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An improved upper bound for the critical value of the contact process on $\mathbb{Z}^d$ with $d \geq 3$

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Abstract

By coupling the basic contact process with a linear system, we give an improved upper bound for the critical value λ_c of the basic contact process on the lattice $\mathbb{Z}^d$ with $d \geq 3$. As a direct corollary of our result, the critical value of the three-dimensional contact process is shown to be at most 0.34.

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1 Introduction

In this paper we are concerned with the basic contact process on $\mathbb{Z}^d$ with $d \geq 3$. First we introduce some notations. For each $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, we use $\|x\|$ to denote the l_1-norm of x, i.e.,

$$\|x\| := \sum_{i=1}^{d} |x_i|.$$

Note that in this paper we use ':=' to mark definitions. For any $x, y \in \mathbb{Z}^d$, we write $x \sim y$ when and only when $\|x - y\| = 1$, i.e., $x \sim y$ means that x and y are neighbors on $\mathbb{Z}^d$. For $1 \leq i \leq d$, we use e_i to denote the ith elementary unit vector of $\mathbb{Z}^d$, i.e.,

$$e_i := (0, \ldots, 0, \underset{i\text{th}}{1}, 0, \ldots, 0). \tag{1.1}$$

We use O to denote the origin of $\mathbb{Z}^d$.

Let $\{0,1\}^{\mathbb{Z}^d}$ be the set of configurations where each vertex on $\mathbb{Z}^d$ is in one of the two states 0 and 1, then for any $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$, we use $\eta(x)$ to denote the state of x and use η^x to denote the configuration where

$$\eta^x(y) := \begin{cases} \eta(y) & \text{if } y \neq x, \\ 1 - \eta(x) & \text{if } y = x. \end{cases}$$

The contact process $\{\eta_t\}_{t \geq 0}$ on $\mathbb{Z}^d$ is a continuous-time Markov process with state space $\{0,1\}^{Z^d}$ evolving as follows. For any $t \geq 0$ and $x \in \mathbb{Z}^d$,

$$\eta_t \text{ flips to } \eta_t^x \text{ at rate } \begin{cases} 1 & \text{if } \eta_t(x) = 1, \\ \lambda \sum_{y \sim x} \eta_t(y) & \text{if } \eta_t(x) = 0, \end{cases}$$

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where $\lambda > 0$ is a constant called the infection rate.

The contact process can be equivalently defined through its generator. According to the evolution of the contact process defined above, the generator $\mathcal{A}$ of the contact process is given by

$$\mathcal{A}f(\eta) = \sum_{x\in\mathbb{Z}^d} c(x,\eta)\big[f(\eta^x) - f(\eta)\big]$$

for any $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $f \in C\big(\{0,1\}^{\mathbb{Z}^d}\big)$, where

$$c(x,\eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum_{y\sim x} \eta(y) & \text{if } \eta(x) = 0 \end{cases} \tag{1.2}$$

while $C\big(\{0,1\}^{\mathbb{Z}^d}\big)$ is the set of continuous functions on $\{0,1\}^{\mathbb{Z}^d}$ with respect to the metric $m(\cdot,\cdot)$ that

$$m(\eta,\xi) = \sum_{x\in\mathbb{Z}^d} J(x)|\eta(x) - \xi(x)|$$

for any $\eta, \xi \in \{0,1\}^{\mathbb{Z}^d}$, where $J : \mathbb{Z}^d \to (0,+\infty)$ is a strictly positive given function on $\mathbb{Z}^d$ such that $\sum_{x\in\mathbb{Z}^d} J(x) < +\infty$.

The contact process belongs to a large class of Markov processes called the spin systems (see Chapter 3 of [9]). $\{c(x,\eta)\}_{x\in\mathbb{Z}^d,\eta\in\{0,1\}^{\mathbb{Z}^d}}$ is called the flip rates function of the spin system, since $c(x,\eta)$ is the rate at which the spin system flips from η to η^x.

Intuitively, the contact process describes the spread of an epidemic on the graph. Vertices in state 1 are infected while those in state 0 are healthy. An infected vertex waits for an exponential time with rate 1 to become healthy while a healthy one is infected at rate proportional to the number of infected neighbors.

The contact process is introduced by Harris in [6]. For a detailed survey of the study of the contact process, see Chapter 6 of [9] and Part one of [11].

In this paper we are mainly concerned with the critical value of the contact process. To give the definition of the critical value, we introduce some notations. For any $\lambda > 0$ and $\eta \in \{0,1\}^{\mathbb{Z}^d}$, we use P_λ^η to denote the probability measure of the contact process with infection rate λ and initial condition $\eta_0 = \eta$. We use δ_1 to denote the configuration where all the vertices are in state 1. Then the contact process has the following property. For $\lambda_1 \geq \lambda_2$ and $t > s$,

$$P_{\lambda_1}^{\delta_1}(\eta_s(O) = 1) \geq P_{\lambda_2}^{\delta_1}(\eta_t(O) = 1). \tag{1.3}$$

A rigorous proof of Equation (1.3) is given in Section 6.1 of [9]. According to Equation (1.3), it is reasonable to define

$$\lambda_c := \sup\big\{\lambda : \lim_{t\to+\infty} P_\lambda^{\delta_1}(\eta_t(O) = 1) = 0\big\}. \tag{1.4}$$

λ_c is called the critical value of the contact process.

Note that in this paper we only deal with the case where $\eta_0 = \delta_1$, so from now on we write $P_\lambda^{\delta_1}$ as P_λ for simplicity.

When $d = 1$, it is shown in Section 6.1 of [9] that $\lambda_c(1) \leq 2$. Liggett improves this result in [10] by showing that $\lambda_c(1) \leq 1.94$. For $d \geq 3$, it is shown in [7] that

$$\lambda_c(d) \leq \frac{1}{\gamma_d} - 1$$

while it is shown in [5] that

$$\lambda_c(d) \leq \frac{1}{2d(2\gamma_d - 1)},$$

where $\gamma_d > 1/2$ is the probability that the simple random walk on $\mathbb{Z}^d$ starting at O never returns to O. Both these two results lead to the conclusion that

$$\limsup_{d\to+\infty} 2d\lambda_c(d) \leq 1$$

according to the fact that

$$1-\gamma_d = \frac{1}{2d}+\frac{1}{2d^2}+o(\frac{1}{d^2}), \tag{1.5}$$

which is given in [8]. It is shown in Section 3.5 of [9] that

$$\lambda_c(d) \geq \frac{1}{2d-1} \tag{1.6}$$

for each $d \geq 1$. As a result,

$$\lim_{d\to+\infty} 2d\lambda_c(d) = 1.$$

When $d=3$, it is shown in [3] and [4] that

$$\gamma_3 = \Big[\frac{\sqrt{6}}{32\pi^3}\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})\Big]^{-1} \in [0.6594, 0.6595],$$

where $\Gamma(z)=\int_0^{+\infty} x^{z-1}e^{-x}dx$. Note that 0.6594 and 0.6595 are rigorous lower and upper bounds for γ_3 respectively, not from simulations. Then

$$\frac{1}{\gamma_3}-1 \in [0.5163, 0.5166] \text{ while } \frac{1}{6(2\gamma_3-1)} \in [0.5224, 0.5228]$$

and hence $\frac{1}{\gamma_3}-1 < \frac{1}{6(2\gamma_3-1)}$.

However, $\frac{1}{2d(2\gamma_d-1)} < \frac{1}{\gamma_d}-1$ for sufficiently large d according to the fact that

$$\frac{1}{\gamma_d}-1 = \frac{1}{2d}+\frac{3}{4d^2}+o(\frac{1}{d^2})$$

while

$$\frac{1}{2d(2\gamma_d-1)} = \frac{1}{2d}+\frac{1}{2d^2}+o(\frac{1}{d^2}),$$

which follows from Equation (1.5).

In this paper, we will give another upper bound $\beta(d)$ for the critical value $\lambda_c(d)$ when $d\geq 3$. $\beta(d)$ satisfies that $\beta(d) < \min\{\frac{1}{2d(2\gamma_d-1)}, \frac{1}{\gamma_d}-1\}$ for each $d\geq 3$. For the precise result, see the next section.

2 Main result

In this section we will give our main result. First we introduce some notations and definitions. From now on we assume that at $t=0$ all the vertices on $\mathbb{Z}^d$ are in state 1 for the contact process, then let λ_c be the critical value of the contact process defined as in Equation (1.4). We write λ_c as $\lambda_c(d)$ when we need to point out the dimension d of the lattice. We denote by $\{S_n\}_{n\geq 0}$ the simple random walk on $\mathbb{Z}^d$, i.e.,

$$P\big(S_{n+1}=y\big|S_n=x\big) = \frac{1}{2d}$$

for each y that $y\sim x$ and $n\geq 0$. We define

$$\gamma := P\big(S_n \neq O \text{ for all } n\geq 1\big|S_0=O\big)$$

as the probability that the simple random walk never return to O conditioned on $S_0=O$. We write γ as γ_d when we need to point out the dimension d of the lattice.

The following theorem gives an upper bound of $\lambda_c(d)$ for $d\geq 3$, which is our main result.

Theorem 2.1. *For each $d \geq 3$,*

$$\lambda_c(d) \leq \frac{2-\gamma_d}{2d\gamma_d}.$$

It is shown in [5] that $\lambda_c(d) \leq \frac{1}{2d(2\gamma_d - 1)}$ for each $d \geq 3$. Since $\gamma_d < 1$,

$$(2-\gamma_d)(2\gamma_d - 1) - \gamma_d = -2(\gamma_d - 1)^2 < 0$$

and hence $\frac{2-\gamma_d}{2d\gamma_d} < \frac{1}{2d(2\gamma_d-1)}$ for each $d \geq 3$. It is shown in [7] that $\lambda_c(d) \leq \frac{1}{\gamma_d} - 1$ for each $d \geq 3$. By direct calculation,

$$\begin{aligned} 1-\gamma &\geq P\big(S_2 = O \big| S_0 = O\big) + P\big(S_4 = O, S_2 \neq O \big| S_0 = O\big) \\ &= \frac{4d^2+4d-3}{8d^3} > \frac{1}{2d-1} \end{aligned}$$

when $d \geq 3$ and hence $\frac{2-\gamma_d}{2d\gamma_d} < \frac{1}{\gamma_d} - 1$ for each $d \geq 3$.

For $d = 3$, since $\gamma_3 \in [0.6594, 0.6595]$, we have the following direct corollary.

Corollary 2.2.

$$\lambda_c(3) \leq \frac{2-\gamma_3}{6\gamma_3} \leq 0.34.$$

This corollary improves the upper bound of $\lambda_c(3)$ given by $\frac{1}{\gamma_3} - 1$, which is about 0.5166. Furthermore, according to Equation (1.6),

$$\lambda_c(3) \geq 0.2$$

and hence $\lambda_c(3) \in [0.2, 0.34]$.

Remark. In [10], Liggett gives three examples to show that in some applications 'a certain degree of precision in the bound is essential' ([10], page 2). We hope our result will be a help if in some further study the fact that the critical value of 3-D contact process is smaller than 0.34 is needed. □

We will prove Theorem 2.1 in the next section. A Markov process $\{\xi_t\}_{t\geq 0}$ with state space $[0, +\infty)^{\mathbb{Z}^d}$ will be introduced as a main auxiliary tool for the proof. The definition of $\{\xi_t\}_{t\geq 0}$ is similar with that of the binary contact path process introduced in [5], except for some modifications in several details.

3 Proof of Theorem 2.1

In this section we give the proof of Theorem 2.1. Throughout this section we assume that the dimension d is fixed and at least 3, which ensures that $\gamma > \frac{1}{2}$. Our aim is to prove the following lemma, Theorem 2.1 follows from which directly.

Lemma 3.1. *If $a, b > 0$ satisfies*

$$2(a+b-1) - (a^2+b^2-1) - 2ab(1-\gamma) > 0$$

then

$$\lambda_c \leq \frac{1}{2d\big(2(a+b-1) - (a^2+b^2-1) - 2ab(1-\gamma)\big)}.$$

If we choose $a = b = 1$, then Lemma 3.1 gives the upper bound of λ_c the same as that given in [5]. However, the best choices of a, b are $a = b = \frac{1}{2-\gamma}$, which gives the following proof of Theorem 2.1.

Proof of Theorem 2.1. Let $L(a,b) = 2(a+b-1) - (a^2+b^2-1) - 2ab(1-\gamma)$, then

$$\sup\left\{L(a,b) : a > 0, b > 0\right\} = L(\frac{1}{2-\gamma}, \frac{1}{2-\gamma}) = \frac{\gamma}{2-\gamma}.$$

As a result, let $a = b = \frac{1}{2-\gamma}$, then

$$\lambda_c \leq \frac{1}{2dL(a,b)} = \frac{2-\gamma}{2d\gamma}$$

according to Lemma 3.1. $\square$

The remainder of this paper is devoted to the proof of Lemma 3.1. From now on we assume that a, b are positive constants which satisfies

$$2(a+b-1) - (a^2+b^2-1) - 2ab(1-\gamma) > 0.$$

Let $\{\xi_t\}_{t\geq 0}$ be a continuous time Markov process with state space $[0,+\infty)^{\mathbb{Z}^d}$ and generator function given by

$$\begin{aligned} \Omega f(\xi) = \sum_{x\in\mathbb{Z}^d} \left[f(\xi^{x,0}) - f(\xi)\right] + \sum_{x\in\mathbb{Z}^d}\sum_{y\sim x} \lambda\left[f(\xi^{x,y}_{a,b}) - f(\xi)\right] \\ + \sum_{x\in\mathbb{Z}^d} f'_x(\xi)\Big(1 - 2d\lambda[(b-1)+a]\Big)\xi(x) \end{aligned} \tag{3.1}$$

for any $\xi \in [0,+\infty)^{\mathbb{Z}^d}$ and sufficiently smooth function f on $[0,+\infty)^{\mathbb{Z}^d}$, where

$$\xi^{x,0}(y) := \begin{cases} \xi(y) & \text{if } y \neq x, \\ 0 & \text{if } y = x, \end{cases}$$

$$\xi^{x,y}_{a,b}(z) := \begin{cases} \xi(z) & \text{if } z \neq x, \\ b\xi(x) + a\xi(y) & \text{if } z = x \end{cases}$$

and f'_x is the partial derivative of $f(\xi)$ with respect to the coordinate $\xi(x)$. According to Theorem 9.1.14 of [9], the domain of Ω is

$$D(\Omega) := \Big\{f : \sup_{x\in\mathbb{Z}^d} \frac{\|f'_x\|}{l(x)} < +\infty\Big\},$$

where $\|f'_x\|$ is the supremum norm of f'_x while $\{l(x)\}_{x\in\mathbb{Z}^d}$ is a strictly positive given function on $\mathbb{Z}^d$ such that

$$\sum_{x\in\mathbb{Z}^d} l(x) < +\infty.$$

If $a = b = 1$ and we drop the last term of Ω involving partial derivatives, then $\{\xi_t\}_{t\geq 0}$ reduces to the binary contact path process introduced in [5] after a time-scaling. $\{\xi_t\}_{t\geq 0}$ belongs to a large class of continuous-time Markov processes called linear systems. For the definition and basic properties of the linear system, see Chapter 9 of [9].

According to the definition of Ω, $\{\xi_t\}_{t\geq 0}$ evolves as follows. For each $x \in \mathbb{Z}^d$ and each neighbor y of x, $\xi_t(x)$ flips to 0 at rate 1 while flips to $b\xi_t(x) + a\xi_t(y)$ at rate λ. Between the jumping moments of $\{\xi_t(x)\}_{t\geq 0}$, $\xi_t(x)$ evolves according to the ODE

$$\frac{d}{dt}\xi_t(x) = \Big(1 - 2d\lambda\big[(b-1)+a\big]\Big)\xi_t(x). \tag{3.2}$$

That is to say, if $\xi(x)$ does not jump during $[t, t+s]$, then

$$\xi_{t+r}(x) = \xi_t(x)\exp\Big\{r\Big(1 - 2d\lambda\big[(b-1)+a\big]\Big)\Big\}$$

for $0 < r < s$.

The linear system $\{\xi_t\}_{t\geq 0}$ and the contact process $\{\eta_t\}_{t\geq 0}$ have the following relationship.

Lemma 3.2. *For any $x \in \mathbb{Z}^d$ and $t \geq 0$, let*

$$\widehat{\eta}_t(x) = \begin{cases} 1 & \text{if } \xi_t(x) > 0, \\ 0 & \text{if } \xi_t(x) = 0, \end{cases}$$

then $\{\widehat{\eta}_t\}_{t\geq 0}$ is a version of the contact process with flip rates function given in Equation (1.2)*.*

Proof of Lemma 3.2. ODE (3.2) can not make $\{\xi_t(x)\}_{t\geq 0}$ flip from 0 to a positive value or flip from a positive value to 0, hence $\widehat{\eta}_t(x)$ stays its value between jumping moments of $\xi(x)$. If $\widehat{\eta}_t(x) = 1$, i.e, $\xi_t(x) > 0$, then $\widehat{\eta}_t(x)$ flips to 0 when and only when $\xi_t(x)$ flips to 0 at some jumping moment. As a result, $\widehat{\eta}_t(x)$ flips from 1 to 0 at rate 1. If $\widehat{\eta}_t(x) = 0$, i.e, $\xi_t(x) = 0$, then $\widehat{\eta}_t(x)$ flips to 1 when and only when $\xi_t(x)$ flips to

$$b\xi_t(x) + a\xi_t(y) = a\xi_t(y)$$

for a neighbor y with $\xi_t(y) > 0$ at some jumping moment. As a result, $\widehat{\eta}_t(x)$ flips from 0 to 1 at rate

$$\lambda \sum_{y\sim x} 1_{\{\xi_t(y)>0\}} = \lambda \sum_{y\sim x} \widehat{\eta}_t(y),$$

where 1_A is the indicator function of the event A. In conclusion, $\{\widehat{\eta}_t\}_{t\geq 0}$ evolves in the same way as a contact process evolves according to the flip rates function given in Equation (1.2). $\square$

By Lemma 3.2, from now on we assume that $\{\eta_t\}_{t\geq 0}$ and $\{\xi_t\}_{t\geq 0}$ are coupled under the same probability space such that $\eta_0(x) = \xi_0(x) = 1$ for each $x \in \mathbb{Z}^d$ and $\eta_t(x) = 1$ when and only when $\xi_t(x) > 0$. Then, P_λ is also the probability measure of $\{\xi_t\}_{t\geq 0}$ while the expectation with respect to P_λ is denoted by E. Note that the initial condition is dropped in these notations since we only deal with case where $\xi_0 = \eta_0 = \delta_1$.

The following two lemmas about expectations of $\xi_t(x)$ and $\xi_t(x)\xi_t(y)$ are important for the proof of Lemma 3.1.

Lemma 3.3. *If $\xi_0(x) = 1$ for any $x \in \mathbb{Z}^d$, then*

$$E\xi_t(x) = 1$$

for any $x \in \mathbb{Z}^d$ and $t \geq 0$.

Lemma 3.4. *For any $x \in \mathbb{Z}^d$ and $t \geq 0$, let $F_t(x) = E\big[\xi_t(O)\xi_t(x)\big]$, then conditioned on $\xi_0(x) = 1$ for all $x \in \mathbb{Z}^d$,*

$$\frac{d}{dt}F_t = \Big(\frac{d}{dt}F_t(x)\Big)_{x\in\mathbb{Z}^d} = G_\lambda F_t, \tag{3.3}$$

where G_λ is a $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix that

$$G_\lambda(x,y) = \begin{cases} -4a\lambda d & \text{if } x \neq 0 \text{ and } x = y, \\ 2a\lambda & \text{if } x \neq 0 \text{ and } x \sim y, \\ 1 - 4d\lambda(b-1) - 4d\lambda a + 2d\lambda(b^2-1) + 2d\lambda a^2 & \text{if } x = y = 0, \\ 4abd\lambda & \text{if } x = 0 \text{ and } y = e_1, \\ 0 & \text{otherwise} \end{cases}$$

and e_1 is defined as in Equation (1.1)*.*

Note that when we say $F_1 = GF_2$ for functions F_1, F_2 on $\mathbb{Z}^d$ and $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix G, we mean

$$F_1(x) = \sum_{y \in \mathbb{Z}^d} G(x,y) F_2(y)$$

for each $x \in \mathbb{Z}^d$, as the product of finite-dimensional matrices.

The proofs of Lemmas 3.3 and 3.4 rely heavily on Theorems 9.1.27 and 9.3.1 of [9]. These two theorems can be seen as the extension of the Hille-Yosida Theorem for the linear system, which ensures that we can execute the calculation

$$\frac{d}{dt} S(t) f = S(t) \Omega f \tag{3.4}$$

for a linear system with generator Ω and semi-group $\{S_t\}_{t \geq 0}$ when f has the form $f(\xi) = \xi(x)$ or $f(\xi) = \xi(x)\xi(y)$. Note that $f_1(\xi) = \xi(x)$ and $f_2(\xi) = \xi(x)\xi(y)$ both belong to the domain of Ω according to the definition of $D(\Omega)$.

Proof of Lemma 3.3. By the generator Ω of $\{\xi_t\}_{t\geq 0}$ and Theorem 9.1.27 of [9] (i.e., Equation (3.4) for $f(\xi) = \xi(x)$),

$$\frac{d}{dt} E\xi_t(x) = -E\xi_t(x) + \lambda \sum_{y \sim x} \Big[(b-1)E\xi_t(x) + aE\xi_t(y)\Big] + \Big(1 - 2d\lambda\big[(b-1)+a\big]\Big) E\xi_t(x)$$

for each $x \in \mathbb{Z}^d$. Since $\xi_0(x) = 1$ for all $x \in \mathbb{Z}^d$, $E\xi_t(x)$ does not depend on the choice of x according to the spatial homogeneity of $\{\xi_t\}_{t \geq 0}$. Therefore,

$$\frac{d}{dt} E\xi_t(x) = -E\xi_t(x) + 2d\lambda(a+b-1)E\xi_t(x) + \big(1 - 2d\lambda(a+b-1)\big)E\xi_t(x) = 0.$$

As a result, $E\xi_t(x) \equiv E\xi_0(x) = 1$. □

Proof of Lemma 3.4. According to the generator Ω of $\{\xi_t\}_{t\geq 0}$ and Theorem 9.3.1 of [9] (i.e., Equation (3.4) for $f(\xi) = \xi(x)\xi(y)$),

$$\begin{aligned} \frac{d}{dt} F_t(x) = &- 2F_t(x) + \lambda \sum_{y \sim O} \Big((b-1)F_t(0) + aE\big[\xi_t(y)\xi_t(x)\big]\Big) \\ &+ \lambda \sum_{y \sim x} \Big((b-1)F_t(0) + aF_t(y)\Big) + 2\Big(1 - 2d\lambda(a+b-1)\Big)F_t(x) \end{aligned} \tag{3.5}$$

when $x \neq O$ while

$$\begin{aligned} \frac{d}{dt} F_t(O) = &- F_t(O) + \lambda \sum_{y \sim O} 2abF_t(y) + 2d\lambda(b^2-1)F_t(O) + \lambda \sum_{y \sim O} a^2 E\big[\xi_t^2(y)\big] \\ &+ 2\big(1 - 2d\lambda(a+b-1)\big)F_t(O). \end{aligned} \tag{3.6}$$

Since $\xi_0(x) = 1$ for any $x \in \mathbb{Z}^d$, according to the spatial homogeneity of $\{\xi_t\}_{t\geq 0}$,

$$E\big[\xi_t(x)\xi_t(y)\big] = F_t(y-x) = F_t(x-y)$$

for any $x, y \in \mathbb{Z}^d$ and

$$F_t(e_i) = F_t(-e_i) = F_t(e_1)$$

for $1 \leq i \leq d$. Therefore, by Equations (3.5) and (3.6),

$$\frac{d}{dt} F_t(x) = \begin{cases} -4ad\lambda F_t(x) + 2a\lambda \sum_{y \sim x} F_t(y) & \text{if } x \neq O, \\ \big[1 - 4d\lambda(a+b-1) + 2d\lambda(b^2-1) + 2da^2\lambda\big]F_t(O) + 4abd\lambda F_t(e_1) & \text{if } x = O. \end{cases} \tag{3.7}$$

Lemma 3.4 follows from Equation (3.7) directly. □

The following lemma shows that if λ ensures the existence of an positive eigenvector of G_λ with respect to the eigenvalue 0, then λ is an upper bound of λ_c, which is crucial for us to prove Lemma 3.1.

Lemma 3.5. *If there exists* $K:\mathbb{Z}^d\rightarrow[0,+\infty)$ *that* $\inf_{x\in\mathbb{Z}^d}K(x)>0$ *and*

$$G_\lambda K=0 \text{ (}\textit{here } 0 \textit{ means the zero function on } \mathbb{Z}^d\text{)},$$

where G_λ *is defined as in Lemma 3.4, then*

$$\lambda\geq\lambda_c.$$

We give the proof of Lemma 3.5 at the end of this section. Now we show how to utilize Lemma 3.5 to prove Lemma 3.1.

Proof of Lemma 3.1. Let $\{S_n\}_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^d$ as we have introduced in Section 2, then we define

$$H(x):=P\big(S_n=O \text{ for some } n\geq 0\big|S_0=x\big)$$

for any $x\in\mathbb{Z}^d$. Then $H(O)=1$ and

$$H(x)=\frac{1}{2d}\sum_{y\sim x}H(y) \tag{3.8}$$

for any $x\neq O$. According to the spatial homogeneity of the simple random walk,

$$\begin{aligned}\gamma&=P\big(S_n\neq O \text{ for all } n\geq 1\big|S_0=O\big)\\&=P\big(S_n\neq O \text{ for all } n\geq 0\big|S_0=e_1\big)=1-H(e_1).\end{aligned} \tag{3.9}$$

For $a,b>0$ that

$$2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)>0$$

and $\lambda>\frac{1}{2d\left[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)\right]}$, we define

$$K(x)=H(x)+\frac{2d\lambda\left[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)\right]-1}{1+2d\lambda(a+b-1)^2}$$

for each $x\in\mathbb{Z}^d$. Then,

$$\inf_{x\in\mathbb{Z}^d}K(x)\geq\frac{2d\lambda\left[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)\right]-1}{1+2d\lambda(a+b-1)^2}>0$$

and $G_\lambda K=0$ according to Equations (3.8), (3.9) and the definition of G_λ. As a result, by Lemma 3.5,

$$\lambda\geq\lambda_c$$

for any $\lambda>\frac{1}{2d\left[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)\right]}$ and hence

$$\lambda_c\leq\frac{1}{2d\left[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)\right]}. \qquad\square$$

At last we give the proof of Lemma 3.5.

Proof of Lemma 3.5. For any $x, y \in \mathbb{Z}^d$, we define

$$G_\lambda^2(x,y) := \sum_{u\in\mathbb{Z}^d} G_\lambda(x,u)G_\lambda(u,y).$$

Note that we interpret G_λ as a $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix while define the product of G_λ and G_λ as that of two finite-dimensional matrices. It is easy to check that the sum in the right-hand side converges since only finitely many terms are non-zero. By induction, if G_λ^k is well-defined for $1 \le k \le n$, then we define

$$G_\lambda^{n+1}(x,y) = \sum_{u\in\mathbb{Z}^d} G_\lambda^n(x,u)G_\lambda(u,y),$$

still as the product of two finite-dimensional matrices. According to the definitions of G_λ and G_λ^2,

$$\{y : G_\lambda(x,y) \neq 0\} \subseteq \{y : \|y-x\| \le 1\} \text{ while } \{y : G_\lambda^2(x,y) \neq 0\} \subseteq \{y : \|y-x\| \le 2\}.$$

Therefore, G_λ^3 is well defined and

$$\{y : G_\lambda^3(x,y) \neq 0\} \subseteq \{y : \|y-x\| \le 3\}.$$

By induction, G_λ^n is well-defined for each $n \ge 1$ and

$$\{y : G_\lambda^n(x,y) \neq 0\} \subseteq \{y : \|y-x\| \le n\}.$$

According to the definition of G_λ,

$$C_3 := \sup_{x\in\mathbb{Z}^d} \sum_{y:y\sim x} |G_\lambda(x,y)|$$

is finite. As a result, by induction,

$$|G_\lambda^n(x,y)| \le C_3^n$$

for each $n \ge 1, x, y \in \mathbb{Z}^d$ and hence

$$\sup_{x,y\in\mathbb{Z}^d} \sum_{n=0}^{+\infty} \frac{t^n |G_\lambda^n(x,y)|}{n!} < +\infty$$

for any $t \ge 0$, where $G_\lambda^0(x,y) = 1_{\{x=y\}}$. Then, it is reasonable to define the $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix e^{tG_λ} as

$$e^{tG_\lambda}(x,y) = \sum_{n=0}^{+\infty} \frac{t^n G_\lambda^n(x,y)}{n!}$$

for $x, y \in \mathbb{Z}^d$ and $t \ge 0$. Since K satisfies $G_\lambda K = 0$,

$$G_\lambda^n K = G_\lambda^{n-1} G_\lambda K = 0$$

for each $n \ge 1$ and hence

$$(e^{tG_\lambda}K)(x) = \sum_{y\in\mathbb{Z}^d} e^{tG_\lambda}(x,y)K(y) = \sum_{y\in\mathbb{Z}^d} G_\lambda^0(x,y)K(y) = K(x) \tag{3.10}$$

for each $x \in \mathbb{Z}^d$ and $t \ge 0$, i.e., K is an eigenvector of e^{tG_λ} with respect to the eigenvalue 1.

For any $\xi \in (-\infty, +\infty)^{\mathbb{Z}^d}$, we define

$$\|\xi\|_\infty = \sup_{x \in \mathbb{Z}^d} |\xi(x)|.$$

Furthermore, we define

$$W = \{\xi \in (-\infty, +\infty)^{\mathbb{Z}^d} : \|\xi\|_\infty < +\infty\},$$

then W is a Banach space with norm $\|\cdot\|_\infty$. By the definition of G_λ, it is easy to check that there exists $M > 0$ such that

$$\|G_\lambda(\xi_1 - \xi_2)\|_\infty \leq M\|\xi_1 - \xi_2\|_\infty$$

for any $\xi_1, \xi_2 \in W$, i.e., ODE (3.3) satisfies Lipschitz condition. As a result, according to the theory of the linear ODE on the Banach space (see page 4–7 on [1]), ODE (3.3) has the unique solution that

$$F_t = e^{tG_\lambda} F_0$$

for any $t \geq 0$. Since $F_0(x) = 1$ for any $x \in \mathbb{Z}^d$,

$$F_t(O) = \sum_{y \in \mathbb{Z}^d} e^{tG_\lambda}(O, y) F_0(y) = \sum_{y \in \mathbb{Z}^d} e^{tG_\lambda}(O, y).$$

Since $G_\lambda(x, y) \geq 0$ when $x \neq y$, $e^{tG_\lambda}(x, y) \geq 0$ for any $x, y \in \mathbb{Z}^d$. Therefore, by Equation (3.10),

$$E(\xi_t^2(O)) = F_t(O) \leq \sum_{y \in \mathbb{Z}^d} e^{tG_\lambda}(O, y) \frac{K(y)}{\inf_{x \in \mathbb{Z}^d} K(x)} = \frac{K(O)}{\inf_{x \in \mathbb{Z}^d} K(x)} \tag{3.11}$$

for any $t \geq 0$. According to Lemmas 3.2, 3.3, Equation (3.11) and Cauchy-Schwartz inequality,

$$\begin{aligned}
\lim_{t \to +\infty} P_\lambda\big(\eta_t(O) = 1\big) &= \lim_{t \to +\infty} P_\lambda\big(\xi_t(O) > 0\big) \\
&\geq \limsup_{t \to +\infty} \frac{(E\xi_t(O))^2}{E(\xi_t^2(O))} = \limsup_{t \to +\infty} \frac{1}{E(\xi_t^2(O))} \\
&\geq \frac{\inf_{x \in \mathbb{Z}^d} K(x)}{K(O)} > 0. \qquad (3.12)
\end{aligned}$$

Note that $\lim_{t \to +\infty} P_\lambda\big(\eta_t(O) = 1\big)$ exists according to Equation (1.3), which shows that $P_\lambda\big(\eta_t(O) = 1\big)$ is decreasing with t. As a result,

$$\lambda \geq \lambda_c$$

for any λ that there exists K which satisfies $\inf_{x \in \mathbb{Z}^d} K(x) > 0$ and $G_\lambda K = 0$. $\square$

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