# A Mermin–Wagner theorem on Lorentzian triangulations with quantum spins

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**Abstract.** We consider infinite random causal Lorentzian triangulations emerging in quantum gravity for critical values of parameters. With each vertex of the triangulation we associate a Hilbert space representing a bosonic particle moving in accordance with the standard laws of Quantum Mechanics. The particles interact via two-body potentials decaying with the graph distance. A Mermin–Wagner type theorem is proven for infinite-volume reduced density matrices related to solutions to DLR equations in the Feynman–Kac (FK) representation.

## 1 Introduction

In this paper, we prove a Mermin–Wagner (MW) type theorem (cf. Mermin and Wagner (1966), Dobrushin and Shlosman (1975), Bonato et al. (1982), Ioffe et al. (2002)) for a system of quantum bosonic particles on an (infinite) random graph represented by a *causal dynamical Lorentzian triangulation* (in brief: CDLT). The CDLTs arise naturally when physicists attempt to define a fundamental path integral in quantum gravity. The reader is referred to Loll et al. (2006) for a review of related publications and to Malyshev et al. (2001) for a rigorous mathematical background behind the model of CDLTs. More precisely, we analyze a quantum system on a random 2D graph $T$ generated by a natural "uniform" measure on the CDLTs corresponding to a "critical" regime (see below).

In modern language, the spirit of the quantum MW theorem is that in a 2D lattice model (more generally, for a model on a countable bi-dimensional graph), any infinite-volume Gibbs state (regardless of whether it is unique or not) is invariant under the action of a Lie group $\mathsf{G}$ provided that the ingredients of local Hamiltonians are $\mathsf{G}$-invariant; see Mermin and Wagner (1966). These ingredients include the kinetic energy part, the single-site potential and the interaction potential. The mathematical constructions used for the proof of this theorem require a certain control over these ingredients: compactness of a configuration space associated

with a single vertex of the lattice (or a more general bi-dimensional graph), a certain smoothness of the interaction potential (or its essential part), sufficiently fast decay of the interaction potential for large distances on the lattice (or on the graph), "regularity" of the lattice (graph) geometry. In particular, the bi-dimensionality of the underlying graph is guaranteed by equation (4.1).

A principal question that needs a careful consideration is about the definition of a quantum Gibbs state in an infinite volume. For the so-called quantum spin systems, with a finite-dimensional phase space of a single spin (and consequently, with bounded local Hamiltonians), such a definition is given within the theory of KMS (Kubo–Martin–Schwinger) states; see, for example, Bratteli and Robinson (2002b). A version of the MW theorem for a model of this type on a 2D square lattice was established in Fröhlich and Pfister (1981), Pfister (1981) and has been generalised in subsequent publications. The KMS-based results (under suitable aforementioned assumptions) can be extended to the model of classical spins on a random graph of the type considered in the present paper; cf. Kelbert et al. (2013). However, the KMS-theory is not efficient for the case of interacting quantum particles where the one-particle kinetic energy operator is equal to $-\Delta/2$ ($\Delta$ stands for a Laplacian on a compact manifold). This is a standard quantum-mechanical model, and the fact that the concept of infinite-volume Gibbs state did not receive so far a properly working definition for such a system was perceived as a regrettable hindrance.

In this paper, we adopt the definition of an infinite-volume Gibbs state (more precisely, of an infinite-volume reduced density matrix (RDM, for short)) from the papers of Kelbert and Suhov (cf. Kelbert and Suhov (2013a, 2013b)). Similar methodologies have been developed in a number of earlier references; see, e.g., Albeverio et al. (2009) and the bibliography therein (viz., Klein and Landau (1981)). In papers of Kelbert and Suhov (2013a, 2013b) a class of so-called FK-DLR states has been introduced, and an MW theorem was established for quantum systems on a bi-dimensional graph $T \sim (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \mathcal{V}(T)$ is the set of vertices and $\mathcal{E} = \mathcal{E}(T)$ the set of edges. As was said above, in the present paper we deal with a random graph $T$ (a CDLT for critical values of parameters). After checking that a typical realization $T = T_\infty$ of the random CDLT satifies certain required properties, we use the constructions from Kelbert and Suhov (2013a, 2013b) (going back to Fröhlich and Pfister (1981), Pfister (1981)) and prove the main results of the paper (see Theorems 2.1, 2.2 and Theorems 3.1, 3.2).

It is appropriate to say that, although we use here some methodology developed in Kelbert and Suhov (2013a, 2013b), the current work deals with a situation different from the above papers, and a number of issues here require specific technical tools. On the other hand, the present paper can be considered as a continuation of Kelbert et al. (2013) where a MW theorem was established for a classical prototype of a quantum system treated here. We believe that models of quantum gravity where various types of quantum matter are incorporated form a natural direction

of research, interesting from both physical and mathematical points of view. Extension of results from Kelbert et al. (2013) to the case of quantum systems is a novel element of the present paper.

## 2 Basic definitions

### 2.1 Lorentzian triangulations in a critical phase

The graph under consideration is a triangulation $T$ of a cylinder $\mathsf{C} = S \times [0, \infty)$ with the base $S$, which is a unit circle in $\mathbb{R}^2$. Physically, $\mathsf{C}$ represents a $(1 + 1)$-dimensional space–time complex. (Pictorially, in the critical case, the graph $T$ develops like a cone, getting "wider" further from the base.) Geometrically, $\mathsf{C}$ can be visualized as a complex plane $\mathbb{C}$ with a family of concentric circles $\{z : |z| = n\}$: here the origin $z = 0$ is treated as a "circle" of infinitesimal radius. The following properties of $T$ are assumed: each triangle belongs to some strip $S \times [\ell, \ell + 1]$, $\ell = 0, 1, 2, \ldots$ such that either (i) two vertices lie on $S \times \{\ell\}$ and one on $S \times \{\ell+1\}$ or (ii) two vertices lie on $S \times \{\ell + 1\}$ and one on $S \times \{\ell\}$ and exactly one edge of triangle is an arc of a circle $S \times \{\ell\}$ in the case (i), or $S \times \{\ell + 1\}$ in the case (ii). In case (i) we speak of an upward triangle, or simply up-triangle, and in case (ii) of an downward triangle, or down-triangle. This includes also a "degenerate" picture where two vertices of a triangle coincide, and the corresponding edge forms the circle. For $\ell = 0$ it is requested that the graph under consideration generates a degenerate picture (i.e., the graph has a single up-triangle in the strip $S \times [0, 1]$, see Figure 1(a). This particular triangle is called the root triangle, and its side represented by the edge along the boundary $S \times \{0\}$ of the strip is called the root edge. Moreover, the (double) vertex lying on $S \times \{0\}$ is called the root vertex.



(a)                                                                 (b)
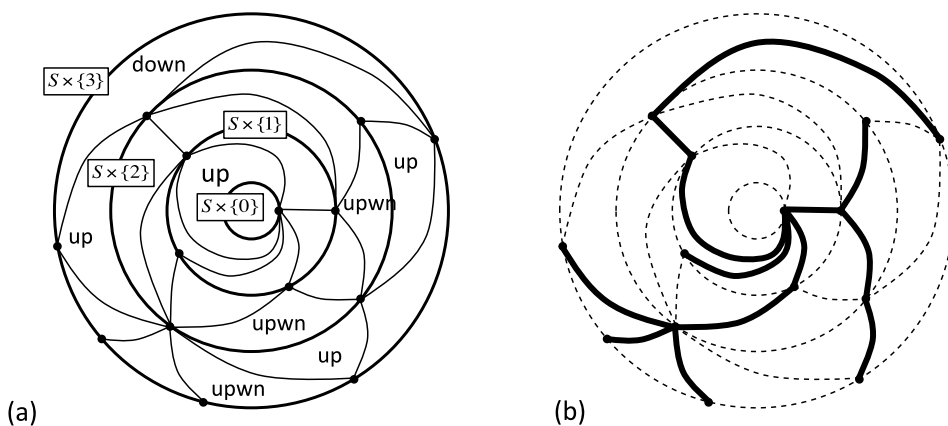
**Figure 1**    (a) *An example of Lorentzian triangulation. Some* (*not all*) *up- and down-triangles are marked.* (b) *The triangulation $T$ is parametrized by the spanning tree $\mathbb{T}$, which is represented by bold lines.*

Finally, we consider graphs modulo an equivalence (that is, up to a homeomorphism of $\mathsf{C}$ preserving all circles $S \times \{\ell\}$, $\ell = 0, 1, 2, \ldots$).

**Definition 1.** A rooted infinite CDLT is defined as an equivalence class of (countable) graphs with the above-listed properties, under the above equivalence. Depending on the context, we use the notation $T = T_\infty$ for a representative or for the whole equivalence class of graphs involved, and speak of the vertex set $\mathcal{V} = \mathcal{V}(T)$ and the edge set $\mathcal{E} = \mathcal{E}(T)$ in the same fashion.

A similar definition can be used to introduce a rooted CDLT on a cylinder $\mathsf{C}_N = S \times [0, N]$ (a rooted CDLT of height $N$). The corresponding notation is $T_N \sim (\mathcal{V}(T_N), \mathcal{E}(T_N))$ or even $T_N \sim (\mathcal{V}_N, \mathcal{E}_N)$. As in Kelbert et al. (2013), we denote by $\mathbb{LT}_N$ and $\mathbb{LT}_\infty$ the sets of CDLTs on $\mathsf{C}_N$ and $\mathsf{C}$, respectively.

To introduce a probability distribution on $\mathbb{LT}_N$ and ultimately on $\mathbb{LT}_\infty$, we use a special 1–1 correspondence between the rooted CDLTs and rooted trees (that is, graphs without cycles and with distinguished vertices). Namely, we extract a subgraph in $T$ by selecting, for each vertex $v \in T$, the leftmost edge going from $v$ downwards and discarding all other edges going from $v$ horizontally or downwards, see Figure 1(b). The graph $\mathbb{T} \subset T$ thus obtained is a spanning tree of $T$, cf. Durhuus et al. (2007) and Malyshev et al. (2001). Moreover, if one indicates, for each vertex of $\mathbb{T}$, its height in $T$ then $T$ can be completely reconstructed when we know $\mathbb{T}$. We call the correspondence $\mathbb{T} \leftrightarrow T$ the *tree parametrization* of the CDLT. It determines a one-to-one bijection $\mathsf{m}$ between the set $\mathbb{LT}_\infty$ and the set of infinite rooted trees $\mathcal{T}_\infty$:

$$\mathsf{m} : \mathcal{T}_\infty \leftrightarrow \mathbb{LT}_\infty.$$

We will use the same symbol $\mathsf{m}$ for the bijection $\mathcal{T}_N \leftrightarrow \mathbb{LT}_N$ where $\mathcal{T}_N$ is the set of all rooted planar trees of height $N$.

By virtue of the tree-parametrization, we will specify a probability distribution on CDLTs by specifying a distribution defined on trees. More precisely, suppose $\mathsf{P}_N^{\text{tree}}$ is a probability measure on $\mathcal{T}_N$. Then the measure $\mathsf{P}_N^{\text{LT}}$ on $\mathbb{LT}_N$ is determined by

$$\mathsf{P}_N^{\text{LT}}\big(\mathsf{m}(\mathbb{T})\big) = \mathsf{P}_N^{\text{tree}}(\mathbb{T}) \qquad \forall \mathbb{T} \in \mathcal{T}_N.$$

Conversely, let $\mathsf{P}^{\text{tree}}$ be a probability distribution on $\mathcal{T}_\infty$. Then the distribution $\mathsf{P}^{\text{LT}}$ on $\mathbb{LT}_\infty$ is given by

$$\mathsf{P}^{\text{LT}}\big(\mathsf{m}(A)\big) = \mathsf{P}^{\text{tree}}(A) \qquad \forall A \in \mathcal{F}(\mathcal{T}_\infty),$$

where $\mathcal{F}(\mathcal{T}_\infty)$ is standard $\sigma$-algebra generated by cylinder sets. In future we omit indices in the notation for the distribution $\mathsf{P}$.

To construct a critical CDLT model we define the corresponding measure on $\mathcal{T}_\infty$ related to a critical Galton–Watson (GW) process $\xi$. For this aim, we set $\mu =$

$\{p_k\}$ to be an offspring distribution on $\mathbb{N} = \{0, 1, \ldots\}$ with mean 1, and define the critical Galton–Watson (GW) branching process $\xi$. Conditional on the event of non-extinction, the GW process becomes the so-called size-biased (SB) process $\hat{\xi} = \{k_n, n = 0, 1, 2, \ldots\}$. In our context, $k_n$ yields the number of vertices on the circle $S \times \{n\}$ in the random infinite rooted CDLT. The reader can consult Lyons et al. (1995) for the formal background for SB branching processes.

In particular, the distribution of an SB process $\hat{\xi}$ is concentrated on the subset $\mathcal{S}$ of $\mathcal{T}_\infty$ formed by the so-called single-spine trees. A single-spine tree consists of a single infinite linear chain $s_0, s_1, \ldots$ called the spine, to each vertex $s_j$ of which there is attached a finite random tree with its root at $s_j$. (Here $s_0$ is the root vertex of the whole tree.) Furthermore, the generating function for the branching number $\nu$ at each vertex $s_j$ is $f'(x)$ where $f(x)$ is the generating function of the initial offspring distribution $\mu$. Moreover, the individual branches are independently and identically distributed in accordance with the original critical GW process Lyons et al. (1995).

Let $\sigma^2$ stand for the variance of the offspring distribution $\nu$. Then

$$\mathsf{E}(k_n \mid k_{n-1}) = k_{n-1} + \sigma^2. \tag{2.1}$$

In fact, let $\nu = \{\tilde{p}_k\}$ be the SB offspring distribution with $\tilde{p}_k = kp_k$ (recall, in the critical case under consideration, the sum $\sum_k kp_k = 1$). Then the distribution for $k_n$ conditioned upon the value $k_{n-1}$ is identified as follows (cf. Lyons et al. (1995)). We choose at random one particle among $k_{n-1}$ particles and generate the number of its descendants according to the distribution $\nu$ with mean $\sigma^2 + 1$. According to (2.1), the number of descendants for the other $k_{n-1} - 1$ particles is generated, independently, by the distribution $\mu$.

Throughout the paper, we assume that the offspring distribution $\mu$ has the mean 1 with finite second moment. Let $\mathsf{P}$ be the corresponding SB Galton–Watson tree distribution.

## 2.2 The local quantum Hamiltonians on CDLTs

Let $M = \mathbb{R}^d/\mathbb{Z}^d$ be a unit $d$-dimensional torus with flat metric and induced volume $v$. A basic quantum model uses the Hilbert space $\mathcal{H} = \mathrm{L}_2(M, v)$ as the phase space of a single quantum particle. The single-particle Hamiltonian $H$ acts in $\mathcal{H}$ as the sum:

$$(H\phi)(x) = -\frac{1}{2}(\Delta\phi)(x) + U(x)\phi(x), \qquad x \in M, \phi \in \mathcal{H}. \tag{2.2}$$

Here $\Delta$ is the Laplacian on $M$ and the function $U : x \in M \mapsto \mathbb{R}$ gives an external potential. Under the assumptions upon $U$ adopted in this paper (see equation (2.10)), $H$ is a self-adjoint operator bounded from below and with a discrete spectrum such that $\forall \beta > 0$, $\exp[-\beta H]$ is a (positive definite) trace class operator.

Given a CDLT $T$ we use the notation $T_N$ for the subgraph in $T$ with the set of vertices $\mathcal{V}_N = \mathcal{V}(T_N)$ of the form $\mathcal{V}_N = \mathcal{V}(T) \cap \{S \times \{0, \ldots, N\}\}$ and the set of

edges $\mathcal{E}(T_N) = \mathcal{E}(T) \cap (\mathcal{V}_N \times \mathcal{V}_N)$. The phase space of a (bosonic) quantum system in $\mathcal{V}_N$ is the tensor product $\mathcal{H}_N = \mathcal{H}^{\otimes \mathcal{V}_N}$; an element $\phi_N \in \mathcal{H}_N$ is a function

$$\phi_N : \mathbf{x}(N) = \mathbf{x}_{\mathcal{V}_N} = \{x(i), i \in \mathcal{V}_N\} \in M^{\times \mathcal{V}_N} \mapsto \mathbb{C} \qquad (2.3)$$

square-integrable in $d\mathbf{x}(N) = \prod_{i \in \mathcal{V}_N} \upsilon(dx(i))$. The Cartesian power $M^{\times \mathcal{V}_N}$ can be considered as the configuration space for the classical prototype of the quantum system in $\mathcal{V}_N$.

The local Hamiltonian $H_N$ of the system in $\mathcal{V}_N$ acts on functions $\phi \in \mathcal{H}^{\otimes \mathcal{V}_N}$: given $\mathbf{x}(N) = (x(j), j \in \mathcal{V}_N) \in M^{\times \mathcal{V}_N}$,

$$(H_N \phi)(\mathbf{x}(N))$$
$$= \left[ \sum_{i \in \mathcal{V}_N} H(i) + \sum_{j, j' \in \mathcal{V}_N \times \mathcal{V}_N} J(\mathrm{d}(j, j')) V(x(j), x(j')) \right] \phi(\mathbf{x}(N)), \qquad (2.4)$$

where $J(\mathrm{d}(j, j')) V(x(j), x(j'))$ represents the interaction between spins $x(j)$ and $x(j')$ at sites $j$ and $j'$. Next, $H(i)$ stands for the copy of operator $H$ acting on variable $x(i) \in M$ and $\mathrm{d}(j, j')$ for the graph distance from vertex $j$ to $j'$.

A more general concept is a Hamiltonian $H_{N|\mathbf{x}^c(N)}$ in the external field generated by an (infinite) configuration $\mathbf{x}^c(N) = \{\overline{x}(j'), j' \in \overline{\mathcal{V}}_N\} \in M^{\times \overline{\mathcal{V}}_N}$ where $\overline{\mathcal{V}}_N = \mathcal{V}(T) \setminus \mathcal{V}_N$. As before, operator $H_{N|\mathbf{x}^c(N)}$ acts in $\mathcal{H}_N$: given $\phi \in \mathcal{H}_N$ and $\mathbf{x}(N) = (x(j), j \in \mathcal{V}_N) \in M^{\times \mathcal{V}_N}$,

$$(H_{N|\mathbf{x}^c(N)} \phi)(\mathbf{x}(N))$$
$$= \left[ H_N + \sum_{(j, j') \in \mathcal{V}_N \times \overline{\mathcal{V}}_N} J(\mathrm{d}(j, j')) V(x(j), \overline{x}(j')) \right] \phi(\mathbf{x}(N)). \qquad (2.5)$$

Again, under assumptions upon $J$ and $V$ described in (2.10)–(2.11), $H_N$ and $H_{N|\mathbf{x}^c(N)}$ are self-adjoint operators bounded from below and with a discrete spectrum such that $\forall \beta > 0$, $G_{\beta, N} = \exp[-\beta H_N]$ and $G_{\beta, N|\mathbf{x}^c(N)} = \exp[-\beta H_{N|\mathbf{x}^c(N)}]$ are (positive definite) trace class operators.

The operators $G_{\beta, N}$ and $G_{\beta, N|\mathbf{x}^c(N)}$ are called the Gibbs operators (in volume $\mathcal{V}_N$ for the inverse temperature $\beta$ and, in the case of $G_{\beta, N|\mathbf{x}^c(N)}$, with the boundary condition $\mathbf{x}^c(N) = \mathbf{x}^c_{\overline{\mathcal{V}}_N}$). The traces

$$\Xi_{\beta, N} = \mathrm{tr}_{\mathcal{H}_N} G_{\beta, N} \quad \text{and} \quad \Xi_{\beta, N|\mathbf{x}^c(N)} = \mathrm{tr}_{\mathcal{H}_N} G_{\beta, N|\mathbf{x}^c(N)} \qquad (2.6)$$

give the corresponding partition functions. The normalized operators

$$R_{\beta, N} = \frac{1}{\Xi_{\beta, N}} G_{\beta, N} \quad \text{and} \quad R_{\beta, N|\mathbf{x}^c(N)} = \frac{1}{\Xi_{\beta, N|\mathbf{x}^c(N)}} G_{\beta, N|\mathbf{x}^c(N)} \qquad (2.7)$$

are called the density matrices (for the corresponding Gibbs ensembles); these are positive definite operators of trace 1. Given $n \in \{0, \ldots, N\}$, the partial traces

$$R^{(n)}_{\beta, N} = \mathrm{tr}_{\mathcal{H}_{N \setminus n}} R_{\beta, N} \quad \text{and} \quad R^{(n)}_{\beta, N|\mathbf{x}^c(N)} = \mathrm{tr}_{\mathcal{H}_{N \setminus n}} R_{\beta, N|\mathbf{x}^c(N)} \qquad (2.8)$$

yield positive definite operators $R_{\beta,N}^{(n)}$ and $R_{\beta,N|\mathbf{x}^c(N)}^{(n)}$ in $\mathcal{H}_n$, of trace 1. Here $\mathcal{H}_{N\backslash n}$ stands for the Hilbert space $\mathcal{H}^{\otimes(\mathcal{V}_N\backslash\mathcal{V}_n)}$. These operators are called the reduced density matrices (RDMs). Note the compatibility relation: $\forall 0 \leq n < n' < N$:

$$R_{\beta,N}^{(n)} = \text{tr}_{\mathcal{H}_{n'\backslash n}}\, R_{\beta,N}^{(n')} \quad \text{and} \quad R_{\beta,N|\mathbf{x}^c(N)}^{(n)} = \text{tr}_{\mathcal{H}_{n'\backslash n}}\, R_{\beta,N|\mathbf{x}^c(N)}^{(n')}. \tag{2.9}$$

## 2.3 Assumptions on the potentials. The group of symmetries

We suppose that the potential $U$ has continuous derivatives whereas $V$ has continuous first and second derivatives: $\forall x, x', x'' \in M$

$$|U(x)|, |\nabla_x U(x)| \leq \overline{U}, \tag{2.10}$$

$$|V(x', x'')|, |\nabla_{x'} V(x', x'')|, |\nabla_{x''} V(x', x'')|, |\nabla_{x'}\nabla_{x''} V(x', x'')| \leq \overline{V}, \tag{2.11}$$

where $\overline{U}, \overline{V} \in (0, \infty)$ are constants.

Next, suppose that a $d' \times d$ matrix $A$ is given, of the row rank $d'$ where $d' \leq d$. We consider a $d'$-dimensional group $\mathsf{G}$ acting on $M$ and preserving the volume $v: (\mathsf{g}, x) \in \mathsf{G} \times M \mapsto \mathsf{g}x \in M$. More precisely, $\mathsf{g}$ is identified with a real $d'$-dimensional vector $\theta = (\theta_1, \ldots, \theta_{d'})$ and the action is given by

$$\mathsf{g}x = x + \theta A \,\text{mod}\, 1. \tag{2.12}$$

**Remark 2.1.** The group $\mathsf{G}$ can be compact (in which case $\mathsf{G}$ is a torus of dimension $d'$) or non-compact (then $\mathsf{G}$ is $\mathbb{R}^{d'}$).

We assume that the functions $U(x)$ and $V(x, x')$ are invariant with respect to the group $\mathsf{G}$: $\forall \mathsf{g} \in \mathsf{G}$ and $x, x' \in M$

$$U(\mathsf{g}x) = U(x), \qquad V(\mathsf{g}x, \mathsf{g}x') = V(x, x'). \tag{2.13}$$

Finally, we assume that the function $r \in (0, \infty) \mapsto J(r)$ in (2.4) and (2.5) is a bounded monotone decreasing function satisfying the condition

$$J(r) \leq \left(\frac{1}{r \ln r}\right)^3, \qquad r \geq 2. \tag{2.14}$$

These assumptions are in place throughout the paper. (We do not analyze the issue of necessity of condition (2.14).)

As usually, the action of the group $\mathsf{G}$ generates unitary operators in $\mathcal{H}$:

$$S(\mathsf{g})\phi(x) = \phi(\mathsf{g}^{-1}x), \qquad x \in M, \phi \in \mathcal{H}. \tag{2.15}$$

Let $S^{(N)}(\mathsf{g})$ be the tensor power of $S(\mathsf{g})$ which acts in $\mathcal{H}_N$: for any $\phi_N \in \mathcal{H}_N$

$$S^{(N)}(\mathsf{g})\phi_N(\mathbf{x}(N)) = \phi_N(\mathsf{g}^{-1}\mathbf{x}(N)), \tag{2.16}$$

where $\mathbf{x}(N) = \{x(i), i \in \mathcal{V}_N\} \in M^{\times\mathcal{V}_N}$ and $\mathsf{g}^{-1}\mathbf{x}(N) = \{\mathsf{g}^{-1}x(i), i \in \mathcal{V}_N\}$.

## 2.4 Limiting RDMs in an infinite volume

We are interested in the "thermodynamic" limit $N \to \infty$. In the absense of phase transitions, one would like to establish a convergence of the RDMs $R_{\beta,N}^{(n)}$ and $R_{\beta,N|\mathbf{x}^c(N)}^{(n)}$ to a limiting RDM in $\mathcal{H}_n$ as $N \to \infty$. A suitable form of convergence is in the trace norm in $\mathcal{H}_n$, guaranteeing that the limiting operator is positive-definite and has trace 1. When phase transitions are not excluded (which is the case under consideration), a more general question is whether the families $\{R_{\beta,N}^{(n)}\}$ and $\{R_{\beta,N|\mathbf{x}^c(N)}^{(n)}\}$ are compact. If we manage to check that $\{R_{\beta,N}^{(n)}\}$ and $\{R_{\beta,N|\mathbf{x}^c(N)}^{(n)}\}$ are compact families for any given $n$ then, invoking a diagonal process, we can consider a family of limiting RDMs $\{R_{\beta}^{(n)}, n = 0, 1, 2, \ldots\}$ (in the case of operators $R_{\beta,N|\mathbf{x}^c(N)}^{(n)}$ the limiting RDMs may depend on the choice of the boundary conditions $\mathbf{x}^c(N)$). The consistency property (2.9) will be inherited in the limit: $\forall 0 \le n < n' < N$,

$$R_{\beta}^{(n)} = \mathrm{tr}_{\overline{\mathcal{H}}_{n' \backslash n}} R_{\beta}^{(n')}. \tag{2.17}$$

A consistent family of RDMs $R_{\beta}^{(n)}$ defines a state of (i.e., a linear positive normalized functional on) the quasilocal C*-algebra constructed as the closure of the inductive limit of $\mathfrak{B}_N$ as $N \to \infty$ where $\mathfrak{B}_N$ is the C*-algebra of the bounded operators in $\mathcal{H}_N$, cf. Bratteli et al. (2002a). This motivates a study of properties of limiting RDM families $\{R_{\beta}^{(n)}\}$. Our results in this direction are summarised in Theorems 2.1 and 2.2.

**Theorem 2.1.** *Fix $\beta > 0$. For P-a.a. CDLT $T \in \mathcal{T}_{\infty}$, $\forall n = 0, 1, 2, \ldots$, the family of the RDMs $\{R_{\beta,N}^{(n)}, N = 1, 2, \ldots\}$ is compact in the trace norm in $\mathcal{H}_n$. Similarly, $\{R_{\beta,N|\mathbf{x}^c(N)}^{(n)}, N = 1, 2, \ldots\}$ is a compact family $\forall$ choice of the boundary conditions $\mathbf{x}^c(N)$.*

**Theorem 2.2.** *Let $R_{\beta}^{(n)}$ be any limiting-point operator for the family $\{R_{\beta,N|\mathbf{x}^c(N)}^{(n)}, N = 1, 2, \ldots\}$. Then, $\forall \mathsf{g} \in \mathsf{G}$, operator $S^{(n)}(\mathsf{g})$ commutes with $R_{\beta}^{(n)}$:*

$$R_{\beta}^{(n)} = S^{(n)}(\mathsf{g}) R_{\beta}^{(n)} \big(S^{(n)}(\mathsf{g})\big)^{-1}. \tag{2.18}$$

**Remark 2.2.** The statement of Theorem 2.2 is straightforward for the limit points $R_{\beta}^{(n)}$ of the family $\{R_{\beta,N}^{(n)}, N = 1, 2, \ldots\}$ but requires a proof for the family $\{R_{\beta,N|\mathbf{x}^c(N)}^{(n)}, N = 1, 2, \ldots\}$.

The main role in the proof of Theorems 2.1 and 2.2 is played by the Feynman–Kac (FK) representation for the RDMs $R_{\beta,N}^{(n)}$ and $R_{\beta,N|\mathbf{x}^c(N)}^{(n)}$ and their limiting counterparts $R_{\beta}^{(n)}$. This representation is discussed in the next section.

## 3 The FK ensembles of paths and loops

### 3.1 The FK representation for the Gibbs operators

The Gibbs operators $G_{\beta,N}$ and $G_{\beta,N|\mathbf{x}^c(N)}$ act as integral operators, with kernels $K_{\beta,N}$ and $K_{\beta,N|\mathbf{x}^c(N)}$

$$(G_{\beta,N}\phi_N)(\mathbf{x}(N)) = \int_{M \times \mathcal{V}_N} K_{\beta,N}(\mathbf{x}(N), \mathbf{y}(N))\phi_N(\mathbf{y}(N))\,\mathrm{d}\mathbf{y}(N),$$

$$(G_{\beta,N|\mathbf{x}^c(N)}\phi_N)(\mathbf{x}(N)) = \int_{M \times \mathcal{V}_N} K_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(N), \mathbf{y}(N))\phi_N(\mathbf{y}(N))\,\mathrm{d}\mathbf{y}(N). \tag{3.1}$$

Here $\mathbf{y}(N) = \{y(i) : i \in \mathcal{V}_N\}$ and we use a shorthand notation $\mathrm{d}\mathbf{y}(N) = \prod_{i \in \mathcal{V}_N} v(\mathrm{d}y(i))$.

Further, the kernels $K_{\beta,N}$ and $K_{\beta,N|\mathbf{x}^c(N)}$ admit the FK-representations summarized in Lemma 3.1. The proof of this lemma follows the standard lines and is omitted. The reader can confer Ginibre (1973) for details.

Given points $x, y \in M$, let $W_{x,y}^\beta$ denote the space of continuous paths $\overline{\omega} = \overline{\omega}_{x,y} : \tau \in [0, \beta] \mapsto \overline{\omega}(\tau) \in M$, of time-length $\beta$, beginning at $x$ and terminating at $y$. Next, let $\mathbb{P}_{x,y}^\beta$ stand for the (unnormalized) Wiener measure on $W_{x,y}^\beta$, with $\mathbb{P}_{x,y}^\beta(W_{x,y}^\beta) = p^\beta(x, y)$ where $p^\beta(x, y)$ is the value of the transition density from $x$ to $y$ in time $\beta$. Furthermore, given particle configurations $\mathbf{x}(N) = \{x(i)\}$, $\mathbf{y}(N) = \{y(i)\} \in M^{\times \mathcal{V}_N}$, we set:

$$W_{\mathbf{x}(N),\mathbf{y}(N)}^\beta = \bigtimes_{i \in \mathcal{V}_N} W_{x(i),y(i)}^\beta, \qquad \mathbb{P}_{\mathbf{x}(N),\mathbf{y}(N)}^\beta = \bigtimes_{i \in \mathcal{V}_N} \mathbb{P}_{x(i),y(i)}^\beta. \tag{3.2}$$

In other words, an element $\overline{\boldsymbol{\omega}}(N) = \overline{\omega}_{\mathbf{x}(N),\mathbf{y}(N)} \in W_{\mathbf{x}(N),\mathbf{y}(N)}^\beta$ is represented by a collection of paths $\{\overline{\omega}_{x(i),y(i)}\}$ where $\overline{\omega}_{x(i),y(i)} \in W_{x(i),y(i)}^\beta$. We call such a collection a path configuration over $\mathcal{V}_N$. Moreover, under measure $\mathbb{P}_{\mathbf{x}(N),\mathbf{y}(N)}^\beta$ the paths $\overline{\omega}_{x(i),y(i)}$ are independent and each of them follows its own marginal measure $\mathbb{P}_{x(i),y(i)}^\beta$.

Further, we need to introduce functionals $h(\overline{\boldsymbol{\omega}}(N))$ and $h(\overline{\boldsymbol{\omega}}(N)|\mathbf{x}^c(N))$ describing an integral energy of the path configuration $\overline{\omega}(N)$ and its energy in the potential field generated by $\mathbf{x}^c(N)$:

$$h(\overline{\boldsymbol{\omega}}(N)) = \sum_{(i,i') \in \mathcal{V}_N \times \mathcal{V}_N} h^{i,i'}(\overline{\omega}(i), \overline{\omega}(i')), \tag{3.3}$$

where $h^{i,i'}(\overline{\omega}(i), \overline{\omega}(i'))$ represents an integral along trajectories $\overline{\omega}(i)$ and $\overline{\omega}(i')$. Namely, for $i = i'$ and $\overline{\omega} \in W_{x(i),y(i)}^\beta$:

$$h^{i,i}(\overline{\omega}, \overline{\omega}) = \int_0^\beta \mathrm{d}\tau\, U(\overline{\omega}(\tau)). \tag{3.4}$$

and for $i \neq i'$ and $\overline{\omega} \in W^{\beta}_{x(i), y(i)}$, $\overline{\omega}' \in W^{\beta}_{x(i'), y(i')}$:

$$h^{i,i'}(\overline{\omega}, \overline{\omega}') = J(\mathrm{d}(i, i')) \int_0^{\beta} \mathrm{d}\tau \, V(\overline{\omega}(\tau), \overline{\omega}'(\tau)). \tag{3.5}$$

Pictorially, $h^{i,i}(\overline{\omega}(i), \overline{\omega}(i))$ yields an energy of the path $\overline{\omega}(i)$ in the external field generated by the potential $U$ and $h^{i,i'}(\overline{\omega}(i), \overline{\omega}(i'))$ the energy of interaction between paths $\overline{\omega}(i)$ and $\overline{\omega}(i')$. Accordingly, $h(\overline{\boldsymbol{\omega}}(N))$ gives a full potential energy of the path configuration $\overline{\boldsymbol{\omega}}(N)$.

Similarly,

$$h(\overline{\boldsymbol{\omega}}(N)|\mathbf{x}^{\mathrm{c}}(N)) = h(\overline{\boldsymbol{\omega}}(N)) + \sum_{i \in \mathcal{V}_N, i' \in \overline{\mathcal{V}}_N} h^{i,i'}(\overline{\omega}(i), \mathbf{x}^{\mathrm{c}}(i')), \tag{3.6}$$

where $h^{i,i'}(\overline{\omega}(i), \mathbf{x}^{\mathrm{c}}(i')) = J(\mathrm{d}(i, i')) \int_0^{\beta} \mathrm{d}\tau \, V(\overline{\omega}(i, \tau), \mathbf{x}^{\mathrm{c}}(i'))$.

**Lemma 3.1.** *The integral kernels $K_{\beta, N}(\mathbf{x}(N), \mathbf{y}(N))$ and $K_{\beta, N|\mathbf{x}^{\mathrm{c}}(N)}(\mathbf{x}(N), \mathbf{y}(N))$ are given by*:

$$K_{\beta, N}(\mathbf{x}(N), \mathbf{y}(N)) = \int_{W^{\beta}_{\mathbf{x}(N), \mathbf{y}(N)}} \mathbb{P}^{\beta}_{\mathbf{x}(N), \mathbf{y}(N)}(\mathrm{d}\overline{\boldsymbol{\omega}}(N)) \exp[-h(\overline{\boldsymbol{\omega}}(N))] \tag{3.7}$$

*and*

$$K_{\beta, N|\mathbf{x}^{\mathrm{c}}(N)}(\mathbf{x}(N), \mathbf{y}(N))$$
$$= \int_{W^{\beta}_{\mathbf{x}(N), \mathbf{y}(N)}} \mathbb{P}^{\beta}_{\mathbf{x}(N), \mathbf{y}(N)}(\mathrm{d}\overline{\boldsymbol{\omega}}(N)) \exp[-h(\overline{\boldsymbol{\omega}}(N)|\mathbf{x}^{\mathrm{c}}(N))]. \tag{3.8}$$

## 3.2 The FK representation for the partition functions and RDMs

Lemma 3.1 implies a working representation for the partition functions $\Xi_{\beta, N}$ and $\Xi_{\beta, N|\mathbf{x}^{\mathrm{c}}(N)}$ (see (2.6)). More precisely, a key ingredient in the corresponding formulas will be the space $W^{\beta}_{x,x} = W^{\beta}_x$ of closed paths (starting and ending up at the same marked point $x \in M$); we will employ the term "loop" to make a distinction with a general case. Accordingly, the notation $\mathbb{P}^{\beta}_{x,x} = \mathbb{P}^{\beta}_x$ will be in place here. Note that measure $\mathbb{P}^{\beta}_x$ in essence does not depend on the choice of the point $x \in M$. Furthermore, the notation $\omega = \omega_x \in W^{\beta}_x$ will be used for a loop with the marked initial/end point $x$, omitting the bar in the previous symbol $\overline{\omega}$. Next, we set:

$$W^{\beta}_{\mathbf{x}(N)} = \bigtimes_{i \in \mathcal{V}_N} W^{\beta}_{x(i)}, \qquad \mathbb{P}^{\beta}_{\mathbf{x}(N)} = \bigtimes_{i \in \mathcal{V}_N} \mathbb{P}^{\beta}_{x(i)}. \tag{3.9}$$

An element $\boldsymbol{\omega}(N) \in W^{\beta}_{\mathbf{x}(N)}$ is represented by a collection of loops $\{\omega_{x(i)}\}$ where $\omega_{x(i)} \in W^{\beta}_{x(i)}$; such a collection is called a loop configuration over $\mathcal{V}_N$. As before, under measure $\mathbb{P}^{\beta}_{\mathbf{x}(N)}$ the loops $\omega_{x(i)}$ are independent and each of them follows its own marginal measure $\mathbb{P}^{\beta}_{x(i)}$.

At this point we apply the Mercer theorem guaranteeing that the traces $\mathrm{tr}_{\mathcal{H}_N} G_{\beta,N}$ and $\Xi_{\beta,N|\mathbf{x}^c(N)} = \mathrm{tr}_{\mathcal{H}_N} G_{\beta,N|\mathbf{x}^c(N)}$ are given by the integrals of the corresponding kernels $K_{\beta,N}(\mathbf{x}(N), \mathbf{y}(N))$ and $K_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(N), \mathbf{y}(N))$ along the diagonal $\mathbf{x}(N) = \mathbf{y}(N)$. This leads to Lemma 3.2 below.

Let us denote:

$$\int \mathrm{d}\boldsymbol{\omega}(N) := \int_{M \times \mathcal{V}_N} \mathrm{d}\mathbf{x}(N) \int_{W^\beta_{\mathbf{x}(N)}} \mathbb{P}^\beta_{\mathbf{x}(N)}(\mathrm{d}\boldsymbol{\omega}(N)). \tag{3.10}$$

**Lemma 3.2.** *The partition functions* $\Xi_{\beta,N}$ *and* $\Xi_{\beta,N|\mathbf{x}^c(N)}$ *are given by*:

$$\Xi_{\beta,N} = \int \mathrm{d}\boldsymbol{\omega}(N) \exp[-h(\boldsymbol{\omega}(N))] \tag{3.11}$$

*and*

$$\Xi_{\beta,N|\mathbf{x}^c(N)} = \int \mathrm{d}\boldsymbol{\omega}(N) \exp[-h(\boldsymbol{\omega}(N)|\mathbf{x}^c(N))]. \tag{3.12}$$

Let us now turn to the RDMs $R^{(n)}_{\beta,N}$ and $R^{(n)}_{\beta,N|\mathbf{x}^c(N)}$. These operators are again given by their integral kernels:

$$\left(R^{(n)}_{\beta,N}\phi_n\right)(\mathbf{x}(n)) = \int_{M \times \mathcal{V}_n} F^{(n)}_{\beta,N}(\mathbf{x}(n), \mathbf{y}(n))\phi_n(\mathbf{y}(n))\, \mathrm{d}\mathbf{y}(n),$$

$$\left(R^{(n)}_{\beta,N|\mathbf{x}^c(N)}\phi_n\right)(\mathbf{x}(n)) = \int_{M \times \mathcal{V}_n} F^{(n)}_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(n), \mathbf{y}(n))\phi_n(\mathbf{y}(n))\, \mathrm{d}\mathbf{y}(n). \tag{3.13}$$

Next, $F^{(n)}_{\beta,N}(\mathbf{x}(n), \mathbf{y}(n))$ and $F^{(n)}_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(n), \mathbf{y}(n))$ are called reduced density matrix kernels (RDMKs). They can be written in the form

$$F^{(n)}_{\beta,N}(\mathbf{x}(n), \mathbf{y}(n)) = \frac{\Xi^{(n)}_{\beta,N}(\mathbf{x}(n), \mathbf{y}(n))}{\Xi_{\beta,N}},$$

$$F^{(n)}_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(n), \mathbf{y}(n)) = \frac{\Xi^{(n)}_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(n), \mathbf{y}(n))}{\Xi_{\beta,N|\mathbf{x}^c(N)}}, \tag{3.14}$$

where quantities $\Xi^{(n)}_{\beta,N}(\mathbf{x}(n), \mathbf{y}(n))$ and $\Xi^{(n)}_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(n), \mathbf{y}(n))$ admit representations similar to (3.11) and (3.12), see Lemma 3.3.

We will use a notation similar to (3.10):

$$\int \mathrm{d}\boldsymbol{\omega}(N \setminus n) := \int_{M \times \mathcal{V}_N \setminus \mathcal{V}_n} \mathrm{d}\mathbf{x}(N \setminus n) \int_{W^\beta_{\mathbf{x}(N \setminus n)}} \mathbb{P}^\beta_{\mathbf{x}(N \setminus n)}(\mathrm{d}\boldsymbol{\omega}(N \setminus n)), \tag{3.15}$$

where $\mathbf{x}(N \setminus n)$ stands for a particle configuration $\{x(j), j \in \mathcal{V}_N \setminus \mathcal{V}_n\}$ and $\boldsymbol{\omega}(N \setminus n)$ for the loop configuration $\{\omega(j), j \in \mathcal{V}_N \setminus \mathcal{V}_n\}$. Symbol $\vee$ will be used for concatenation of particle configurations and for concatenation of path and loop configurations (originally defined over disjoint sets). Accordingly, for a path configuration

$\overline{\omega}(n) \in W^{\beta}_{\mathbf{x}(n),\mathbf{y}(n)}$ over $\mathcal{V}_n$ and a loop configuration $\omega(N \setminus n)$ over $\mathcal{V}_N \setminus \mathcal{V}_n$, the energies $h(\overline{\omega}(n) \vee \omega(N \setminus n))$ and $h(\overline{\omega}(n) \vee \omega(N \setminus n)|\mathbf{x}^c(N))$ are defined as in (3.2)–(3.5).

**Lemma 3.3.** *The numerators* $\Xi^{(n)}_{\beta,N}(\mathbf{x}(n), \mathbf{y}(n))$ *and* $\Xi^{(n)}_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(n), \mathbf{y}(n))$ *are given by*:

$$\Xi^{(n)}_{\beta,N}(\mathbf{x}(n), \mathbf{y}(n)) = \int_{W^{\beta}_{\mathbf{x}(n),\mathbf{y}(n)}} \mathbb{P}^{\beta}_{\mathbf{x}(n),\mathbf{y}(n)}(d\overline{\omega}(n))$$
$$\times \int d\omega(N \setminus n) \exp[-h(\overline{\omega}(n) \vee \omega(N \setminus n))] \tag{3.16}$$

*and*

$$\Xi^{(n)}_{\beta,N|\mathbf{x}^c(N)}(\mathbf{x}(n), \mathbf{y}(n)) = \int_{W^{\beta}_{\mathbf{x}(n),\mathbf{y}(n)}} \mathbb{P}^{\beta}_{\mathbf{x}(n),\mathbf{y}(n)}(d\overline{\omega}(n))$$
$$\times \int d\omega(N \setminus n) \exp[-h(\overline{\omega}(n) \vee \omega(N \setminus n)|\mathbf{x}^c(N))]. \tag{3.17}$$

The proof of Lemma 3.3 consists in translating the partial traces into the integrals of the kernels $F^{(n)}_{\beta,N}$ and $F^{(n)}_{\beta,N|\mathbf{x}^c(N)}$ of the operators $R^{(n)}_{\beta,N}$ and $R^{(n)}_{\beta,N|\mathbf{x}^c(N)}$. We omit it from the paper.

### 3.3 The FK-DLR equations

The representations (3.11)–(3.12) suggest introducing probability distributions $\mu_N$ and $\mu_{N|\mathbf{x}^c(N)}$ on loop configurations $\omega_N$, with the densities (the Radon–Nikodym derivatives)

$$p_N(\omega(N)) := \frac{\mu_N(d\omega(N))}{d\omega(N)} = \frac{\exp[-h(\omega(N))]}{\Xi_{\beta,N}},$$
$$p_{N|\mathbf{x}^c(N)}(\omega(N)) := \frac{\mu_{N|\mathbf{x}^c(N)}(d\omega(N))}{d\omega(N)} = \frac{\exp[-h(\omega(N)|\mathbf{x}^c(N))]}{\Xi_{\beta,N|\mathbf{x}^c(N)}}. \tag{3.18}$$

A crucial property is that the measures $\mu_N$ and $\mu_{N|\mathbf{x}^c(N)}$ satisfy DLR (Dobrushin–Lanford–Ruelle)-type equations. Namely, let $p^{(n)}_N(\omega(n)|\omega(N \setminus n))$ and $p^{(n)}_{N|\mathbf{x}^c(N)}(\omega(n)|\omega(N \setminus n))$ stand for the conditional densities generated by $\mu_N$ and $\mu_{N|\mathbf{x}^c(N)}$, respectively, for the loop configuration $\omega(n)$ over $\mathcal{V}_n$ given a loop configuration $\omega(N \setminus n)$ over $\mathcal{V}_N \setminus \mathcal{V}_n$. Then

$$p^{(n)}_N(\omega(n)|\omega(N \setminus n)) := \frac{\mu_N(d\omega(n)|\omega(N \setminus n))}{d\omega(N)}$$
$$= \frac{\exp[-h(\omega(n)|\omega(N \setminus n))]}{\Xi_{\beta,n}(\omega(N \setminus n))}, \tag{3.19}$$

$$p_{N|\mathbf{x}^c(n)}^{(n)}\big(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n)\big) := \frac{\mu_{N|\mathbf{x}^c(N)}(\mathrm{d}\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n))}{\mathrm{d}\boldsymbol{\omega}(N)}$$

$$= \frac{\exp[-h(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^c(N))]}{\Xi_{\beta,n|\mathbf{x}^c(N)}(\boldsymbol{\omega}(N\setminus n))}.$$

Here $h(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n))$ and $h(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^c(N))$ stand for "conditional" energies and $\Xi_{\beta,n}(\boldsymbol{\omega}(N\setminus n))$ and $\Xi_{\beta,n|\mathbf{x}^c(N)}(\boldsymbol{\omega}(N\setminus n))$ for "conditional" partition functions:

$$h\big(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n)\big) = h\big(\boldsymbol{\omega}(n)\vee\boldsymbol{\omega}(N\setminus n)\big) - h\big(\boldsymbol{\omega}(N\setminus n)\big), \quad (3.20)$$

$$h\big(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^c(N)\big) = h\big(\boldsymbol{\omega}(n)\vee\boldsymbol{\omega}(N\setminus n)|\mathbf{x}^c(N)\big)$$
$$- h\big(\boldsymbol{\omega}(N\setminus n)|\mathbf{x}^c(N)\big), \quad (3.21)$$

$$\Xi_{\beta,n}\big(\boldsymbol{\omega}(N\setminus n)\big) = \int \mathrm{d}\boldsymbol{\omega}(n)\exp\big[-h\big(\boldsymbol{\omega}(n)\big)|\boldsymbol{\omega}(N\setminus n)\big] \quad (3.22)$$

and

$$\Xi_{\beta,n|\mathbf{x}^c(N)}\big(\boldsymbol{\omega}(N\setminus n)\big) = \int \mathrm{d}\boldsymbol{\omega}(n)\exp\big[-h\big(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^c(N)\big)\big]. \quad (3.23)$$

We call equation (3.19) the FK-DLR equation in volume $\mathcal{V}_N$.

Concluding this section, we give an expression for the kernels $F_{\beta,N}^{(n)}(\mathbf{x}(n),\mathbf{y}(n))$ and $F_{\beta,N|\mathbf{x}^c(N)}^{(n)}(\mathbf{x}(n),\mathbf{y}(n))$: $\forall 0 < n \leq n' < N$:

$$F_{\beta,N}^{(n)}\big(\mathbf{x}(n),\mathbf{y}(n)\big)$$

$$= \int \mathrm{d}\boldsymbol{\omega}(N\setminus n')\frac{p_N^{(N\setminus n')}(\boldsymbol{\omega}(N\setminus n'))}{\Xi_{\beta,n'}(\boldsymbol{\omega}(N\setminus n'))}\int \mathrm{d}\boldsymbol{\omega}(n'\setminus n)$$

$$\times \int_{W_{\mathbf{x}(n),\mathbf{y}(n)}^\beta} \mathbb{P}_{\mathbf{x}(n),\mathbf{y}(n)}^\beta\big(\mathrm{d}\overline{\boldsymbol{\omega}}(n)\big)$$

$$\times \exp\big[-h\big(\overline{\boldsymbol{\omega}}(n)\vee\boldsymbol{\omega}(n'\setminus n)|\boldsymbol{\omega}(N\setminus n')\big)\big],$$

$$F_{\beta,N|\mathbf{x}^c(N)}^{(n)}\big(\mathbf{x}(n),\mathbf{y}(n)\big)$$
$$(3.24)$$

$$= \int \mathrm{d}\boldsymbol{\omega}(N\setminus n')\frac{p_{N|\mathbf{x}^c(N)}^{(N\setminus n')}(\boldsymbol{\omega}(N\setminus n'))}{\Xi_{\beta,n'}(\boldsymbol{\omega}(N\setminus n')\vee\mathbf{x}^c(N))}\int \mathrm{d}\boldsymbol{\omega}(n'\setminus n)$$

$$\times \int_{W_{\mathbf{x}(n),\mathbf{y}(n)}^\beta} \mathbb{P}_{\mathbf{x}(n),\mathbf{y}(n)}^\beta\big(\mathrm{d}\overline{\boldsymbol{\omega}}(n)\big)$$

$$\times \exp\big[-h\big(\overline{\boldsymbol{\omega}}(n)\vee\boldsymbol{\omega}(n'\setminus n)|\boldsymbol{\omega}(N\setminus n')\vee\mathbf{x}^c(N)\big)\big].$$

For $n = n'$, the integral $\int \mathrm{d}\boldsymbol{\omega}(n'\setminus n)$ is omitted.

Our next goal is to write down FK-DLR equations for the whole of $\mathcal{V} = \mathcal{V}(T)$. Here we consider a probability measure $\mu = \mu_{\mathcal{V}}$ on infinite loop configurations $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\mathcal{V}}$ over $\mathcal{V}$ (for a formal background, see Kelbert and Suhov (2013a)). The equation is written for $p^{(n)}(\boldsymbol{\omega}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n))$, the conditional probability density for a loop configuration $\boldsymbol{\omega}(n)$ over $\mathcal{V}_n$, given a loop configuration $\boldsymbol{\Omega}^{\mathrm{c}}(n)$ over $\mathcal{V} \setminus \mathcal{V}_n$. This density should be given by

$$p^{(n)}(\boldsymbol{\omega}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n)) := \frac{\mu(\mathrm{d}\boldsymbol{\omega}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n))}{\mathrm{d}\boldsymbol{\omega}(n)} = \frac{\exp[-h(\boldsymbol{\omega}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n))]}{\Xi_{\beta,n}(\boldsymbol{\Omega}^{\mathrm{c}}(n))}. \qquad (3.25)$$

Like $h(\boldsymbol{\omega}(n)|\boldsymbol{\omega}(N \setminus n))$ and $\Xi_{\beta,n}(\boldsymbol{\omega}(N \setminus n))$ before, the quantities $h(\boldsymbol{\omega}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n))$ and $\Xi_{\beta,n}(\boldsymbol{\Omega}^{\mathrm{c}}(n))$ represent the conditional energy and the conditional partition function. They can be defined as the limits

$$h(\boldsymbol{\omega}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n)) = \lim_{N \to \infty} h(\boldsymbol{\omega}(n)|\boldsymbol{\Omega}(N \setminus n)), \qquad (3.26)$$

$$\Xi_{\beta,n}(\boldsymbol{\Omega}^{\mathrm{c}}(n)) = \lim_{N \to \infty} \Xi_{\beta,n}(\boldsymbol{\Omega}(N \setminus n)), \qquad (3.27)$$

where $\boldsymbol{\Omega}(N \setminus n)$ stands for the restriction of $\boldsymbol{\Omega}^{\mathrm{c}}(n)$ to $\mathcal{V}_N \setminus \mathcal{V}_n$. The existence of the limit will be guaranteed by the assumption (2.14) $\forall \boldsymbol{\omega}(n)$ and $\boldsymbol{\Omega}^c(n)$ for P-a.a. $T \in \mathcal{T}_{\infty}$.

Formulas (3.24) admit a generalization to the infinite-volume situation: $\forall 0 \leq n < n'$,

$$F_{\beta}^{(n)}(\mathbf{x}(n), \mathbf{y}(n))$$

$$= \int \frac{\mu(\mathrm{d}\boldsymbol{\Omega}^{\mathrm{c}}(n'))}{\Xi_{\beta,n'}(\boldsymbol{\Omega}^{\mathrm{c}}(n'))} \int \mathrm{d}\boldsymbol{\omega}(n' \setminus n) \qquad (3.28)$$

$$\times \int_{W_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}} \mathbb{P}_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}(\mathrm{d}\overline{\boldsymbol{\omega}}(n)) \exp[-h(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))];$$

owing to the FK-DLR propety, the RHS in (3.28) does not depend on the choice of $n' > n$. Moreover, the integral

$$\int \mathrm{d}\mathbf{x}(n) F_{\beta}^{(n)}(\mathbf{x}(n), \mathbf{x}(n)) = \mu(\mathcal{V}) = 1.$$

Consider the operator $R_{\beta}^{(n)}$ in $\mathcal{H}(n) = \mathcal{H}^{\otimes \mathcal{V}_n}$ with the integral kernel $F_{\beta}^{(n)}(\mathbf{x}(n), \mathbf{y}(n))$ given by (3.21). The aforementioned properties imply that the trace $\mathrm{tr}_{\mathcal{H}(n)} R_{\beta}^{(n)} = 1$ and the following compatibility relation holds true:

$$R_{\beta}^{(n)} = \mathrm{tr}_{\mathcal{H}(n' \setminus n)} R_{\beta}^{(n')}. \qquad (3.29)$$

Thus, were the operators $R_{\beta}^{(n)}$ positive definite, we could speak of an infinite-volume state of the quasilocal C*-algebra $\mathfrak{B}$. Cf. Remark 2.2. Notwithstanding, we state our main result.

**Theorem 3.1.** *Under the above assumptions, any limit-point operator $R_\beta^{(n)}$ from Theorem 2.1 is a positive definite trace-class integral operator of trace 1 and with the kernel $F_\beta^{(n)}$ admitting the representation (3.21) where probability distribution $\mu$ satisfies the infinite-volume FK-DLR equations (3.25).*

**Theorem 3.2.** *Let an integral operator $R_\beta^{(n)}$ admit the representation (3.28) where probability distribution $\mu$ satisfies the infinite-volume FK-DLR equations (3.25). Then $\forall \mathsf{g} \in \mathsf{G}$*

$$R_\beta^{(n)} = S^{(n)}(\mathsf{g}) R_\beta^{(n)} \big( S^{(n)}(\mathsf{g}) \big)^{-1}. \tag{3.30}$$

## 4 The proofs: The compactness and the tuned-action arguments

The proof of Theorems 2.1 and 3.1 is based on a compactness argument (cf. Kelbert and Suhov (2013a, 2013b)). We want to note that this argument does not depend upon the dimensionality of the system.

### 4.1 Proof of Theorems 2.1 and 3.1

As in Kelbert and Suhov (2013a, 2013b), we first prove that, $\forall n \geq 0$, the sequences of RDMKs $\{F_{\beta,N}^{(n)}, N = n+1, n+2, \ldots\}$ and $\{F_{\beta,N|\mathbf{x}^c(N)}^{(n)}, N = n+1, n+2, \ldots\}$ are compact in the space $C^0(M^{\times \mathcal{V}_n} \times M^{\times \mathcal{V}_n})$. Applying Lemma 1.5 from Kelbert and Suhov (2013a) (this lemma goes back to Suhov (1970)), we will obtain that the sequences of RDMs $\{R_{\beta,N}^{(n)}\}$ and $\{R_{\beta,N|\mathbf{x}^c(N)}^{(n)}\}$ are compact in the trace-norm topology in $\mathcal{H}(n)$. This yields the statement of Theorem 2.1. A straightforward consequence of the convergence will be that any limiting RDMK $F_\beta^{(n)}$ admits the representation (3.21) where $\mu$ satisfies the infinite-volume FK-DLR equation, that is, the assertion of Theorem 3.1.

To verify compactness of the RDMKs $\{F_{\beta,N}^{(n)}\}$ and $\{F_{\beta,N|\mathbf{x}^c(N)}^{(n)}\}$, we follow the same line as in Kelbert and Suhov (2013a, 2013b), that is, employ the Ascoli–Arzela theorem. To this end, we need to check the properties of uniform boundedness and equicontinuity. For definiteness, we focus on the (slightly more complex) case of the sequence $\{F_{\beta,N|\mathbf{x}^c(N)}^{(n)}\}$.

More precisely, to show uniform boundedness, we first use an upper bound for the number of vertices $k_i$ on $\mathcal{V}_i \setminus \mathcal{V}_{i-1}$ under the measure $\mathsf{P}$; cf. Kelbert et al. (2013), equation (4.1). Namely, $\forall \varepsilon \in (0, 1)$, for $\mathsf{P}$-a.a. $T \in \mathcal{T}_\infty \exists$ a constant $C = C(T)$ such that

$$k_i \leq C i (\ln i)^{1/2+\varepsilon}, \qquad i = 2, 3, \ldots \tag{4.1}$$

(see Kelbert et al. (2013)). This yields that

$$\sum_{i=1}^{\infty} k_i J(i) < C_1(T) + C(T) \sum_{i=2}^{\infty} i (\ln i)^{1/2+\varepsilon} \left( \frac{1}{i \ln i} \right)^3 := C(T) J^*. \tag{4.2}$$

We use (4.1) and (4.2) to bound the quantity

$$q\big(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N)\big)$$
$$:=\frac{\exp[-h(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N))]}{\Xi_{\beta,n}(\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N))};$$

(4.3)

cf. (3.24) for $n'=n$. Namely, (4.2) implies that, for P-a.a. $T\in\mathcal{T}_{\infty}$, $\forall n\geq 0$ and $N>n$,

$$\exp\big[-\beta(\overline{U}+C(T)J^{*}\overline{V})\sharp\mathcal{V}_{n}\big]$$
$$\leq\exp\big[-h(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N))\big]$$
$$\leq\exp\big[\beta(\overline{U}+C(T)J^{*}\overline{V})\sharp\mathcal{V}_{n}\big]$$

(4.4)

for all path configurations $\overline{\boldsymbol{\omega}}\in W_{\mathbf{x}(n),\mathbf{y}(n)}$ and loop configurations $\boldsymbol{\omega}(N\setminus n)\in W_{\mathcal{V}_{N}\setminus\mathcal{V}_{n}}$. Here $\sharp\mathcal{V}_{n}=\sum_{i=1}^{n}k_{i}$ stands for the number of vertices in the set $\mathcal{V}_{n}$, cf. (4.1).

The lower bound in (4.4) yields that

$$\Xi_{\beta,n}\big(\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N)\big)\geq\exp\big[-\beta(\overline{U}+C(T)J^{*}\overline{V})\sharp\mathcal{V}_{n}\big]\times(\overline{p}_{M}^{\beta})^{\sharp\mathcal{V}_{n}},\quad(4.5)$$

where

$$\overline{p}_{M}^{\beta}=\frac{1}{(2\pi\beta)^{d/2}}\sum_{\underline{n}=(n_{1},\ldots,n_{d})\in\mathbb{Z}^{d}}\exp(-|\underline{n}|^{2}/2\beta)$$

(4.6)

is the probability density of transition from $x\in M$ to $x$ in time $\beta$ in the Brownian motion on $M$. Next, (4.5) and the upper bound in (4.4) imply that

$$q\big(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N)\big)\leq\frac{1}{(\overline{p}_{M}^{\beta})^{\sharp\mathcal{V}_{n}}}\exp\big[2\beta(\overline{U}+C(T)J^{*}\overline{V})\sharp\mathcal{V}_{n}\big].\quad(4.7)$$

Substituting (4.7) in (3.24), we obtain that

$$F_{\beta,N|\mathbf{x}^{\mathrm{c}}(N)}^{(n)}\big(\mathbf{x}(n),\mathbf{y}(n)\big)\leq\exp\big[2\beta(\overline{U}+C(T)J^{*}\overline{V})\sharp\mathcal{V}_{n}\big]$$

(4.8)

which gives the desired uniform upper bound.

To check equicontinuity, we analyze the derivatives $\nabla_{x(i)}F_{\beta,N|\mathbf{x}^{\mathrm{c}}(N)}^{(n)}(\mathbf{x}(n),\mathbf{y}(n))$ and $\nabla_{y(i)}F_{\beta,N|\mathbf{x}^{\mathrm{c}}(N)}^{(n)}(\mathbf{x}(n),\mathbf{y}(n))$, $i\in\mathcal{V}_{n}$. Again we use the representation (3.24) with $n=n'$. We need to differentiate the integral

$$\int_{W_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}}\mathbb{P}_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}\big(\mathrm{d}\overline{\boldsymbol{\omega}}(n)\big)\exp\big[-h\big(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N)\big)\big].$$

(4.9)

For definiteness, consider one of the gradients $\nabla_{y(i)}$. It is convenient to represent the integral (4.8) in the form

$$\prod_{j\in\mathcal{V}_{n}}p_{M}^{\beta}\big(x(j),y(j)\big)\frac{1}{[\overline{p}_{M}^{\beta}]^{\#\mathcal{V}_{n}}}$$
$$\times\int_{W_{\mathbf{x}(n),\mathbf{x}(n)}^{\beta}}\mathbb{P}_{\mathbf{x}(n),\mathbf{x}(n)}^{\beta}\big(\mathrm{d}\boldsymbol{\omega}(n)\big)\exp\big\{-h\big[(\boldsymbol{\omega}(n)+\overline{\boldsymbol{\eta}}(n))|\boldsymbol{\omega}(N\setminus n)\vee\mathbf{x}^{\mathrm{c}}(N)\big]\big\}.$$

(4.10)

Here $p_M^\beta(x, y)$ denotes the transition probability density

$$p_M^\beta(x, y) = \frac{1}{(2\pi\beta)^{d/2}} \sum_{\underline{n}=(n_1,\dots,n_d)\in\mathbb{Z}^d} \exp(-|x - y + \underline{n}|^2/2\beta), \qquad (4.11)$$

and $\overline{p}_M^\beta$ has been determined in (4.6).

Next, $\overline{\eta}(n) = \{\eta(j), j \in \mathcal{V}_n\}$ is a collection of linear paths

$$\eta(j, \tau) = \frac{\tau}{\beta}(y(j) - x(j)), \qquad j \in \mathcal{V}_n, \qquad (4.12)$$

and the component-wise addition in $\omega(n) + \overline{\eta}(n)$:

$$\omega(n) + \overline{\eta}(n) = \{\omega(j) + \eta(j), j \in \mathcal{V}_n\}$$
$$\text{where } \omega(j) + \eta(j) : \tau \in [0, \beta] \mapsto (\omega(j, \tau) + \eta(j, \tau)) \bmod 1. \qquad (4.13)$$

It is now clear that there will be two contributions into $\nabla_{y(i)} F_{\beta,N|\mathbf{x}^c(N)}^{(n)}(\mathbf{x}(n), \mathbf{y}(n))$: one coming from

$$\nabla_{y(i)} p_M^\beta(x(j), y(j)),$$

the other from

$$\nabla_{y(i)} \exp\{-h[(\omega(n) + \overline{\eta}(n))|\omega(N \setminus n) \vee \mathbf{x}^c(N)]\}$$
$$= -\nabla_{y(i)} h[(\omega(n) + \overline{\eta}(n))|\omega(N \setminus n) \vee \mathbf{x}^c(N)]$$
$$\times \exp\{-h[(\omega(n) + \overline{\eta}(n))|\omega(N \setminus n) \vee \mathbf{x}^c(N)]\}.$$

The uniform bound

$$\left|\nabla_{y(i)} p_M^\beta(x(j), y(j))\right| \le C(\beta) \in (0, +\infty) \qquad (4.14)$$

is straightforward. Next, we have the estimate

$$\left|\nabla_{y(i)} h[(\omega(n) + \overline{\eta}(n))|\omega(N \setminus n) \vee \mathbf{x}^c(N)]\right|$$
$$\le \beta(\#\mathcal{V}_n)[\overline{U} + C(T)J^*\overline{V}].$$

Together with (4.4) it implies that

$$\nabla_{y(i)} \exp\{-h[(\omega(n) + \overline{\eta}(n))|\omega(N \setminus n) \vee \mathbf{x}^c(N)]\}$$
$$\le (\#\mathcal{V}_n)[\overline{U} + C(T)J^*\overline{V}] \exp[\beta(\overline{U} + C(T)J^*\overline{V})\sharp\mathcal{V}_n]. \qquad (4.15)$$

The bounds (4.14) and (4.16) lead to a uniform bound upon $|\nabla_{y(i)} F_{\beta,N|\mathbf{x}^c(N)}^{(n)}(\mathbf{x}(n), \mathbf{y}(n))|$. This completes the proof of Theorems 2.1 and 3.1.

### 4.2 Proof of Theorems 2.2 and 3.2

Theorem 2.2 follows from Theorem 3.2; therefore, we focus on the proof of Theorem 3.2. Equation (3.30) follows from the property

$$\lim_{n' \to \infty} \frac{q_{n'}^{(n)}(S(\mathsf{g})\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))}{q_{n'}^{(n)}(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))} = 1, \qquad \mathsf{g} \in \mathsf{G}, \tag{4.16}$$

uniformly in the path configurations $\overline{\boldsymbol{\omega}}(n) = \{\omega(j), j \in \mathcal{V}_n\} \in W_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}$ and the loop configurations $\boldsymbol{\Omega}^{\mathrm{c}}(n')$ over $\mathcal{V} \setminus \mathcal{V}_{n'}$. Here, the functional $q_{n'}^{(n)}(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))$ emerges from representation (3.28):

$$q_{n'}^{(n)}(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))(\mathbf{x}(n), \mathbf{y}(n))$$
$$= \frac{1}{\Xi_{\beta,n'}(\boldsymbol{\Omega}^{\mathrm{c}}(n'))} \int \mathrm{d}\boldsymbol{\omega}(n' \setminus n) \tag{4.17}$$
$$\times \int_{W_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}} \mathbb{P}_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}(\mathrm{d}\overline{\boldsymbol{\omega}}(n)) \exp[-h(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))],$$

and

$$S(\mathsf{g})\overline{\boldsymbol{\omega}}(n) = \{S(\mathsf{g})\omega(i), i \in \mathcal{V}_n\}$$

$$\text{where } S(\mathsf{g})\omega(i) : \tau \in [0, \beta] \mapsto S(\mathsf{g})\omega(i, \tau).$$

To check (4.16), we again follow the argument used in Kelbert and Suhov (2013a, 2013b) (which goes back to Pfister (1981) and Fröhlich and Pfister (1981); cf. also Georgii (1988)). The backbone of the argument is the following inequality: $\forall$ given $a > 1$, $\mathsf{g} \in \mathsf{G}$ and positive integer $n$, if $n'$ is large enough then, $\forall \overline{\boldsymbol{\omega}}(n) \in W_{\mathbf{x}(n),\mathbf{y}(n)}^{\beta}$, $\mathbf{x}(n), \mathbf{y}(n) \in M^{\times \mathcal{V}_n}$ and the loop configurations $\boldsymbol{\Omega}^{\mathrm{c}}(n')$ over $\mathcal{V} \setminus \mathcal{V}_{n'}$,

$$aq_{n'}^{(n)}(\mathsf{g}\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n')) + aq_{n'}^{(n)}(\mathsf{g}^{-1}\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))$$
$$\geq 2q_{n'}^{(n)}(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n')). \tag{4.18}$$

The verification of equation (4.18) is based on a special construction related to a family of "tuned" actions $\mathsf{g}_{n'\setminus n}\boldsymbol{\omega}(n' \setminus n)$ on loop configurations $\boldsymbol{\omega}(n' \setminus n)$; see equations (4.20), (4.21) below. (A tuned action can be described as an interpolation between the unity (identity) and the group action by $g$.) A particular feature of the tuned action $\mathsf{g}_{n'\setminus n}$ is that it "decays" to $\mathrm{e}$, the unit element of $\mathsf{G}$ (which generates a "trivial" identity action), when we move the vertex of the tree $T$ from $\mathcal{V}_n$ towards

$\mathcal{V} \setminus \mathcal{V}_{n'}$. Formally, (4.18) is implied by the following estimate: $\forall$ given $n$, $\overline{\boldsymbol{\omega}}(n) \in \overline{W}^{\beta}_{\mathbf{x}(n), \mathbf{y}(n)}$, $\mathsf{g} \in \mathsf{G}$ and $a \in (1, \infty)$, for any $n'$ large enough, $\boldsymbol{\omega}(n' \setminus n)$ and $\boldsymbol{\Omega}^{\mathrm{c}}(n')$,

$$\frac{a}{2} \exp\big[-h\big(\mathsf{g}\overline{\boldsymbol{\omega}}(n) \vee \mathsf{g}_{n' \setminus n}\boldsymbol{\omega}(n' \setminus n) | \boldsymbol{\Omega}^{\mathrm{c}}(n')\big)\big]$$

$$+ \frac{a}{2} \exp\big[-h\big(\mathsf{g}^{-1}\overline{\boldsymbol{\omega}}(n) \vee \mathsf{g}^{-1}_{n' \setminus n}\boldsymbol{\omega}(n' \setminus n) | \boldsymbol{\Omega}^{\mathrm{c}}(n')\big)\big] \qquad (4.19)$$

$$\geq \exp\big[-h\big(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n) | \boldsymbol{\Omega}^{\mathrm{c}}(n')\big)\big].$$

Indeed, (4.18) follows from (4.19) by integrating in $\mathrm{d}\boldsymbol{\omega}(n' \setminus n)$ and normalizing by $\Xi_{\beta, n'}(\boldsymbol{\Omega}^{\mathrm{c}}(n'))$; cf. (4.17). Here it is important that the Jacobian of the map $\boldsymbol{\omega}(n' \setminus n) \mapsto \mathsf{g}_{n' \setminus n}\boldsymbol{\omega}(n' \setminus n)$ is equal to 1.

The rest of the argument concentrates on verifying (4.19). The tuned family $\mathsf{g}_{n' \setminus n}$ consists of individual actions $\mathsf{g}^{(n')}_j \in \mathsf{G}$ at vertices $j \in \mathcal{V}_{n'} \setminus \mathcal{V}_n$:

$$\mathsf{g}_{n' \setminus n} = \big\{\mathsf{g}^{(n')}_j, j \in \mathcal{V}_{n'} \setminus \mathcal{V}_n\big\}. \qquad (4.20)$$

We use the representation (2.12) and identify the element $\mathsf{g} \in \mathsf{G}$ with a vector $\underline{\theta} = \theta A \in \mathbb{R}^d$. Then the actions $\mathsf{g}^{(n')}_j \in \mathsf{G}$ correspond to multiples of the vector $\underline{\theta}$; cf. equation (4.21) below. It is convenient to fix a positive integer $\overline{r} > n$ and identify

$$\mathsf{g}^{(n')}_j \qquad \text{with } \underline{\theta}\gamma(n', k), \qquad (4.21)$$

where $k = \mathsf{d}(s_0, j)$ (recall, $s_0$ is the root of $T$) and

$$\gamma(n', k) = \begin{cases} 1, & k \leq \overline{r}, \\ \vartheta(k - \overline{r}, n' - \overline{r}), & k > \overline{r}. \end{cases} \qquad (4.22)$$

In turn, the function $\vartheta(a, b)$ is determined by

$$\vartheta(a, b) = \mathbf{1}(a \leq 0) + \frac{\mathbf{1}(0 < a < b)}{Q(b)} \int_a^b z(u)\, \mathrm{d}u, \qquad a, b \in \mathbb{R}, \qquad (4.23)$$

with the same functions $Q(b)$ and $z(u)$ as in Fröhlich and Pfister (1981)

$$Q(b) = \int_0^b z(u)\, \mathrm{d}u,$$

$$\text{where } z(u) = \mathbf{1}(u \leq 2) + \mathbf{1}(u > 2)\frac{1}{u \ln u}, b > 0. \qquad (4.24)$$

Next, $\mathsf{g}^{-1}_{n' \setminus n}$ is the collection of the inverse elements:

$$\mathsf{g}^{-1}_{n' \setminus n} = \big\{\mathsf{g}^{(n')^{-1}}_j, j \in \mathcal{V}_{n'} \setminus \mathcal{V}_n\big\}.$$

It will be convenient to use formulas (4.21)–(4.23) for $\mathsf{g}^{(n)}_j$ for $j \in \mathcal{V}_n$, or even for $j \in \mathcal{V}$, as these formulas agree with the requirement that $\mathsf{g}^{(n')}_j \equiv \mathsf{g}$ when $j \in \mathcal{V}_n$

and $g_j^{(n')} \equiv e$ for $j \in \mathcal{V} \setminus \mathcal{V}_{n'}$. Accordingly, we will employ the notation $g_{n'} = \{g_j^{(n')}, j \in \mathcal{V}_{n'}\}$.

Next, we use the invariance property (2.13). The Taylor formula for the function $V \in \mathbf{C}^2(\mathbb{R}^2)$ yields for $j, j' \in \mathcal{V}_n$:

$$\begin{aligned}
&\big|V\big(g_j^{(n')}\omega(j), g_{j'}^{(n')}\omega(j')\big) \\
&\quad + V\big(g_j^{(n')^{-1}}\omega(j), g_{j'}^{(n')^{-1}}\omega(j')\big) - 2V\big(\omega(j), \omega(j')\big)\big| \qquad (4.25) \\
&\quad \leq C|\underline{\theta}|^2 |\gamma(n', j) - \gamma(n', j')|^2 \overline{V}.
\end{aligned}$$

Here $C \in (0, \infty)$ is a constant, the upper bound $\overline{V}$ is taken from (2.11), and we use the notation from (4.22).

The bound (4.25) is crucial: this where the structure of the group action is exploited. It is based on the fact that the first-order terms in the expansion in the LHS of (4.25) cancel each other, due to the presence of elements $g_j^{(n')}$ and $g_{j'}^{(n')}$ and their inverses, $g_j^{(n')^{-1}}$ and $g_{j'}^{(n')^{-1}}$. This idea can be traced back to Pfister (1981) and Fröhlich and Pfister (1981).

Further, the term $|\gamma(n', j) - \gamma(n', j')|^2$ can be specified as

$$|\gamma(n', k) - \gamma(n', k')|^2 = \begin{cases} 0, & \text{if } k, k' \leq \overline{r}, \text{ or } k, k' \geq n', \\ [\vartheta(k - \overline{r}, n' - \overline{r}) - \vartheta(k' - \overline{r}, n' - \overline{r})]^2, \\ \qquad \text{if } \overline{r} < k, k' \leq n', \\ \vartheta(k - \overline{r}, n' - \overline{r})^2, \\ \qquad \text{if } \overline{r} < k \leq n', k' \notin \,]\overline{r}, n'[, \\ \vartheta(k' - \overline{r}, n' - \overline{r})^2, \\ \qquad \text{if } \overline{r} < k' \leq n', k \notin \,]\overline{r}, n'[ \end{cases} \qquad (4.26)$$

with notations $k = \mathrm{d}(j, s_0), k' = \mathrm{d}(j', s_0)$.

The convexity property of the function exp, together with equation (4.25), yield that, $\forall a > 1$,

$$\begin{aligned}
&\frac{a}{2}\exp\big[-h\big(g_{n'}(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n))|\boldsymbol{\Omega}^{\mathrm{c}}(n')\big)\big] \\
&\quad + \frac{a}{2}\exp\big[-h\big(g_{n'}^{-1}(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n))|\boldsymbol{\Omega}^{\mathrm{c}}(n')\big)\big] \\
&\quad \geq a\exp\Big[-\frac{1}{2}h\big(g_{n'}(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n))|\boldsymbol{\Omega}^{\mathrm{c}}(n')\big) \qquad (4.27) \\
&\qquad\qquad - \frac{1}{2}h\big(g_{n'}^{-1}(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n))|\boldsymbol{\Omega}^{\mathrm{c}}(n')\big)\Big] \\
&\quad \geq a\exp\big[-h\big(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n)|\boldsymbol{\Omega}^{\mathrm{c}}(n')\big)\big]e^{-C\Phi/2}.
\end{aligned}$$

Here

$$\Phi = \Phi(n, \mathsf{g}) = |\underline{\theta}|^2 \sum_{(j, j') \in \mathcal{V}_n \times \mathcal{V}} J(\mathsf{d}(j, j')) |\gamma(n', k) - \gamma(n', k')|^2. \tag{4.28}$$

The series in (4.28) converges for P-a.a. $T$, owing to condition (2.14) and estimate (4.31) below.

The next observation is that

$$\Phi \le 3|\underline{\theta}|^2 \sum_{(j, j') \in \mathcal{V}_{n'} \times \mathcal{V}} \mathbf{1}(k \le k') J(\mathsf{d}(j, j'))$$

$$\times \left[ \vartheta(k - \bar{r}, n' - \bar{r}) - \vartheta(k' - \bar{r}, n' - \bar{r}) \right]^2, \tag{4.29}$$

where, by virtue of the triangle inequality, for all $j, j'$: $k \le k'$

$$0 \le \vartheta(k - \bar{r}, n' - \bar{r}) - \vartheta(k' - \bar{r}, n' - \bar{r})$$

$$\le \mathsf{d}(j, j') \frac{z(k - \bar{r})}{Q(n' - \bar{r})}. \tag{4.30}$$

Thus,

$$\Phi \le \frac{3|\underline{\theta}|^2}{Q(n' - \bar{r})^2} \sum_{(j, j') \in \mathcal{V}_{n'} \times \mathcal{V}} J(\mathsf{d}(j, j')) \mathsf{d}(j, j')^2 z(k - \bar{r})^2$$

$$\le \frac{3|\underline{\theta}|^2}{Q(n' - \bar{r})^2} \left[ \sup_{j \in \mathcal{V}} \sum_{j' \in \mathcal{V}} J(\mathsf{d}(j, j')) \mathsf{d}(j, j')^2 \right] \sum_{j \in \mathcal{V}_{n' + \bar{r}}} z(k - \bar{r})^2.$$

Owing to (2.14), it remains to bound the sum $\sum_{j \in \mathcal{V}_{n + \bar{r}}} z(k - \bar{r})^2$. Note that $u(\ln u)^{1/2+\varepsilon} z(u) < 1$ when $u \in (u_0(\varepsilon), \infty)$. Next, we use the bound (4.1) on the number of vertices in $\mathcal{V}_n \setminus \mathcal{V}_{n-1}$. Therefore,

$$\sum_{j \in \mathcal{V}_{n' + \bar{r}}} z(k - \bar{r})^2 = \sum_{1 \le k \le n' + \bar{r}} z(k - \bar{r}) \sum_{j \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} z(k - \bar{r})$$

$$\le C_0 \sum_{1 \le k \le n' + \bar{r}} z(k - \bar{r}) \le C_1 Q(n' - \bar{r})$$

and

$$\Phi \le \frac{C(T)}{Q(n' - \bar{r})} \to \infty, \qquad \text{as } n' \to \infty. \tag{4.31}$$

Hence, given $a > 1$ for $n'$ large enough, the term $ae^{-C\Phi/2}$ in the RHS of (4.27) becomes $> 1$. Consequently,

$$\frac{a}{2} \exp\left[-h\left(\mathsf{g}_{n'}(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n)) | \boldsymbol{\Omega}^{\mathrm{c}}(n')\right)\right]$$

$$+ \frac{a}{2} \exp\left[-h\left(\mathsf{g}_{n'}^{-1}(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n)) | \boldsymbol{\Omega}^{\mathrm{c}}(n')\right)\right] \tag{4.32}$$

$$\ge \exp\left[-h\left(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n' \setminus n) | \boldsymbol{\Omega}^{\mathrm{c}}(n')\right)\right].$$

Equation (4.32) implies that

$$q_{n'}^{(n)}(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))$$

$$= \int_{W_{\mathbf{x}(n'\setminus n)}^{\beta}} \mathrm{d}\boldsymbol{\omega}(n'\setminus n) \frac{\exp[-h(\overline{\boldsymbol{\omega}}(n) \vee \boldsymbol{\omega}(n'\setminus n)|\boldsymbol{\Omega}^{\mathrm{c}}(n))]}{\Xi_{\beta,n'}(\boldsymbol{\Omega}^{\mathrm{c}}(n'))} \tag{4.33}$$

for any $n$ and $n'$ large enough obeys

$$a\big[q_{n'}^{(n)}(\mathsf{g}\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n')) + q_{n'}^{(n)}(\mathsf{g}^{-1}\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n'))\big]$$

$$\geq 2q_{n'}^{(n)}(\overline{\boldsymbol{\omega}}(n)|\boldsymbol{\Omega}^{\mathrm{c}}(n')) \tag{4.34}$$

uniformly in the boundary condition $\boldsymbol{\Omega}^{\mathrm{c}}(n')$. Thus, (4.18) is established, which completes the proof of Theorem 3.2.

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