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Abrupt convergence for a family of Ornstein–Uhlenbeck processes

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Abstract. We consider a family of Ornstein–Uhlenbeck processes. Under some suitable assumptions on the behaviour of the drift and diffusion coefficients, we prove profile cut-off phenomenon with respect to the total variation distance in the sense of the definition given by Barrera and Ycart [*ALEA Lat. Am. J. Probab. Math. Stat.* **11** (2014) 445–458]. We compute explicitly the cut-off time, the window time, and the profile function. Moreover, we prove that the average process satisfies a profile cut-off phenomenon with respect to the total variation distance. Also, a sample of N Ornstein–Uhlenbeck processes has a window cut-off with respect to the total variation distance in the sense of the definition given by Barrera and Ycart [*ALEA Lat. Am. J. Probab. Math. Stat.* **11** (2014) 445–458]. The cut-off time and the cut-off window for the average process and for the sampling process are the same.

1 Introduction

The Ornstein–Uhlenbeck process is a well-known stochastic process. It has been widely studied for the past seventy years. It was introduced into Physics in 1930 by Uhlenbeck and Ornstein (1930). It has been used in financial mathematics to model prices in markets in Jeanblanc and Rutkowski (2000) and in biology to model neural activity in Lánský, Sacerdote and Tomassetti (1995).

The cut-off phenomenon has been widely investigated in the past thirty years; see Diaconis (1996) and Saloff-Coste (2004). The term "cut-off" was introduced by Aldous and Diaconis (1986) in the early 1980s to describe the phenomenon of the abrupt convergence of Markov chains introduced as models of shuffling cards. This phenomenon refers to an asymptotically drastic convergence of a family of stochastic processes. The term cut-off is naturally associated to switching phenomena, *that is*, "all/nothing" or "1/0" behaviour. Alternative names are threshold phenomenon and abrupt convergence. Since the appearance of Aldous and Diaconis (1986), many families of stochastic processes have been shown to have similar properties. For more details, see Barrera and Ycart (2014) and Diaconis (1996). Saloff-Coste (2004) gives an extensive list of random walks for which the phenomenon occurs. Now, it is a well-studied feature of Markov processes. Barrera and Jara (2016) study the cut-off phenomenon in a continuous setting.

Key words and phrases. Cut-off phenomenon, total variation distance, Ornstein–Uhlenbeck processes.

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Lachaud (2005) proved that the average process and the sampling process of the Ornstein–Uhlenbeck process satisfy a window cut-off with the same cut-off time and window time. This is surprising, since the sample process comprises a large number of processes.

In the present paper, we consider a family of Ornstein–Uhlenbeck processes. Under some mild assumptions on the behaviour of the drift and diffusion coefficients, we prove that the family of Ornstein–Uhlenbeck processes has a profile cut-off in the sense of the definition given by Barrera and Ycart (2014) with an explicit profile function, cut-off time, and cut-off window. Following the spirit of Lachaud (2005), we prove that the average process of the Ornstein–Uhlenbeck process has a profile cut-off with an explicit profile function, cut-off time and cut-off window. Moreover, the sampling process of the Ornstein–Uhlenbeck process has a window cut-off in the sense of the definition given by Barrera and Ycart (2014) with an explicit cut-off time and cut-off window. We also note that in this case, the average process and the sampling process have the same cut-off time and the window time.

Consider a one-parameter family of stochastic processes in continuous time $\{x^N\}_{N\in\mathbb{N}}$ indexed by $N \in \mathbb{N}$, $x^N := \{x_t^N\}_{t\geq 0}$, each one converging to an asymptotic distribution ν^N when t goes to infinity. Let us denote by $d_N(t)$ the distance between the distribution at time t of the Nth process, $\mathbb{P}(x_t^N \in \cdot)$, and its asymptotic distribution as $t \to +\infty$, ν^N, where the "distance" can be taken to be the total variation, separation, Hellinger, relative entropy, Wasserstein, L^p distances, etc. Following Barrera and Ycart (2014), the cut-off phenomenon for $\{x^N\}_{N\in\mathbb{N}}$ can be expressed at three increasingly sharp levels. Let us denote by M the diameter of the metric space of probability measures in which we are working. In general, M could be infinite. For any Ornstein–Uhlenbeck process, we have explicit formulas for its mean, its variance, and its distribution. We can also give explicit expressions for the total variation distance between normal distributions in terms of the cumulative distribution function of the standard normal distribution. In our case, we will focus on the total variation distance, so throughout this paper, $M := 1$.

The cut-off phenomenon refers to the abrupt convergence to an asymptotic probability measure (asymptotic distribution). When the distance is the total variation distance, this asymptotic distribution can be interpreted as the survival function of a certain positive "random variable" concentrated around the cut-off time. This "random variable" could be viewed as the instant at which the family of processes reaches the equilibrium.

Definition 1.1 (Cut-off). The family $\{x^N\}_{N\in\mathbb{N}}$ of stochastic processes has a cut-off at $\{t_N\}_{N\in\mathbb{N}}$ if

$$\lim_{N\to+\infty} d_N(ct_N) = \begin{cases} M & \text{if } 0 < c < 1, \\ 0 & \text{if } c > 1. \end{cases}$$

Definition 1.2 (Window cut-off). The family $\{x^N\}_{N\in\mathbb{N}}$ of stochastic processes has a window cut-off at $\{(t_N, w_N)\}_{N\in\mathbb{N}}$ if $t_N \to +\infty$, $w_N = o(t_N)$ as $N \to +\infty$ and

$$\lim_{c\to-\infty} \liminf_{N\to+\infty} d_N(t_N + cw_N) = M, \qquad \lim_{c\to+\infty} \limsup_{N\to+\infty} d_N(t_N + cw_N) = 0.$$

Definition 1.3 (Profile cut-off). The family $\{x^N\}_{N\in\mathbb{N}}$ of stochastic processes has a profile cut-off at $\{(t_N, w_N)\}_{N\in\mathbb{N}}$ with profile function G if $t_N \to +\infty$, $w_N = o(t_N)$ as $N \to +\infty$,

$$G(c) := \lim_{N\to+\infty} d_N(t_N + cw_N)$$

exists for all $c \in \mathbb{R}$ and

$$\lim_{c\to-\infty} G(c) = M, \qquad \lim_{c\to+\infty} G(c) = 0.$$

This material is organized as follows. In Section 2, we describe the model, state the main result as well as establish the basic notation, and give the proof of the main result. In Section 3, we prove that the average process of a sampling of the Ornstein–Uhlenbeck process has a profile cut-off and the sampling process of the Ornstein–Uhlenbeck process has a window cut-off. In Section 4, we draw some conclusions about the results obtained within Section 2 and Section 3. In the Appendix, we give some basic results that we use throughout Section 2 and Section 3, in order to improve the readability.

2 Main result

We will establish some basic notation. Take $\mu \in \mathbb{R}$ and let $\sigma^2 \in]0, +\infty[$ be fixed numbers. We denote by $\mathcal{N}(\mu, \sigma^2)$ the normal distribution with mean μ and variance σ^2. Given two probability measures $\mathbb{P}$ and $\mathbb{Q}$ that are defined on the same measurable space $(\Omega, \mathcal{F})$, we define the total variation distance between $\mathbb{P}$ and $\mathbb{Q}$ by

$$\|\mathbb{P} - \mathbb{Q}\|_{\mathbb{TV}} := \sup_{A\in\mathcal{F}} \left|\mathbb{P}(A) - \mathbb{Q}(A)\right|.$$

Theorem 2.1 (Main theorem). *Let $\{k_N\}_{N\in\mathbb{N}}$ and $\{b_N\}_{N\in\mathbb{N}}$ be sequences of non-negative numbers and let $\{c_N\}_{N\in\mathbb{N}}$ be a sequence of real numbers. Let us consider the family of processes indexed by $N \in \mathbb{N}$, $x^N = \{x_t^N\}_{t\geq 0}$ that are given by the solution of the following linear non-homogeneous stochastic differential equation*:

$$\begin{aligned} dx_t^N &= -k_N x_t^N \, dt + c_N f(t) \, dt + \sqrt{b_N} \, dW_t, \qquad t \geq 0, \\ x_0^N &= x_0, \end{aligned} \tag{2.1}$$

where x_0 is a non-zero deterministic initial condition, $f : [0, +\infty[\to \mathbb{R}$ is a continuous function, and $\{W_t\}_{t\geq 0}$ is a standard one dimensional Brownian motion. We assume that

$$f_N := \lim_{t\to+\infty}\left(e^{-k_N t}\int_0^t e^{k_N s} f(s)\,ds\right) \in \mathbb{R}, \tag{2.2}$$

$$\lim_{N\to+\infty}\frac{k_N}{b_N} = +\infty, \tag{2.3}$$

$$\lim_{N\to+\infty}\frac{1}{k_N}\ln\left(\frac{2x_0^2 k_N}{b_N}\right) = +\infty, \tag{2.4}$$

and

$$\lim_{N\to+\infty}\frac{k_N c_N^2 \sup_{t\geq 0}|f_N(t)|^2}{b_N} = 0, \tag{2.5}$$

where $f_N(t) := e^{-k_N t}\int_0^t e^{k_N s} f(s)\,ds$ for every $N \in \mathbb{N}$ and $t \geq 0$. For every $N \in \mathbb{N}$ and $t \geq 0$, we define $d^N(t) := \|\mathbb{P}(x_t^N \in \cdot) - \mathbb{P}(x_\infty^N \in \cdot)\|_{\mathbb{TV}}$, where x_∞^N represents the asymptotic distribution of x_t^N as $t \to +\infty$ Then the family $\{x^N\}_{N\in\mathbb{N}}$ has a profile cut-off in the sense of Definition 1.3 with respect to the total variation distance when $N \to +\infty$. The profile function $G : \mathbb{R} \to [0, 1]$ is given by

$$\lim_{N\to+\infty} d^N(t_N + b w_N) = G(b) := \|\mathcal{N}(e^{-b}, 1) - \mathcal{N}(0, 1)\|_{\mathbb{TV}}. \tag{2.6}$$

The cut-off time t_N and the window time w_N are given by

$$t_N := \frac{1}{2k_N}\ln\left(\frac{2x_0^2 k_N}{b_N}\right) \tag{2.7}$$

and

$$w_N := \frac{1}{k_N} \tag{2.8}$$

for every N large enough.

Remark 2.2. By the hypothesis (2.4), we can take $N_0 := N_0(x_0^2) \in \mathbb{N}$ large enough in order that the cut-off time t_N given by the relation (2.7) satisfies $t_N > 0$ for every $N \geq N_0$. Also, by the hypothesis (2.3) we have that the window time w_N given by the relation (2.8) satisfies $w_N = o(t_N)$ as $N \to +\infty$. Now, by Lemma A.1, we have that the function G defined by the relation (2.6) satisfies

$$\lim_{b\to-\infty} G(b) = 1, \qquad \lim_{b\to+\infty} G(b) = 0.$$

Theorem 2.1 provides computable assumptions to verify in order to obtain a profile cut-off.

Remark 2.3. By the Fundamental Theorem of Calculus, we have

$$\lim_{b\to+\infty} \frac{G(b)}{(\frac{e^{-b}}{\sqrt{2\pi}})} = 1.$$

Using the Mill ratio, we obtain

$$1 + \frac{4e^{-\frac{e^{-2b}}{8}-b}}{\sqrt{2\pi}(4+e^{-2b})} \le G(b) \le 1 + \frac{4e^{-\frac{e^{-2b}}{8}+b}}{\sqrt{2\pi}}$$

for every $b \in \mathbb{R}$. Therefore,

$$\lim_{b\to-\infty} \frac{G(b)-1}{(\frac{4e^{-\frac{e^{-2b}}{8}}}{\sqrt{2\pi}})} = 0.$$

Proof of the Main Theorem 2.1. Fix $N \in \mathbb{N}$ and $t > 0$. Using the Itô isometry, we have that x_t^N has normal distribution with mean

$$\mu_t^N := x_0 e^{-k_N t} + c_N e^{-k_N t} \int_0^t e^{k_N s} f(s)\, ds$$

and variance $\frac{b_N}{2k_N}(1-e^{-2k_N t})$. Then by hypothesis (2.2), we have that the asymptotic probability measure associated to the transition kernel of the linear non-homogeneous stochastic differential equation (2.1) is a normal distribution with mean $c_N f_N$ and variance $\frac{b_N}{2k_N}$. Define

$$d^N(t) := \left\| \mathcal{N}\left(\mu_t^N, \frac{b_N}{2k_N}(1-e^{-2k_N t})\right) - \mathcal{N}\left(c_N f_N, \frac{b_N}{2k_N}\right) \right\|_{\mathbb{TV}},$$

$$D^N(t) := \left\| \mathcal{N}\left(\sqrt{\frac{2k_N}{b_N}}(\mu_t^N - c_N f_N), 1\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}}$$

and

$$R^N(t) := \|\mathcal{N}(0, 1-e^{-2k_N t}) - \mathcal{N}(0,1)\|_{\mathbb{TV}}.$$

By the triangle inequality, we have

$$d^N(t) \le \left\| \mathcal{N}\left(\mu_t^N, \frac{b_N}{2k_N}(1-e^{-2k_N t})\right) - \mathcal{N}\left(\mu_t^N, \frac{b_N}{2k_N}\right) \right\|_{\mathbb{TV}} + \left\| \mathcal{N}\left(\mu_t^N, \frac{b_N}{2k_N}\right) - \mathcal{N}\left(c_N f_N, \frac{b_N}{2k_N}\right) \right\|_{\mathbb{TV}}.$$

Using the translation invariance property of the total variation distance, we obtain

$$d^N(t) \le \left\| \mathcal{N}\left(0, \frac{b_N}{2k_N}(1-e^{-2k_N t})\right) - \mathcal{N}\left(0, \frac{b_N}{2k_N}\right) \right\|_{\mathbb{TV}} + \left\| \mathcal{N}\left(\mu_t^N - c_N f_N, \frac{b_N}{2k_N}\right) - \mathcal{N}\left(0, \frac{b_N}{2k_N}\right) \right\|_{\mathbb{TV}}.$$

Using the scaling invariance property of the total variation distance, we get

$$d^N(t) \le \|\mathcal{N}(0, 1 - e^{-2k_N t}) - \mathcal{N}(0,1)\|_{\mathbb{TV}} + \left\| \mathcal{N}\left(\sqrt{\frac{2k_N}{b_N}}(\mu_t^N - c_N f_N), 1\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}}. \tag{2.9}$$

By the inequality (2.9), we have

$$d^N(t) \le D^N(t) + R^N(t). \tag{2.10}$$

Using the same ideas, we can obtain

$$D^N(t) \le d^N(t) + R^N(t). \tag{2.11}$$

Therefore, using the inequality (2.10) and the inequality (2.11), we get

$$|d^N(t) - D^N(t)| \le R^N(t).$$

Using Lemma A.3 and the hypothesis (2.3), we obtain

$$\lim_{N \to +\infty} |d^N(t_N + bw_N) - D^N(t_N + bw_N)| = 0 \tag{2.12}$$

for every $b \in \mathbb{R}$. We can note that

$$D^N(t_N + bw_N) = \left\| \mathcal{N}\left(\operatorname{sgn}(x_0)e^{-b} + \sqrt{\frac{2k_N}{b_N}} c_N (f_N(t) - f_N), 1\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}}$$

for every $b \in \mathbb{R}$, where $\operatorname{sgn}(x_0)$ denotes the sign of x_0 and $f_N(t) = e^{-k_N t} \int_0^t e^{k_N s} \times f(s)\, ds$. Using hypotheses (2.2), (2.5) and Lemma A.2, we obtain

$$\lim_{N \to +\infty} D^N(t_N + bw_N) = \|\mathcal{N}(\operatorname{sgn}(x_0)e^{-b}, 1) - \mathcal{N}(0,1)\|_{\mathbb{TV}}$$

for every $b \in \mathbb{R}$. Using the relation (2.12) and the scaling invariance property of the total variation distance, we get

$$\lim_{N \to +\infty} d^N(t_N + bw_N) = \|\mathcal{N}(e^{-b}, 1) - \mathcal{N}(0,1)\|_{\mathbb{TV}} =: G(b)$$

for every $b \in \mathbb{R}$. □

3 The profile cut-off phenomenon for the average process and window cut-off for the sampling process

In this section, we consider two natural stochastic processes associated to the stochastic differential equation defined by (2.1). Roughly speaking, these processes could be used to model the average and the sampling of a population. As a corollary of Theorem 2.1, we have a statement for the average process that it will define by the relation (3.1).

Corollary 3.1 (Average process). *Let* $\{(x_t^{N,1},\dots,x_t^{N,N})\}_{t\geq 0}$ *be a sample of* N *independent stochastic processes governed by the linear non-homogeneous stochastic differential equation* (2.1). *Fix* $N \in \mathbb{N}$. *Define the uniform average process* $\{s_t^N\}_{t\geq 0}$ *by*

$$s_t^N := \frac{1}{N}\sum_{i=1}^{N} x_t^{N,i}, \qquad t \geq 0. \tag{3.1}$$

We assume that

$$f_N := \lim_{t\to+\infty}\left(e^{-k_N t}\int_0^t e^{k_N s} f(s)\,ds\right) \in \mathbb{R},$$

$$\lim_{N\to+\infty} \frac{N k_N}{b_N} = +\infty,$$

$$\lim_{N\to+\infty} \frac{1}{k_N}\ln\left(\frac{2x_0^2 N k_N}{b_N}\right) = +\infty,$$

$$\lim_{N\to+\infty} \frac{N k_N c_N^2 \sup_{t\geq 0}|f_N(t)|^2}{b_N} = 0,$$

where $f_N(t) := e^{-k_N t}\int_0^t e^{k_N s} f(s)\,ds$ *for every* $N \in \mathbb{N}$ *and* $t \geq 0$. *For every* $N \in \mathbb{N}$ *and* $t \geq 0$, *define* $d^N(t) := \|\mathbb{P}(s_t^N \in \cdot) - \mathbb{P}(s_\infty^N \in \cdot)\|_{\mathbb{TV}}$, *where* s_∞^N *represents the asymptotic distribution of* s_t^N *as* $t \to +\infty$. *Then, the family* $\{s^N := \{s_t^N\}_{t\geq 0}\}_{N\in\mathbb{N}}$ *has a profile cut-off in the sense of Definition* 1.3 *with respect to the total variation distance when* $N \to +\infty$. *The profile function* $G : \mathbb{R} \to [0, 1]$ *is given by*

$$\lim_{N\to+\infty} d^N(t_N + b w_N) = G(b) := \|\mathcal{N}(e^{-b}, 1) - \mathcal{N}(0, 1)\|_{\mathbb{TV}}.$$

The cut-off time t_N *and the window time* w_N *are given by*

$$t_N := \frac{1}{2k_N}\ln\left(\frac{2x_0^2 N k_N}{b_N}\right), \qquad w_N := \frac{1}{k_N}$$

for every N *large enough.*

Proof of the Corollary 3.1. Fix $N \in \mathbb{N}$. The average process $\{s_t^N\}_{t\geq 0}$ satisfies the following stochastic differential equation,

$$\begin{aligned} ds_t^N &= -k_N s_t^N\,dt + c_N f(t)\,dt + \sqrt{\frac{b_N}{N}}\,d\tilde{W}_t, \qquad t \geq 0,\\ s_0^N &= x_0, \end{aligned} \tag{3.2}$$

where x_0 is a non-zero deterministic initial condition and $\{\tilde{W}_t\}_{t\geq 0}$ is a standard one dimensional Brownian motion. The proof follows from the relation (3.2) and Theorem 2.1. □

The next proposition does not follow directly from Theorem 2.1, as did Corollary 3.1. Nevertheless, with some slight modifications in the proof of Theorem 2.1, we can prove that there is a window cut-off for the sampling process associated to the linear non-homogeneous stochastic differential equation (2.1). In the sampling process, the dimension of the sampling vector goes to infinity and this fact does not permit giving a profile. For this reason, we introduce the Hellinger (A.2) distance because it is computable for product measures of normal distributions.

Proposition 3.2 (Sampling process). *Let $\{\pi_t^N := (x_t^{N,1}, \ldots, x_t^{N,N})\}_{t\geq 0}$ be a sample of N independent stochastic processes governed by the linear non-homogeneous stochastic differential equation* (2.1). *We assume that*

$$f_N := \lim_{t\to+\infty}\left(e^{-k_N t}\int_0^t e^{k_N s} f(s)\,ds\right) \in \mathbb{R}, \tag{3.3}$$

$$\lim_{N\to+\infty} \frac{k_N}{b_N} = +\infty, \tag{3.4}$$

$$\lim_{N\to+\infty} \frac{1}{k_N}\ln\left(\frac{2x_0^2 N k_N}{b_N}\right) = +\infty, \qquad \lim_{N\to+\infty} \frac{N k_N c_N^2 \sup_{t\geq 0}|f_N(t)|^2}{b_N} = 0,$$

where $f_N(t) := e^{-k_N t}\int_0^t e^{k_N s} f(s)\,ds$ for every $N\in\mathbb{N}$ and $t\geq 0$. For every $N\in\mathbb{N}$ and $t\geq 0$, define $d^N(t) := \|\mathbb{P}(\pi_t^N \in \cdot) - \mathbb{P}(\pi_\infty^N \in \cdot)\|_{\mathbb{TV}}$, where π_∞^N represents the asymptotic distribution of π_t^N as $t\to+\infty$ Then, the family $\{\pi^N := \{\pi_t^N\}_{t\geq 0}\}_{N\in\mathbb{N}}$ has a window cut-off in the sense of Definition 1.2 with respect to the total variation distance when $N\to+\infty$, i.e.,

$$\lim_{b\to-\infty}\liminf_{N\to+\infty} \Delta^N(t_N + b w_N) = 1, \qquad \lim_{b\to+\infty}\limsup_{N\to+\infty} \Delta^N(t_N + b w_N) = 0$$

for every $b\in\mathbb{R}$, where the cut-off time t_N and the window time w_N are given by

$$t_N := \frac{1}{2k_N}\ln\left(\frac{2x_0^2 N k_N}{b_N}\right), \qquad w_N := \frac{1}{k_N}$$

for every N large enough.

Proof of Theorem 3.2. Fix $N\in\mathbb{N}$ and $t>0$. The distribution of π_t^N is given by $\mathcal{N}(\mu_t^N, \frac{b_N}{2k_N}(1-e^{-2k_N t}))^{\otimes N}$, where $\mu_t^N := x_0 e^{-k_N t} + c_N e^{-k_N t}\int_0^t e^{k_N s} f(s)\,ds$. Using the hypothesis (3.3), we have that the asymptotic distribution as $t\to+\infty$ is given by $\mathcal{N}(c_N f_N, \frac{b_N}{2k_N})^{\otimes N}$. Define

$$\delta^N(t) := \left\|\mathcal{N}\left(\mu_t^N, \frac{b_N}{2k_N}(1-e^{-2k_N t})\right)^{\otimes N} - \mathcal{N}\left(c_N f_N, \frac{b_N}{2k_N}\right)^{\otimes N}\right\|_{\mathbb{TV}},$$

$$\Delta^N(t) := \left\|\mathcal{N}\left(\sqrt{\frac{2k_N}{b_N}}(\mu_t^N - c_N f_N), 1\right)^{\otimes N} - \mathcal{N}(0,1)^{\otimes N}\right\|_{\mathbb{TV}},$$

and

$$r^N(t) := \|\mathcal{N}(0, (1 - e^{-2k_N t}))^{\otimes N} - \mathcal{N}(0, 1)^{\otimes N}\|_{\mathbb{TV}}.$$

Then

$$\left|\delta^N(t) - \Delta^N(t)\right| \le r^N(t). \tag{3.5}$$

Using the inequality (A.1) in Remark A.4 and the hypothesis (3.4), we get

$$\lim_{N\to+\infty} r^N(t_N + bw_N) = 0 \tag{3.6}$$

for every $b \in \mathbb{R}$. Therefore,

$$\lim_{N\to+\infty} \left|\delta^N(t_N + bw_N) - \Delta^N(t_N + bw_N)\right| = 0 \tag{3.7}$$

for every $b \in \mathbb{R}$. We can note that

$$\begin{aligned}
&\Delta^N(t_N + bw_N) \\
&\quad = \Bigg\| \mathcal{N}\Bigg(\frac{\operatorname{sgn}(x_0)e^{-b}}{\sqrt{N}} \\
&\qquad + \sqrt{\frac{2k_N}{b_N}} c_N\big(f_N(t_N + bw_N) - f_N\big), 1\Bigg)^{\otimes N} - \mathcal{N}(0, 1)^{\otimes N} \Bigg\|_{\mathbb{TV}}
\end{aligned}$$

for every $b \in \mathbb{R}$, where $\operatorname{sgn}(x_0)$ denotes the sign of x_0 and $f_N(t) = e^{-k_N t} \int_0^t e^{k_N s} \times f(s)\, ds$. Using Lemma A.7, item (i), item (ii) and item (iii), we have

$$\begin{aligned}
&\lim_{b\to-\infty} \liminf_{N\to+\infty} \Delta^N(t_N + bw_N) = 1, \\
&\lim_{b\to+\infty} \limsup_{N\to+\infty} \Delta^N(t_N + bw_N) = 0
\end{aligned} \tag{3.8}$$

for every $b \in \mathbb{R}$. Now the relations (3.6), (3.7) and (3.8) imply the statement. □

4 Conclusions

In the Main Theorem 2.1, in order that there be a profile cut-off phenomenon, we need to assume the relations (2.2), (2.3), (2.4) and (2.5). Those relations guarantee the abrupt convergence to the equilibrium measure for the process given by (2.1). From this fact, we can obtain immediately a profile cut-off in the following cases:

- *Constant drift coefficient, zero in-homogeneous coefficient and noise coefficient going to vanish.* In the stochastic differential equation (2.1), put $k_N = k \in]0, +\infty[$, $c_N = 0$ for N large enough and $b_N = o(1)$ as $N \to +\infty$. Then, by the Main Theorem 2.1, we obtain a profile cut-off.

- f *bounded*: In the stochastic differential equation (2.1), assume the conditions (2.2), (2.3) and (2.4). Then (2.5) will be fulfilled if $\lim_{N\to+\infty} \frac{c_N^2}{b_N k_N} = 0$. Consequently, we obtain a profile cut-off.
- $f \in L^2([0, +\infty[)$: In the stochastic differential equation (2.1), assume the conditions (2.2), (2.3) and (2.4). Then, using the Cauchy–Schwarz inequality, the condition (2.5) will be fulfilled if $\lim_{N\to+\infty} \frac{c_N^2}{b_N k_N} = 0$. Therefore, we obtain a profile cut-off.

Therefore, we can build several examples of sequences $\{k_N\}_{N\in\mathbb{N}}$, $\{b_N\}_{N\in\mathbb{N}}$, $\{c_N\}_{N\in\mathbb{N}}$ and a function f such that the assumptions of Theorem 2.1 are fulfilled. Consequently, by the Main Theorem 2.1, we can obtain immediately a profile cut-off.

As in Theorem 2.1, we have several examples in which the average process has a profile cut-off and the sampling process has a window cut-off with the same cut-off time and window time. We also obtain immediately the results of Lachaud (2005) taking the sequences $\{k_N\}_{N\in\mathbb{N}}$, constants $\{b_N\}_{N\in\mathbb{N}}$, and $c_N = 0$ for any $N \in \mathbb{N}$. Moreover, we obtain an explicit profile function.

Appendix: Some properties of the total variation distance

Lemma A.1. *Fix $\mu \in \mathbb{R}$. Then*

$$\|\mathcal{N}(\mu, 1) - \mathcal{N}(0, 1)\|_{\mathbb{TV}} = \frac{2}{\sqrt{2\pi}} \int_0^{\frac{|\mu|}{2}} e^{-\frac{x^2}{2}}\, dx \le \frac{|\mu|}{\sqrt{2\pi}}.$$

Lemma A.2. *Let $\{\mu_N\}_{N\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\lim_{N\to+\infty} \mu_N = \mu \in \mathbb{R}$. Then*

$$\lim_{N\to+\infty} \|\mathcal{N}(\mu_N, 1) - \mathcal{N}(0, 1)\|_{\mathbb{TV}} = \|\mathcal{N}(\mu, 1) - \mathcal{N}(0, 1)\|_{\mathbb{TV}}.$$

Lemma A.3. *Suppose $\{\sigma_N^2\}_{N\in\mathbb{N}} \subset]0, +\infty[$ is a sequence such that $\lim_{N\to+\infty} \sigma_N^2 = \sigma^2 \in]0, +\infty[$. Then*

$$\lim_{N\to+\infty} \|\mathcal{N}(0, \sigma_N^2) - \mathcal{N}(0, \sigma^2)\|_{\mathbb{TV}} = 0.$$

Remark A.4. Suppose that $\sigma^2 \in]0, 1[$. Then

$$\begin{aligned} \|\mathcal{N}(0, \sigma^2) - \mathcal{N}(0, 1)\|_{\mathbb{TV}} &= \frac{2}{\sqrt{2\pi}} \int_0^{x(\sigma^2)} \left(\frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} - e^{-\frac{x^2}{2}}\right) dx \\ &\le \frac{\sqrt{2\pi}}{\pi} \frac{(1-\sigma^2)}{\sigma^2}, \end{aligned} \tag{A.1}$$

where $x(\sigma^2) := \sqrt{\frac{\sigma^2 \ln(\sigma^2)}{\sigma^2 - 1}}$. The relation (A.1) now follows from a straightforward calculation.

Lemma A.5. *Let $N \in \mathbb{N}$ and take $\mu_1, \mu_2, \ldots, \mu_N \in \mathbb{R}$ and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2 \in]0, +\infty[$. Then*

$$\|\mathcal{N}(\mu_1, \sigma_1^2) \otimes \cdots \otimes \mathcal{N}(\mu_N, \sigma_N^2) - \mathcal{N}(\tilde{\mu}_1, \tilde{\sigma}_1^2) \otimes \cdots \otimes \mathcal{N}(\tilde{\mu}_N, \tilde{\sigma}_N^2)\|_{\mathbb{TV}}$$
$$\leq \sum_{i=1}^{N} \|\mathcal{N}(\mu_k, \sigma_k^2) - \mathcal{N}(\tilde{\mu}_k, \tilde{\sigma}_k^2)\|_{\mathbb{TV}}.$$

Definition A.6 (Hellinger distance). Let $\mathbb{P}$, $\mathbb{Q}$ and Λ be probability measures defined on the measurable space $(\Omega, \mathcal{F})$. Suppose that $\mathbb{P} \ll \Lambda$ and $\mathbb{Q} \ll \Lambda$. Then we define the Hellinger distance between $\mathbb{P}$ and $\mathbb{Q}$ by

$$d_H(\mathbb{P}, \mathbb{Q}) := \left(\frac{1}{2} \int_{\Omega} \left(\sqrt{\frac{d\mathbb{P}}{d\Lambda}} - \sqrt{\frac{d\mathbb{Q}}{d\Lambda}} \right)^2 d\Lambda \right)^{1/2}, \tag{A.2}$$

where $\frac{d\mathbb{P}}{d\Lambda}$ and $\frac{d\mathbb{Q}}{d\Lambda}$ are the Radon–Nikodym derivatives of $\mathbb{P}$ and $\mathbb{Q}$ with respect to Λ, respectively.

Lemma A.7 (The relation between the Hellinger distance and total variation distance). *Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures defined in the measurable space $(\Omega, \mathcal{F})$. Then*

(i)

$$d_H^2(\mathbb{P}, \mathbb{Q}) \leq \|\mathbb{P} - \mathbb{Q}\|_{\mathbb{TV}} \leq \sqrt{2} d_H(\mathbb{P}, \mathbb{Q}).$$

(ii) *Suppose $\mathbb{P}^N = \underbrace{\mathbb{P} \otimes \cdots \otimes \mathbb{P}}_{N\text{-times}}$ and $\mathbb{Q}^N = \underbrace{\mathbb{Q} \otimes \cdots \otimes \mathbb{Q}}_{N\text{-times}}$ are two product probability measures. Then*

$$d_H^2(\mathbb{P}^N, \mathbb{Q}^N) = 1 - (1 - d_H^2(\mathbb{P}, \mathbb{Q}))^N.$$

(iii) *Suppose $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1^2, \sigma_2^2 \in]0, +\infty[$. Then*

$$d_H^2(\mathcal{N}(\mu_1, \sigma_1^2), \mathcal{N}(\mu_2, \sigma_2^2)) = 1 - \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}} e^{-\frac{(\mu_1 - \mu_2)^2}{4(\sigma_1^2 + \sigma_2^2)}}.$$

Proof. For the item (i), see Proposition 2.2 of Barrera, Lachaud and Ycart (2006). For the item (ii), see Proposition 2.3 of Barrera, Lachaud and Ycart (2006). For the item (iii), see relation (5) of Lachaud (2005). □

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