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On free and classical type G distributions

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Abstract. There is a one-to-one correspondence between classical one-dimensional infinitely divisible distributions and free infinitely divisible distributions. In this work we study the free infinitely divisible distributions corresponding to the one-dimensional type G distributions. A new characterization of classical type G distributions is given first and the class of type A classical infinitely divisible distributions is introduced. The corresponding free type A distributions are studied and the role of a special symmetric beta distribution is shown as a building block for free type A distributions. It is proved that this symmetric beta distribution is the free multiplicative convolution of an arcsine distribution with the Marchenko–Pastur distribution.

1 Introduction

The distribution of a one-dimensional random variable X is said to be a variance mixture of the normal distribution if it is the distribution of $V^{1/2}Z$, where Z and V are independent random variables with V being positive and Z having the standard normal distribution. When the mixing distribution of V is infinitely divisible, X is also infinitely divisible [12,16]. In this case we say that X has a *type G distribution or belongs to the class CTG* of classical type G distributions, and write $\mathcal{L}(V^{1/2}Z) \in CTG$. Such distributions are also called $B(V)$ distributions in [20] and Gaussian transforms in [10]. A type G distribution is the law of the subordinated Brownian motion B_{V_t} at time $t = 1$, where $\{B_t : t \geq 0\}$ is a Brownian motion independent of a nondecreasing Lévy process (subordinator) $\{V_t : t \geq 0\}$, such that V_1 has the same distribution as V.

Many important examples of classical infinitely divisible symmetric distributions are of type G: symmetric α-stable distributions, $0 < \alpha < 2$, where V is a positive $\alpha/2$-stable random variable, and more generally, convolutions of symmetric stable distributions of different stability indices; the Laplace distribution, where V has the exponential distribution, and more generally, symmetric gamma distributions, where V has the gamma distribution; Student t, where V has the distribution of the reciprocal chi-square distribution; symmetric normal inverse

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Gaussian, with inverse Gaussian distribution for V; and more generally, all the symmetric generalized hyperbolic distributions.

The Bercovici–Pata [8] bijection is a homeomorphism Λ from the set of classical one-dimensional infinitely divisible distributions $I(*)$ to the set of free infinitely divisible distributions $I(\boxplus)$. The bijection Λ is such that if μ is a distribution in $I(*)$ with classical characteristic triplet (a, ν, c), then $\Lambda(\mu)$ has free characteristic triplet (a, ν, c); see also [7]. For an introduction to free probability see the monographs [11,14].

The purpose of this paper is to study the class of free infinitely divisible distributions corresponding to the classical type G distributions under the bijection Λ, which we call free type G distributions. An important role is played by the theory of Upsilon transformations of classical infinitely divisible distributions as recently studied in [2,4,6,13,18] and [19].

We first take a new look to classical type G distributions. This is done by introducing the class CTA of type A distributions on $\mathbb{R}$ as those classical infinitely divisible distributions whose Lévy measures are mixtures of the symmetric arcsine distribution. The building block example is the symmetric compound Poisson arcsine distribution which is the distribution of the random variable $R = \sum_{i=1}^{N} Y_i$, where N is a random variable with classical Poisson distribution of mean one and independent of a sequence $Y_1, Y_2, \ldots$ of random variables with the arcsine distribution on $(-1, 1)$. We then prove a new characterization of Lévy measures in CTG showing that CTG is the image of (the ancestor class) CTA under an appropriate Upsilon type transformation. We conclude that any classical type G random variable has a random integral representation with respect to a Lévy process with type A distribution at time 1.

In the second part of this work we introduce the class of free type G (denoted by FTG) and free type A (denoted by FTA) distributions as free infinitely divisible distributions which are the image of CTG and CTA under Λ, respectively. Analogous characterizations as for the classical case are given in terms of the Lévy measure and the free cumulant transform. We identify the symmetric beta distribution with shape parameter $\alpha = 3/2$ and scale parameter $\beta = 1/2$ as a free infinitely divisible distribution in FTA, but not in FTG, being the image of the classical compound Poisson arcsine distribution under Λ. Moreover, this distribution is identified as the free multiplicative convolution of an arcsine distribution with the Marchenko–Pastur distribution. Its role as a building block for free type G distributions is shown.

The paper is organized as follows. Section 2 recalls basic facts about Upsilon transformations of Lévy measures, free infinite divisibility, multiplicative convolutions and free compound Poisson distributions. In Section 3 we take a new look to classical type G distributions and introduce a new class of classical type A distributions. Section 4 characterizes the free infinitely divisible distributions given by the image of the Bercovici–Pata bijection of CTA and CTG. Finally, in Section 5 we present examples of free type G and free type A distributions, as well as their interpretations as free multiplicative convolutions.

2 Background and notation

2.1 Upsilon transformations and ancestors

Let $I(*)$ be the class of all infinitely divisible distributions on $\mathbb{R}$. For $\mu \in I(*)$, the Lévy–Khintchine representation of its *classical cumulant function* $\mathcal{C}_\mu(z) = \log \widehat{\mu}(z)$ is given by

$$\mathcal{C}_\mu(t) = ict - \frac{1}{2}at^2 + \int_{\mathbb{R}} (e^{itx} - 1 - itx1_{\{|x|\le 1\}})\nu(dx), \qquad t \in \mathbb{R}, \tag{1}$$

where $c \in \mathbb{R}$, $a \ge 0$, and ν is a measure on $\mathbb{R}$ (called the *Lévy measure*) satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(1, |x|^2)\nu(dx) < \infty. \tag{2}$$

The triplet (a, ν, c) is called the generating triplet of $\mu \in I(*)$. Let $\mathfrak{M}_L(\mathbb{R})$ be the class of all Lévy measures of elements in $I(*)$ and let $\mathfrak{M}_L(\mathbb{R}_+)$ be the class of Lévy measures on $\mathbb{R}_+$ with $\nu(-\infty, 0) = 0$ and $\int_{\mathbb{R}_+}(1 \wedge x)\nu(dx) < \infty$. See [17] for a detailed study of classical infinitely divisible distributions on $\mathbb{R}^d$.

The Upsilon transformation Υ_0, introduced in Barndorff-Nielsen and Thorbjørnsen [5,6] and studied further in [2–4], is defined as the mapping on $\mathfrak{M}_L(\mathbb{R})$ into $\mathfrak{M}_L(\mathbb{R})$, given by

$$\Upsilon_0(\nu)(dx) = \int_0^\infty \nu(s^{-1}dx)e^{-s}\, ds. \tag{3}$$

This mapping is one-to-one, smooth and strongly regularizing, and hence $\Upsilon_0(\mathfrak{M}_L(\mathbb{R}))$ is a proper subset of $\mathfrak{M}_L(\mathbb{R})$. For $\rho \in \mathfrak{M}_L(\mathbb{R}_+)$, $\Upsilon_0(\rho) \in \mathfrak{M}_L(\mathbb{R}_+)$ and $\Upsilon_0(\mathfrak{M}_L(\mathbb{R}_+)) = \mathrm{B}(\mathbb{R}_+)$, which is the smallest class which contains all mixtures of classical exponential distributions and is closed under convolution and weak convergence. Moreover, $\mathrm{B}(\mathbb{R}_+)$ is the class of distributions whose Lévy measures have completely monotone densities and it is called the Goldie–Steutel–Bondesson class. In this work we call an infinitely divisible distribution with Lévy measure ν the Υ_0-*ancestor* of the infinitely divisible distribution with Lévy measure $\Upsilon_0(\nu)$.

More specifically, we have the following result, which is used several times in this work. It is a special case of (2.16) and (2.18) in Theorem A of [2].

Lemma 1. (a) *Let $\rho \in \mathfrak{M}_L(\mathbb{R}_+)$. Then the Lévy measure $\Upsilon_0(\rho)$ is absolutely continuous with completely monotone density $\eta(s;\rho)$ given by*

$$\eta(s;\rho) = \int_{\mathbb{R}_+} \tau^{-1}e^{-s\tau^{-1}}\rho(d\tau). \tag{4}$$

Moreover,

$$\eta(s;\rho) = \int_{\mathbb{R}_+} e^{-sr}Q(dr), \tag{5}$$

where Q is a measure given by

$$Q(dr) = r \underset{\leftarrow}{\rho}(dr), \tag{6}$$

and $\underset{\leftarrow}{\rho}$ is the measure on $\mathbb{R}_+$ induced from ρ by the mapping $s \to s^{-1}$.

(b) *Conversely, let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a completely monotone function satisfying*

$$\int_0^\infty \min(1, r)\eta(r)\, dr < \infty. \tag{7}$$

Then there exist a Lévy measure $\rho \in \mathfrak{M}_L(\mathbb{R}_+)$ such that $\Upsilon_0(\rho)$ has density η and therefore η is represented by (4).

The theory of general Upsilon transformations of Lévy measures is discussed in [4]. Of particular interest in the present work is the generalized Upsilon transformation $\Upsilon_{1-\alpha}$, $0 < \alpha \le 1$, defined on $\mathfrak{M}_L(\mathbb{R})$ into $\mathfrak{M}_L(\mathbb{R})$ by

$$\Upsilon_{1-\alpha}(\nu)(dx) = \int_0^\infty \nu(s^{-\alpha}\, dx)e^{-s}\, ds. \tag{8}$$

Like for Υ_0 we call ν the $\Upsilon_{1-\alpha}$-*ancestor* of $\Upsilon_{1-\alpha}(\nu)$.[1]

2.2 Free infinite divisibility

In this section we present several facts about free additive and multiplicative convolutions as well as free infinitely divisible distributions and, in particular, some useful results about free compound Poisson distributions. We refer to [1,7,9,11] and [14] for details and further material.

2.2.1 *Transforms of probability measures.* We first recall several transforms of probability measures that are useful in the analytic theory of free probability. The basic one is the *Cauchy transform* of a probability measure μ on $\mathbb{R}$, defined, for $z \in \mathbb{C} \setminus \mathbb{R}$, by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x}\mu(dx). \tag{9}$$

It is known that G_μ is analytic in $\mathbb{C} \setminus \mathbb{R}$, $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$, where $\mathbb{C}^+ := \{z : \mathrm{Im}(z) > 0\}$, $\mathbb{C}^- := \{z : \mathrm{Im}(z) < 0\}$, and that $\lim_{y\to\infty} iyG_\mu(iy) = 1$. Moreover, the following *inversion formula* holds when the density u of μ exists

$$u(x) = -\frac{1}{\pi} \lim_{y\to 0} \mathrm{Im}\, G_\mu(x + iy). \tag{10}$$

[1] In [13] what is here denoted by $\Upsilon_{1/2}$ is called $\Psi_{-2,2}$.

In connection to free infinite divisibility, the *reciprocal of the Cauchy transform* is useful. It is the function $F_\mu(z) : \mathbb{C}^+ \to \mathbb{C}^+$ defined by

$$F_\mu(z) = \frac{1}{G_\mu(z)}.$$

The *free cumulant transform of a probability* measure μ is

$$\mathcal{C}_\mu^{\boxplus}(z) = zF_\mu^{-1}(z^{-1}) - 1 \tag{11}$$

for z in a domain D_μ of $\mathbb{C}^-$.

On the other hand, for the study of multiplicative convolutions of measures, the Ψ_μ-transform and the S_μ-transform are useful. The first one is defined by

$$\Psi_\mu(z) = z^{-1}G_\mu(z^{-1}) - 1. \tag{12}$$

It was proved in [9] that for probability measures with support on $\mathbb{R}_+$ and such that $\mu(\{0\}) < 1$, the function $\Psi_\mu(z)$ has a unique inverse $\chi_\mu(z)$ in the left-half plane $i\mathbb{C}_+$ and $\Psi_\mu(i\mathbb{C}_+)$ is a region contained in the circle with diameter $(\mu(\{0\}) - 1, 0)$. In this case the S-transform of μ is defined as

$$S_\mu(z) = \chi_\mu(z)\frac{1+z}{z}.$$

Let

$$H = \{z \in \mathbb{C}_+; |\operatorname{Re}(z)| < \operatorname{Im}(z)\}, \qquad \widetilde{H} = \{z \in \mathbb{C}_-; |\operatorname{Re}(z)| < |\operatorname{Im}(z)|\}.$$

Recently, it was proved in [1] that when μ is a symmetric probability measure on $\mathbb{R}$ with $\mu(\{0\}) < 1$, the transform Ψ_μ has a unique inverse on H, $\chi_\mu : \Psi_\mu(H) \to H$ and a unique inverse on $\widetilde{H}$, $\widetilde{\chi}_\mu : \Psi_\mu(\widetilde{H}) \to \widetilde{H}$. In this case there are two S-transforms for μ given by

$$S_\mu(z) = \chi_\mu(z)\frac{1+z}{z} \quad \text{and} \quad \widetilde{S}_\mu(z) = \widetilde{\chi}_\mu(z)\frac{1+z}{z}$$

and these are such that

$$S_\mu^2(z) = \frac{1+z}{z}S_{\mu^2}(z) \quad \text{and} \quad \widetilde{S}_\mu^2(z) = \frac{1+z}{z}S_{\mu^2}(z) \tag{13}$$

for z in $\Psi_\mu(H)$ and $\Psi_\mu(\widetilde{H})$, respectively, where μ^2 is the measure on $\mathbb{R}_+$ induced by the transformation $t \to t^2$. Moreover,

$$z = \mathcal{C}_\mu^{\boxplus}(zS_\mu(z)) \quad \text{and} \quad z = \mathcal{C}_\mu^{\boxplus}(z\widetilde{S}_\mu(z)) \tag{14}$$

for z in $\Psi_\mu(H)$ and $\Psi_\mu(\widetilde{H})$, respectively.

2.2.2 *Additive and multiplicative free convolutions.* The *free additive convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$ such that

$$\mathcal{C}^{\boxplus}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}^{\boxplus}_{\mu_1}(z) + \mathcal{C}^{\boxplus}_{\mu_2}(z) \tag{15}$$

for z in the common domain where $\mathcal{C}^{\boxplus}_{\mu_1}$ and $\mathcal{C}^{\boxplus}_{\mu_2}$ are defined.

On the other hand, following [1], the *free multiplicative convolution* of a probability measure μ_1 supported on $\mathbb{R}_+$ with a symmetric probability measure μ_2 on $\mathbb{R}$ is defined as the symmetric probability measure $\mu_1 \boxtimes \mu_2$ on $\mathbb{R}$ such that

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z) S_{\mu_2}(z) \quad \text{and} \quad \widetilde{S}_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z) \widetilde{S}_{\mu_2}(z). \tag{16}$$

2.2.3 *Relation between free and classical infinite divisibility.* A probability measure μ on $\mathbb{R}$ is *free infinitely divisible* if for $n > 1$ there exists a probability measure $\mu_{1/n}$ on $\mathbb{R}$ such that $\mu = \mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}$ (n times). There is a *free Lévy–Khintchine formula* for the free cumulant transform similar to the classical cumulant function (1). Specifically, μ is free infinitely divisible if and only if

$$\mathcal{C}^{\boxplus}_{\mu}(z) = cz + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - zx} - 1 - zx 1_{[-1,1]}(x) \right) \nu(dx), \qquad z \in \mathbb{C}^-, \tag{17}$$

where $c \in \mathbb{R}, a \geq 0$, and ν is a Lévy measure [7], Proposition 4.16. As in the classical case, the triplet (a, ν, c) is unique.

The *Bercovici–Pata bijection* Λ between classical and free infinitely divisible distributions is such that if μ is infinitely divisible in the classical sense with Lévy–Khintchine representation (1) and characteristic triplet (a, ν, c), then $\Lambda(\mu)$ is free infinitely divisible with free Lévy–Khintchine representation (17) and free characteristic triplet (a, ν, c). This bijection is such that $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$, Λ is preserved under affine transforms, and it is a homeomorphism w.r.t. weak convergence, that is, $\mu_n \Rightarrow \mu$ if and only if $\Lambda(\mu_n) \Rightarrow \Lambda(\mu)$. The Cauchy distribution is a fixed point of Λ.

The most well-known examples of free infinitely divisible distributions are the semicircle and the free Poisson distributions. More specifically, if Z is a classical random variable with the standard Gaussian distribution $\mathcal{L}(Z)$, then $\mathrm{w} = \Lambda(\mathcal{L}(Z))$ is the standard *semicircle distribution* on $(-2, 2)$ given by

$$\mathrm{w}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2} 1_{[-2,2]}(x)\, dx.$$

In this case $\mathcal{C}^{\boxplus}_{\mathrm{w}}(z) = z^2$ and $S_{\mathrm{w}}(z) = 1/\sqrt{z}$.

When $\mathcal{L}(N)$ is the classical Poisson distribution with mean one, $\mathrm{m} = \Lambda(\mathcal{L}(N))$ is the *Marchenko–Pastur distribution* (a special case of the so-called free Poisson distribution)

$$\mathrm{m}(dx) = \frac{1}{2\pi x} \sqrt{x(4 - x)} 1_{[0,4]}(x)\, dx. \tag{18}$$

The free cumulant and S-transforms are given, respectively, by $\mathcal{C}^{\boxplus}_{\mathrm{m}}(z) = \frac{z}{1-z}$ and

$$S_{\mathrm{m}}(z) = \frac{1}{z+1}. \tag{19}$$

2.2.4 *Free compound Poisson distributions.* When μ is a classical compound Poisson distribution, the Lévy measure ν is a finite measure and the classical Lévy–Khintchine representation (1) takes the form

$$\mathcal{C}_{\mu}(t) = \int_{\mathbb{R}} (e^{itx} - 1)\nu(dx).$$

In this case we say that μ has CCP(ν) distribution. The corresponding free infinitely divisible distribution $\Lambda(\mu)$ is called the *free compound Poisson distribution*—denoted by FCP(ν)— with free cumulant transform

$$\mathcal{C}^{\boxplus}_{\Lambda(\mu)}(z) = \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1\right)\nu(dx). \tag{20}$$

The first part of the following theorem is useful to identify some free compound Poisson distributions. The second part gives an interpretation of some symmetric FCP(ν) distributions as the multiplicative convolution of the Marchenko–Pastur distribution $\mathrm{m} = \Lambda(\mathcal{L}(N))$ with the measure ν. The latter is an extension of Proposition 12.18 in [14] for compactly supported probability measures to symmetric probability measures with unbounded support.

Theorem 2. *Let ν be a probability distribution on $\mathbb{R}$.*

(a) *If μ has the classical compound distribution* FCP(ν), *the Cauchy transform $G_{\Lambda(\mu)}$ of the free compound Poisson distribution* FCP(ν) *satisfies the equation*

$$G_{\Lambda(\mu)}(z^2 G_{\nu}(z)) = z^{-1}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{21}$$

(b) *If ν is a symmetric probability distribution on $\mathbb{R}$, the distribution* FCP(ν) *is symmetric and* FCP(ν) $= \mathrm{m} \boxtimes \nu$.

Proof. (a) From (20), since ν is a probability measure we have

$$\mathcal{C}^{\boxplus}_{\Lambda(\mu)}(z) = \int_{\mathbb{R}} \frac{1}{1-zx}\nu(dx) - 1 \tag{22}$$

$$= z^{-1}\int_{\mathbb{R}} \frac{1}{z^{-1}-x}\nu(dx) - 1$$

$$= z^{-1} G_{\nu}(z^{-1}) - 1 \tag{23}$$

$$= \Psi_{\nu}(z). \tag{24}$$

Taken together with (11) this gives

$$F^{-1}_{\Lambda(\mu)}(z) = z^2 G_\nu(z) \tag{25}$$

which is equivalent to (21).

(b) From (16) and (19) the S-transform of $\lambda = \Lambda(\mathcal{L}(N)) \boxtimes \nu$ is given by

$$S_\lambda(z) = \frac{1}{z+1} S_\nu = \frac{\chi_\nu(z)}{z}.$$

Then

$$zS_\lambda(z) = \chi_\nu(z),$$
$$\Psi_\nu(zS_\lambda(z)) = \Psi_\nu(\chi_\nu(z)) = z$$

and similarly for $\widetilde{S}_\nu$ and $\widetilde{\chi}_\nu$. This means by equation (14) that

$$\mathcal{C}^{\boxplus}_\lambda(z) = \Psi_\nu(z)$$

which by (23) and the uniqueness of the Lévy–Khintchine representation gives that $\lambda = \Lambda(\mu)$. □

3 A new look to classical type G distributions

3.1 Alternative expression for the Lévy measure

Let X have a classical type G distribution $\mathcal{L}(V^{1/2}Z)$ and let ρ be the Lévy measure of V. When V is nondegenerate, X has a symmetric non-Gaussian infinitely divisible distribution with characteristic triplet $(0, \nu, 0)$, where ν has a symmetric Lévy density υ given by

$$\upsilon(x) = \int_{\mathbb{R}_+} \varphi(x;\tau)\rho(d\tau), \qquad x \in \mathbb{R}, \tag{26}$$

and where $\varphi(x,\tau)$ denotes the density function of the one-dimensional Gaussian distribution with zero mean and variance τ (see [16,20]). Moreover, it is well known that the classical cumulant transform of a type G distribution is given by

$$\mathcal{C}^*_\mu(t) = \int_{\mathbb{R}_+} (e^{-t^2\tau/2} - 1)\rho(d\tau), \qquad t \in \mathbb{R}. \tag{27}$$

The following result expresses the Lévy measure of a type G distribution in terms of mixtures of arcsine distributions and the Upsilon transformation Υ_0 of the Lévy measure ρ. Recall

$$\mathrm{a}(x,s) = \begin{cases} \dfrac{1}{\pi}(s-x^2)^{-1/2}, & |x| < \sqrt{s}, \\ 0, & |x| > \sqrt{s}, \end{cases} \tag{28}$$

is the density of the arcsine distribution a_s on $(-\sqrt{s}, \sqrt{s})$; in particular, a_2 is the standard arcsine distribution, that is, a_2 has zero mean and variance one.

Following the notation in Lemma 1, for $\rho \in \mathfrak{M}_L(\mathbb{R}_+)$, we denote by $\eta(\cdot; \rho)$ the Lévy density of $\Upsilon_0(\rho)$.

Proposition 3. *Let μ be a classical type G distribution $\mathcal{L}(V^{1/2}Z)$, with V having Lévy measure ρ. The Lévy measure ν of μ has a density υ given by*

$$\upsilon(x) = \int_0^\infty \mathrm{a}(x; s)\eta(s; \rho)\, ds. \tag{29}$$

Proof. From a well-known result by Box and Muller, the standard Gaussian distribution is the distribution of $E^{1/2}A$, where A and E are independent random variables with E having the exponential distribution with mean 2 and A having the arcsine distribution on $(-1, 1)$. In fact, it is easy to prove that

$$\varphi(x; 1) = \int_0^\infty \frac{1}{2} e^{-s/2} \mathrm{a}(s^{1/2}x, 1)s^{-1}\, ds = \int_0^\infty e^{-s} \mathrm{a}(x, s)\, ds$$

and therefore for $t > 0$

$$\varphi(x; t) = \int_0^\infty \frac{1}{t} e^{-s/t} \mathrm{a}(x, s)\, ds, \qquad x \in \mathbb{R}. \tag{30}$$

Thus, using the last expression in (26), we obtain

$$\begin{aligned} \upsilon(x) &= \int_{\mathbb{R}_+} \frac{1}{\tau} \int_0^\infty e^{-s/\tau} \mathrm{a}(x, s)\, ds\, \rho(d\tau) \\ &= \int_0^\infty \mathrm{a}(x, s) \int_{\mathbb{R}_+} \frac{1}{\tau} e^{-s/\tau} \rho(d\tau)\, ds \\ &= \int_0^\infty \mathrm{a}(x, s)\eta(s; \rho)\, ds, \end{aligned}$$

which proves the result. □

3.2 Characterization

Rosinski [16] proved the following characterization for Lévy measures of type G distributions.

Theorem 4. *A symmetric probability measure μ on $\mathbb{R}$ is of type G if and only if it is infinitely divisible and its Lévy measure ν is either zero or*

$$\nu(dr) = g(r^2)\, dr, \tag{31}$$

where $g(r)$ is a completely monotone function in $r \in (0, \infty)$ such that

$$\int_0^\infty \min(1, r^2) g(r^2)\, dr < \infty.$$

The following result is a new characterization of Lévy measures of type G distributions. It is useful to identify the ancestor class of type G distributions under the transformation $\Upsilon_{1/2}$.

Theorem 5. *A symmetric probability measure μ on $\mathbb{R}$ is of type G if and only if it is infinitely divisible and its Lévy measure ν is either zero or has a Lévy density υ representable as*

$$\upsilon(dx) = \int_0^\infty \mathrm{a}(x; s)\eta(s)\, ds, \tag{32}$$

where η is a completely monotone function in $s \in (0, \infty)$ such that

$$\int_0^\infty \min(1, s)\eta(s)\, ds < \infty. \tag{33}$$

Proof. The only if part is given by Proposition 3 and Lemma 1(a). On the other hand, suppose υ is the Lévy density of an infinitely divisible symmetric distribution μ satisfying (32) and (33). From Lemma 1(b), η is the Lévy density of $\Upsilon_0(\rho)$ for a Lévy measure ρ in $\mathfrak{M}_L(\mathbb{R}_+)$. Therefore from (26) and (29), μ is the distribution $\mathcal{L}(V^{1/2}Z)$ where V is infinitely divisible with Lévy measure ρ. $\square$

Since a completely monotone function does not have bounded support, we obtain:

Corollary 6. *A type G Lévy measure is not a finite range mixture of arcsine measures.*

3.3 The $\Upsilon_{1/2}$-ancestor class of CTG: Type A distributions

We now define the class of *type A distributions* on $\mathbb{R}$ as those symmetric infinitely divisible distributions whose Lévy measure ν is either zero or has a density υ of the form

$$\upsilon(x) = \int_{\mathbb{R}_+} \mathrm{a}(x; s)\rho(ds) \tag{34}$$

for some Lévy measure $\rho \in \mathfrak{M}_L(\mathbb{R}_+)$. We denote by CTA the class of type A distributions on $\mathbb{R}$ and observe that it is indeed well defined. When we talk about free type A distributions, which will appear later, we refer to type A distributions as classical type A distributions.

Lemma 7. *For any $\rho \in \mathfrak{M}_L(\mathbb{R}_+)$ the function υ given by the expression (34) defines a symmetric Lévy density.*

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} \min(1, x^2)\upsilon(x)\,dx &= \int_{-\infty}^{\infty} \min(1, x^2) \int_{\mathbb{R}_+} \mathrm{a}(x; s)\rho(ds)\,dx \\ &= \frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} \int_{\mathbb{R}_+} \min(1, sy^2)\rho(ds)\,dy \\ &\leq \frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} \int_{\mathbb{R}_+} \min(1, s)\rho(ds)\,dy \\ &= \int_{\mathbb{R}_+} \min(1, s)\rho(ds) < \infty,\end{aligned}$$

since ρ is a Lévy measure on $\mathbb{R}_+$. □

From Proposition 3(a) we have that CTG is a subclass of CTA. Examples of type A distributions that are not type G are obtained when the Lévy measure ρ has bounded support. This follows from Corollary 6. Concrete examples of this are obtained when υ is the arcsine distribution $\mathrm{a}(x, \tau)$ corresponding to $\rho = \delta_\tau$, or υ is the semicircle distribution $\mathrm{w}(x, 1)$ obtained from (34) when ρ is the distribution of $U^{1/2}$, where U has uniform distribution on (0, 1).

The following result gives a characterization of type G distributions as the image of type A distributions under the transformation $\Upsilon_{1/2}$. We recall that the mapping $\Upsilon_{1/2}$ is defined in (8).

Theorem 8. $CTG = \Upsilon_{1/2}(CTA)$.

Proof. Let $\nu \in CTA$. Using (8), (34) and Lemma 1(a)

$$\begin{aligned}\Upsilon_{1/2}(\nu)(dx) &= \int_0^{\infty} \upsilon(r^{-1/2}x)r^{-1/2}e^{-r}\,dr\,dx \\ &= \int_0^{\infty}\int_{\mathbb{R}_+} r^{-1/2}\mathrm{a}(r^{-1/2}x; s)e^{-r}\rho(ds)\,dr\,dx \\ &= \int_0^{\infty}\int_{\mathbb{R}_+} \mathrm{a}(x; rs)e^{-r}\rho(ds)\,dr\,dx \qquad (35) \\ &= \int_0^{\infty} \mathrm{a}(x; y)\int_0^{\infty} e^{-r}\rho(r^{-1}dy)\,dr\,dx \\ &= \int_0^{\infty} \mathrm{a}(x; s)\eta(s; \rho)\,ds\,dx. \qquad (36)\end{aligned}$$

Since, by the second part of Lemma 1(a), $\eta(s; \rho)$ is completely monotone in s, on account of Theorem 5, $\Upsilon_{1/2}(\nu)$ is the Lévy measure of a type G distribution. Hence $\Upsilon_{1/2}(CTA) \subset CTG$.

On the other hand, let μ be a type G distribution $\mathcal{L}(V^{1/2}Z)$ with V having Lévy measure ρ and let $\widetilde{\nu}$ be the Lévy measure of μ. From Proposition 3(a) we have that the Lévy density $\widetilde{\upsilon}$ of $\widetilde{\nu}$ is given by

$$\widetilde{\upsilon}(x) = \int_0^\infty \mathrm{a}(x; s)\eta(s; \rho)\, ds.$$

Let ν be the Lévy measure whose density is given by (34). Then from (36) and the uniqueness of Lévy measures we have $\widetilde{\nu} = \Upsilon_{1/2}(\nu)$, that is, $CTG \subset \Upsilon_{1/2}(CTA)$. $\square$

As a consequence of the above theorem and Theorem 2.1 in [13] we deduce the following integral representation of type G distributions.

Theorem 9. *An infinitely divisible random variable X is of type G if and only if there is a Lévy process $\{A_t; t \geq 0\}$ with type A distribution at time* 1 *such that*

$$\mathcal{L}(X) = \mathcal{L}\left(\int_0^{1/2} (-\log(2t))^{1/2}\, dA_t\right).$$

Proof. Let ν be a Levy measure on $\mathbb{R}$. Using change of variables $y = s^2$ and (8) with $\alpha = 1/2$ we have

$$\int_0^\infty \nu(s^{-1}\, dx) s e^{-s^2}\, ds = \frac{1}{2}\int_0^\infty \nu(s^{-1/2}\, dx) e^{-s}\, ds = \frac{1}{2}\Upsilon_{1/2}(\nu)(dx). \tag{37}$$

If ν is the Lévy measure of a type G, from Theorem 8 there is a Lévy measure ν_0 of a type A distribution such that $\nu = \Upsilon_{1/2}(\nu_0)$. Using the same notation $\Upsilon_{1/2}$ for mappings of Lévy measures and mappings of their corresponding classical infinitely divisible distributions, we observe that in the notation in [13], $g(u) = G_{-2,2}(u) = \int_u^\infty x e^{-x^2}\, dx = \frac{1}{2}\exp(-u^2)$ and $\Upsilon_{1/2} = 2\Psi_{-2,2}$. Let $\{A_t; t \geq 0\}$ be a Lévy process such that $\mathcal{L}(A_1) = \mathcal{L}(A)$. From Theorem 2.1 in [13] $\Upsilon_{1/2}(\mathcal{L}(A)) = \mathcal{L}(\int_0^{1/2} f(t)\, dA_t)$, where f is the inverse function of $g(u)$, that is, $f(t) = (-\log(2t))^{1/2}$. The result follows by the relation between Upsilon transformations of Lévy measures and the random integral representations of their corresponding distributions, as explained in Section 9 of [4] and by observing that the triplet $(0, \nu, 0)$ of a type G distribution and the triplet $(0, \nu_0, 0)$ of a type A distribution only depend on their Lévy measures. $\square$

4 Free type A and free type G distributions

We say that a free infinitely divisible distribution λ is *free* type G if there is a classical type G probability measure μ such that $\lambda = \Lambda(\mu)$. Similarly, we say that a free infinitely divisible distribution λ is *free* type A if there is a classical type A probability measure μ such that $\lambda = \Lambda(\mu)$. We denote by FTG and FTA the classes of free type G and free type A distributions on $\mathbb{R}$, respectively.

The semicircle distribution is a free type G and a free type A distribution. Additional examples are provided in the last section of this paper.

By the Bercovici–Pata bijection Λ, the characterization of Lévy measures of classical type G and classical type A distributions given in Section 3 hold as well in the free case. We next present a characterization in terms of the free cumulant transform. In view of Theorem 8 we first consider the ancestor class of free type A distributions.

4.1 The free $\Upsilon_{1/2}$-ancestor class

The arcsine probability measure plays an important role in the study of free type A distributions. It is then of interest to study the corresponding free compound Poisson arcsine distribution which we are able to identify in an explicit manner.

Let λ_s be the free compound Poisson distribution FCP(a_s), for $s > 0$. We first derive the Cauchy transform and then the density of FCP(a_s). Recall (see, e.g., [11]) that the Cauchy transform of the arcsine measure a_s on $(-\sqrt{s}, \sqrt{s})$ is given by

$$G_{\mathrm{a}_s}(z) = \left(\sqrt{z^2 - s}\right)^{-1}. \tag{38}$$

Lemma 10. *The Cauchy transform of the probability measure λ_s of the free compound Poisson distribution* FCP(a_s) *is given by*

$$G_{\lambda_s}(z) = \frac{1}{\sqrt{2s}}\sqrt{1 - \sqrt{z^{-2}(z^2 - 4s)}}. \tag{39}$$

Proof. Using (38) in (21), we have that λ_s is such that

$$G_{\lambda_s}\left(\frac{z^2}{\sqrt{z^2 - s}}\right) = z^{-1}.$$

Making the change of variable

$$r = \frac{z^2}{\sqrt{z^2 - s}},$$

we observe that $r \in \mathbb{C}^+$ when $z \in \mathbb{C}^+$ and that r and z satisfy

$$z^4 - z^2 r^2 + r^2 s = 0.$$

Solving for z^2,

$$z^2 = \frac{r^2 \pm \sqrt{r^2(r^2 - 4s)}}{2}$$

and hence

$$z = \pm\sqrt{\frac{r^2 \pm \sqrt{r^2(r^2-4s)}}{2}}.$$

Then, the potential candidates for G_λ are

$$\frac{\sqrt{2}}{\pm\sqrt{z^2 \pm \sqrt{z^2(z^2-4s)}}}.$$

Since G_{λ_s} must be such that $G_{\lambda_s} : \mathbb{C}^+ \to \mathbb{C}^-$ and $|z|G_{\lambda_s}(z) \to 1$ when $|z| \to \infty$, we deduce that

$$G_{\lambda_s}(z) = \frac{\sqrt{2}}{\sqrt{z^2 + \sqrt{z^2(z^2-4s)}}}.$$

Then, multiplying and dividing by $(\sqrt{z^2 - (z^2(z^2-4s))^{1/2}})$,

$$\begin{aligned} G_{\lambda_s}(z) &= \frac{\sqrt{2}\sqrt{z^2 - \sqrt{z^2(z^2-4s)}}}{\sqrt{z^4 - z^2(z^2-4s)}} \\ &= \frac{\sqrt{2}\sqrt{z^2 - \sqrt{z^2(z^2-4s)}}}{\sqrt{4sz^2}} \\ &= \frac{1}{\sqrt{2s}}\sqrt{1 - \sqrt{z^{-2}(z^2-4s)}}, \end{aligned}$$

as we wanted to prove. □

We next identify the distribution λ_s as the symmetric beta distribution $\mathrm{SB}_s(\frac{3}{2}, \frac{1}{2})$ on $(-2\sqrt{s}, 2\sqrt{s})$. Recall that for $\alpha, \beta > 0$, a probability measure has *the symmetric beta distribution* $\mathrm{SB}_s(\alpha, \beta)$ on $(-2\sqrt{s}, 2\sqrt{s})$ if it is absolutely continuous with density

$$g(x) = \frac{1}{2B(\alpha,\beta)\sqrt{s}}|x|^{\alpha-1}\left(2\sqrt{s} - |x|\right)^{\beta-1}, \qquad |x| < 2\sqrt{s}.$$

Proposition 11. *The probability measure λ_s with Cauchy transform* (39) *has the symmetric beta distribution* $\mathrm{SB}_s(\frac{3}{2}, \frac{1}{2})$ *with density*

$$g_{\lambda_s}(x) = \frac{1}{2\pi\sqrt{s}}|x|^{-1/2}\left(2\sqrt{s} - |x|\right)^{1/2}, \qquad |x| < 2\sqrt{s}. \tag{40}$$

Proof. From the inversion formula (10), the density of G_{λ_s} is given by

$$g_{\lambda_s}(x) = -\frac{1}{\pi\sqrt{2s}}\,\mathrm{Im}\sqrt{1 - \sqrt{x^{-2}(x^2-4s)}}.$$

We notice that there is an imaginary part when $|x| < 2\sqrt{s}$ and $x \neq 0$. Thus, we are looking for $b < 0$ such that

$$\sqrt{1 - \sqrt{x^{-2}(x^2 - 4s)}} = a + ib.$$

That is,

$$1 - i\sqrt{x^{-2}(4s - x^2)} = a^2 - b^2 + 2iab;$$

so $a^2 = b^2 + 1$, and

$$\sqrt{x^{-2}(4s - x^2)} = -2ab$$

if and only if

$$x^{-2}(4s - x^2) = 4a^2b^2 = 4b^4 + 4b^2.$$

Then, solving for b^2 in the equation

$$b^4 + b^2 - \tfrac{1}{4}x^{-2}(4s - x^2) = 0$$

we obtain

$$b^2 = \frac{-1 \pm \sqrt{1 + x^{-2}(4s - x^2)}}{2}.$$

Since b^2 is real and nonnegative, we have

$$b^2 = \tfrac{1}{2}|x|^{-1}(\sqrt{4s} - |x|)$$

and therefore

$$b = -\sqrt{\tfrac{1}{2}|x|^{-1}(\sqrt{4s} - |x|)},$$

from where we obtain (40). $\square$

The above result gives a concrete example of a free infinitely divisible distribution which belongs to FTA but not to FTG. This follows from Corollary 6 since $FTG = \Lambda(CTG)$. An interpretation of this distribution as the multiplicative convolution of the Marchenko–Pastur and an arcsine distribution is proved in Section 5.

Since the symmetric beta distribution has been derived as the symmetric free infinitely divisible distribution with arcsine Lévy measure, we trivially have the following result.

Proposition 12. *For each $s > 0$ the symmetric beta distribution $\mathrm{SB}_s(\frac{3}{2}, \frac{1}{2})$ has free cumulant transform*

$$\mathcal{C}^{\boxplus}_{\lambda_s}(z) = \frac{1}{\sqrt{1 - sz^2}} - 1. \tag{41}$$

Proof. Taking $\nu = \mathrm{a}_s$ in (23) and using (38) we obtain

$$\begin{aligned}\mathcal{C}^{\boxplus}_{\lambda_s}(z) &= \int_{\mathbb{R}}\left(\frac{1}{1-zx} - 1\right)\mathrm{a}(x;s)\,dx\\ &= z^{-1}G_{\nu}(z^{-1}) - 1\\ &= \frac{1}{\sqrt{1-sz^2}} - 1\end{aligned}$$

as we wanted to prove. □

4.2 Free type *A* distributions

We begin with a characterization of free infinitely divisible distributions in FTA. It gives the free cumulant of a distribution in FTA as a mixture of the free cumulant transforms of the symmetric beta distributions $\mathrm{SB}_s(\frac{3}{2}, \frac{1}{2})$, $s > 0$.

Theorem 13. *A symmetric free probability distribution* λ *belongs to* FTA *if and only if its cumulant transform is expressible as*

$$\mathcal{C}^{\boxplus}_{\lambda}(z) = \int_{\mathbb{R}_+}\mathcal{C}^{\boxplus}_{\lambda_s}(z)\rho(ds) = \int_{\mathbb{R}_+}\left(\frac{1}{\sqrt{1-sz^2}} - 1\right)\rho(ds) \tag{42}$$

for some Lévy measure $\rho \in \mathfrak{M}_L(\mathbb{R}_+)$, *and where* λ_s *is the symmetric beta distribution* $\mathrm{SB}_s(1/2, 3/2)$ *with density* (40). *Moreover*

$$\mathcal{C}^{\boxplus}_{\lambda}(z) = \int_{\mathbb{R}_+}\mathcal{C}^{\boxplus}_{\mathrm{m}\boxtimes\mathrm{a}_s}(z)\rho(ds), \tag{43}$$

where m *is the Marchenko–Pastur distribution and* a_s *the arcsine distribution on* $(-\sqrt{s}, \sqrt{s})$.

Proof. Let V be a nonnegative infinitely divisible random variable with Lévy measure ρ. Let $\mu \in CTA$ with Lévy measure given by (34). Then, from (22) and (41) we obtain

$$\begin{aligned}\mathcal{C}^{\boxplus}_{\Lambda(\mu)}(z) &= \int_{\mathbb{R}}\left(\frac{1}{1-zx} - 1\right)\nu(dx)\\ &= \int_{\mathbb{R}_+}\left\{\int_{\mathbb{R}}\left(\frac{1}{1-zx} - 1\right)\mathrm{a}(x;s)\,dx\right\}\rho(ds)\\ &= \int_{\mathbb{R}_+}\mathcal{C}^{\boxplus}_{\lambda_s}(z)\rho(ds)\\ &= \int_{\mathbb{R}_+}\left(\frac{1}{\sqrt{1-sz^2}} - 1\right)\rho(ds)\end{aligned}$$

which, by the uniqueness of the free cumulant transform, shows that $\lambda = \Lambda(\mu)$, and conversely. To prove (43) we use Proposition 17 in Section 5 to obtain $\mathcal{C}^{\boxplus}_{\lambda_s}(z) = \mathcal{C}^{\boxplus}_{\mathrm{m}\boxtimes\mathrm{a}_s}(z)$. The result then follows from the first equality in (42). □

4.3 Free type G distributions

The free cumulant transforms of free type G distributions are also mixtures of the free cumulant transforms of the symmetric beta distributions $\mathrm{SB}_s(\frac{3}{2}, \frac{1}{2})$, $s > 0$.

Theorem 14. *A symmetric probability measure τ in $\mathbb{R}$ is FTG if and only if it is the semicircle distribution or its free cumulant transform is given by*

$$\mathcal{C}_\tau^{\boxplus}(z) = \int_{\mathbb{R}_+} \mathcal{C}_{\lambda_s}^{\boxplus}(z)\rho(ds) = \int_0^\infty \left(\frac{1}{\sqrt{1-sz^2}} - 1\right)\eta(s)\,ds, \tag{44}$$

where η is a completely monotone function in $\mathbb{R}_+$ such that $\int_0^\infty \min(1, r)\eta(r)\,dr < \infty$. Moreover

$$\mathcal{C}_\tau^{\boxplus}(z) = \int_{\mathbb{R}_+} \mathcal{C}_{\mathrm{m}\boxtimes\mathrm{a}_s}^{\boxplus}(z)\eta(s)\,ds, \tag{45}$$

where m *is the Marchenko–Pastur distribution and* a_s *the arcsine distribution on* $(-\sqrt{s}, \sqrt{s})$.

Proof. Suppose $\tau \in FTG$. Then $\tau = \Lambda(\mu)$, where $\mu \in CTG$, that is, μ is the distribution of $\mathcal{L}(V^{1/2}Z)$ with V having Lévy measure ρ. Then the Lévy measure ν of μ (and τ) has a Lévy density υ giving by (29). Therefore using (22) and (41) we obtain

$$\begin{aligned}
\mathcal{C}_{\Lambda(\mu)}^{\boxplus}(z) &= \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1\right)\upsilon(x)\,dx \\
&= \int_{\mathbb{R}} \int_0^\infty \left(\frac{1}{1-zx} - 1\right)\mathrm{a}(x;s)\eta(s;\rho)\,ds\,dx \\
&= \int_0^\infty \eta(s;\rho)\,ds \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1\right)\mathrm{a}(x;s)\,dx \\
&= \int_0^\infty \left(\frac{1}{\sqrt{1-sz^2}} - 1\right)\eta(s;\rho)\,ds,
\end{aligned}$$

where from Lemma 1(a), $\eta(s;\rho)$ is completely monotone in $\mathbb{R}_+$. Conversely, if (44) is satisfied with η a completely monotone Lévy density, then by Lemma 1(b) there is a Lévy measure ρ in $\mathbb{R}_+$ such that η is the density of $\Upsilon_0(\rho)$. Taking $\mu \in CTG$ of the form $\mathcal{L}(V^{1/2}Z)$ with V having Lévy measure ρ, from the above calculations we have that (44) is the free cumulant transform of $\Lambda(\mu)$ and therefore $\Lambda(\mu)$ belongs to FTG. To prove (45) we use Proposition 17 in Section 5 to obtain $\mathcal{C}_{\lambda_s}^{\boxplus}(z) = \mathcal{C}_{\mathrm{m}\boxtimes\mathrm{a}_s}^{\boxplus}(z)$. The result then follows from the first equality in (44). $\square$

5 Examples and multiplicative convolutions

In this section we describe some free type A distributions in terms of multiplicative convolutions. As before we use the notation $\mathcal{L}(X)$ for the distribution of a classical random variable X. In particular, we denote by Z, N and A_s, $s > 0$, classical random variables with standard Gaussian distribution $\mathcal{L}(Z)$, Poisson distribution $\mathcal{L}(N)$ of mean one and arcsine distribution $\mathrm{a}_s = \mathcal{L}(A_s)$ on $(-\sqrt{s}, \sqrt{s})$. Then $\mathrm{w} = \Lambda(\mathcal{L}(Z))$ is the semicircle distribution on $(-2, 2)$ and $\mathrm{m} = \Lambda(\mathcal{L}(N))$ is the Marchenko–Pastur distribution on $(0, 4)$ given by (18).

Using the characterization Theorem 5 of classical type G distributions, we first obtain the following interpretation of some symmetric free compound Poisson type G distributions and some symmetric free compound Poisson type A distributions. Recall that distributions in CTG and in CTA are symmetric and therefore they have symmetric Lévy measures.

Proposition 15. *Let $\mathcal{L}(V^{1/2}Z) \in CTG$ have a Lévy measure ν with density*

$$\upsilon(x) = \int_0^\infty \mathrm{a}(x; s)\eta(s)\, ds, \tag{46}$$

where η is a probability density. Then

$$\Lambda(\mathcal{L}(V^{1/2}Z)) = \mathrm{m} \boxtimes \mathcal{L}(AH), \tag{47}$$

where H is a random variable with distribution η independent of the arcsine random variable A on $(-1, 1)$.

Proof. The measure ν is a (symmetric) probability measure on $\mathbb{R}$, since η is a probability density on $\mathbb{R}_+$. Then we can apply Theorem 2 to obtain

$$\Lambda(\mathcal{L}(V^{1/2}Z)) = \mathrm{m} \boxtimes \nu.$$

Finally, υ given by (46) is the density of the scale mixture $A_1 H$, where H has distribution with density η independent of the arcsine random variable A on $(-1, 1)$. □

In a very similar manner we can obtain the more general result for free compound Poisson type A distributions.

Proposition 16. *Let $\mu \in FTA$ have a symmetric Lévy measure ν with density*

$$\upsilon(dx) = \int_0^\infty \mathrm{a}(x; s)\rho(ds) \tag{48}$$

for some probability measure ρ on $\mathbb{R}_+$. Then $\Lambda(\mu) = \mathrm{m} \boxtimes \mathcal{L}(AH)$, where H is a random variable with distribution ρ independent of the arcsine random variable A.

Next, we use Proposition 15 to describe some free type A distributions in terms of multiplicative convolution.

5.1 A symmetric beta distribution

In Proposition 11 we showed that the FCP(a_s) distribution has the density of the symmetric beta distribution $\mathrm{SB}_s(3/2, 1/2)$ on $(-2\sqrt{s}, 2\sqrt{s})$ given by

$$g_{\lambda_s}(x) = \frac{1}{2\pi\sqrt{s}}|x|^{-1/2}\left(2\sqrt{s} - |x|\right)^{1/2}, \qquad |x| < 2\sqrt{s}.$$

The following result shows that the free type A $\Upsilon_{1/2}$-ancestor given by the symmetric beta distribution $\mathrm{SB}_s(3/2, 1/2)$ on $(-2\sqrt{s}, 2\sqrt{s})$, has a representation as the multiplicative convolution of the Marchenko–Pastur distribution on $(0, 4)$ with the arcsine distribution a_s on $(-\sqrt{s}, \sqrt{s})$.

Proposition 17. *Let* λ_s *be the symmetric beta distribution* $\mathrm{SB}_s(3/2, 1/2)$ *on* $(-2\sqrt{s}, 2\sqrt{s})$. *Then* $\lambda_s = \mathrm{m} \boxtimes \mathrm{a}_s$.

Proof. From Theorem 2 we have that $\mathrm{m} \boxtimes \mathrm{a}_s$ has the same distribution as FCP(a_s) which is a $\mathrm{SB}_s(3/2, 1/2)$ distribution on $(-2\sqrt{s}, 2\sqrt{s})$ by Proposition 11. □

Remark 18. (a) From the above proposition we are able to give an interpretation of the symmetric beta distribution in terms of nonconmutative random variables. Let s and a be nonconmutative random variables in free relation in a nonconmutative probability space (A, τ), where a is a Haar unitary element with $*$-distribution a_4 on $(-2, 2)$ and s is a semicircular element with $*$-distribution w on $(-2, 2)$; see, for example, [15]. Then the above proposition says that $\mathrm{SB}_4(3/2, 1/2)$ is the $*$-distribution of the nonconmutative random variable $s^2(a + a^*)$.

(b) The symmetric beta $\mathrm{SB}_s(3/2, 1/2)$ is an explicit example of a distribution in FTA which is not in FTG. This is a consequence of Corollary 6. since m has finite range.

5.2 Free Normal Poisson distribution

Consider the Normal Poisson type G distribution $\mathcal{L}(N^{1/2}Z)$. We will show that the corresponding free type G distribution $\Lambda(\mathcal{L}(N^{1/2}Z))$, called the *free Normal Poisson distribution*, has an interpretation as the multiplicative convolution of the Marchenko–Pastur distribution with the Gaussian distribution.

Recall that $N^{1/2}Z$ is a symmetric CCP(ν) distribution with Lévy measure ν equal to the standard Gaussian measure.

Proposition 19. *The following two representations of the free Normal Poisson distribution* $\Lambda(\mathcal{L}(N^{1/2}Z))$ *hold*:

(a)

$$\Lambda(\mathcal{L}(N^{1/2}Z)) = \mathrm{m} \boxtimes \mathcal{L}(Z),$$

(b)

$$\Lambda(\mathcal{L}(N^{1/2}Z)) = \mathrm{m} \boxtimes \mathcal{L}(E^{1/2}A),$$

where E *is an exponential random variable of mean* 2 *independent of the arcsine random variable* A *on* $(-1, 1)$.

Proof. (a) Since $\mathcal{L}(N^{1/2}Z)$ is a symmetric CCP(ν) distribution with the Gaussian distribution as its Lévy measure ν, by the Bercovici–Pata bijection $\Lambda(\mathcal{L}(N^{1/2}Z))$ is a FCP(ν) with the same Gaussian Lévy measure ν. From Theorem 2 $\Lambda(\mathcal{L}(N^{1/2}Z))$ is the free multiplicative convolution of $\mathcal{L}(N)$ and the Gaussian distribution $\mathcal{L}(Z)$. The proof of (b) follows from (a) and the fact that the standard Gaussian distribution $\mathcal{L}(Z)$ is the distribution of $E^{1/2}A$, where A and E are independent random variables, E having the exponential distribution with mean 2 and A with the arcsine distribution on $(-1, 1)$. □

5.3 Semicircle Marchenko–Pastur distribution

Finally, we present another example of a distribution in FTA which is not in FTG. It is the multiplicative convolution of the Marchenko–Pastur distribution with the semicircle distribution, recently considered in [15]. Here we show that this distribution is free type A. We recall that semicircle distribution w_s on $(-2s, 2s)$ is given by

$$\mathrm{w}_s(dx) = \frac{1}{2\pi s^2}\sqrt{4s^2 - x^2}\,1_{[-2s,2s]}(x)\,dx.$$

Proposition 20. *For each* $s > 0$, *let* μ_s *be the* FCP(w_s) *distribution where* w_s *is the semicircle distribution on* $(-2s, 2s)$. *Then* $\Lambda(\mu_s) \in FTA$ *and* $\Lambda(\mu_s) = \mathrm{m} \boxtimes \mathrm{w}_s$. *Furthermore*, $\Lambda(\mu_s)$ *does not belong to* FTG.

Proof. Using Theorem 2 we have $\Lambda(\mu_s) = \mathrm{m} \boxtimes \mathrm{w}_s$. The fact that $\Lambda(\mu_s)$ is in FTA but not in FTG follows from Corollary 6, since it is well known and easy to check that the semicircle distribution w_1 is the law of $\mathcal{L}(U^{1/2}A)$, where U is a random variable with uniform distribution on $(0, 1)$ independent of A, and since U has finite range. □

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