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ON THE COMPOSITION IDEALS OF LIPSCHITZ MAPPINGS

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Abstract. We study some properties of Lipschitz mappings which admit factorization through an operator ideal. Lipschitz cross norms have been established from known tensor norms in order to represent certain classes of Lipschitz mappings. Inspired by the definition of p-summing linear operators, we derive a new class of Lipschitz mappings that is called *strictly Lipschitz p-summing*.

1. Introduction and preliminaries

Let X be a metric space, and let E be a Banach space. Every Lipschitz mapping $T : X \longrightarrow E$ admits a factorization of the form

$$T = \widehat{T} \circ \delta_X, \tag{1.1}$$

where $\widehat{T}$ is the linearization of T, and δ_X is the canonical embedding. Let $\mathcal{I}$ be an operator ideal. There is a constructive method for defining new classes of Lipschitz mappings which consists of the composing of linear operators of $\mathcal{I}$ and Lipschitz mappings, and the resulting space is denoted by $\mathcal{I} \circ \mathrm{Lip}_0$. This technique is usually used to generate some ideals of multilinear mappings and homogeneous polynomials (see [2], [11], and [12]). The study of the space $\mathcal{I} \circ \mathrm{Lip}_0$ is well motivated; many interesting spaces resulting from this technique belong to famous classes of Lipschitz mappings: for example, the spaces of Lipschitz–Cohen strongly p-summing, Lipschitz compact, Lipschitz weakly compact, strongly Lipschitz p-integral, and

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strongly Lipschitz p-nuclear operators. Moreover, the appearance of a linear operator and a Lipschitz mapping in the formula (1.1) motivates us to investigate the connection between the Lipschitz operator T and its linearization. Given an operator ideal $\mathcal{I}$, by considering the correspondence $T \leftrightarrow \widehat{T}$, we can obtain the following identification:

$$\mathcal{I} \circ \mathrm{Lip}_0(X; E) = \mathcal{I}\big(\mathcal{F}(X); E\big). \tag{1.2}$$

In this article, our main objective is to derive and study new classes of Lipschitz mappings which satisfy (1.2). We are interested in representing these classes by using Lipschitz cross norms, which were recently studied by Cabrera-Padilla et al. in [3]. First, we establish some relations between tensor norms defined on tensor product and Lipschitz cross norms, and then we prove that every tensor norm generates a Lipschitz cross norm. Thus, if $\mathcal{I}$ is an operator ideal which admits a representation through a tensor norm α, that is,

$$\mathcal{I}(E; F^*) = (E \widehat{\otimes}_\alpha F)^*$$

for every Banach space E, F, then there is a Lipschitz cross norm α^L for which the space $\mathcal{I} \circ \mathrm{Lip}_0$ admits a Lipschitz tensor representation; that is,

$$\mathcal{I} \circ \mathrm{Lip}_0(X; E^*) = (X \widehat{\boxtimes}_{\alpha^L} E)^*,$$

for every metric space X and every Banach space E. Among our results, we will investigate the Lipschitz cross norms corresponding with Chevet–Saphar norms. We will define a new concept in the category of Lipschitz operators, which is Lipschitz strictly p-summing. The operators of this class have a strong relationship with their linearizations for the concept of p-summing. Certain results and properties of this new class will be obtained.

This paper is organized as follows. First, we recall some standard notations which will be used throughout. In Section 2, we define for a given operator ideal $\mathcal{I}$ the class $\mathcal{I} \circ \mathrm{Lip}_0$ of Lipschitz mappings satisfying that their linearizations belong to $\mathcal{I}$. Some examples of classes of Lipschitz mappings which are represented by this procedure are given. Section 3 contains the main results; we start by studying Lipschitz cross norms generated by tensor norms. We then consider the Chevet–Saphar norms, and we study the corresponding Lipschitz cross norms. Inspired by the definition of p-summing, we introduce the concept of Lipschitz strictly p-summing for which we prove that the Lipschitz mapping T is strictly p-summing if and only if its linearization $\widehat{T}$ is p-summing. This notion coincides with the notions of p-summing and Lipschitz p-summing operators when we are considering only linear operators.

Now, we recall briefly some basic notation and terminology. Throughout this paper, the letters E, F will denote Banach spaces and X, Y will denote metric spaces with a distinguished point (pointed metric spaces), which we denote by 0. Let E be a Banach space, and let $n \in \mathbb{N}$. We denote by $l_p^n(E)$, $(1 \leq p \leq \infty)$, the space of all sequences $(x_i)_{1\leq i\leq n}$ in E equipped with the norm

$$\big\|(x_i)_{1\leq i\leq n}\big\|_{l_p^n(E)} = \Big(\sum_1^n \|x_i\|^p\Big)^{\frac{1}{p}},$$

and by $l_p^{n\omega}(E)$ we denote the space of all sequences $(x_i)_{1\leq i\leq n}$ in E equipped with the norm

$$\left\|(x_i)_{1\leq i\leq n}\right\|_{l_p^{n\omega}(E)} = \sup_{\|x^*\|_{E^*}=1}\Big(\sum_1^n\left|x^*(x_i)\right|^p\Big)^{\frac{1}{p}}.$$

Let X be a pointed metric space. We denote by $X^\#$ the Banach space of all Lipschitz functions $f : X \longrightarrow \mathbb{R}$ which vanish at 0 under the Lipschitz norm given by

$$\mathrm{Lip}(f) = \sup\Big\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in X, x \neq y\Big\}.$$

We denote by $\mathcal{F}(X)$ the free Banach space over X; that is, the norm closed linear subspace of $\mathrm{Lip}_0(X)^*$ spanned by the functions $\delta_{(x,y)}$ with $x, y \in X$ and $\delta_{(x,y)} : X^\# \to \mathbb{R}$ defined as

$$\delta_{(x,y)}(f) = f(x) - f(y).$$

We have $\mathcal{F}(X)^* = X^\#$. (For a general theory of free Banach space, see [10], [11], and [16].) Let X be a metric space, and let E be a Banach space. We denote by $\mathrm{Lip}_0(X;E)$ the Banach space of all Lipschitz functions $T : X \to E$ such that $T(0) = 0$ with pointwise addition and Lipschitz norm. Note that, for any $T \in \mathrm{Lip}_0(X;E)$, there exists a unique linear map (linearization of T) $\widehat{T} : \mathcal{F}(X) \longrightarrow E$ such that $\widehat{T}\circ\delta_X = T$, and $\|\widehat{T}\| = \mathrm{Lip}(T)$; that is, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ \delta_X \downarrow & \nearrow \widehat{T} & \\ \mathcal{F}(X) & & \end{array} \tag{1.3}$$

where δ_X is the canonical embedding so that $\langle\delta_X(x), f\rangle = \langle\delta_{(x,0)}, f\rangle = f(x)$ for $f \in X^\#$. The Lipschitz transpose map of a Lipschitz operator $T : X \to E$ is a linear operator $T^t : E^* \to X^\#$, which is defined by

$$T^t(e^*)(x) = e^*\big(T(x)\big).$$

We have

$$T^t = Q_X^{-1}\circ\widehat{T}^*,$$

where Q_X is the isomorphism isometric between $X^\#$ and $\mathcal{F}(X)^*$ such that

$$Q_X(f)(m) = m(f) \quad \text{for every } f \in X^\# \text{ and } m \in \mathcal{F}(X).$$

If X is a Banach space and $T : X \to E$ is a linear operator, then the corresponding linear operator $\widehat{T}$ is given by

$$\widehat{T} = T\circ\beta_X, \tag{1.4}$$

where $\beta_X : \mathcal{F}(X) \to X$ is the linear quotient map which verifies $\beta_X\circ\delta_X = \mathrm{id}_X$, and $\|\beta_X\| \leq 1$ (see [13, p. 124] for more details about the operator β_X). Let X be a metric space, and let E be a Banach space. By $X\boxtimes E$, we denote the Lipschitz tensor product of X and E. This is the vector space spanned by the linear functional $\delta_{(x,y)}\boxtimes e$ on $\mathrm{Lip}_0(X;E^*)$ defined by

$$\delta_{(x,y)}\boxtimes e(f) = \big\langle f(x)-f(y), e\big\rangle.$$

Let α be a norm on $X \boxtimes E$; α is considered a Lipschitz cross norm if it satisfies the condition

$$\alpha(\delta_{(x,y)} \boxtimes e) = d(x, y)\|e\|.$$

A Lipschitz cross norm α is called *dualizable* if, given $f \in X^{\#}$ and $e^* \in E^*$ for all $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$\Big|\sum_{i=1}^{n} \big(f(x_i) - f(y_i)\big)\langle e^*, e_i\rangle\Big| \leq \mathrm{Lip}(f)\|e^*\|\alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big).$$

Every Lipschitz mapping $T : X \to E^*$ admits a linear functional φ_T defined on the Lipschitz tensor product $X \boxtimes E$ by

$$\varphi_T\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \sum_{i=1}^{n} \big\langle f(x_i) - f(y_i), e_i\big\rangle.$$

As in [1], a subclass $\mathcal{A}$ of Lip_0 is said to be a *normed (Banach) Lipschitz ideal* if, for every pointed metric space X and every Banach space E, the pair $(\mathcal{A}(X; E), \|\cdot\|_{\mathcal{A}})$ is a normed (Banach) space, and if the following hold:

(a) For every $f \in X^{\#}$ and $e \in E$, the Lipschitz operator $f \boxtimes e : X \to E$ defined by $f \boxtimes e(x) = f(x)e$ is in $\mathcal{A}(X; E)$ and

$$\|f \boxtimes e\|_{\mathcal{A}} \leq \mathrm{Lip}(f)\|e\|.$$

(b) For all $T \in \mathcal{A}(X; E)$, we have

$$\mathrm{Lip}(T) \leq \|T\|_{\mathcal{A}}.$$

(c) *Ideal property*: Let Z be a metric space, and let F be a Banach space. The composed operator $u \circ T \circ g$ is in $\mathcal{A}(Z; F)$, and

$$\|u \circ T \circ g\|_{\mathcal{A}} \leq \|u\|\|T\|_{\mathcal{A}} \,\mathrm{Lip}(g)$$

for every $g \in \mathrm{Lip}_0(Z; X)$, $T \in \mathcal{A}(X; E)$, and $u \in \mathcal{B}(E; F)$ ($\mathcal{B}(E; F)$ is the Banach space of all linear operators from E into F).

2. Lipschitz spaces generated by the composition method

In this section, we apply composition ideals technique to generate new classes of Lipschitz mappings. Given an operator ideal $\mathcal{I}$, let X be a pointed metric space, and let E be a Banach space. A Lipschitz operator $T \in \mathrm{Lip}_0(X; E)$ is said to be of type $\mathcal{I}\circ \mathrm{Lip}_0$ if there exist a Banach space Z, a Lipschitz operator $L \in \mathrm{Lip}_0(X; Z)$, and a linear operator $u \in \mathcal{I}(Z; E)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ L \searrow & & \nearrow u \\ & Z & \end{array}$$

On the other hand, $T = L \circ u$. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) ideal, then the space $\mathcal{I} \circ \mathrm{Lip}_0(X; E)$ is a normed (Banach) Lipschitz ideal with respect to the following norm:

$$\|T\|_{\mathcal{I}\circ\mathrm{Lip}_0} = \inf \mathrm{Lip}(L)\|u\|_{\mathcal{I}}.$$

In [1], the connection between the Lipschitz operators of $\mathcal{I} \circ \mathrm{Lip}_0$ and their linearizations is established.

Theorem 2.1 ([1, Proposition 3.2]). *Let $\mathcal{I} \circ \mathrm{Lip}_0$ be the space of Lipschitz mappings generated by the normed operator ideal $\mathcal{I}$. The following properties are equivalent.*

(1) *The Lipschitz operator T belongs to $\mathcal{I} \circ \mathrm{Lip}_0(X; E)$.*
(2) *The linearization $\widehat{T}$ belongs to $\mathcal{I}(\mathcal{F}(X); E)$.*

In this case we have $\|T\|_{\mathcal{I}\circ\mathrm{Lip}_0} = \|\widehat{T}\|_{\mathcal{I}}$, and then

$$\mathcal{I} \circ \mathrm{Lip}_0(X; E) = \mathcal{I}\big(\mathcal{F}(X); E\big)$$

holds isometrically.

Proposition 2.2. *Let $\mathcal{I}_2$, $\mathcal{I}_2$ be two operator ideals, and then we proceed as follows.*

(1) *If $\mathcal{I}_1 \circ \mathrm{Lip}_0(X; E) \subset \mathcal{I}_2 \circ \mathrm{Lip}_0(X; E)$, then $\mathcal{I}_1(\mathcal{F}(X); E) \subset \mathcal{I}_2(\mathcal{F}(X); E)$.*
(2) *If $\mathcal{I} \circ \mathrm{Lip}_0(X; E) = \mathrm{Lip}_0(X; E)$, then $\mathcal{I}(\mathcal{F}(X); E) = \mathcal{B}(\mathcal{F}(X); E)$.*

Proof. (1) If we let $u \in \mathcal{I}_1(\mathcal{F}(X); E)$, then the Lipschitz operator $T = u \circ \delta_X : X \longrightarrow E$ verifies that

$$\widehat{T} = u.$$

By Theorem 2.1, $T \in \mathcal{I}_1 \circ \mathrm{Lip}_0(X; E)$; hence $T \in \mathcal{I}_2 \circ \mathrm{Lip}_0(X; E)$. Consequently, $\widehat{T} = u \in \mathcal{I}_2(\mathcal{F}(X); E)$.

(2) If we let $u \in \mathcal{B}(\mathcal{F}(X); E)$, then $T = u \circ \delta_X \in \mathrm{Lip}_0(X; E)$; hence

$$\widehat{T} = u \in \mathcal{I}\big(\mathcal{F}(X); E\big). \qquad \square$$

The next Proposition follows directly from the previous one.

Proposition 2.3. *Let E be a Banach space. The following properties are equivalent:*

(1) $\mathrm{id}_E \in \mathcal{I}(E; E)$,
(2) *$\mathcal{I} \circ \mathrm{Lip}_0(X; E) = \mathrm{Lip}_0(X; E)$ for every pointed metric space X.*

As in the linear case studied in [1], we give the definition of a Lipschitz dual of a given operator ideal.

Definition 2.4 ([1, Definition 3.8]). The Lipschitz dual of a given operator ideal $\mathcal{I}$ is defined by

$$\mathcal{I}^{\mathrm{Lip}_0\,-\,\mathrm{dual}}(X; E) = \big\{T \in \mathrm{Lip}_0(X; E) : T^t \in \mathcal{I}(E^*; X^{\#})\big\}.$$

If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) ideal, we define

$$\|T\|_{\mathcal{I}^{\mathrm{Lip}_0\,-\,\mathrm{dual}}} = \|T^t\|_{\mathcal{I}};$$

then the space $\mathcal{I}^{\mathrm{Lip}_0\,-\,\mathrm{dual}}(X; E)$ becomes a normed (Banach) Lipschitz ideal.

An operator ideal $\mathcal{I}$ is symmetric if

$$\mathcal{I} = \mathcal{I}^{\text{dual}} = \{u \in \mathcal{I}(F;G) : u^* \in \mathcal{I}(G^*;F^*)\}.$$

If $\mathcal{I}$ is symmetric, then we have the following coincidence between a Lipschitz ideal and its dual.

Proposition 2.5. *The following properties are equivalent:*

(1) *$\mathcal{I}$ is symmetric,*
(2) *$\mathcal{I}^{\mathrm{Lip}_0\,-\,\mathrm{dual}}(X;E) = \mathcal{I}\circ \mathrm{Lip}_0(X;E)$ for every pointed metric space X and Banach space E.*

In the rest of this section, we present some examples of classes of Lipschitz mappings which were generated by known operator ideals using the composition method. In [15], we introduced the class $\mathcal{D}_p^L$ of Lipschitz–Cohen strongly p-summing operators. Proposition 3.1 in [15] asserts that $T : X \to E$ is Lipschitz–Cohen strongly p-summing if and only if its linearization $\widehat{T}$ is strongly p-summing. Combining this with Theorem 2.1, the class $\mathcal{D}_p^L$ can be interpreted in terms of the composition method as follows.

Theorem 2.6. *Let X be a pointed metric space, and let E be a Banach space. Let $1 < p \leq \infty$, and let p^* be its conjugate ($\frac{1}{p} + \frac{1}{p^*} = 1$). We have*

$$\mathcal{D}_p^L(X;E) = \mathcal{D}_p \circ \mathrm{Lip}_0(X;E) = \Pi_{p^*}^{\text{dual}} \circ \mathrm{Lip}_0(X;E) = \Pi_{p^*}^{\mathrm{Lip}_0\,-\,\mathrm{dual}}(X;E),$$

where Π_p and $\mathcal{D}_p$ are the classes of p-summing and strongly p-summing linear operators, respectively.

The classes of Lipschitz compact and weakly compact operators have been introduced in [14]. By $\mathrm{Lip}_{0\mathcal{K}}$ and $\mathrm{Lip}_{0\mathcal{W}}$, we denote the Banach–Lipschitz ideals of Lipschitz compact and Lipschitz weakly compact operators, respectively. In [14, Propositions 2.1, 2.2], a similar relation has been established, as in Theorem 2.1.

A simple consequence of the linear result given in [8, Corollary 1] asserts that the Banach space F is reflexive if and only if, for every Banach space G and linear operator $v : F \to G$, v is weakly compact. We have the next characterization.

Theorem 2.7. *Let X be a pointed metric space. The following properties are equivalent.*

(1) *The metric space X is finite.*
(2) *For all Banach space E, we have $\mathrm{Lip}_{0\mathcal{W}}(X;E) = \mathrm{Lip}_0(X;E)$.*

Proof. (1) $\Rightarrow$ (2): This is immediate.

(2) $\Rightarrow$ (1): Let E be a Banach space, and let $v : \mathcal{F}(X) \to E$ be a linear operator. We will show that v is weakly compact. By (2), the Lipschitz operator $T = v\circ\delta_X : X \to E$ is Lipschitz weakly compact; hence $\widehat{T} = v$ is weakly compact. Consequently, the space $\mathcal{F}(X)$ is reflexive; however, by [7, Theorem 1.1] the space $\mathcal{F}(X)$ is never reflexive if X is an infinite metric space. $\square$

Next we recall the definition of strongly p-nuclear operators introduced in [5].

Definition 2.8. Let $1 \leq p < \infty$. A Lipschitz operator $T \in \mathrm{Lip}_0(X; E)$ is regarded as strongly Lipschitz p-nuclear ($1 \leq p < \infty$) if there exist operators $A \in B(l_p; E)$, $b \in \mathrm{Lip}_0(X; l_\infty)$, and a diagonal operator $M_\lambda \in B(l_\infty; l_p)$ induced by a sequence $\lambda \in l_p$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ b \downarrow & & \uparrow A \\ l_\infty & \xrightarrow{M_\lambda} & l_p \end{array}$$

The triple (A, b, λ) is called a *strongly Lipschitz p-nuclear factorization* of T. We denote $\mathcal{N}_p^{\mathrm{SL}}(X; E)$ the Banach space of all strongly Lipschitz p-nuclear operators from X into E with the norm

$$sv_p^L(T) = \inf \|A\| \|M_\lambda\| \, \mathrm{Lip}(b),$$

where the infimum is taken over all the above factorizations. For Banach spaces F, G, we denote by $\mathcal{N}_p(F; G)$ the space of all p-nuclear linear operators which admit a factorization as in the Lipschitz case, with the operator b being linear.

Proposition 2.9. *Let $1 \leq p < \infty$. The Lipschitz operator $T : X \to E$ is strongly Lipschitz p-nuclear if and only if its linearization $\widehat{T}$ is p-nuclear. Consequently,*

$$\mathcal{N}_p^{\mathrm{SL}}(X; E) = \mathcal{N}_p \circ \mathrm{Lip}_0(X; E).$$

Proof. If we let T be a strongly Lipschitz p-nuclear operator, then we have

$$T = A \circ M_\lambda \circ b.$$

If we use the Lipschitz factorization of T and b,

$$\widehat{T} \circ \delta_X = A \circ M_\lambda \circ \widehat{b} \circ \delta_X,$$

then by the uniqueness of linearization we obtain

$$\widehat{T} = A \circ M_\lambda \circ \widehat{b};$$

hence $\widehat{T}$ is p-nuclear. The converse is immediate. $\square$

This definition of strongly Lipschitz p-nuclear operator is analogous to the definition of strongly Lipschitz p-integral operator introduced in [14]. In fact, the same definition was also introduced in [5, p. 5275]. In the first definition [14], the authors have considered a factorization in which the left operator is linear and the right is Lipschitz. In the second definition [5], the role of these operators has been changed.

Definition 2.10 ([14, Definition 2.4]). Let $1 \leq p < \infty$. A Lipschitz operator $T \in \mathrm{Lip}_0(X; E)$ is called *strongly Lipschitz p-integral* if there exist a finite measure space (Ω, Σ, μ), a bounded linear operator $A \in \mathcal{B}(L_p(\mu); E^{**})$, and a Lipschitz operator $b \in \mathrm{Lip}_0(X; L_\infty(\mu))$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{T} & E & \xrightarrow{\mathcal{K}_E} & E^{**} \\ b \downarrow & & & & \uparrow A \\ L_\infty(\mu) & & \xrightarrow{I_{\infty,p}} & & L_p(\mu) \end{array}$$

where $I_{\infty,p} : L_\infty(\mu) \to L_p(\mu)$ is the formal inclusion operator. The triple (A, b, μ) is considered a strongly Lipschitz p-integral factorization of T. We denote by $\mathfrak{I}_p^{\mathrm{SL}}(X; E)$ the Banach space of all strongly Lipschitz p-integral operators from X into E with the norm

$$si_p^L(T) = \inf \mathrm{Lip}(b)\|A\|.$$

For Banach spaces F, G we denote by $\mathfrak{I}_p(F; G)$ the space of all p-integral linear operators. Using the same argument in the proof of Proposition 2.9, we can prove the following.

Proposition 2.11. *Let $1 \leq p < \infty$. The Lipschitz operator $T : X \to E$ is strongly Lipschitz p-integral if and only if its linearization $\widehat{T}$ is p-integral. Consequently*

$$\mathfrak{I}_p^{\mathrm{SL}}(X; E) = \mathfrak{I}_p \circ \mathrm{Lip}_0(X; E)$$

holds isometrically. If $p = 1$, then we have

$$\begin{aligned}\mathfrak{I}_1^{\mathrm{SL}}(X; E) &= \mathfrak{I}_1 \circ \mathrm{Lip}_0(X; E)\\ &= \mathfrak{I}_1^{\mathrm{dual}} \circ \mathrm{Lip}_0(X; E).\end{aligned}$$

As in the linear case, we give a factorization result for strongly Lipschitz p-nuclear operators. For the proof, we use the linearization operators and the result [9, Theorem 5.27].

Theorem 2.12. *Let $1 \leq p < \infty$. A Lipschitz operator $T : X \to E$ is strongly Lipschitz p-nuclear if and only if there exist a Banach space Z, a compact linear operator $v : Z \to E$, and a strongly Lipschitz p-integral operator $L : X \to Z$ such that*

$$T = v \circ L.$$

In this case

$$sv_p^L(T) = \inf \|v\| si_p^L(L).$$

Proof. If we let $T : X \to E$ be a strongly Lipschitz p-nuclear operator, then $\widehat{T} : \mathcal{F}(X) \to E$ is p-nuclear. Theorem 2.27 in [9] asserts that there exist a Banach space Z, a compact operator linear $v : Z \to E$, and a p-integral operator $w : \mathcal{F}(X) \to Z$ such that $\widehat{T} = v \circ w$. Hence

$$\widehat{T} \circ \delta_X = v \circ w \circ \delta_X \Rightarrow T = v \circ L,$$

where $L = w \circ \delta_{X,}$ which is strongly Lipschitz p-nuclear by Proposition 2.11.

Conversely, if we suppose that

$$T = v \circ L,$$

where v is a compact operator and L is strongly Lipschitz p-integral, then

$$\begin{array}{ccc} X & \overset{T}{\to} & E \\ \delta_X \downarrow & & \uparrow v \\ \mathcal{F}(X) & \overset{\widehat{L}}{\to} & Z \end{array}$$

that is, $\widehat{T} = v \circ \widehat{L}$, with $\widehat{L}$ is p-integral. Thus $\widehat{T} = v \circ \widehat{L}$ is p-nuclear, and T is strongly Lipschitz p-nuclear. $\square$

3. Main results

3.1. **Results on Lipschitz tensor product.** Let F, G be two Banach spaces. We denote by $F \otimes G$ its algebraic tensor product. There are various tensor norms that may be defined on the tensor product $F \otimes G$. If we consider a pointed metric space X and a Banach space E, there is an attempt to generalize the definition of tensor product to the category of metric spaces. The authors in [3] have studied the space $X \boxtimes E$ which is called *Lipschitz tensor product.* Some Lipschitz cross norms have been defined on this space. In this section we give some relations between Lipschitz cross norms and tensor norms. In the following section, we will use the terminology of Lipschitz cross norms for norms defined on $X \boxtimes E$ and tensor norms for norms defined on $F \otimes G$.

Theorem 3.1. *Every tensor norm α generates a dualizable Lipschitz cross norm α^L such that, for all pointed metric space X and Banach space E, we have*

$$\alpha^L\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\Big), \tag{3.1}$$

where $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$. In this case, the linear map $\Phi : X \boxtimes_{\alpha^L} E \to \mathcal{F}(X) \otimes_\alpha E$ defined by

$$\Phi\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big) = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i$$

is well defined and is an isometry.

Proof. Let α be a tensor norm. The properties of the norm α^L have been inherited from those of α. Let X be a pointed metric space, and let E be a Banach space. Let $x, y \in X$, and let $e \in E$. Then

$$\begin{aligned}\alpha^L(\delta_{(x,y)} \boxtimes e) &= \alpha(\delta_{(x,y)} \otimes e) = \|\delta_{(x,y)}\|\|e\| \\ &= d(x,y)\|e\|.\end{aligned}$$

Hence α^L is a Lipschitz cross norm. Let $f \in X^{\#}$ $(= \mathcal{F}(X)^*)$, and let $e^* \in E^*$. We have

$$\begin{aligned}\Big|\sum_{i=1}^{n}\big(f(x_i) - f(y_i)\big)\langle e^*, e_i\rangle\Big| &= \Big|f \otimes e^*(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i)\Big| \\ &\le \mathrm{Lip}(f)\|e^*\|\alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\Big) \\ &\le \mathrm{Lip}(f)\|e^*\|\alpha^L\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i\Big).\end{aligned}$$

Then we see that α^L is dualizable. Now it is easy to show that Φ is linear. Let $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i = 0$, and we will show that $\Phi(u) = 0$. Indeed, let $f \in \mathcal{F}(X)^*$, and let $e^* \in E^*$. Since $\mathcal{F}(X)^* = X^{\#}$, we then have by [3, Proposition 1.6]

$$\sum_{i=1}^{n} \big(f(x_i) - f(y_i)\big) e^*(e_i) = 0.$$

Thus

$$\sum_{i=1}^{n} f(\delta_{(x_i,y_i)}) e^*(e_i) = 0.$$

Hence $\Phi(u) = 0$ tells us that Φ is well defined. Let $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes_{\alpha^L} E$. By (3.1) we have

$$\begin{aligned} \alpha\big(\Phi(u)\big) &= \alpha\Big(\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i\Big) \\ &= \alpha^L(u); \end{aligned}$$

thus Φ is an isometry. $\square$

Since Φ is a linear isometry, its range $\Phi(X \boxtimes_{\alpha^L} E)$ is closed. On the other hand, the tensors of the form $\sum_{i=1}^{n} \delta_{(x_i,y_i)} \otimes e_i$ are dense in $\mathcal{F}(X) \widehat{\otimes}_\alpha E$. This shows that the range $\Phi(X \boxtimes_{\alpha^L} E)$ is dense in $\mathcal{F}(X) \widehat{\otimes}_\alpha E$; thus $X \widehat{\boxtimes}_{\alpha^L} E$ is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\alpha E$.

Corollary 3.2. *For every pointed metric space X and Banach space E, we have that*

$$X \widehat{\boxtimes}_{\alpha^L} E = \mathcal{F}(X) \widehat{\otimes}_\alpha E \tag{3.2}$$

holds isometrically.

As a consequence of Theorem 2.1 and Corollary 3.2, we get the following result.

Corollary 3.3. *Let $\mathcal{I} \circ L$ be a Lipschitz ideal generated by the operator ideal $\mathcal{I}$. Suppose that $\mathcal{I}$ can be interpreted through a tensor product; that is, there is a tensor norm α such that for every Banach spaces F, G we have*

$$\mathcal{I}(F; G^*) = (F \widehat{\otimes}_\alpha G)^*.$$

Then there is a Lipschitz cross norm α^L defined as in (3.1) such that

$$\mathcal{I} \circ \mathrm{Lip}_0(X; E^*) = (X \widehat{\boxtimes}_{\alpha^L} E)^*.$$

If we consider the projective tensor norm π and injective tensor norm ε, by using (3.2) and the last corollary, it is not hard to see that

$$\mathrm{Lip}_0(X; E^*) = (X \widehat{\boxtimes}_{\pi^L} E)^* = \big(\mathcal{F}(X) \widehat{\otimes}_\pi E\big)^*$$

and that

$$\mathfrak{I}_1\big(\mathcal{F}(X); E^*\big) = (X \widehat{\boxtimes}_{\varepsilon^L} E)^* = \big(\mathcal{F}(X) \widehat{\otimes}_\varepsilon E\big)^* = \mathfrak{I}_1^{\mathrm{SL}}(X; E^*).$$

3.2. **Lipschitz Chevet–Saphar norms.** We will consider the Chevet–Saphar tensor norms, and we will discuss their corresponding Lipschitz cross norms. Let F, G be two Banach spaces. The Chevet–Saphar norms g_p and d_p are defined on a tensor product $F \otimes G$ for $1 \leq p \leq \infty$ as

$$d_p(u) = \inf\Big\{ \big\|(x_i)_i\big\|_{l_p^{n,w}(F)} \big\|(g_i)_i\big\|_{l_{p^*}^n(G)} : u = \sum_{i=1}^n x_i \otimes g_i \Big\}$$

and as

$$g_p(u) = \inf\Big\{ \big\|(x_i)_i\big\|_{l_{p^*}^n(F)} \big\|(g_i)_i\big\|_{l_p^{n,w}(G)} : u = \sum_{i=1}^n x_i \otimes g_i \Big\}.$$

These norms are mainly introduced to study the classes of p-summing and strongly p-summing linear operators. The dual spaces of the corresponding tensor products coincide with these last spaces; that is,

$$\mathcal{D}_p(F;G^*) = (F\widehat{\otimes}_{g_{p^*}} G)^* \quad \text{and} \quad \Pi_p(F;G^*) = (F\widehat{\otimes}_{d_p} G)^*.$$

We recall that a linear operator $T : F \to G$ is p-summing if there exists a positive constant C such that, for every $x_1, \ldots, x_n \in F$ and $g_1^*, \ldots, g_n^* \in G^*$, we have

$$\Big| \sum_{i=1}^n \big\langle T(x_i), g_i^* \big\rangle \Big| \leq C d_p(u), \tag{3.3}$$

where $u = \sum_{i=1}^n x_i \otimes g_i^*$. The space $\Pi_p(F,G)$ stands for the Banach space of all p-summing linear operators and

$$\|T\|_{\Pi_p} = \inf\{C, \text{verifying the equality (3.3)}\}.$$

Moreover, for the definition of strongly p-summing linear operators, we substitute $d_p(u)$ with $g_{p^*}(u)$ in (3.3). Again $\mathcal{D}_p(F;G)$ stands for the Banach space of all strongly p-summing linear operators with the norm $\|T\|_{D_p}$. (For more details about these notions see [6] and [9].) Now let X be a pointed metric space, and let E be a Banach space. We define d_p^L, the corresponding norm of d_p, as follows: for every $u = X \boxtimes E$ we have

$$d_p^L(u) = d_p\big(\Phi(u)\big) = \inf\big\{ \|m_i\|_{l_p^{n,w}(\mathcal{F}(X))} \big\|(e_i)_i\big\|_{l_{p^*}^n(E)} \big\},$$

where the infimum is taken over all representations of the form $\sum_{i=1}^n m_i \otimes e_i \in \mathcal{F}(X) \otimes E$ such that $\Phi(u) = \sum_{i=1}^n m_i \otimes e_i$. By (3.2), we obtain the following identification for $1 \leq p < \infty$

$$X \widehat{\boxtimes}_{d_p^L} E = \mathcal{F}(X) \widehat{\otimes}_{d_p} E.$$

Let $T \in \mathrm{Lip}_0(X;E)$ be a Lipschitz operator. The operator T can be see as a linear functional on $X \boxtimes E^*$ whose action on a tensor $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i^*$ is given by

$$\langle T, u \rangle = \sum_{i=1}^n \big\langle T(x_i) - T(y_i), e_i^* \big\rangle.$$

Inspired by the definition of p-summing linear operators (3.3), we introduce a new definition in the category of Lipschitz mappings.

Definition 3.4. Let $1 \leq p < \infty$. Let X be a pointed metric space, and let E be a Banach space. A Lipschitz operator $T : X \to E$ is said to be *strictly Lipschitz p-summing* if there exists a positive constant C such that, for every $x_i, y_i \in X$ and $e_i^* \in E^*$ $(1 \leq i \leq n)$, we have

$$\Big|\sum_{i=1}^{n}\big\langle T(x_i) - T(y_i), e_i^*\big\rangle\Big| \leq C d_p^L(u), \tag{3.4}$$

where $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i^*$. We denote by $\Pi_p^{\mathrm{SL}}(X;E)$ the Banach space of all strictly Lipschitz p-summing operators from X into E of which its norm $\|T\|_{\Pi_p^{\mathrm{SL}}}$ is the smallest constant C verifying (3.4).

In [4], the author has studied the class $\Pi_p^L(X;E)$ of Lipschitz p-summing operators. He has defined a norm on the space of molecules $\mathcal{F}(X;E)$ of which we have the next duality

$$\Pi_p^L(X;E^*) = \mathcal{F}_{cs_p}(X;E)^*,$$

where cs_p is defined by

$$cs_p(u) = \inf\big\{\|\delta_{(x_i,y_i)}\|_{l_{p^*}^{n,w}(\mathcal{F}(X))}\big\|(e_i)_i\big\|_{l_p^n(E)}\big\},$$

and where the infimum is taken over all representations of u of the form $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in \mathcal{F}(X;E)$. Note that the space of molecules $\mathcal{F}(X;E)$ plays the same role of Lipschitz tensor product $X \boxtimes E$ whose norms can be defined on both spaces. Definitions cs_p and d_p^L look very similar; however, they do not coincide. In the definition of cs_p we are only using elements in $\mathcal{F}(X)$ of the form $\delta_{(x,y)}$, but in the second case we have to consider all elements of $\mathcal{F}(X)$. Thus the infimum in d_p^L will in general be smaller. This means that

$$\Pi_p^{\mathrm{SL}}(X;E) \subset \Pi_p^L(X;E).$$

In [15], we have seen that, if the linearization $\widehat{T}$ of T is p-summing, then T is Lipschitz p-summing, but the converse is not true in general. In our case, we show that it is true for the concept of strictly Lipschitz p-summing.

Theorem 3.5. *Let $1 \leq p < \infty$. Let X be a metric space, and let E be a Banach space. The following properties are equivalent.*

(1) *The Lipschitz operator T belongs to $\Pi_p^{\mathrm{SL}}(X;E)$.*
(2) *The linearization $\widehat{T}$ belongs to $\Pi_p(\mathcal{F}(X);E)$.*

Proof. (2) $\Rightarrow$ (1): Suppose that $\widehat{T} \in \Pi_p(\mathcal{F}(X);E)$. Let $x_i, y_i \in X$, and let $e_i^* \in E^*$ $(1 \leq i \leq n)$. If we put $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i^*$, then

$$\begin{aligned}\Big|\sum_{i=1}^{n}\big\langle T(x_i) - T(x_i), e_i^*\big\rangle\Big| &= \Big|\sum_{i=1}^{n}\big\langle \widehat{T}(\delta_{(x_i,y_i)}), e_i^*\big\rangle\Big| \\ &\leq \|\widehat{T}\|_{\Pi_p} d_p\big(\Phi(u)\big) = \|\widehat{T}\|_{\Pi_p} d_p^L(u);\end{aligned}$$

hence T is strictly Lipschitz p-summing, and

$$\|T\|_{\Pi_p^{\mathrm{SL}}} \leq \|\widehat{T}\|_{\Pi_p}.$$

(1) $\Rightarrow$ (2): Suppose that $T \in \Pi_p^{\mathrm{SL}}(X;E)$. Let $m_i \in \mathcal{F}(X)$ ($m_i = \sum_{j=1}^{k_i} \lambda_i^j \delta_{(x_i^j, y_i^j)}$: $\lambda_i^j \in \mathbb{R}$), and let $e_i^* \in E$ ($1 \leq i \leq n$)

$$\Big|\sum_{i=1}^{n} \big\langle \widehat{T}(m_i), e_i^* \big\rangle\Big| = \Big|\sum_{i=1}^{n}\sum_{j=1}^{k_i} \big\langle T(x_i^j) - T(y_i^j), \lambda_i^j e_i^* \big\rangle\Big|$$
$$\leq \|T\|_{\Pi_p^{\mathrm{SL}}} d_p^L(u) = \|T\|_{\Pi_p^{\mathrm{SL}}} d_p\big(\Phi(u)\big),$$

where

$$u = \sum_{i=1}^{n}\sum_{j=1}^{k_i} \lambda_i^j \delta_{(x_i^j, y_i^j)} \boxtimes e_i^*,$$

and $\Phi(u) = \sum_{i=1}^{n} m_i \otimes e_i^*$. Hence $\widehat{T}$ is p-summing, and $\|\widehat{T}\|_{\Pi_p} \leq \|T\|_{\Pi_p^{\mathrm{SL}}}$. $\square$

As immediate consequences, we have the following results.

Corollary 3.6. *For every pointed metric space X and Banach space E, we have*

$$\Pi_p^{\mathrm{SL}}(X;E) = (X \widehat{\boxtimes}_{d_p^L} E^*)^* = \big(\mathcal{F}(X) \widehat{\otimes}_{d_p} E^*\big)^* = \Pi_p\big(\mathcal{F}(X);E\big).$$

Corollary 3.7. *The next inclusion is strict:*

$$\Pi_p^{\mathrm{SL}}\big(\mathbb{R}; l_1(\mathbb{R})\big) \subset \Pi_p^{L}\big(\mathbb{R}; l_1(\mathbb{R})\big).$$

Proof. We know that $\delta_{\mathbb{R}} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$, ($\mathcal{F}(\mathbb{R}) = l_1(\mathbb{R})$, in fact) is Lipschitz p-summing. Its linearization is the identity on $\mathcal{F}(\mathbb{R})$ which cannot be p-summing because $\mathcal{F}(\mathbb{R})$ is infinite-dimensional. Hence $\delta_{\mathbb{R}}$ is not strictly Lipschitz p-summing. $\square$

If X is a Banach space, and $T : X \to E$ is a linear operator, then we have the following result.

Proposition 3.8. *Let X, E be two Banach spaces, and let $T : X \to E$ be a linear operator. The following properties are equivalent.*

1. *T is Lipschitz p-summing.*
2. *T is p-summing.*
3. *T is strictly Lipschitz p-summing.*

In this case we have

$$\|T\|_{\Pi_p} = \|T\|_{\Pi_p^L} = \|T\|_{\Pi_p^{\mathrm{SL}}}.$$

Proof. The equivalence (1) $\Leftrightarrow$ (2) has been proved by Farmer and Johnson in [11], and we have $\|T\|_{\Pi_p} = \|T\|_{\Pi_p^L}$. Now if T is p-summing, then by (1.4), T is strictly Lipschitz p-summing, and we have

$$\|T\|_{\Pi_p^{\mathrm{SL}}} = \|T \circ \beta_X\|_{\Pi_p} \leq \|T\|_{\Pi_p}.$$

The last implication is immediate with $\|T\|_{\Pi_p^L} \leq \|T\|_{\Pi_p^{\mathrm{SL}}}$. $\square$

In the linear case, every p-integral operator linear is p-summing. Then, by Proposition 2.11 and Theorem 3.5, we conclude that every strongly Lipschitz p-integral operator is strictly Lipschitz p-summing. In the next results, we give some coincidence situations as in the linear case. For the proof, we use the linearization of both classes and the linear results given in [9, p. 99].

Corollary 3.9. *Let $1 \leq p < \infty$. Let Xbe a pointed metric space, and let E be an injective Banach space. Then*

$$\Pi_p^{\mathrm{SL}}(X;E) = \mathfrak{I}_p^{\mathrm{SL}}(X;E),$$

with equality of norms.

Corollary 3.10. *Let X be a pointed metric space, and let$E$ be a Banach space. Then*

$$\Pi_2^{\mathrm{SL}}(X;E) = \mathfrak{I}_2^{\mathrm{SL}}(X;E),$$

with equality of norms.

Corollary 3.11. *If E is a subspace of an $\mathcal{L}_p$-space, then $1 \leq p \leq 2$, and, for every pointed metric space X,*

$$\Pi_q^{\mathrm{SL}}(X;E) = \mathfrak{I}_q^{\mathrm{SL}}(X;E) = \mathfrak{I}_2^{\mathrm{SL}}(X;E),$$

for all $2 \leq q < \infty$.

We next show a Lipschitz version of a well-known characterization of an $\mathcal{L}_\infty$-space which states that a Banach space X is an $\mathcal{L}_\infty$-space if and only if, for every Banach space E and 1-summing linear operator $T : X \to E$, T is 1-integral (see [9, Corollary 6.24] for more details about this characterization).

Theorem 3.12. *Let X be a pointed metric space. The following properties are equivalent.*

(1) *The space $\mathcal{F}(X)$ is an $\mathcal{L}_\infty$-space.*
(2) *For all Banach space E we have $\Pi_1^{\mathrm{SL}}(X;E) = \mathfrak{I}_1^{\mathrm{SL}}(X;E)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathcal{F}(X)$ is an $\mathcal{L}_\infty$-space. Let E be a Banach space, and let $T \in \Pi_1^{\mathrm{SL}}(X;E)$; then $\widehat{T} : \mathcal{F}(X) \to E$ is 1-summing. By the characterization of an $\mathcal{L}_\infty$-space, $\widehat{T}$ is 1-integral. Consequently, T is in $\mathfrak{I}_1^{\mathrm{SL}}(X;E)$.

(2) $\Rightarrow$ (1): Let $v : \mathcal{F}(X) \to E$ be 1-summing linear operator. It is apparent that v is 1-integral. If we let $T = v \circ \delta_X$, then T is strictly Lipschitz 1-summing, and $\widehat{T} = v$. By (2), T is strongly Lipschitz 1-integral, and then its linearization is 1-integral. □

We finish this section by discussing the Lipschitz tensor norm associated with the Chevet–Saphar norm g_p. For every $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E$, we have

$$g_p^L(u) = g_p\big(\Phi(u)\big).$$

Then

$$(X \widehat{\boxtimes}_{g_p^L} E)^* = \big(\mathcal{F}(X) \widehat{\otimes}_{g_p} E\big)^*.$$

Proposition 3.1 in [15] states that there is an equivalence between a Lipschitz mapping T and its linearization $\widehat{T}$ for the concept of strongly p-summing. Moreover, we have

$$\mathcal{F}_{\mu_p}(X;E)^* = \mathcal{D}_p^L(X;E^*),$$

where the norm μ_p is defined as follows:

$$\mu_p(u) = \inf\Big\{\|\delta_{(x_i,y_i)}\|_{l_p^n(\mathcal{F}(X))}\big\|(e_i)_i\big\|_{l_{p^*}^{n,w}(E)} : u = \sum_{i=1}^n \delta_{(x_i,y_i)} e_i\Big\}.$$

Combining this definition with Theorem 2.1, we obtain the following identification.

Theorem 3.13. *Let X be a metric space, and let Ebe a Banach space. We have*

$$(X\widehat{\boxtimes}_{g_p^L} E)^* = \big(\mathcal{F}(X)\widehat{\otimes}_{g_p} E\big)^* = \mathcal{D}_{p^*}^L(X;E^*) = \mathcal{D}_{p^*}\big(\mathcal{F}(X);E^*\big).$$

Corollary 3.14. *The norms g_p^L and $\mu_p$are the same.*

Proof. Let $u \in X \boxtimes E$. By the definition of $g_{p^*}^L$, we have

$$g_p^L(u) \leq \mu_{p^*}(u).$$

On the other hand,

$$\begin{aligned}
\mu_{p^*}(u) &= \sup_{T\in B_{\mathcal{F}_{\mu_{p^*}}(X;E)^*}} \big|\langle T,u\rangle\big| = \sup_{\widehat{T}\in B_{\mathcal{D}_{p^*}(\mathcal{F}(X);E^*)}} \big|\langle \widehat{T},u\rangle\big| \\
&= \sup_{\widehat{T}\in B_{\mathcal{D}_{p^*}(\mathcal{F}(X);E^*)}} \Big|\sum_{i=1}^n \big\langle \widehat{T}(\delta_{(x_i,y_i)}), e_i\big\rangle\Big| \\
&\leq \sup_{\widehat{T}\in B_{\mathcal{D}_{p^*}(\mathcal{F}(X);E^*)}} \|\widehat{T}\|_{\mathcal{D}_{p^*}} g_p\Big(\sum_{i=1}^n \delta_{(x_i,y_i)} \otimes e_i\Big) \\
&\leq g_p\big(\Phi(u)\big) = g_p^L(u).
\end{aligned}$$

□

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