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The cone percolation on $\mathbb{T}_d$

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Abstract. We study a rumor model from a percolation theory and a branching process point of view. The existence of a giant component is related to the event where a rumor spreads out trough an infinite number of individuals. We present sharp lower and upper bounds for the probability of that event, according to the distribution of the random variables defining the radius of influence of each individual.

1 Introduction

We study long range dependent oriented percolation processes on a tree through its most basic property: the existence of a giant connected component. The starting point to rigorous percolation theory, beyond the nearest neighbor independent setup on $\mathbb{Z}^d$, is due to several authors such as Grimmett and Newman (1990), Burton and Meester (1993), Lyons (2010) and Benjamini and Schram (1996), to name a few. More recently Lebensztayn and Rodriguez (2008), proposed a model on general graphs called *disk percolation*. In that model a reaction chain starting from the origin of the graph, based on independent copies of a geometric random variables with parameter $q \in [0, 1]$, defines the existence or not of a giant component. They obtain a sufficient condition for the existence of phase transition based on q, which means that the existence of a non-empty subcritical (no giant components) and supercritical (giant components with positive probability) phases. They associate their model to a rumor or an epidemic process. Here, instead of working on a general family of graphs we focus on homogeneous trees and instead of fixing the random variable which defines the radius of infection or the radius of influence of each vertex to be geometric, we consider general random variables. So, as a result, instead of having a phase transition phenomena depending on an element of a parametric space, we have that phenomena depending on the family of general positive random variables. A variation of this model for $\mathbb{N}$, instead of $\mathbb{T}_d$, can be found in Junior et al. (2011).

We consider a process which allows us to associate the dynamic activation on the set of vertices to a discrete rumor process. Individuals become spreaders as soon as they heard about the rumor. Next time, they propagate the rumor within

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their *radius of influence* and immediately become stiflers. Our main interest is to establish whether the process has positive probability of involving an infinite set of individuals. We present sharp lower and upper bounds for the probability of that event, according to the general distribution of the random variables that define the *radius of influence* of each individual. We say that the process *survives* if the amount of vertices involved is infinite. Otherwise we say the process *dies out*.

Consider $\mathbb{T}_d$ the homogeneous tree such that each vertex has $d+1$ neighbours, $d \geq 2$. Let $\mathcal{V}(\mathbb{T}_d)$ be the set of vertices of $\mathbb{T}_d$. We single out one vertex from $\mathcal{V}(\mathbb{T}_d)$ and call this $\mathcal{O}$, the origin. For each two vertices $u, v \in \mathcal{V}(\mathbb{T}_d)$, we say that $u \leq v$ if u belongs to the path connecting $\mathcal{O}$ to v. For two vertices u, v let $d(u, v)$, be the distance between u and v, that is the number of edges the path from u to v has.

Now, let us define

$$\mathbb{T}_d^+(u) = \{v \in \mathcal{V}(\mathbb{T}_d) : u \leq v\}.$$

We define the *Cone Percolation Process* in $\mathbb{T}_d$. Let $\{\bar{R}_v\}_{\{v \in \mathcal{V}(\mathbb{T}_d)\}}$ and R be a set of independent and identically distributed random variables. We define $p_k = \mathbb{P}[R = k]$ for $k = 0, 1, \ldots$. To avoid trivialities, we assume $p_0 \in (0, 1)$. Furthermore, for each $u \in \mathcal{V}(\mathbb{T}_d)$, we define the random sets

$$B_u = \{v \in \mathcal{V}(\mathbb{T}_d) : u \leq v \text{ and } d(u, v) \leq \bar{R}_u\} \tag{1.1}$$

and consider the non-decreasing sequence of random sets $I_0 \subset I_1 \subset \cdots$ defined as $I_0 = \{\mathcal{O}\}$ and inductively $I_{n+1} = \bigcup_{u \in I_n} B_u$ for all $n \geq 0$. Let $I = \bigcup_{n \geq 0} I_n$ be the connected component of the origin. Under the rumor process interpretation, I is the set of vertices which heard about the rumor. We say that the process survives if $|I| = \infty$, referring to the surviving event as V.

Pick a $v \in \mathcal{V}(\mathbb{T}_d)$ such that $d(\mathcal{O}, v) = 1$ and consider $\mathbb{T}_d^+ = \mathbb{T}_d \setminus \mathbb{T}_d^+(v)$. Consider $\mathbb{P}_+$ and $\mathbb{P}$ the probability measures associated to the processes on $\mathbb{T}_d^+$ and $\mathbb{T}_d$ (we do not mention the random variable R unless absolutely necessary). By a coupling argument, one can see that for a fixed distribution of R

$$\mathbb{P}_+[V] \leq \mathbb{P}[V]. \tag{1.2}$$

Furthermore, by the definition of $\mathbb{T}_d^+$ and its relation with $\mathbb{T}_d$ we have that for a fixed distribution of R

$$\mathbb{P}_+[V] = 0 \quad \text{if and only if} \quad \mathbb{P}[V] = 0. \tag{1.3}$$

The paper is organized as follows. Section 2 presents the main results. Section 3 brings the proofs for the main results together with auxiliary lemmas and handy inequalities. Section 4 presents results for the heterogeneous setup of the *Cone Percolation Process*. Finally, in Section 5 we present examples where some conditions can be verified.

2 Main results

Theorem 2.1. *Consider the Cone Percolation Process on $\mathbb{T}_d^+$ with radius of influence R*

(I) *If* $(1-p_0)d > 1$, *then* $\mathbb{P}_+[V] > 0$,
(II) *If* $(1-p_0)d \le 1$ *and* $\mathbf{E}(d^R) > 1 + p_0$, *then* $\mathbb{P}_+[V] > 0$,
(III) *If* $(1-p_0)d \le 1$ *and* $\mathbf{E}(d^R) \le 2 - \frac{1}{d}$, *then* $\mathbb{P}_+[V] = 0$.

Let ρ and ψ be, respectively, the smallest nonnegative roots of the equations

$$\mathbf{E}\big(\rho^{d^R}\big) + (1-\rho)p_0 = \rho, \tag{2.1}$$

$$\mathbf{E}\big(\psi^{(d/(d-1))(d^R-1)}\big) = \psi. \tag{2.2}$$

Theorem 2.2. *Consider the Cone Percolation Process on $\mathbb{T}_d^+$. Then*

$$1-\rho \le \mathbb{P}_+(V) \le 1-\psi.$$

Theorem 2.3. *For the Cone Percolation Process on $\mathbb{T}_d$ with radius of influence R, it holds that*

$$\begin{aligned} &-\big(1-\rho^{(d+1)/d}\big)p_0 - \mathbf{E}\big(\rho^{((d+1)/d)d^R}\big) \\ &\quad \le \mathbb{P}[V] \le 1 - \mathbf{E}\big(\psi^{((d+1)/(d-1))(d^R-1)}\big). \end{aligned} \tag{2.3}$$

3 Proofs

3.1 Auxiliary processes

Let us define two auxiliary branching process, being the first one $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$. For this process, each individual has a number of offspring distributed as the random variable X, assuming values in $\{0, d, d^2, \dots\}$ such that

$$\mathbf{P}[X=0] = p_0,$$
$$\mathbf{P}\big[X=d^k\big] = p_k \qquad \text{for } k = 1, 2, \dots$$

whose expectation is

$$\mathbf{E}[X] = \mathbf{E}\big[d^R\big] - p_0 \tag{3.1}$$

and whose generating function is

$$\varphi_X(s) = \mathbf{E}\big[s^X\big] = \mathbf{E}\big[s^{d^R}\big] + (1-s)p_0. \tag{3.2}$$

The second auxiliary process is $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. For this process, each individual has a number of offsprings distributed as the random variable Y, assuming values in $\{0, d, d+d^2, \dots, \sum_{i=1}^{k} d^i\}$ such that

$$\mathbf{P}\left[Y = \frac{d(d^k-1)}{d-1}\right] = p_k \qquad \text{for } k = 0, 1, 2, \dots$$

whose expectation is

$$\mathbf{E}[Y] = \frac{d}{d-1}\left(\mathbf{E}\left[d^R\right] - 1\right) \tag{3.3}$$

and whose generating function is

$$\varphi_Y(s) = \mathbf{E}\left[s^Y\right] = \mathbf{E}\left[s^{(d/(d-1))(d^R-1)}\right]. \tag{3.4}$$

3.2 Proofs

Proof of Theorem 2.1. First, we can assure (I) by a comparison with a supercritical branching process. In order to prove (II) one can see that our process dominates $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$. This process survives as long as $\mathbf{E}[X] > 1$ therefore from (3.1) our process survives if $\mathbf{E}[d^R] > 1 + p_0$.

Second, also by a coupling argument, our process is dominated by $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$. That process dies out provided $\mathbf{E}[Y] \le 1$ and $\mathbb{P}[Y = 1] \neq 1$, therefore from (3.3) our process dies out if $\mathbf{E}[d^R] \le 2 - \frac{1}{d}$, proving (III). □

Proof of Theorem 2.2. In order to find the extinction probability of $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ (Grimmett and Stirzaker (2001, p. 173)), let us consider the smallest nonnegative root of the equation $\rho = \varphi_X(\rho)$. Therefore, from (3.2)

$$\mathbf{E}\left[\rho^{d^R}\right] + (1-\rho)p_0 = \rho$$

and by construction of the processes, as $\mathbb{P}_+[V^c] \le \rho$, we have that

$$1 - \rho \le \mathbb{P}_+[V].$$

In order to find the extinction probability of $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ (Grimmett and Stirzaker (2001, p. 173)), let us consider the smallest nonnegative root of the equation $\psi = \varphi_Y(\psi)$. Therefore, from (3.4)

$$\mathbf{E}\left[\psi^{(d/(d-1))(d^R-1)}\right] = \psi$$

and by the construction of the processes, as $\mathbb{P}_+[V^c] \ge \psi$, we have that

$$\mathbb{P}_+[V] \le 1 - \psi. \qquad □$$

Proof of Theorem 2.3. Observe that except for the root, all vertices see towards infinity a tree like $\mathbb{T}_d^+$. So, assuming $\bar{R}_{\mathcal{O}} = k$ the probability for the process to survive is greater or equal than the probability of the process to survive from at least one of the $d^{k-1}(d+1)$ trees that have as root the furthest infected vertices. Now note that, still assuming $\bar{R}_{\mathcal{O}} = k$, the probability for the process to survive on $\mathbb{T}_d$ is smaller or equal than the probability for the process to survive from at least one of the $(d+1)(d^k-1)(d-1)^{-1}$ vertices which are in the radius of influence $(\bar{R}_{\mathcal{O}})$ of the origin of the tree.

$$1-\left(1-\mathbb{P}_+[V]\right)^{(d+1)d^{k-1}} \leq \mathbb{P}[V|\bar{R}_{\mathcal{O}}=k] \leq 1-\left(1-\mathbb{P}_+[V]\right)^{((d+1)/(d-1))[d^k-1]}.$$

From this and from Theorem 2.2 follows (2.3). □

4 Heterogeneous cone percolation on $\mathbb{T}_d^+$

Suppose we have two sets of independent random variables, $\{R_z\}_{\{z\in\mathbb{N}\}}$ and $\{\bar{R}_v\}_{\{v\in\mathcal{V}(\mathbb{T}_d^+)\}}$, such that for all $z \in \mathbb{N}$ and all $u \in \mathcal{V}$ such that $d(\mathcal{O},u)=z$, $\bar{R}_u$ and R_z are identically distributed. We assume $\mathbb{P}[R_z=0]<1$ for all $z\in\mathbb{N}$.

We define the *Heterogeneous Cone Percolation Process* from the set of B_u presented in (1.1).

Let $u \leq v \in \mathcal{V}(\mathbb{T}_d^+)$. Now consider the event

$$V_{u,v}: \text{Process starting from } u \text{ reaches } v.$$

For $\mathbb{T}_d^+$ we define

$$\partial T_n^u = \left\{v \in \mathbb{T}_d^+ : d(u,v)=n\right\}.$$

Given a fixed integer n, let $X_0^n = \{\mathcal{O}\}$. For $j=1,2,\dots$ we define

$$X_j^n = \bigcup_{u\in X_{j-1}^n} \left\{v \in \partial T_n^u : V_{u,v} \text{ occurs}\right\}.$$

Again, for all $j=1,2,\dots$ consider

$$Z_j^n = \left|X_j^n\right|.$$

So, for all fixed positive integer n, $\{Z_j^n\}_{j\geq 0}$ is a branching process dominated by the number of vertices $v \in \partial T_{jn}^{\mathcal{O}}$ which are activated.

Lemma 4.1. *Consider n fixed. For μ_j, the mean number of offspring of one individual of generation j for the process $\{Z_j^n\}_{j\geq 0}$, it holds that*

$$\mu_j = d^n \rho_j^{(n)},$$

where $\rho_j^{(n)} = \mathbb{P}_+[V_{u,v}]$, for any fixed pair $u \leq v$ such that $d(\mathcal{O},u)=jn$ and $d(\mathcal{O},v)=(j+1)n$.

Proof. For fixed j and n, consider for some u such that $d(\mathcal{O}, u) = jn$, $\partial T_n^u = \{v_1, v_2, \dots, v_{d^n}\}$. So we can write the number of offspring of u as $\sum_{i=1}^{d^n} I_{\{V_{u,v_i}\}}$. Taking expectation, it finishes the proof. □

Lemma 4.2. *Consider n fixed and $\rho_j^{(n)} = \mathbb{P}_+[V_{u,v}]$, for any fixed pair $u \le v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$,*

$$\rho_j^{(n)} \ge \prod_{k=0}^{n-1} \left[1 - \prod_{i=0}^{k} \mathbb{P}_+[R_{jn+i} < k+1-i] \right].$$

Proof. For any fixed pair $u \le v$ such that $d(\mathcal{O}, u) = jn$ and $d(\mathcal{O}, v) = (j+1)n$, we have that

$$V_{u,v} = \bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^{k} \{\bar{R}_{u(i)} \ge k+1-i\} \right],$$

where $u(i)$ is the vertex from the path connecting u to v such that $d(\mathcal{O}, u(i)) = jn + i$. From this follows

$$\begin{aligned} \rho_j^{(n)} &= \mathbb{P}_+\left(\bigcap_{k=0}^{n-1} \left[\bigcup_{i=0}^{k} \{R_{jn+i} \ge k+1-i\} \right] \right) \\ &\ge \prod_{k=0}^{n-1} \mathbb{P}_+\left(\bigcup_{i=0}^{k} \{R_{jn+i} \ge k+1-i\} \right). \end{aligned}$$

The inequality is a consequence of the FKG inequality (Alon and Spencer (2008, p. 89)). □

Theorem 4.3. *The Heterogeneous Cone Percolation Process in $\mathbb{T}_d^+$ has a giant component with positive probability if for some fixed n,*

$$\liminf_{j\to\infty} d^n \prod_{k=0}^{n-1} \left[1 - \prod_{i=0}^{k} \mathbb{P}_+[R_{jn+i} < k+1-i] \right] > 1. \tag{4.1}$$

Proof. From Souza and Biggins (1992, p. 40) a branching process in varying environments is *uniformly supercritical* if there exists constants $a > 0$ and $c > 1$ such that

$$\prod_{k=i}^{j+i-1} \mu_k \ge ac^j, \qquad \text{for all } i \ge 0 \text{ and } j \ge 0. \tag{4.2}$$

From Lemma 4.1 and Lemma 4.2, inequality (4.1) implies that

$$\liminf_{j\to\infty} \mu_j > 1,$$

and this guarantees (4.2).

Now note that we can write

$$Z_{j+1} = \sum_{i=1}^{Z_j} Y_{j,i}^n,$$

where $Y_{j,i}^n$ are i.i.d. copies of Y_j^n, being the number of offspring from the ith individual of the jth generation. By considering Lemma 4.1, we have for all j that

$$\frac{Y_j^n}{\mu_j} \le \frac{d^n}{\mu_j} = \frac{1}{\rho_j^{(n)}} \le m^{-n},$$

where

$$m = \min_{\{u \in \mathcal{V}(\mathbb{T}_d^+),\, jn \le |u| \le (j+1)n\}} \mathbb{P}_+[\bar{R}_u > 0] \ge \min_{u \in \mathcal{V}(\mathbb{T}_d^+)} \mathbb{P}_+[\bar{R}_u > 0] > 0.$$

So, from Theorem 1 in Souza and Biggins (1992, p. 40), we conclude that the *Heterogeneous Cone Percolation Process* has a giant component with positive probability if

$$\liminf_{j \to \infty} d^n \prod_{k=0}^{n-1} \left[1 - \prod_{i=0}^{k} \mathbb{P}_+[R_{jn+i} < k + 1 - i] \right] > 1. \qquad \square$$

5 Examples

Example 5.1. Consider a *Cone Percolation Process* in $\mathbb{T}_d$, assuming

$$\mathbb{P}[R = 1] = p = 1 - \mathbb{P}[R = 0].$$

In words, consider R following a Bernoulli distribution with parameter p ($R \sim \mathcal{B}(p)$).

- If $p > d^{-1}$, then $\mathbb{P}[V] > 0$.
- If $p \le d^{-1}$, then $\mathbb{P}[V] = 0$.

By the definition, one can see that

$$\mathbb{P}[V^c] = (1 - p) + \big(\mathbb{P}_+[V^c]\big)^{d+1} p.$$

Observing that the upper and lower process presented by $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{Y}_n\}_{n \in \mathbb{N}}$ presented in Section 3.1 are the same, we see that

$$\mathbb{P}[V] = p\big(1 - \psi^{d+1}\big),$$

being ψ the solution of

$$p\psi^d - \psi + 1 - p = 0.$$

Example 5.2. Consider a *Cone Percolation Process* in $\mathbb{T}_d$, assuming

$$\mathbb{P}(R = k) = (1 - p)p^k, \qquad k = 0, 1, 2, \ldots.$$

In other words, assume R following a Geometric distribution with parameter $1 - p$ ($R \sim \mathcal{G}(1 - p)$). From Theorem 2.3

- If $dp^2 - 2dp + 1 < 0$, then $\mathbb{P}[V] > 0$.
- If $2pd \leq 1$, then $\mathbb{P}[V] = 0$.

As a consequence of this and (1.3), for d fixed

$$\frac{1}{2d} < \inf\{p : \mathbb{P}[V] > 0\} \leq 1 - \sqrt{1 - \frac{1}{d}}.$$

Example 5.3. Consider a *Cone Percolation Process* in $\mathbb{T}_d$, assuming

$$\mathbb{P}(R = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

- If $(pd + 1 - p)^n - (1 - p)^n > 1$, then $\mathbb{P}[V] > 0$.
- If $2d - d(pd + 1 - p)^n \geq 1$, then $\mathbb{P}[V] = 0$.

Consider $d = 2$ and R following a Binomial distribution with parameters 4 and $\frac{1}{2}$ ($R \sim \mathcal{B}(4, \frac{1}{2})$).

Therefore ρ and ψ are, respectively, solutions of

$$x^{16} + 4x^8 + 6x^4 + 4x^2 - 16x + 1 = 0,$$

$$x^{30} + 4x^{14} + 6x^6 + 4x^2 - 16x + 1 = 0.$$

So ρ = 0.0635146 and ψ = 0.06350850, which implies that

$$0.937435919 \leq \mathbb{P}[V] \leq 0.937435962.$$

Consider $d = 4$ and R following a Binomial distribution with parameters 4 and $\frac{1}{4}$ ($R \sim \mathcal{B}(4, \frac{1}{4})$).

Therefore ρ and ψ are, respectively, solutions of

$$x^{256} + 12x^{64} + 54x^{16} + 108x^4 - 256x + 81 = 0,$$

$$x^{340} + 12x^{84} + 54x^{20} + 108x^4 - 256x + 81 = 0.$$

So ρ = 0.3208787235 and ψ = 0.3208787200, which implies that

$$0.682158629 \leq \mathbb{P}[V] \leq 0.682158630.$$

Example 5.4. Consider a *Heterogeneous Cone Percolation Process* on $\mathbb{T}_d^+$, assuming that R_j are Bernoullis, that is,

$$\mathbb{P}_+[R_j = 1] = 1 - \mathbb{P}_+[R_j = 0] \qquad \text{for } j = 0, 1, 2, \ldots.$$

By applying Theorem 4.3 with $n = 1$, one can see that the *Heterogeneous Cone Percolation Process* on $\mathbb{T}_d^+$ survives with positive probability if

$$\liminf_{j\to\infty} d\mathbb{P}_+[R_j = 1] > 1.$$

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