rangle (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) dt \\
 & - \sum_k \int_0^u \langle \cdot, (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) \rangle n_t^a dB_t^k \\
 & - \int_0^u \langle \cdot, \text{trace} (\| \cdot \|_t^a)^{-1} \nabla^2 \nu(Y_t^a) \rangle n_t^a dt.
 \end{aligned} \tag{7.33}$$

Also,

$$\begin{aligned}
 v_u^{a,\nu} = & -\int_0^u \frac{1}{2} (\| \cdot \|_t^a)^{-1} \left(\text{Ric}^\#(\nu(Y_t^a)) \right)^T dt - \int_0^u (\| \cdot \|_t^a)^{-1} \nabla_{\sigma_k} \nu(Y_t^a) dB_t^k \\
 & - \frac{1}{2} \int_0^u (\| \cdot \|_t^a)^{-1} \text{trace } \nabla^2 \nu(Y_t^a) dt + \int_0^u \| \nabla \nu(Y_t^a) \|^2 n_t^a dt.
 \end{aligned} \tag{7.34}$$

With these notations and the expression for $f_a(t)$ in formula (7.18) we have

$$dw_t^{a,T} = dv_t^{a,T}(w_t^{a,T}) + f_a(t)dv_t^{a,\nu}$$

and so

$$dw_t^{a,T} = dv_t^{a,T}(w_t^{a,T}) + \left(f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t e^{-\int_s^t \tilde{c}_a(u) du} d\tilde{r}_s^a \right) dv_t^{a,\nu}. \tag{7.35}$$

We also have

$$dw_t^T = dv_t^T(w_t^T) + \left(f(0)e^{-\tilde{C}(t)} + \int_0^t e^{-\tilde{C}(s,t)} d\tilde{r}_s \right) dv_t^\nu, \tag{7.36}$$

(recall that $e^{-\tilde{C}(s,t)} = 0$ if $s < \alpha_t$), where

$$\begin{aligned} v_u^\nu &= -\frac{1}{2} \int_0^u \text{ric}_t(n_t)^T dt + \int_0^u \|\nabla \nu(Y_t)\|^2 n_t dt - \int_0^u (\//t)^{-1} \nabla_{\sigma(Y_t)} dB_t \nu \\ &\quad - \frac{1}{2} \int_0^u (\//t)^{-1} \Delta^{h,T} \nu(Y_t) dt, \end{aligned} \tag{7.37}$$

$$\begin{aligned} v_u^T(\cdot) &= -\frac{1}{2} \int_0^u \text{ric}_t(\cdot) dt - \int_0^u s_t(\cdot) dL_t - \int_0^u \langle \cdot, (\//t)^{-1} \nabla_{\sigma^T(Y_t)} dB_t \nu \rangle n_t \\ &\quad - \frac{1}{2} \int_0^u \langle \cdot, (\//t)^{-1} \Delta^h \nu(Y_t) \rangle n_t dt \\ &\quad - \sum_{j \geq 2} \int_0^u \langle \cdot, \nabla_{\sigma_j(Y_t)} \nu \rangle (\//t)^{-1} \nabla_{\sigma_j(Y_t)} \nu dt. \end{aligned} \tag{7.38}$$

We investigate further (7.35)

$$\begin{aligned} dw_t^{a,T} &= dv_t^{a,T}(w_t^{a,T}) + f_a(0)e^{-\tilde{C}_a(t)} dv_t^{a,\nu} \\ &\quad + \left(\int_0^t e^{-\int_s^t \tilde{c}_a(u) du} \langle w_s^{a,T}, \nabla_{\sigma_k(Y_s^a)} \nu \rangle dB_s^k \right) dv_t^{a,\nu} \\ &\quad - \frac{1}{2} \left(\int_0^t e^{-\int_s^t \tilde{c}_a(u) du} \langle \text{ric}_s^a(w_s^{a,T}), n_s^a \rangle ds \right) dv_t^{a,\nu} \\ &\quad + \frac{1}{2} \left(\int_0^t e^{-\int_s^t \tilde{c}_a(u) du} \langle w_s^{a,T}, (\//s^a)^{-1} \text{trace} \nabla^2 \nu(Y_s^a) \rangle ds \right) dv_t^{a,\nu}. \end{aligned}$$

From this the required convergence should follow: when a approaches zero, $v_t^{a,\nu}$ approaches v_t^ν and $v_t^{a,T}$ approaches v_t^T . If furthermore if $f_a(0) \rightarrow f(0)$, then

$$\lim_{a \rightarrow 0} f_a(0)e^{-\tilde{C}_a(t)} = f(0)e^{-\tilde{C}(t)}.$$

Hence the components of $w_t^{a,T}$ is the solution to a system of non-Markovian stochastic differential equations whose coefficients converge, and furthermore $v_t^{a,T}(w_t^{a,T})$ converges only in UCP, not in $\mathcal{H}_p([0, T])$. These factors explain why the proof below is long given this simple explanation. To prove that $v_t^{a,T} \rightarrow v_t^T$ in UCP topology, c.f. (7.33) and (7.38), we only need to prove that $\int_0^t s_t^a(\cdot) dL_t^a \rightarrow \int_0^t s_t(\cdot) dL_t$ in UCP topology. This is a consequence of Lemma 7.8, together with the facts that UCP topology is equivalent to local convergence in \mathcal{S}_p and that the random variables L_S^a are uniformly bounded in L^2 .

To make the rest of the proof more transparent let us define

$$\begin{aligned} \tilde{u}_t &= \int_0^t e^{-\tilde{C}(s,t)} \langle \cdot, \nabla_{\sigma^T(Y_s)} dB_s \nu \rangle \\ &\quad - \frac{1}{2} \int_0^t e^{-\tilde{C}(s,t)} \langle \text{ric}_s(\cdot), n_s \rangle ds + \frac{1}{2} \int_0^t e^{-\tilde{C}(s,t)} \langle \cdot, (\//s)^{-1} \Delta^{h,T} \nu(Y_s) \rangle ds; \end{aligned} \tag{7.39}$$

$$\begin{aligned} \tilde{u}_t^a &= \int_0^t e^{-\tilde{C}_a(s,t)} \langle \cdot, \nabla_{\sigma^T(Y_s^a)} dB_s^\nu \rangle \\ &\quad - \frac{1}{2} \int_0^t e^{-\tilde{C}_a(s,t)} \langle \text{ric}_s^a(\cdot), n_s^a \rangle ds + \frac{1}{2} \int_0^t e^{-\tilde{C}_a(s,t)} \langle \cdot, //s^{-1} \Delta^{h,T} \nu(Y_s^a) \rangle ds \end{aligned} \tag{7.40}$$

Then, by Lemma 7.3, we may write

$$f_a(t) = f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t d\tilde{u}_s^a(w_s^{a,T}), \tag{7.41}$$

$$f(t) = f(0)e^{-\tilde{C}(t)} + \int_0^t d\tilde{u}_s(w_s^T). \tag{7.42}$$

Take these equalities back to equations (7.35) and (7.36), we see

$$dw_t^{a,T} = dv_t^{a,T}(w_t^{a,T}) + \left(f_a(0)e^{-\tilde{C}_a(t)} + \int_0^t d\tilde{u}_s^a(w_s^{a,T}) \right) dv_t^{a,\nu}.$$

We also have

$$dw_t^T = dv_t^T(w_t^T) + \left(f(0)e^{-\tilde{C}(t)} + \int_0^t d\tilde{u}_s(w_s^T) \right) dv_t^\nu.$$

Let us simply compute the difference of the two matrices:

$$\begin{aligned} &d(w_t^{a,T} - w_t^T) \\ &= d(v_t^{a,T} - v_t^T)(w_t^{a,T}) + dv_t^T(w_t^{a,T} - w_t^T) + f_a(t) d(v_t^{a,\nu} - v_t^\nu) \\ &\quad + dv_t^\nu \left(f_a(0)e^{-\tilde{C}_a(t)} - f(0)e^{-\tilde{C}(t)} \right) \\ &\quad + dv_t^\nu \int_0^t d(\tilde{u}_s^a - \tilde{u}_s)(w_s^{a,T}) + dv_t^\nu \int_0^t d\tilde{u}_s(w_s^{a,T} - w_s^T). \end{aligned} \tag{7.43}$$

Now we recall that convergence in UCP topology is implied by local convergence in \mathcal{S}_1 . For a stopping time S' smaller than S we have

$$\left\| \left(\int_0^\cdot dv_t^T (w_t^{a,T} - w_t^T) \right)^{S'} \right\|_{\mathcal{S}_1} \leq \| (v^T)^{S'} \|_{\mathcal{H}_\infty} \cdot \| (w^{a,T} - w^T)^{S'} \|_{\mathcal{S}_1} \tag{7.44}$$

Since $v_0^T = 0$ and v^T has locally bounded \mathcal{H}_∞ norm we can split the time interval and we only have to make the proof on $[0, S']$ where $S' \leq S$ is a stopping time so that

$$\| (v^T)^{S'} \|_{\mathcal{H}_\infty} < 1. \tag{7.45}$$

Then using an argument analogous to that for (7.44) we see

$$\begin{aligned} &\left\| \left(\int_0^\cdot dv_t^\nu \int_0^t d\tilde{u}_s(w_s^{a,T} - w_s^T) \right)^{S'} \right\|_{\mathcal{S}_1} \\ &\leq \| (v^\nu)^{S'} \|_{\mathcal{H}_\infty} \cdot \left\| \left(\int_0^\cdot d\tilde{u}_t(w_t^{a,T} - w_t^T) \right)^{S'} \right\|_{\mathcal{S}_1} \\ &\leq \| (v^\nu)^{S'} \|_{\mathcal{H}_\infty} \cdot \| (\tilde{u})^{S'} \|_{\mathcal{H}_\infty} \cdot \| (w^{a,T} - w^T)^{S'} \|_{\mathcal{S}_1} \end{aligned} \tag{7.46}$$

Since v^ν and \tilde{u} have locally bounded \mathcal{H}_∞ norms, with the same argument we can take S' so that

$$\| (v^\nu)^{S'} \|_{\mathcal{H}_\infty} \cdot \| (\tilde{u})^{S'} \|_{\mathcal{H}_\infty} < 1. \tag{7.47}$$

We want to prove that $\|(w^{a,T} - w^T)^{S'}\|_{\mathcal{S}_1} \rightarrow 0$ as $a \rightarrow 0$. Using (7.43-7.45), (7.46), (7.47) and Gronwall lemma, it is sufficient to prove that

$$\lim_{a \rightarrow 0} \left\| \left(\int_0^\cdot d(v_t^{a,T} - v_t^T)(w_t^{a,T}) \right)^{S'} \right\|_{\mathcal{S}_1} = 0, \tag{7.48}$$

$$\lim_{a \rightarrow 0} \left\| \left(\int_0^\cdot dv_t^\nu \left(f_a(0)e^{-\tilde{C}_a(t)} - f(0)e^{-\tilde{C}(t)} \right) \right)^{S'} \right\|_{\mathcal{S}_1} = 0. \tag{7.49}$$

and

$$\lim_{a \rightarrow 0} \left\| \left(\int_0^\cdot dv_t^\nu \int_0^t d(\tilde{u}_s^a - \tilde{u}_s)(w_s^{a,T}) \right)^{S'} \right\|_{\mathcal{S}_1} = 0. \tag{7.50}$$

For (7.48) we write

$$\int_0^t d(v_s^{a,T} - v_s^T)(w_s^{a,T}) = w_t^{a,T}(v_t^{a,T} - v_t^T) - \int_0^t v_s^{a,T} - v_s^T dw_s^{a,T}. \tag{7.51}$$

From (7.32) and Lemma 7.5 we see that the processes $w^{a,T}$ are uniformly bounded in \mathcal{H}_2 . Since $v_t^{a,T} \rightarrow v_t^T$ in UCP topology, $v_t^{a,T} \rightarrow v_t^T$ locally in \mathcal{S}_∞ . We have

$$\left\| \left(\int_0^\cdot v_s^{a,T} - v_s^T dw_s^{a,T} \right)^{S'} \right\|_{\mathcal{S}_2} \leq \|(v^{a,T} - v^T)^{S'}\|_{\mathcal{S}_\infty} \cdot \|w_s^{a,T}\|_{\mathcal{H}_2}$$

and

$$\begin{aligned} \|w^{a,T}(v^{a,T} - v^T)\|_{\mathcal{S}_2} &\leq \|(v^{a,T} - v^T)^{S'}\|_{\mathcal{S}_\infty} \cdot \|w_s^{a,T}\|_{\mathcal{S}_2} \\ &\leq 3\|(v^{a,T} - v^T)^{S'}\|_{\mathcal{S}_\infty} \cdot \|w_s^{a,T}\|_{\mathcal{H}_2}. \end{aligned}$$

From this, (7.51) and the fact that \mathcal{S}_1 norm is smaller than \mathcal{S}_2 norm, we obtain (7.48).

For (7.49) it is sufficient to compute the \mathcal{H}_2 norm of

$$\left(\int_0^\cdot dv_t^\nu \left(f_a(0)e^{-\tilde{C}_a(t)} - f(0)e^{-\tilde{C}(t)} \right) \right)^{S'}$$

and to use the dominated convergence theorem.

Finally let us prove (7.50). This can be done by modifying S' , using the facts that the processes $W^{a,T}$ have uniformly bounded \mathcal{S}_2 norms and $\tilde{u}^a \rightarrow \tilde{u}$ in UCP topology. For this last point, use (7.29) in Corollary 7.10 and Corollary B.4. \square

7.5 Convergence of the normal parts

Lemma 7.12. For all $p \in [1, \infty)$ and $T > 0$, as $a \rightarrow 0$,

$$\mathbf{E} \left[\int_0^T |f_a(t) - f(t)|^p dt \right] \rightarrow 0. \tag{7.52}$$

Proof. Write

$$\begin{aligned} f_a(t) - f(t) &= (f_a(0) - f(0))e^{-\tilde{C}_a(t)} + f(0) \left(e^{-\tilde{C}_a(t)} - e^{-\tilde{C}(t)} \right) \\ &\quad + \int_0^t \left(e^{-\tilde{C}_a(s,t)} - e^{-\tilde{C}(s,t)} \right) d\tilde{r}_s^a + \int_0^t e^{-\tilde{C}(s,t)} d(\tilde{r}_s^a - \tilde{r}_s). \end{aligned}$$

The first term in the right converges to 0 in $L^p([0, T] \times \mathbb{P})$ due to the positiveness of $\tilde{C}_a(t)$. The second term in the right converges to 0 due to (7.28). For the last term

in the right we use boundedness of $e^{-\tilde{C}(s,t)}$ and the fact that $\tilde{r}_t^a \rightarrow \tilde{r}_t$ in $\mathcal{H}_p([0, T])$ due to (7.19) and (7.31) together with (B.2) and Lemmas 7.11 and 7.5 which allow to prove that $w_t^{a,T} \rightarrow w_t^T$ in $\mathcal{S}_q([0, T])$, $q \in [1, \infty)$.

We are left to prove that

$$\int_0^t \left(e^{-\tilde{C}_a(s,t)} - e^{-\tilde{C}(s,t)} \right) d\tilde{r}_s^a \rightarrow 0 \quad \text{in } L^p([0, T] \times \mathbb{P}).$$

Here it is easier to replace S by $T \geq S$ which is deterministic. We have

$$\begin{aligned} & \mathbf{E} \left[\int_0^T \left| \int_0^s \left(e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)} \right) d\tilde{r}_u^a \right|^p ds \right] \\ &= \int_0^T \mathbf{E} \left[\left| \int_0^s \left(e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)} \right) d\tilde{r}_u^a \right|^p \right] ds \\ &\leq C(p, T) \int_0^T \mathbf{E} \left[\int_0^s \left| e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)} \right|^p du \right] ds. \end{aligned}$$

The last inequality comes from the fact that the identity map from $\mathcal{S}_p([0, s])$ to $\mathcal{H}_p([0, s])$ is continuous and bounded by $C(p, s)$ satisfying $0 < C(p, s) \leq C(p, T)$. Notice that the fact that $u \mapsto e^{-\tilde{C}_a(u,s)} - e^{-\tilde{C}(u,s)}$ is not adapted is not a problem since in $d\tilde{r}_u^a$ there is no integration with respect to B^1 . We conclude with (7.29) which is easily seen to be true with S replaced by T . \square

Lemma 7.13. *For all $p \in [1, \infty)$, $T > 0$ and all smooth $\phi : M \rightarrow \mathbf{R}_+$ vanishing in a neighbourhood of ∂M , $\phi(Y_t^a)f_a(t) \rightarrow \phi(Y_t)f(t)$ in $\mathcal{S}_p([0, T])$.*

Proof. Since ϕ is bounded and the processes $f_a(t)$ are uniformly bounded in $\mathcal{S}_p([0, T])$ independently of a , it is sufficient to prove convergence in UCP topology.

We have

$$\phi(R_t^a)f_a(t) - \phi(R_t)f(t) = (\phi(R_t^a) - \phi(R_t))f_a(t) + \phi(R_t)(f_a(t) - f(t)).$$

Since the processes $f_a(t)$ are uniformly bounded in $\mathcal{S}_p([0, T])$ independently of a and $R_t^a \rightarrow R_t$ in $\mathcal{S}_p([0, T])$, the first term in the right converges to 0 in UCP topology. Let us consider the second term:

$$\begin{aligned} d(\phi(R_t)(f_a(t) - f(t))) &= (f_a(t) - f(t))d\phi(R_t) + \phi(R_t)d(f_a(t) - f(t)) \\ &\quad + d\phi(R_t)d(f_a(t) - f(t)). \end{aligned}$$

The integral of the first term in the right converges to 0 in UCP topology, due to (7.29) and the fact that $\phi(R_t)$ has uniformly bounded absolutely continuous local characteristics.

On the other hand

$$\begin{aligned} & \phi(R_t)d(f_a(t) - f(t)) \\ &= -\tilde{c}_a(t)\phi(R_t)(f_a(t) - f(t))dt + \phi(R_t)(\tilde{c}(t) - \tilde{c}_a(t))f(t)dt \\ &\quad + \phi(R_t)(f_a(t) - f(t))d\tilde{r}_a(t) + \phi(R_t)f(t)d(\tilde{r}_t^a - \tilde{r}_t). \end{aligned}$$

From subsection 7.4 together with (7.19) and (B.2) we get that $\tilde{r}^a \rightarrow \tilde{r}$ in semimartingale topology.

So due to the presence of $\phi(R_t)$ which vanishes in a neighbourhood of ∂M all the terms behave nicely, with the help of (7.52).

Finally the covariance term can be treated with similar methods. \square

With this we completed the proof of Theorem 5.6 and close this section.

A The half line example

On the half line we select a reflected Brownian motion with ‘good’ sample path properties. To begin with, we consider two reflected Brownian motions: the solution to the Skorohod problem associated with a Brownian motion $x + B_t$ and the solution to the Tanaka problem associated with $x + \int_0^t \text{sign}(x + B_s)dB_s$. The first is a stochastic flow, see Lemma A.1 below, while the second is not.

The solution and the derivative flow to the Skorohod problem for $x + B_t$ is approximated by solutions and derivative flows to a family of SDEs with explicit drifts. Furthermore, its derivative flow is shown to coincide with the damped parallel translation introduced in Appendix C.

Denote the space of real valued continuous function with $f(0) = 0$ by $C_0(\mathbf{R}; \mathbf{R})$ and its subset of non-negative valued functions by $C_0(\mathbf{R}; \mathbf{R}_+)$. To each $x \geq 0$ and $f \in C_0(\mathbf{R}, \mathbf{R})$ there exists a unique nondecreasing function $h \in C_0(\mathbf{R}, \mathbf{R}_+)$ such that, for $g(t) := x + f(t) + h(t)$, $\int_0^t \chi_{\{0\}}(g(s))dh(s) = h(t)$. The pair (g, h) is the solution to the Skorohod problem associated to (x, f) and is denoted by

$$\Phi.(x, f) = (g, h). \tag{A.1}$$

It is well known that $h(t) = -\inf_{0 \leq s \leq t} \{x + f(s)\} \wedge 0$.

If B_t is a standard real valued Brownian motion, then the Skorohod problem defines the pair of stochastic processes $(X_t(x), L_t(x))$, and $L_t(x)$ is called the local time at 0 of $X_t(x)$ and

$$X_t(x) = x + B_t + L_t(x). \tag{A.2}$$

On the other hand, the process $(|x + B_t|)$ is also a reflected Brownian motion. In fact, by Tanaka’s formula, $|x + B_t| = x + \beta_t + 2\ell_t^0(\omega)$ where $\beta_t = \int_0^t \text{sign}(x + B_s)dB_s$ is a Brownian motion and ℓ_t^0 is the local time of $x + B_t$. The local time ℓ_t^0 is also the boundary time, i.e. the total time spent by $x + B_t$ on the boundary $\{0\}$ before time t . The local time of a brownian motion was introduced by P. Lévy (1940) as a Borel measurable function $\Omega \times \mathbf{R}_+ \times \mathbf{R} \rightarrow \mathbf{R}_+$ such that (1) for all $f \in \mathcal{B}_b(\mathbf{R}; \mathbf{R})$, $\int_0^t f(x + B_s)ds = \int_{\mathbf{R}} f(a)\ell_t^a da$, and (2) $(t, a) \mapsto \ell_t^a(\omega)$ is continuous a.s.. It is also well known that $\ell_t^0 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \chi_{[0, \varepsilon)}(x + B_s)ds = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \chi_{(-\varepsilon, \varepsilon)}(x + B_s)ds$. It is clear that $(|x + B_t(\omega)|, 2\ell_t^0(\omega))$ is the solution to the Skorohod problem associated with $x + \beta_t(\omega)$, and $|x + B_t|$ is not a stochastic flow.

It turns out that $X_t(x) = x + B_t + L_t(x)$ has many nice properties. Despite that the probability distribution of $X_t(x)$ is that of a reflecting Brownian, on a sample path level it is not at all the reflected path! It is rather, a lifted path, by ‘the lower envelope’ curve. The lower envelope curve is the unique continuous decreasing curve that is below the given curve (B_t) . Let $0 < x < y$. Let $\tau(y) = \inf\{t > 0 : X_t(y) = 0\}$. It is clear that $X_t(y) - X_t(x) = y - x$ until $X_t(x)$ reaches zero and the two stochastic processes coalesce when $X_t(y)$ reaches zero. If we compensate x by $L_t(x)$, the two processes $X_t(x + L_t(x))$ and $X_t(x)$ are equal for all t .

In Lemma A.1 we summarise the sample properties of $X_t(x)$ and discuss differentiability of $X_t(x)$ with respect to x . These properties are elementary and not surprising. It is perhaps more surprising that these elementary properties of $X_t(x)$ are passed to the reflected Brownian motion on a manifold with boundary. We should mention that differentiability with respect to the initial value was studied in [11, K. Burdzy] and [3, S. Andres] for domains in \mathbf{R}^2 and polygons.

For $s < t$ define $\theta_s B = B_{s+} - B_s$. Let ξ be an \mathcal{F}_s measurable random variable and $(X_{s,t}(\xi, \theta_s B), L_{s,t}(\xi, \theta_s B))$ the solution to the Skorohod problem for $(\xi, \theta_s B)$,

$$X_{s,t}(\xi, \theta_s B) = \xi + (\theta_s B)_{t-s} + L_{s,t}(\xi, \theta_s B).$$

Define $L_{s,t}(\xi, \theta_s B) = 0$ for $0 \leq t \leq s$. For simplicity we also omit B in the flow, and write $X_t(x)$ for $X_t(x, B)$. Let $T(x, y) = \inf\{t > 0, X_t(x) = X_t(y)\}$ be the first time $X_t(x)$ and $X_t(y)$ meet.

Lemma A.1. *The following statements hold pathwise.*

1. For all $0 \leq s < t, x \in \mathbf{R}$,

$$X_{s,t}(X_s(x, B), \theta_s B) = X_t(x, B), \quad L_t(x, B) = L_s(x, B) + L_{s,t}(X_s(x, B), \theta_s B).$$

2. Let $0 < x < y$, then $X_t(x)$ and $X_t(y)$ coalesce at the finite time $T(x, y)$. Furthermore $T(x, y) = \tau(y)$ and $L_{\tau(y)}(x) = y - x$.
3. For all $t \geq 0$ and $x > 0$, $X_t(x + L_t(x)) = X_t(x)$.
4. For all $x \geq 0$ and $t \geq 0$,

$$\partial_x X_t(x) = \begin{cases} 1, & t < \tau(x) \\ 0, & t > \tau(x) \end{cases}.$$

Proof. For part (1), we observe that,

$$X_{s,t}(X_s(x, B), \theta_s B) = x + B_t + L_s(x, B) + L_{s,t}(X_s(x), \theta_s B).$$

Define $\tilde{L}(r) = L_r(x, B)$ when $r \leq s$ and $\tilde{L}(t) = L_s(x, B) + L_{s,t}(X_s(x), \theta_s B)$ for $t > s$. Then $\tilde{L} \in C_0(\mathbf{R}_+, \mathbf{R}_+)$, and $(X_{s,t}(X_s(x, B), \theta_s B), \tilde{L})$ solves the Skorohod problem for (x, B) . By the uniqueness of the Skorohod problem, $X_t(x, B) = X_{s,t}(X_s(x, B), \theta_s B)$ and $\tilde{L}(t) = L_t(x, B)$.

Part (2). From the construction of the solution of the Skorohod problem, it is easy to see that $\tau(x) < \tau(y)$ and $X_t(y) - X_t(x) = y - x$ on $\{t < \tau(x)\}$, and $X_{\tau(y)}(y) = X_{\tau(x)}(x)$ on $\{t = \tau(y)\}$. By the flow property, $0 \leq X_t(x) \leq X_t(y)$ a.s. for all time. In other words, the two curves $\{X_s(x), s \leq t\}$ and $\{X_s(y), s \leq t\}$ are parallel on $\{t < \tau(x)\}$, until the lower curve hits zero after which the distance between the two curves decreases until $X_t(y)$ reaches zero, upon which point the two curves meet. The accumulated upward lift that $X_t(x)$ receives up to $\tau(y)$ is

$$- \inf_{0 \leq s \leq \tau(y, \omega)} \{(x - y + y + B_s(\omega)) \wedge 0\} = y - x.$$

This shows that $X_{\tau(y)}(x) = 0$ and together with the flow property we see the coalescence. We completed the proof that $T(x, y) = \tau(y)$ and $L_{\tau(y)}(x) = y - x$.

Part (3). On $\{t < \tau(x)\}$, $X_t(x + L_t(x)) = X_t(x)$ trivially. If $t \geq \tau(x + L_t(x))$,

$$\begin{aligned} X_t(x + L_t(x)) &= x + L_t(x) + B_t(x) - \inf_{0 \leq s \leq t} ((x + L_t(x) + B_s) \wedge 0) \\ &= x + L_t(x) + B_t(x) - \inf_{0 \leq s \leq t} ((x + B_s) \wedge 0) - L_t(x) = X_t(x). \end{aligned}$$

If $\tau(x) \leq t < \tau(x + L_t(x))$, $X_t(x + L_t(x)) = x + L_t(x) + B_t$ while $X_t(x)$ receives the kick of the size $L_t(x)$: $X_t(x) = x + B_t + L_t(x)$.

Part (4). Take $t < \tau(x)$. Then $t < \tau(x + \varepsilon)$ for $\varepsilon > t - \tau(x)$ and $X_t(x + \varepsilon) = X_t(x) + \varepsilon$, consequently $\partial_x X_t(x) = 1$. Suppose $t > \tau(x)$. Then by part (2), $X_t(x) = X_t(x - \varepsilon)$ for any $\varepsilon < 0$. If $0 < \varepsilon < L_t(x)$, $0 \leq X_t(x + \varepsilon) - X_t(x) \leq X_t(x + L_t(x)) - X_t(x) = 0$. We used part (3) in the last step. Hence $\partial_x X_t(x) = 0$ for $t > \tau(x)$. This completes the proof. \square

A consequence of Lemma A.1 is the following. If we pick up a time $t > \tau(x)$, then $X_t(x + L_t(x))$ must reach 0 between $\tau(x)$ and t .

In the following we construct a family of stochastic processes $\{X^a(x), a > 0\}$ with the properties stated below illustrating the general construction. (1) For each a , X^a is a

stochastic flow and $x \mapsto X_t^a(x)$ is a diffeomorphism on its image; (2) they approximate the reflected Brownian motion; (3) their derivatives approximate $\partial_x X_t(x)$.

Let $\phi(x) = \int_0^x e^{-\frac{y^2}{2}} dy$. For $x > 0$ and $a > 0$ let

$$u^a(x) = P(\tau(x) > a) = \sqrt{\frac{2}{\pi}} \phi\left(\frac{x}{\sqrt{a}}\right) = 2P_a \chi_{(0,x)}(0),$$

where P_t denotes the heat semigroup. Thus $\partial_x u^a = 2p_a$ where p_a is the Gaussian kernel. Formally $u^0(0) = P(\tau(0) > 0) = 0$ and for $x > 0$, $u^0(x) = P(\tau(x) > 0) = 1$, and $\frac{\partial}{\partial x} \ln u^0(x) = \frac{\partial}{\partial x} \chi_{(-\infty,x)} = \delta_0(x)$, the Dirac mass at 0. Note that $\ln u^a$ is a concave function with positive gradient:

$$\partial_x \ln u^a = \frac{1}{\sqrt{a}} (\ln \phi)' \left(\frac{x}{\sqrt{a}} \right) = \frac{1}{\sqrt{a}} \frac{e^{-\frac{x^2}{2a}}}{\phi\left(\frac{x}{\sqrt{a}}\right)} > 0; \tag{A.3}$$

$$\partial_x^2 (\ln u^a) = \frac{1}{a} (\ln \phi)'' \left(\frac{x}{\sqrt{a}} \right) = -\frac{x e^{-\frac{x^2}{2a}}}{a^{\frac{3}{2}} \phi\left(\frac{x}{\sqrt{a}}\right)} - \frac{e^{-\frac{x^2}{2a}}}{a \phi\left(\frac{x}{\sqrt{a}}\right)} < 0. \tag{A.4}$$

Proposition A.2. Let $X_t^a(x)$ be the solution to

$$X_t^a(x) = x + B_t + \int_0^t \partial_x \ln u^a(X_s^a(x)) ds. \tag{A.5}$$

Then $x \mapsto X_t^a(x)$ is an increasing function, $a \mapsto X_t^a(x)$ decreases as a decreases to zero. For every (t, x, ω) , $\lim_{a \downarrow 0} X_t^a(x)$ exists. For every $x \geq 0$, the following holds for almost surely all ω : $\lim_{a \downarrow 0} X_t^a(x) = X_t(x)$ for all t .

Proof. That $X_t^a(x)$ increases with x follows from the comparison theorem one dimensional SDEs. We also observe that the drift $\partial_x \ln u^a(x)$ in (A.5) increases with a .

$$\partial_a \partial_x \ln u^a = -\frac{1}{2a^{\frac{3}{2}}} (\ln \phi)' \left(\frac{x}{\sqrt{a}} \right) - \frac{x}{2\sqrt{a} a^{\frac{3}{2}}} (\ln \phi)'' \left(\frac{x}{\sqrt{a}} \right) > 0.$$

For $y > 0$, define

$$F(y) = y (\ln \phi)''(y) + (\ln \phi)'(y).$$

It is clear that $F(y)$ is negative for y sufficiently large. By the comparison theorem, $X_t^a(x)$ increases with a and $\bar{X}_t(x) = \lim_{a \downarrow 0} X_t^a(x)$ exists for every t, x, ω . Consequently

$$A_t^x := \lim_{a \downarrow 0} \int_0^t \partial_x \ln u^a(X_s^a(x)) ds$$

exists and

$$\bar{X}_t(x) = x + B_t + A_t^x.$$

Let $f(t, a) = \int_0^t \partial_x \ln u^a(X_s^a(x)) ds$, which is positive and increasing with t . Thus A_t^x is non-negative and nondecreasing in t .

Note that $\lim_{a \rightarrow 0} \partial_x \ln u^a(x) = 0$ for $x > 0$, but the convergence is not uniform in x . For $x \in (0, \sqrt{a}]$,

$$\partial_x \ln u^a \geq \frac{1}{\sqrt{a}} \frac{e^{-\frac{x^2}{2a}}}{\frac{x}{\sqrt{a}}} \geq \frac{1 - \frac{x^2}{2a}}{x} > \frac{1}{2x}.$$

By comparison with the Bessel square process Bes^2 or standard criterion for diffusion process, for almost surely all ω , $X_t^a(x)$ cannot reach 0. Next we observe that, $X_t(y)$ is a

flow, $X_t^a(y) > X_t^a(x)$ whenever $y > x$. Thus $X_t^a(x) > 0$ for all $a > 0$. The limiting process $\bar{X}_t(x)$ has the property:

$$\bar{X}_0(x) = x, \quad \bar{X}_t(x) \geq 0.$$

Let $S < T$ be random times with $\bar{X}_t(x, \omega) > 0$ for $t \in [S, T]$. Let

$$\delta(\omega) = \inf\{\bar{X}_t(x, \omega), t \in [S(\omega), T(\omega)]\} > 0.$$

The function $x \mapsto \partial_x \ln u^a(x)$ decreases,

$$\int_S^T \partial_x \ln u^a(\bar{X}_s(x)) ds \leq \int_S^T \partial_x \ln u^a(\delta(\omega)) ds.$$

Then, since $X_s^a(x) \geq \bar{X}_s(x)$,

$$\begin{aligned} A_T(\omega) - A_S(\omega) &= \lim_{a \downarrow 0} \int_S^T \partial_x \ln u^a(X_s^a(\omega)) ds \\ &\leq \lim_{a \downarrow 0} \int_S^T \partial_x \ln u^a(\bar{X}_s(\omega)) ds \\ &\leq \lim_{a \downarrow 0} \int_S^T \partial_x \ln u^a(\delta(\omega)) ds = 0. \end{aligned}$$

This implies that

$$\int_0^t \chi_{\{\bar{X}_s(x) > 0\}} dA_s^x = 0$$

and $(\bar{X}_t(x), A_t^x)$ solves the Skorohod problem associated to $x + B_t$. □

Lemma A.3. For all $x > 0$ and $a > 0$, $\partial_x^3 \ln u^a > 0$.

Proof. It is clearly sufficient to consider the case $a = 1$.

$$A(x) = \partial_x \ln u^1(x) = \phi'(x).$$

We have from (A.4)

$$A'(x) = -xA(x) - A^2(x)$$

and this implies

$$\phi'''(x) = A''(x) = (x^2 - 1)A(x) + 3xA^2(x) + 2A^3(x). \tag{A.6}$$

It is clearly positive when $x \geq 1$. For $0 \leq x < 1$,

$$A(x) = \frac{e^{-x^2/2}}{\int_0^x e^{-y^2/2} dy} > \frac{1 - x^2/2}{x} > \frac{1 - x^2}{3x}.$$

Hence

$$\phi'''(x) \geq A(x) ((x^2 - 1) + 3xA(x)) > 0.$$

This completes the proof. □

Since $\ln u_t^a(x)$ is smooth, the derivative flow $V_t^a(x) = \partial_x X_t^a(x)$ exists and satisfies the linear equation $\dot{V}_t^a = (\partial_x^2 \ln u^a) V_t^a$. We prove that V_t^a converges to 1 when $t < \tau(x)$ and converges to 0 when $t > \tau(x)$. In the sequel, by $V_{\tau(x+h)}^a(x)$ we mean $\partial_x X_s^a(x)|_{s=\tau(x+h)}$, and $\tau(x+h)$ is not differentiated.

Theorem A.4. Let $X_t^a(x)$ be the solution to (A.5). Let $V_t^a(x) = \partial_x X_t^a(x)$. Then the following holds.

1. For all positive a and t , $x \mapsto V_t^a(x)$ is increasing and $t \mapsto V_t^a(x)$ decreases. For any $y > x$,

$$\lim_{a \rightarrow 0} V_{\tau(y)}^a(x) = 0.$$

2. For almost all ω the following holds for all $x > 0$ and $t \geq 0$ such that $t \neq \tau(x)$:

$$\lim_{a \downarrow 0} V_t^a(x) = \partial_x X_t(x).$$

Furthermore,

$$\lim_{a \rightarrow 0} \mathbf{E} \int_0^T |V_t^a(x) - \partial_x X_t(x)| dt = 0.$$

Proof. We observe that

$$\frac{d}{dt} V_t^a(x) = \partial_x^2 (\ln u_t^a)(X_t^a(x)) V_t^a(x),$$

and $V_0^a(x) = 1$, leading to the formula,

$$V_t^a(x) = e^{\int_0^t \partial_x^2 (\ln u_s^a)(X_s^a(x)) ds}. \tag{A.7}$$

- (1) Since $\partial_x^2 (\ln u^a) < 0$, $V_t^a(x)$ decreases with t . We differentiate (A.7) to see that

$$\partial_x V_t^a(x) = V_t^a(x) \int_0^t (\partial_x^3 \ln u_s^a)(X_s^a(x)) V_s^a(x) ds.$$

Firstly, letting $V_0^a = 1$. By Lemma A.3, $\partial_x^3 \ln u^a > 0$, so $x \mapsto V_t^a(x)$ is increasing.

Let x, ω be fixed. Let $t \neq \tau(x, \omega)$ be a non-negative number and $h > 0$. There is a number $\theta(\omega) \in [0, 1]$ s.t.

$$\frac{X_t^a(x+h) - X_t^a(x)}{h} = \partial_x X_t^a(x + \theta h) \geq \partial_x X_t^a(x).$$

Since $\tau(x+h, \omega) \neq \tau(x, \omega)$ for a.e. ω , for almost all ω we may set $t = \tau(x, \omega)$:

$$0 \leq \partial_x X_t^a(x)|_{t=\tau(x+h)} \leq \frac{X_{\tau(x+h)}^a(x+h) - X_{\tau(x+h)}^a(x)}{h} \leq \frac{X_{\tau(x+h)}^a(x+h)}{h}.$$

Take $h = L_t(x)$. By Proposition A.2

$$\lim_{a \rightarrow 0} X_{\tau(x+L_t(x))}^a(x + L_t(x)) = X_{\tau(x+L_t(x))}(x + L_t(x)) = 0.$$

Thus for any $h > 0$,

$$\lim_{a \downarrow 0} V_{\tau(x+h)}^a(x) = 0.$$

- (2) Let $x > 0$. By Lemma A.1, $X_t(x + L_t(x)) = X_t(x)$ for all $t \geq 0$. So if $t > \tau(x)$, then $t \geq \tau(x + L_t(x))$. Suppose that $t > \tau(x)$. Since $V_t^a(x)$ decreases with t ,

$$0 \leq V_t^a(x) \leq V_{\tau(x+L_t(x))}^a(x).$$

By the conclusion of part (1), the right hand side converges to 0 as $a \rightarrow 0$.

If $t < \tau(x)$, $X_t^a(x) > X_t(x) > 0$ by comparison theorem for SDEs. Also $\partial_x^2 \ln u^a < 0$,

$$1 \geq \lim_{a \downarrow 0} \exp \left(\int_0^t \partial_x^2 \ln u_s^a(X_s^a(x)) ds \right) \geq \exp \left(\lim_{a \downarrow 0} \int_0^t \partial_x^2 \ln u_s^a(X_s(x)) ds \right).$$

On the other hand for every y , $\partial_x^2 \ln u^a(y) \rightarrow 0$ and $\inf_{s \in [0, t]} X_s(x) > 0$ for $t < \tau(x)$. This concludes that $\lim_{a \downarrow 0} \exp \left(\int_0^t \partial_x^2 \ln u_s^a(X_s^a(x)) ds \right) = 1$. Note that $V_t^a(x)$ is uniformly bounded to conclude the convergence in L^1 . \square

B Convergence in semimartingale spaces

Let $a_0 > 0$ and let $\{(Y_t^a, t < \xi^a), a \in [0, a_0]\}$ be a family of continuous semi-martingales with values in a manifold M . If U is an open domain in M , set:

$$\tau^{U,a} = \inf\{t > 0 : Y_t^a \notin U\}.$$

Definition B.1. (1) We say that Y^a converges to Y^0 in the topology of uniform convergence in probability on compact time sets (UCP) if

(1a) for all relatively compact open domain $U \subset M$,

$$\liminf_{a \rightarrow 0} \tau^{U,a} \geq \tau^{U,0},$$

(1b) for all $t > 0$, the following convergence holds in probability:

$$\lim_{a \rightarrow 0} \sup_{s \leq t \wedge \tau^{U,a} \wedge \tau^{U,0}} \rho(Y_s^a, Y_s^0) \stackrel{(P)}{=} 0$$

(2) Let $p \in [1, \infty)$. We say that Y^a converges to Y^0 locally in \mathcal{S}_p if there exists an increasing sequence of stopping times $(T_n)_{n \geq 1}$ with $\lim_{n \rightarrow \infty} T_n = \xi^0$ such that for some $a_1 > 0$ and for all $a < a_1$ and all $n \in \mathbf{N}$, $T_n < \xi^a$ a.s. and

$$\lim_{a \rightarrow 0} \mathbf{E} \left[\sup_{t \leq T_n} \rho^p(Y_t^a, Y_t^0) \right] = 0 \tag{B.1}$$

Given (1b), condition (1a) is equivalent to $\liminf_{a \rightarrow 0} \xi^a \geq \xi^0$.

Let \mathbb{D} denote the space of real-valued C\`adl\`ag semimartingales defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.

For two real valued semi-martingales $X, Y \in \mathbb{D}$ we define the distance functions:

$$r(X, Y) = \sum_{n > 0} 2^{-n} \mathbf{E} \left(1 \wedge \sup_{0 \leq t \leq n} |X_t - Y_t| \right),$$

$$\hat{r}(X, Y) = \sup_{|H| \leq 1} r \left(\int_0^t H_s d(X_s - Y_s) \right),$$

where the supremum is taken over all predictable processes H bounded by 1. The distance r is compatible with UCP:

$$\sup_{0 \leq s \leq t} |X_s^{(n)} - X_s| \rightarrow 0 \quad (\text{in probability})$$

for each $t > 0$ if and only if $r(X^n - X)$ converges to 0. The distance \hat{r} induces the semi-martingale topology on the vector space of semi-martingales.

Define

$$\mathcal{S}^p = \{X \in \mathbb{D} : \|X\|_{\mathcal{S}^p} = \|\sup_t |X_t|\|_{L^p} < \infty\},$$

$$\mathcal{H}^p = \{X \in \mathbb{D} : \|X\|_{\mathcal{H}^p} = \inf \left\{ |X_0| + [M, M]_{\infty}^{\frac{1}{2}} + \int_0^{\infty} |dA_s| \Big|_{L^p} < \infty \right\},$$

where the infimum is taken over all semi-martingale decompositions $X = X_0 + M + A$. When the time interval is restricted to a finite time interval $[0, T]$ the notations will be $\mathcal{S}^p([0, T])$ and $\mathcal{H}^p([0, T])$.

A semi-martingale is locally in \mathcal{S}^p and \mathcal{H}^p if there exists a sequence of stopping times T_n increasing to infinity such that $X^{T_n} \chi_{\{T_n > 0\}}$ are in these spaces. It is prelocally

in these spaces if all $X^{T_n-} \chi_{\{T_n > 0\}}$ are, where $X^{T-}(t) = X_t \chi_{[0, T)}(t) + X_{T-} \chi_{[T, \infty)}(t)$. Let $(X^{(n)})$ and X be semi-martingales. Let $1 \leq p < \infty$. If $X^{(n)}$ converges to X in the semi-martingale topology, then there exists a subsequence that converges prelocally in \mathcal{H}^p . If $X^{(n)}$ converges to X prelocally in \mathcal{H}^p then it converges in the semi-martingale topology. Extension to \mathbf{R}^k -valued processes is done by considering the components.

The following estimate of M. Emery is useful: If Y is a semi-martingale and H a left continuous process with right limit, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ where $p, q \in [1, \infty]$, then

$$\left\| \int_0^\infty H_s dZ_s \right\|_{\mathcal{H}^r} \leq \|H\|_{\mathcal{H}^p} \|Z\|_{\mathcal{H}^q}. \tag{B.2}$$

We review these convergence in the settings that the semi-martingales may have finite life times or take values in a manifold. See [7, M. Arnaudon and A. Thalmaier] for details.

Let $Z_t = Z_0 + M_t + A_t$ be a semi-martingale in \mathbf{R}^k with lifetime ξ and the canonical decomposition of Z_t into the sum of its starting point, a local martingale M_t starting at 0, and a finite variation process A_t starting at 0. Define

$$v(Z)_t = \sum_{i=1}^k \left(|Z_0^i| + \langle M^i, M^i \rangle_t^{1/2} + \int_0^t |dA^i|_s \right), \quad t < \xi. \tag{B.3}$$

Let $T > 0$. We say that a family of semi-martingales $Z^{(n)}$ converges to 0 in $\mathcal{S}_p([0, T])$ if $\mathbf{E} \left[\sup_{s \leq T} |Z_s^{(n)}|^p \right] \rightarrow 0$. It converges to 0 in $\mathcal{H}_p([0, T])$ if $v(Z^{(n)}) \rightarrow 0$ in $\mathcal{S}_p([0, T])$.

To define this for a manifold valued stochastic process, we will use an embedding $\Phi : M \rightarrow \mathbf{R}^k$. The definition will in fact be independent of this embedding.

Definition B.2. Let (Y_t^a) be a family of semi-martingales indexed by a .

1. We say that Y^a converges to Y^0 in semi-martingale topology or in SM topology if the semi-martingale norm of $\Phi(Y^a) - \Phi(Y^0)$, $v(\Phi(Y^a) - \Phi(Y^0))$, converges to 0 in UCP topology.
2. Let $p \in [1, \infty)$. We say that Y^a converges to Y^0 locally in \mathcal{H}_p if the processes $v(\Phi(Y^a) - \Phi(Y^0))$ converge to 0 locally in \mathcal{S}_p .

The convergence in the semi-martingale topology is stronger than convergence in the UCP topology. However it is a remarkable fact that they coincide on the subset of martingales in the manifold. The following characterisations of convergence will be very useful (see [7, M. Arnaudon and A. Thalmaier]).

- Proposition B.3.**
- If $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in UCP topology then for all $p \in [0, \infty)$ there exists a sequence $a_k \rightarrow 0$ such that $Y^{a_k} \rightarrow Y^0$ as $k \rightarrow \infty$ locally in \mathcal{S}_p .
 - If $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in SM topology then for all $p \in [0, \infty)$ there exists a sequence $a_k \rightarrow 0$ such that $Y^{a_k} \rightarrow Y^0$ as $k \rightarrow \infty$ locally in \mathcal{H}_p .
 - If for some $p \in [1, \infty)$ $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ locally in \mathcal{S}_p then $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in UCP topology.
 - If for some $p \in [1, \infty)$ $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ locally in \mathcal{H}_p then $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in SM topology.

As a consequence, a standard way to establish UCP or SM convergence given by the following:

- Corollary B.4.**
- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in UCP topology if and only if there exists $p \in [1, \infty)$ such that for any $a_k \rightarrow 0$ there exists a subsequence a_{k_ℓ} such that $Y^{a_{k_\ell}} \rightarrow Y^0$ locally in \mathcal{S}_p .

- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in SM topology if and only if there exists $p \in [1, \infty)$ such that for any $a_k \rightarrow 0$ there exists a subsequence a_{k_ℓ} such that $Y^{a_{k_\ell}} \rightarrow Y^0$ locally in \mathcal{H}_p .

For processes which take their values in a compact manifold and which are defined in bounded times, we have the following easy relations.

Corollary B.5. *Assume that M is compact and that all processes are defined on some deterministic time interval $[0, T]$. The following equivalences hold:*

- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in UCP topology;
- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in \mathcal{S}_p for some $p \in [1, \infty)$;
- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in \mathcal{S}_p for all $p \in [1, \infty)$.

Similarly, we have the equivalences

- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in SM topology;
- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in \mathcal{H}_p for some $p \in [1, \infty)$;
- $Y^a \rightarrow Y^0$ as $a \rightarrow 0$ in \mathcal{H}_p for all $p \in [1, \infty)$.

C Ikeda and Watanabe’s damped parallel translation

The parallel transport P_t along a semi-martingale (Z_t) is the semi-martingale with values in $L(T_{Z_0}M, T_{Z_t}M)$ solving the Stratonovich SDE

$$\circ dP_t = \mathfrak{h}_{P_t}(\circ dZ_t), \quad P_0 = \text{Id}_{T_{Z_0}M} \tag{C.1}$$

where \mathfrak{h}_{P_t} denotes horizontal lift to the orthonormal frame bundle. We have identified \mathbf{R}^d with $T_{Z_0}M$. Parallel transport is an isometry, a proof for its existence on manifolds with boundary can be found in [29, N. Ikeda and S. Watanabe]. For simplicity we also use the notation $\parallel_t(Z)$.

If Z_t is a diffusion process with generator $\mathcal{L} = \frac{1}{2}\Delta + U$, where U is a time dependent vector field, remaining in the interior of M for all time (which happens if M has no boundary or if U is sufficiently strong in a neighbourhood of the boundary), then the parallel transport P_t along Z_t is the diffusion process whose generator on differential 1-forms is $\frac{1}{2}$ trace $\nabla^2 + \nabla_U$. If $\Delta^1 = -(d^*d + dd^*)$ is the Hodge Laplacian, $\text{trace}\nabla^2 = \Delta^1 + \text{Ric}$. The damped parallel translation W_t along Z_t is the solution to the equation

$$DW_t = \left(\nabla_{W_t}U - \frac{1}{2}\text{Ric}^\sharp(W_t) \right) dt, \quad W_0 = \text{Id}_{T_{Z_0}M}, \tag{C.2}$$

where the covariant derivative DW_t is defined to be $\parallel_t d(\parallel_t^{-1}W_t)$. The process (W_t) is a diffusion process with generator on 1-forms \mathcal{L}^W :

$$\mathcal{L}^W \alpha = \frac{1}{2}\Delta^1 \alpha + \nabla_U \alpha + \alpha(\nabla.U). \tag{C.3}$$

The fundamental property of \mathcal{L}^W is its commutation with differentiation:

$$d(\mathcal{L}f) = \mathcal{L}^W(df), \quad f \in C^\infty(M). \tag{C.4}$$

As a consequence, if $F \in C^{1,2}([0, T] \times M, \mathbf{R})$ is such that $F(t, Z_t)$ is a local martingale, then $dF(t, W_t)$ is also a local martingale, where dF is the differential of F in the second variable. On the other hand, (C.2) together with the fact that P_t is an isometry yield estimations on the norm of W_t . This allows to estimate the norm of dF . Another fundamental property is that W_t is the derivative of the flow corresponding to parallel couplings of L -diffusions.

We construct a damped parallel transport along Brownian motion in a manifold with boundary. The covariant derivative DW_t has three components: one coming from the behaviour in M^0 (the usual one), one tangential to ∂M absolutely continuous with respect to dL_t and involving the shape operator of ∂M , and the third is normal to ∂M and has jumps. This is similar to what happens for the half line. Concerning the half line case the flow corresponding to parallel coupling of reflected Brownian motion is $\partial_x X_t$, so in this case $W_t = \partial_x X_t$ and the study is complete.

We also define the second fundamental form and shape operator for level sets of the distance function to the boundary. Let $S(r) = \{y \in F_0, R(y) = r\}$. Within a tubular neighbourhood F_0 of the boundary, R is smooth around x . Let $r = R(x)$ and $\nu_x = \sigma_1(x) = \nabla R(x)$. For $w \in T_x M, w' \in T_x S(r)$ we define $\Pi_x : T_x M \times T_x S(r) \rightarrow \mathbf{R}$ and $\mathcal{S}_x : T_x M \rightarrow T_x S(r)$ by

$$\Pi(w, w') = \langle \mathcal{S}(w), w' \rangle = -\langle \nabla_w \nu, w' \rangle = -\nabla dR(w, w') \tag{C.5}$$

The bilinear map Π is said to be the second fundamental form of $S(r)$ and \mathcal{S}_x its shape operator or the Weingarten map.

Assume that there exists $\delta_0 > 0$ and a tubular neighbourhood of ∂M with radius $3\delta_0$. If D is a set denote τ_D the exit time of Y from D .

Lemma C.1. *Let U be a relatively compact set of M . Let $R_0 < \delta_0, Y_0 \in U, Y_t$ the reflected Brownian motion constructed in Theorem 3.3. Set*

$$\tau_{2\delta_0} = \inf\{t : R_t = 2\delta_0\}.$$

Then under a probability measure equivalent to $P, \{R_t, t < \tau_{\delta_0} \wedge \tau_U \wedge T\}$ is the solution to a Skorohod problem for a one dimensional Brownian motion on \mathbf{R}_+ .

Proof. Within E_0 , because of Proposition 3.2 (2),

$$R_t = R_0 + B_t^1 + \int_0^t \Delta R(Y_s) ds + L_t.$$

Let Q be the probability measure whose density with respect to P is the exponential martingale of $-\int_0^t \Delta R(Y_s) dB_s^1$. Then under $Q, \tilde{B}_t^1 := B_t^1 + \int_0^t \Delta R(Y_s) ds$ is a Brownian motion. On the other hand, L_t is nondecreasing and dL_t vanishes when $R_t \neq 0$. Since $R_t \geq 0$ we have $(R, L) = \Phi(0, \tilde{B}^1)$, the solution to Skorohod problem. See (A.1). By the uniqueness of the Skorohod problem, under $Q, \{R_t, t < \tau_{2\delta_0} \wedge \tau_U \wedge T\}$ has the law of a one dimensional reflected Brownian motion. \square

Let $\mathcal{L}(\omega) = \{t \geq 0 : Y_t(\omega) \in \partial M\}$ be the set of times that Y_t spends on the boundary. It has Lebesgue measure zero for a.s. all ω and its complement

$$(0, \infty) \setminus \mathcal{L}(\omega) = \cup_{\alpha} (l_{\alpha}(\omega), r_{\alpha}(\omega))$$

is the union of countably many disjoint open intervals, the excursion intervals. Denote the set of right end times of excursions by $\mathcal{R}(\omega)$:

$$\mathcal{R}(\omega) = \cup_{\alpha} \{r_{\alpha}(\omega)\}.$$

We are interested in defining a damped parallel translation W_t along Y_t , which agrees with the usual one during an excursion, and pick up a change of direction when exiting the boundary. The normal direction on the boundary is zero: we remove $-\langle W_t, \nu(Y_t) \rangle \nu(Y_t)$

upon the process entering the boundary. We would have liked to define a stochastic processes (W_t) , if it were possible, with values in $n \times n$ matrices, satisfying

$$\begin{aligned} (\//t)^{-1}W_t &= (\//r_\alpha)^{-1}W_{r_\alpha} - \frac{1}{2} \int_{r_\alpha}^t (\//s)^{-1} \text{Ric}^\#(\//s-) ((\//s)^{-1}W_s) ds \\ &\quad - \int_{r_\alpha}^t (\//s)^{-1} \mathcal{S}(\//s-) ((\//s)^{-1}W_s^\varepsilon) dL_s, \quad t \in (r_\alpha, r_{\alpha+1}) \end{aligned}$$

$$\begin{aligned} (\//r_{\alpha+1})^{-1}W_{r_{\alpha+1}} \\ = (\//r_{\alpha+1})^{-1}W_{(r_{\alpha+1})-} - \langle W_{(r_{\alpha+1})-}, \nu_{Y_{(r_{\alpha+1})-}} \rangle (\//r_{\alpha+1})^{-1} \nu_{Y_{(r_{\alpha+1})-}}. \end{aligned}$$

Given (l_α, r_α) , for any $\epsilon > 0$ there is an excursion $(l_{\alpha'}, r_{\alpha'})$ such that $0 < l_{\alpha'} - r_\alpha < \epsilon$, and so the heuristic definition given above does not make sense.

We remedy this problem with an approximation adding jumps only on excursions of size greater or equal to ϵ . We consider the set of excursions of lengths greater or equal to a given size $\varepsilon > 0$ and define

$$\mathcal{R}_\varepsilon(\omega) = \{s = r_\alpha(\omega) \in \mathcal{R}(\omega) : r_\alpha(\omega) - l_\alpha(\omega) \geq \varepsilon\},$$

where the excursions of size greater than or equal to ε are ordered with l_1 the first time Y_t hits the boundary and we consider only $\alpha \in \mathbf{N}$. If $r_\alpha - l_\alpha \geq \varepsilon$,

$$\begin{aligned} (\//t)^{-1}W_t^\varepsilon &= (\//r_\alpha)^{-1}W_{r_\alpha}^\varepsilon - \frac{1}{2} \int_{r_\alpha}^t (\//s)^{-1} \text{Ric}^\#(\//s-) ((\//s)^{-1}W_s^\varepsilon) ds \\ &\quad - \int_{r_\alpha}^t (\//s)^{-1} \mathcal{S}(\//s-) ((\//s)^{-1}W_s^\varepsilon) dL_s, \quad t \in (r_\alpha, r_{\alpha+1}) \end{aligned}$$

$$\begin{aligned} (\//r_{\alpha+1})^{-1}W_{r_{\alpha+1}}^\varepsilon \\ = (\//r_{\alpha+1})^{-1}W_{(r_{\alpha+1})-}^\varepsilon - \langle W_{(r_{\alpha+1})-}, \nu_{Y_{(r_{\alpha+1})-}} \rangle (\//r_{\alpha+1})^{-1} \nu_{Y_{(r_{\alpha+1})-}}. \end{aligned}$$

where $(W^{\varepsilon, T})$ denotes the tangential part of W_t . This takes into consideration those times slightly before $(l_{\alpha+1}, r_{\alpha+1})$ and is relevant to the integration with respect to L_t .

Since Y_t spends Lebesgue time 0 on the boundary, for integration with respect to a continuous process we could ignore the boundary process. We would like to simply remove the normal part of W_t upon it touches down to the boundary. We are lead to the following alternative description. Let $v \in T_{Y_0}M$ and $t > 0$, for almost surely all ω , the following holds,

$$\begin{aligned} (\//t)^{-1}W_t^\varepsilon &= Id - \frac{1}{2} \int_0^t (\//s)^{-1} \text{Ric}^\#(\//s-) ((\//s)^{-1}W_s^\varepsilon) ds \\ &\quad - \int_0^t (\//s)^{-1} \mathcal{S}(\//s-) ((\//s)^{-1}W_s^\varepsilon) dL_s, \quad t \notin \mathcal{R}_\varepsilon(\omega) \\ (\//t)^{-1}W_t^\varepsilon &= (\//t-)^{-1}W_{t-}^\varepsilon - \sum_{s \leq t, s \in \mathcal{R}_\varepsilon(\omega)} \langle W_{t-}^\varepsilon, \nu_{Y_{t-}} \rangle (\//t-)^{-1} \nu_{Y_{t-}}. \end{aligned}$$

In other words, W_t^ε is continuous at any time t that is not an element of $\mathcal{R}_\varepsilon(\omega)$, and satisfies the following covariant equation

$$DW_t = -\frac{1}{2} \text{Ric}^\#(W_t) dt - \mathcal{S}(W_t) dL_t.$$

If t is the right hand side of an excursion, we remove the normal part of its component. This description will be used in Theorem C.3.

Define

$$\mathcal{T} = \{\alpha_t \in \mathcal{H}_\infty([0, T]) : \alpha_t \in T_{Y_t}^* M, \alpha_t = 0 \text{ on } \{Y_t \notin D\} \text{ for some } D\}. \quad (\text{C.6})$$

where D is relatively compact subset of M . This is the set of bounded semi-martingale with values in the pull back cotangent bundle by Y_t , with the property that there is relatively compact subset D of M such that $\alpha_t = 0$ whenever $Y_t \notin D$. Denote by τ^D the first exit time from D by Y_t .

Definition C.2. The limit process $W_t \in L(T_{Y_0} M, T_{Y_t} M)$, below in Theorem C.3, is said to be a solution to the following equation

$$DW_t = -\frac{1}{2} \text{Ric}^\sharp(W_t) dt - \mathcal{S}(W_t) dL_t - \chi_{\{t \in \mathcal{R}(\omega)\}} \langle W_{t-}, \nu_{Y_t} \rangle \nu_{Y_t}, \quad W_0 = \text{Id}_{T_{Y_0} M}. \quad (\text{C.7})$$

Theorem C.3. Let W_t^ε be the solution to

$$DW_t^\varepsilon = -\frac{1}{2} \text{Ric}^\sharp(W_t^\varepsilon) dt - \mathcal{S}(W_t^\varepsilon) dL_t - \chi_{\{t \in \mathcal{R}_\varepsilon(\omega)\}} \langle W_{t-}^\varepsilon, \nu_{Y_t} \rangle \nu_{Y_t}, \quad W_0^\varepsilon = \text{Id}_{T_{Y_0} M}. \quad (\text{C.8})$$

There exists an adapted right continuous stochastic process W_t such that $\lim_{\varepsilon \rightarrow 0} W_t^\varepsilon = W_t$ in UCP and in \mathcal{S}_p if M is compact, and for any $p \geq 1$. Furthermore for any $\alpha \in \mathcal{T}$,

$$\lim_{\varepsilon \rightarrow 0} \left(\int_0^{\cdot \wedge \tau_D} \alpha_s(DW_s^\varepsilon) \right) \stackrel{\mathcal{S}_2}{\rightrightarrows} \left(\int_0^{\cdot \wedge \tau_D} \alpha_s(DW_s) \right).$$

The same result but with different formulation can be found in [29, N. Ikeda and S. Watanabe]. We give a proof close to [29, N. Ikeda and S. Watanabe], which will be used for our approximation result (Theorem 5.6 and Corollary 5.7).

Proof. Since the definition and convergence are local in Y , we can assume that M is compact. Since $Y_t(\omega)$ has a finite number of excursions larger than ε , the process W_t^ε is a well defined right continuous process. We first prove that as $\varepsilon \rightarrow 0$, $(W_t^\varepsilon)_{t \in [0, T]}$ converges in \mathcal{S}_p to a process which we will call $(W_t)_{t \in [0, T]}$.

Using the parallel translation process \parallel_t along Y_t , we reformulate the equation as an equation in the linear space $L(T_{Y_0} M, T_{Y_0} M)$. Set

$$\text{ric}_t = \parallel_t^{-1} \circ \text{Ric}_{Y_t}^\sharp \circ \parallel_t, \quad \mathfrak{s}_t = \parallel_t^{-1} \circ \mathcal{S}_{Y_t} \circ \parallel_t, \quad \vec{n}_t = \parallel_t^{-1}(\nu_{Y_t}). \quad (\text{C.9})$$

Then W_t^ε is a solution to (C.8) if and only if $w_t^\varepsilon = (\parallel_t)^{-1} W_t^\varepsilon$ satisfies the following equations. For any $r_\alpha \in \mathcal{R}_\varepsilon(\omega)$,

$$\begin{aligned} w_t^\varepsilon &= -w_{r_\alpha}^\varepsilon - \frac{1}{2} \int_{r_\alpha}^t (\parallel_s)^{-1} \text{ric}_s(w_s^\varepsilon) ds - \int_{r_\alpha}^t \mathfrak{s}_s(w_s^\varepsilon) dL_s, \quad t \in (r_\alpha, r_{\alpha+1}) \\ w_{r_{\alpha+1}}^\varepsilon &= w_{(r_{\alpha+1})^-}^\varepsilon - \langle w_{(r_{\alpha+1})^-}^\varepsilon, \vec{n}_{(r_{\alpha+1})^-} \rangle \vec{n}_{(r_{\alpha+1})^-}, \end{aligned}$$

This means (w_t^ε) satisfies the following equation:

$$dw_t^\varepsilon = -\frac{1}{2} \text{ric}_t(w_t^\varepsilon) dt - \mathfrak{s}_t(w_t^\varepsilon) dL_t - \chi_{\{t \in \mathcal{R}_\varepsilon(\omega)\}} \langle w_{t-}^\varepsilon, n_t \rangle n_t, \quad w_0 = \text{Id}. \quad (\text{C.10})$$

Let $0 < \varepsilon' < \varepsilon$, the difference between w_t^ε and $w_t^{\varepsilon'}$ is given by

$$\begin{aligned} w_t^\varepsilon - w_t^{\varepsilon'} &= \int_0^t \left(-\frac{1}{2} \text{ric}_s(w_s^\varepsilon) + \frac{1}{2} \text{ric}_s(w_s^{\varepsilon'}) \right) ds + \int_0^t \left(-s_s(w_s^\varepsilon) + s_s(w_s^{\varepsilon'}) \right) dL_s \\ &\quad - \sum_{\{s \in \mathcal{R}_\varepsilon(\omega) \cap [0, t]\}} \langle w_{s-}^\varepsilon, n_s \rangle n_s + \sum_{\{s \in \mathcal{R}_{\varepsilon'}(\omega) \cap [0, t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle n_s \\ &= -\frac{1}{2} \int_0^t \text{ric}_s(w_s^\varepsilon - w_s^{\varepsilon'}) ds - \int_0^t s_s(w_s^\varepsilon - w_s^{\varepsilon'}) dL_s \\ &\quad - \sum_{\{s \in \mathcal{R}_\varepsilon(\omega) \cap [0, t]\}} \langle w_{s-}^\varepsilon - w_{s-}^{\varepsilon'}, n_s \rangle n_s + \sum_{\{s \in (\mathcal{R}_{\varepsilon'}(\omega) \setminus \mathcal{R}_\varepsilon) \cap [0, t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle n_s. \end{aligned}$$

Consequently,

$$\begin{aligned} \|w_t^\varepsilon - w_t^{\varepsilon'}\|^2 &= -\int_0^t \langle \text{ric}_s(w_s^\varepsilon - w_s^{\varepsilon'}), w_s^\varepsilon - w_s^{\varepsilon'} \rangle ds - 2 \int_0^t \langle s_s(w_s^\varepsilon - w_s^{\varepsilon'}), w_s^\varepsilon - w_s^{\varepsilon'} \rangle dL_s \\ &\quad - 2 \sum_{\{s \in \mathcal{R}_\varepsilon(\omega) \cap [0, t]\}} \langle w_{s-}^\varepsilon - w_{s-}^{\varepsilon'}, n_s \rangle^2 \\ &\quad + 2 \sum_{\{s \in (\mathcal{R}_{\varepsilon'}(\omega) \setminus \mathcal{R}_\varepsilon) \cap [0, t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle \langle n_s, w_{s-}^\varepsilon - w_{s-}^{\varepsilon'} \rangle \end{aligned}$$

which yields

$$\begin{aligned} \|w_t^\varepsilon - w_t^{\varepsilon'}\|^2 &\leq 2 \sum_{\{s \in (\mathcal{R}_{\varepsilon'}(\omega) \setminus \mathcal{R}_\varepsilon) \cap [0, t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle \langle n_s, w_{s-}^\varepsilon - w_{s-}^{\varepsilon'} \rangle \\ &\quad + \int_0^t \|w_s^\varepsilon - w_s^{\varepsilon'}\|^2 (-\underline{\text{Ric}}(Y_s) ds - 2\underline{\mathcal{S}}(Y_s) dL_s) \end{aligned}$$

and

$$\|w_t^\varepsilon - w_t^{\varepsilon'}\|^2 \leq K_t + \int_0^t \|w_s^\varepsilon - w_s^{\varepsilon'}\|^2 (\rho ds + 2C dL_s) \tag{C.11}$$

where $\rho, C \geq 0$, $-\rho$ is a lower bound for the Ricci curvature, C is an upper bound for the norm of the shape operator, and

$$K_t = \sup_{s \leq t} \left| 2 \sum_{\{r \in (\mathcal{R}_{\varepsilon'}(\omega) \setminus \mathcal{R}_\varepsilon) \cap [0, s]\}} \langle w_{r-}^{\varepsilon'}, n_r \rangle \langle n_r, w_{r-}^\varepsilon - w_{r-}^{\varepsilon'} \rangle \right|. \tag{C.12}$$

So using Gronwall lemma we get

$$\sup_{s \leq t} \|w_s^\varepsilon - w_s^{\varepsilon'}\|^2 \leq K_t e^{\rho t + 2CL_t}. \tag{C.13}$$

On the other hand it is a remarkable but not surprising fact that each term

$$\langle w_{s-}^{\varepsilon'}, n_s \rangle \langle n_s, w_{s-}^\varepsilon - w_{s-}^{\varepsilon'} \rangle$$

can be written as a stochastic integral over an interval not containing any excursion of size larger than ε . This comes from the fact that the normal part of $w_t^{\varepsilon'}$ is set to zero at the end of each excursion of size at least ε' . More precisely,

$$\langle w_{s-}^{\varepsilon'}, n_s \rangle \langle n_s, w_{s-}^\varepsilon - w_{s-}^{\varepsilon'} \rangle = \int_u^s d \langle w_{r-}^{\varepsilon'}, n_r \rangle \langle n_r, w_{r-}^\varepsilon - w_{r-}^{\varepsilon'} \rangle$$

where u is the last vanishing time of $\langle w_{r-}^{\varepsilon'}, n_r \rangle$ before s . Now since we are outside excursions of size larger than ε' the process inside the integral has no jumps, and

since the range of s_r is orthogonal to n_r , the process inside the integral is a continuous semi-martingale whose drift is absolutely continuous with respect to ds with bounded derivative, see (C.10) and (7.11). Consequently letting

$$C_t := 2 \sum_{\{s \in (\mathcal{R}_{\varepsilon'}(\omega) \setminus \mathcal{R}_\varepsilon) \cap [0, t]\}} \langle w_{s-}^{\varepsilon'}, n_s \rangle \langle n_s, w_{s-}^\varepsilon - w_{s-}^{\varepsilon'} \rangle$$

we can write

$$C_t = \int_0^t a_s ds + b_s dB_s \tag{C.14}$$

where a_s, b_s are uniformly bounded, B_s is a real-valued Brownian motion and a_s, b_s vanish outside $U(\varepsilon) \cap [0, t]$ where $U(\varepsilon)$ is the set of times not contained in excursions larger than ε .

From this we get for $q > 1$

$$\mathbf{E}[K_T^q]^{1/q} \leq C_q \mathbf{E}[\lambda(U(\varepsilon) \cap [0, T])] \tag{C.15}$$

for some $C_q > 0$, where λ is the Lebesgue measure in \mathbf{R} . On the other hand by (3.23) in Corollary 3.5 the random variable $e^{\rho T + 2CL_T}$ has finite moments of any order. As a consequence, using (C.13) and Hölder inequality, for any $p \geq 1$

$$\mathbf{E}[\sup_{t \leq T} |w_t^\varepsilon - w_t^{\varepsilon'}|^p]^{1/p} \leq C'_p \mathbf{E}[\lambda(U(\varepsilon) \cap [0, T])].$$

for some $C'_p > 0$. The left hand side goes to 0 as $\varepsilon \rightarrow 0$, so for $\varepsilon_n \rightarrow 0$, w^{ε_n} is a Cauchy sequence in \mathcal{S}_p , it converges to some process w . Clearly w does not depend on the sequence. Letting $W_t = \int_0^t w_s ds$ then W_t^ε converges to W_t in \mathcal{S}_p .

Let us now prove that DW^ε converges to DW in the sense given by theorem C.3. Let $(\alpha_t)_{t \in [0, T]}$ be a \mathcal{H}_∞ semimartingale taking its values in $T_{Y_t}^* M$, bounded by 1. We have for $0 < \varepsilon' < \varepsilon$

$$\begin{aligned} & \int_0^t \alpha_s (DW_s^\varepsilon) - \int_0^t \alpha_s (DW_s^{\varepsilon'}) \\ &= \alpha_t (W_t^\varepsilon - W_t^{\varepsilon'}) - \int_0^t D\alpha_s (W_s^\varepsilon - W_s^{\varepsilon'}). \end{aligned}$$

By (B.2), a result of M. Emery, we see that

$$\begin{aligned} & \mathbf{E} \left[\sup_{t \leq T} \left\| \int_0^t \alpha_s (DW_s^\varepsilon) - \int_0^t \alpha_s (DW_s^{\varepsilon'}) \right\|^2 \right] \\ & \leq \|\alpha\|_{\mathcal{S}_\infty([0, T])} \|W^\varepsilon - W^{\varepsilon'}\|_{\mathcal{S}_2([0, T])} + \|W^\varepsilon - W^{\varepsilon'}\|_{\mathcal{S}_2([0, T])} \|\alpha\|_{\mathcal{H}_\infty([0, T])}. \end{aligned}$$

From the first part of the proof and the assumption on α , we get that for $\varepsilon_n \rightarrow 0$, $(\int_0^\cdot \alpha_s (DW_s^{\varepsilon_n}))_{t \in [0, T]}$ is a Cauchy sequence in $\mathcal{S}_2([0, T])$, so it converges to some process which is linear in α_s . Consequently we denote it by $(\int_0^\cdot \alpha_s (DW_s))_{t \in [0, T]}$. \square

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