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INEQUALITIES ON THE SPECTRAL RADIUS AND THE OPERATOR NORM OF HADAMARD PRODUCTS OF POSITIVE OPERATORS ON SEQUENCE SPACES

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Abstract. K. M. R. Audenaert (2010), R. A. Horn and F. Zhang (2010), Z. Huang (2011), A. R. Schep (2011), A. Peperko (2012), and D. Chen and Y. Zhang (2015) have proved inequalities on the spectral radius and the operator norm of Hadamard products and ordinary matrix products of finite and infinite nonnegative matrices that define operators on sequence spaces. In the present article, we extend and refine several of these results, and we also prove some analogues for the numerical radius.

1. Introduction

In [20], X. Zhan conjectured that, for nonnegative $(n \times n)$-matrices A and B, the spectral radius $\rho(A \circ B)$ of the Hadamard product satisfies

$$\rho(A \circ B) \leq \rho(AB),$$

where AB denotes the usual matrix product of A and B. This conjecture was confirmed by K. M. R. Audenaert in [1] by proving

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}\big((A \circ A)(B \circ B)\big) \leq \rho(AB). \tag{1.1}$$

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These inequalities were established via a trace description of the spectral radius. Using the fact that the Hadamard product is a principal submatrix of the Kronecker product, R. A. Horn and F. Zhang proved in [12] the inequalities

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}(AB \circ BA) \le \rho(AB) \tag{1.2}$$

and also the right-hand side of the inequality in (1.1). Applying the techniques of [12], Z. Huang proved that

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \le \rho(A_1 A_2 \cdots A_m) \tag{1.3}$$

for nonnegative $(n \times n)$-matrices $A_1, A_2, \ldots, A_m$ (see [13]). A related inequality for nonnegative $(n \times n)$-matrices was shown in [8]:

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \le \rho(A_1)\rho(A_2)\cdots\rho(A_m). \tag{1.4}$$

In [17] and [18], A. R. Schep extended inequalities (1.1) and (1.2) to nonnegative matrices that define bounded operators on sequence spaces (in particular, on l^p spaces, $1 \le p < \infty$). In Schep's proofs, certain results on the Hadamard product from [5] were used. It was claimed in [17, Theorem 2.7] that

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}\big((A \circ A)(B \circ B)\big) \le \rho^{\frac{1}{2}}(AB \circ BA) \le \rho(AB). \tag{1.5}$$

However, the proof of [17, Theorem 2.7] actually demonstrates that

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}\big((A \circ A)(B \circ B)\big) \le \rho^{\frac{1}{2}}(AB \circ AB) \le \rho(AB). \tag{1.6}$$

It turned out that $\rho(AB \circ BA)$ and $\rho(AB \circ AB)$ may in fact be different and that (1.5) is false in general. This error was corrected in [18] and [16]. Moreover, it was proved in [16] that, for nonnegative matrices that define bounded operators on sequence spaces, the inequalities

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}\big((A \circ A)(B \circ B)\big) \le \rho(AB \circ AB)^{\frac{1}{4}}\rho(BA \circ BA)^{\frac{1}{4}} \le \rho(AB) \tag{1.7}$$

and (1.3) hold.

In [4], by applying the techniques of [1], the inequality (1.3) in the case of nonnegative $(n \times n)$-matrices was interpolated as

$$\begin{aligned} &\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \\ &\quad \le \big[\rho(A_1 \circ A_2 \circ \cdots \circ A_m)\big]^{1-\frac{2}{m}} \\ &\qquad \times \big[\rho\big((A_1 \circ A_1)(A_2 \circ A_2)\cdots(A_m \circ A_m)\big)\big]^{\frac{1}{m}} \\ &\quad \le \rho(A_1 A_2 \cdots A_m) \end{aligned} \tag{1.8}$$

for $m \ge 2$.

The article is organized as follows. In Section 2, we introduce some definitions and facts, and we recall some results from [5] and [15], which we will need in our proofs. In Section 3, we extend and/or refine several inequalities from [13], [16], [4], [5], and [15] (including the inequalities (1.3) and (1.8)) to nonnegative matrices that define bounded operators on sequence spaces. More precisely, in Theorem 3.1 we prove a version of inequality (1.3) which is valid for arbitrary positive kernel operators on Banach function spaces. In Theorem 3.2, we refine inequality (1.3) and prove analogues for the operator norm and the numerical radius. Consequently, Corollary 3.4 generalizes and refines (1.8). In Theorem 3.6, we refine the inequality (1.4) and prove analogue results for the operator norm and the numerical radius. We generalize and refine some additional results from [13] and [4] in Corollary 3.5 and Theorem 3.9. We conclude the article by applying the spectral mapping theorem to obtain additional results (Theorem 3.13, Corollaries 3.14 and 3.15). Several inequalities in the article appear to be new, even in the case of nonnegative $(n \times n)$-matrices.

2. Preliminaries

Let R denote either the set $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or the set $\mathbb{N}$ of all natural numbers. Let $S(R)$ be the vector lattice of all complex sequences $(x_i)_{i \in R}$. A Banach space $L \subseteq S(R)$ is called a *Banach sequence space* if $x \in S(R)$, $y \in L$ and $|x| \leq |y|$ imply that $x \in L$ and $\|x\|_L \leq \|y\|_L$. The cone of all nonnegative elements in L is denoted by L_+.

Let us denote by $\mathcal{L}$ the collection of all Banach sequence spaces L satisfying the property that $e_i = \chi_{\{i\}} \in L$ and $\|e_i\|_L = 1$ for all $i \in R$. Standard examples of spaces from $\mathcal{L}$ are Euclidean spaces, the well-known spaces $l^p(R)$ $(1 \leq p \leq \infty)$, and the space c_0 of all null-convergent sequences, equipped with the usual norms. The set $\mathcal{L}$ also contains all Cartesian products $L = X \times Y$ for $X, Y \in \mathcal{L}$, equipped with the norm $\|(x, y)\|_L = \max\{\|x\|_X, \|y\|_Y\}$.

A matrix $A = [a_{ij}]_{i,j \in R}$ is called *nonnegative* if $a_{ij} \geq 0$ for all $i, j \in R$. Given matrices A and B, we write $A \leq B$ if the matrix $B - A$ is nonnegative. Note that the matrices here need not be finite-dimensional.

By an *operator* on a Banach sequence space L we always mean a linear operator on L. We say that a *nonnegative matrix* A defines an operator on L if $Ax \in L$ for all $x \in L$, where $(Ax)_i = \sum_{j \in R} a_{ij} x_j$. Then $Ax \in L_+$ for all $x \in L_+$, and so A defines a *positive* operator on L. Recall that this operator is always bounded; that is, its operator norm

$$\|A\| = \sup\big\{\|Ax\|_L : x \in L, \|x\|_L \leq 1\big\} = \sup\big\{\|Ax\|_L : x \in L_+, \|x\|_L \leq 1\big\} \quad (2.1)$$

is finite. Also, its spectral radius $\rho(A)$ is always contained in the spectrum. We will frequently use the equality $\rho(ST) = \rho(TS)$ that holds for all bounded operators S and T on a Banach space.

If $A = [a_{ij}]$ is a nonnegative matrix that defines an operator on $l^2(R)$, then the matrix $A^T = [a_{ji}]$ defines its adjoint operator on a Hilbert space $l^2(R)$, so that we have

$$\|A\|^2 = \|AA^T\| = \|A^T A\| = \rho(AA^T) = \rho(A^T A). \quad (2.2)$$

Given nonnegative matrices $A = [a_{ij}]_{i,j \in R}$ and $B = [b_{ij}]_{i,j \in R}$, let $A \circ B = [a_{ij} b_{ij}]_{i,j \in R}$ be the *Hadamard* (or *Schur*) *product* of A and B, and let $A^{(t)} = [a_{ij}^t]_{i,j \in R}$ be the *Hadamard* (or *Schur*) *power* of A for $t \geq 0$. Here we use the convention $0^0 = 1$.

The following result was proved in [5, Theorem 3.3] and [15, Theorem 5.1 and Remark 5.2] by using only basic analytic methods and elementary facts.

Theorem 2.1. *Given $L \in \mathcal{L}$, let $\{A_{ij}\}_{i=1,j=1}^{k,m}$ be nonnegative matrices that define operators on L. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are positive numbers such that $\sum_{j=1}^m \alpha_j \geq 1$, then the matrix $A := (A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}) \cdots (A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)})$ also defines an operator on L and it satisfies the inequalities*

$$A \leq (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)}, \tag{2.3}$$

$$\|A\| \leq \|A_{11} \cdots A_{k1}\|^{\alpha_1} \cdots \|A_{1m} \cdots A_{km}\|^{\alpha_m}, \tag{2.4}$$

$$\rho(A) \leq \rho(A_{11} \cdots A_{k1})^{\alpha_1} \cdots \rho(A_{1m} \cdots A_{km})^{\alpha_m}. \tag{2.5}$$

The following special case of Theorem 2.1 ($k = 1$) was considered in the finite-dimensional case by several authors using different methods (see, e.g., [8], [6], [5], [15] for references).

Corollary 2.2. *Given $L \in \mathcal{L}$, let $A_1, \ldots, A_m$ be nonnegative matrices that define operators on L, and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i \geq 1$. Then we have*

$$\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}\| \leq \|A_1\|^{\alpha_1} \|A_2\|^{\alpha_2} \cdots \|A_m\|^{\alpha_m} \tag{2.6}$$

and

$$\rho(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \leq \rho(A_1)^{\alpha_1} \rho(A_2)^{\alpha_2} \cdots \rho(A_m)^{\alpha_m}. \tag{2.7}$$

The following special case of Theorem 2.1 was also proved in [5, Proposition 3.1] and [15, Lemma 4.2].

Proposition 2.3. *Given L in $\mathcal{L}$, let $A_1, \ldots, A_m$ be nonnegative matrices that define operators on L. Then, for any $t \geq 1$ and $i = 1, \ldots, m$, $A_i^{(t)}$ also defines an operator on L, and the following inequalities hold:*

$$A_1^{(t)} \cdots A_m^{(t)} \leq (A_1 \cdots A_m)^{(t)}, \tag{2.8}$$

$$\|A_1^{(t)} \cdots A_m^{(t)}\| \leq \|A_1 \cdots A_m\|^t, \tag{2.9}$$

$$\rho(A_1^{(t)} \cdots A_m^{(t)}) \leq \rho(A_1 \cdots A_m)^t. \tag{2.10}$$

Note that Theorem 2.1 and its special cases proved to be quite useful in different contexts (see, e.g., [7], [8], [5], [15], [6], [17], [16], [4]). It will also be one of the main tools in the current article.

Banach sequence spaces are special cases of Banach function spaces. As proved in [5] and [15], the inequalities in Theorem 2.1 and Corollary 2.2 can be extended to positive kernel operators on Banach function spaces, provided that $\sum_{i=1}^m \alpha_i = 1$. Since our first theorem in the next section gives an inequality for these general spaces, we shortly recall some basic definitions and results from [5] and [15].

Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ of subsets of a nonvoid set X. Let $M(X, \mu)$ be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on X. A Banach space $L \subseteq M(X, \mu)$ is called a *Banach function space* if $f \in L$, $g \in M(X, \mu)$, and $|g| \leq |f|$ imply that $g \in L$ and $\|g\| \leq \|f\|$. We will assume that X is the carrier of L; that is, there is no subset Y of X of strictly positive measure with the property that $f = 0$ almost everywhere on Y for all $f \in L$ (see [19]). Observe that a Banach sequence space is a Banach function space over a measure space (R, μ), where μ denotes the counting measure on R (and for $L \in \mathcal{L}$, the set R is the carrier of L).

As before, by an *operator* on a Banach function space L we always mean a linear operator on L. An operator T on L is said to be *positive* if it maps nonnegative functions to nonnegative ones. Given operators S and T on L, we write $S \geq T$ if the operator $S - T$ is positive.

In the special case $L = L^2(X, \mu)$, we can define the *numerical radius* $w(T)$ of a bounded operator T on $L^2(X, \mu)$ by

$$w(T) = \sup\big\{ \big|\langle Tf, f\rangle\big| : f \in L^2(X, \mu), \|f\|_2 = 1 \big\}.$$

If, in addition, T is positive, then it is easy to prove that

$$w(T) = \sup\big\{ \langle Tf, f\rangle : f \in L^2(X, \mu)_+, \|f\|_2 = 1 \big\}.$$

From this it follows easily that $w(S) \leq w(T)$ for all positive operators S and T on $L^2(X, \mu)$ with $S \leq T$.

An operator K on a Banach function space L is called a *kernel operator* if there exists a $(\mu \times \mu)$-measurable function $k(x, y)$ on $X \times X$ such that, for all $f \in L$ and for almost all $x \in X$,

$$\int_X \big|k(x, y) f(y)\big| \, d\mu(y) < \infty \qquad \text{and} \qquad (Kf)(x) = \int_X k(x, y) f(y) \, d\mu(y).$$

One can check that a kernel operator K is positive if and only if its kernel k is nonnegative almost everywhere. For the theory of Banach function spaces we refer the reader to the book [19].

Let K and H be positive kernel operators on L with kernels k and h, respectively, and $\alpha \geq 0$. The *Hadamard* (or *Schur*) *product* $K \circ H$ of K and H is the kernel operator with kernel equal to $k(x, y)h(x, y)$ at point $(x, y) \in X \times X$ that can be defined (in general) only on some order ideal of L. Similarly, the *Hadamard* (or *Schur*) *power* $K^{(\alpha)}$ of K is the kernel operator with kernel equal to $(k(x, y))^\alpha$ at point $(x, y) \in X \times X$ that can be defined only on some ideal of L.

Let $K_1, \ldots, K_n$ be positive kernel operators on a Banach function space L, and let $\alpha_1, \ldots, \alpha_n$ be positive numbers such that $\sum_{j=1}^n \alpha_j = 1$. Then the *Hadamard weighted geometric mean* $K = K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)}$ of the operators $K_1, \ldots, K_n$ is a positive kernel operator defined on the whole space L, since $K \leq \alpha_1 K_1 + \alpha_2 K_2 + \cdots + \alpha_n K_n$ by the inequality between the weighted arithmetic and geometric means. Let us recall the following result, which was proved in [5, Theorem 2.2] and [15, Theorem 5.1].

Theorem 2.4. *Let $\{A_{ij}\}_{i=1,j=1}^{k,m}$ be positive kernel operators on a Banach function space L. If α_1, $\alpha_2, \ldots,$ α_m are positive numbers such that $\sum_{j=1}^m \alpha_j = 1$, then the inequalities (2.3), (2.4), and (2.5) hold.*

If, in addition, $L = L^2(X, \mu)$, then

$$\begin{aligned} &w\big((A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}) \cdots (A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)})\big) \\ &\quad \leq w(A_{11} \cdots A_{k1})^{\alpha_1} \cdots w(A_{1m} \cdots A_{km})^{\alpha_m}. \end{aligned} \tag{2.11}$$

The following result is a special case of Theorem 2.4.

Theorem 2.5. *Let $A_1, \dots, A_m$ be positive kernel operators on a Banach function space L, and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $\sum_{j=1}^m \alpha_j = 1$. Then the inequalities (2.6) and (2.7) hold.*

If, in addition, $L = L^2(X, \mu)$, then

$$w(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \le w(A_1)^{\alpha_1} w(A_2)^{\alpha_2} \cdots w(A_m)^{\alpha_m}. \tag{2.12}$$

3. Results

We begin with a new proof of (1.3) that is based on the inequality (2.7).

Theorem 3.1. *Let $A_1, \dots, A_m$ be positive kernel operators on a Banach function space L. Then*

$$\rho(A_1^{(\frac{1}{m})} \circ A_2^{(\frac{1}{m})} \circ \cdots \circ A_m^{(\frac{1}{m})}) \le \rho(A_1 A_2 \cdots A_m)^{\frac{1}{m}}. \tag{3.1}$$

If, in addition, $L \in \mathcal{L}$ (and so $A_1, \dots, A_m$ can be considered as nonnegative matrices that define operators on L), then

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \le \rho(A_1 A_2 \cdots A_m). \tag{3.2}$$

Proof. The block matrix

$$T = T(A_1, A_2, \dots, A_m) := \begin{bmatrix} 0 & A_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & A_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & A_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & A_{m-1} \\ A_m & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

defines a positive kernel operator on the Cartesian product of m copies of L. Since T^m has a diagonal form,

$$\begin{aligned} T^m = \operatorname{diag}(&A_1 A_2 \cdots A_m, A_2 A_3 \cdots A_m A_1, A_3 A_4 \cdots A_m A_1 A_2, \dots, \\ &A_m A_1 A_2 \cdots A_{m-1}), \end{aligned}$$

we have $\rho(T)^m = \rho(T^m) = \rho(A_1 A_2 \cdots A_m)$.

Now define $T_k := T(A_k, A_{k+1}, \dots, A_m, A_1, \dots, A_{k-1})$ for $k = 1, 2, \dots, m$. Then $\rho(T_k)^m = \rho(A_1 A_2 \cdots A_m)$ for each k. Using the inequality (2.7) we obtain that

$$\rho(T_1^{(\frac{1}{m})} \circ T_2^{(\frac{1}{m})} \circ \cdots \circ T_m^{(\frac{1}{m})}) \le \big(\rho(T_1)\rho(T_2) \cdots \rho(T_m)\big)^{\frac{1}{m}} = \rho(A_1 A_2 \cdots A_m)^{\frac{1}{m}}.$$

Since

$$\rho(T_1^{(\frac{1}{m})} \circ T_2^{(\frac{1}{m})} \circ \cdots \circ T_m^{(\frac{1}{m})}) = \rho(A_1^{(\frac{1}{m})} \circ A_2^{(\frac{1}{m})} \circ \cdots \circ A_m^{(\frac{1}{m})}),$$

the inequality (3.1) is proved.

If, in addition, $L \in \mathcal{L}$, then we apply the inequality

$$\rho(T_1 \circ T_2 \circ \cdots \circ T_m) \le \rho(T_1)\rho(T_2) \cdots \rho(T_m),$$

which is a special case of the inequality (2.7). We then observe that $\rho(T_1 \circ T_2 \circ \cdots \circ T_m) = \rho(A_1 \circ A_2 \circ \cdots \circ A_m)$ and $\rho(T_1)\rho(T_2)\cdots\rho(T_m) = \rho(A_1 A_2 \cdots A_m)$. This completes the proof. □

It should be mentioned that the special case of inequality (3.1) for pairs of operators on L^p-spaces was already given in [17, Theorem 2.8].

The following theorem generalizes the inequalities (1.6) to several matrices, and it provides an alternative proof of the inequality (3.2). We also establish related inequalities for the operator norm and the numerical radius.

Theorem 3.2. *Given $L \in \mathcal{L}$, let $A_1, \ldots, A_m$ be nonnegative matrices that define operators on L. For $t \in [1, m]$ and $i = 1, \ldots, m$, put $P_i = A_i^{(t)} A_{i+1}^{(t)} \cdots A_m^{(t)} A_1^{(t)} A_2^{(t)} \cdots A_{i-1}^{(t)}$. Then*

$$\begin{aligned} \rho(A_1 \circ \cdots \circ A_m) &\le \rho(P_1^{(\frac{1}{t})} \circ \cdots \circ P_m^{(\frac{1}{t})})^{\frac{1}{m}} \\ &\le \rho(A_1^{(t)} \cdots A_m^{(t)})^{\frac{1}{t}} \\ &\le \rho\big((A_1 \cdots A_m)^{(t)}\big)^{\frac{1}{t}} \le \rho(A_1 \cdots A_m) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\big\|(A_1 \circ \cdots \circ A_m)^m\big\| \\ &\quad\le \|P_1^{(\frac{1}{t})} \circ \cdots \circ P_m^{(\frac{1}{t})}\| \\ &\quad\le \big(\|P_1\| \cdots \|P_m\|\big)^{\frac{1}{t}} \\ &\quad\le \Big(\big\|(A_1 A_2 \cdots A_m)^{(t)}\big\|\big\|(A_2 \cdots A_m A_1)^{(t)}\big\| \cdots \big\|(A_m A_1 \cdots A_{m-1})^{(t)}\big\|\Big)^{\frac{1}{t}} \\ &\quad\le \|A_1 A_2 \cdots A_m\|\|A_2 \cdots A_m A_1\| \cdots \|A_m A_1 \cdots A_{m-1}\|. \end{aligned} \tag{3.4}$$

If, in addition, $L = l^2(R)$ and $t = m$, then

$$\begin{aligned} &w\big((A_1 \circ \cdots \circ A_m)^m\big) \\ &\quad\le w(P_1^{(\frac{1}{m})} \circ \cdots \circ P_m^{(\frac{1}{m})}) \le \big(w(P_1) \cdots w(P_m)\big)^{\frac{1}{m}} \\ &\quad\le \Big(w\big((A_1 A_2 \cdots A_m)^{(m)}\big) w\big((A_2 \cdots A_m A_1)^{(m)}\big) \cdots \\ &\qquad \times w\big((A_m A_1 \cdots A_{m-1})^{(m)}\big)\Big)^{\frac{1}{m}}. \end{aligned} \tag{3.5}$$

Proof. Similarly as for P_i, we define the Hadamard product

$$\begin{aligned} H_i &= A_i^{(t)} \circ A_{i+1}^{(t)} \circ \cdots \circ A_m^{(t)} \circ A_1^{(t)} \circ A_2^{(t)} \circ \cdots \circ A_{i-1}^{(t)} \\ &= (A_i \circ A_{i+1} \circ \cdots \circ A_m \circ A_1 \circ A_2 \circ \cdots \circ A_{i-1})^{(t)} = (A_1 \circ \cdots \circ A_m)^{(t)}, \end{aligned}$$

so that, in fact, $H_1 = H_2 = \cdots = H_m$. Let us prove the inequalities (3.3). Since $\frac{m}{t} \ge 1$, we apply the inequality (2.3) to obtain the inequality

$$(A_1 \circ \cdots \circ A_m)^m = H_1^{(\frac{1}{t})} \cdots H_m^{(\frac{1}{t})} \le P_1^{(\frac{1}{t})} \circ \cdots \circ P_m^{(\frac{1}{t})}.$$

Therefore, we have

$$\rho(A_1 \circ \cdots \circ A_m)^m = \rho\big((A_1 \circ \cdots \circ A_m)^m\big) \le \rho(P_1^{(\frac{1}{t})} \circ \cdots \circ P_m^{(\frac{1}{t})}),$$

proving the first inequality in (3.3). Since $\frac{m}{t} \geq 1$, for the proof of the second inequality in (3.3) we can apply the inequality (2.7) to obtain that

$$\rho(P_1^{(\frac{1}{t})} \circ \cdots \circ P_m^{(\frac{1}{t})}) \leq \big(\rho(P_1) \cdots \rho(P_m)\big)^{\frac{1}{t}} = \rho(A_1^{(t)} \cdots A_m^{(t)})^{\frac{m}{t}}.$$

Using the inequalities (2.8) and (2.10) we prove the remaining inequalities in (3.3):

$$\rho(A_1^{(t)} \cdots A_m^{(t)}) \leq \rho\big((A_1 \cdots A_m)^{(t)}\big) \leq \rho(A_1 \cdots A_m)^t.$$

The inequalities (3.4) and (3.5) are proved in a similar way. □

Corollary 3.3. *Given $L \in \mathcal{L}$, let A and B be nonnegative matrices that define operators on L. Then, for every $t \in [1, 2]$,*

$$\rho(A\circ B) \leq \rho\big((A^{(t)}B^{(t)})^{(\frac{1}{t})}\circ(B^{(t)}A^{(t)})^{(\frac{1}{t})}\big)^{\frac{1}{2}} \leq \rho(A^{(t)}B^{(t)})^{\frac{1}{t}} \leq \rho\big((AB)^{(t)}\big)^{\frac{1}{t}} \leq \rho(AB)$$

and

$$\begin{aligned}\big\|(A \circ B)^2\big\| &\leq \big\|(A^{(t)}B^{(t)})^{(\frac{1}{t})} \circ (B^{(t)}A^{(t)})^{(\frac{1}{t})}\big\| \leq \big(\|A^{(t)}B^{(t)}\|\|B^{(t)}A^{(t)}\|\big)^{\frac{1}{t}} \\ &\leq \big(\big\|(AB)^{(t)}\big\|\big\|(BA)^{(t)}\big\|\big)^{\frac{1}{t}} \leq \|AB\|\|BA\|.\end{aligned}$$

If, in addition, $L = l^2(R)$, then

$$\begin{aligned}w\big((A \circ B)^2\big) &\leq w\big((A^{(2)}B^{(2)})^{(\frac{1}{2})} \circ (B^{(2)}A^{(2)})^{(\frac{1}{2})}\big) \\ &\leq \big(w(A^{(2)}B^{(2)})w(B^{(2)}A^{(2)})\big)^{\frac{1}{2}} \leq \big(w\big((AB)^{(2)}\big)w\big((BA)^{(2)}\big)\big)^{\frac{1}{2}}.\end{aligned}$$

As a consequence of Theorem 3.2, we obtain the following infinite-dimensional generalization and refinement of (1.8), which was the main result of [4].

Corollary 3.4. *Given $L \in \mathcal{L}$, let $A_1, \ldots, A_m$ be nonnegative matrices that define operators on L. For $t \in [1, m]$ and $i = 1, \ldots, m$, put $P_i = A_i^{(t)}A_{i+1}^{(t)} \cdots A_m^{(t)}A_1^{(t)}A_2^{(t)} \cdots A_{i-1}^{(t)}$. Then*

$$\begin{aligned}\rho(A_1 \circ A_2 \circ \cdots \circ A_m) &\leq \rho(A_1 \circ A_2 \circ \cdots \circ A_m)^{1-\frac{t}{m}}\rho(P_1^{(\frac{1}{t})} \circ \cdots \circ P_m^{(\frac{1}{t})})^{\frac{t}{m^2}} \\ &\leq \rho(A_1 \circ A_2 \circ \cdots \circ A_m)^{1-\frac{t}{m}}\rho(A_1^{(t)} \cdots A_m^{(t)})^{\frac{1}{m}} \\ &\leq \rho(A_1 \circ A_2 \circ \cdots \circ A_m)^{1-\frac{t}{m}}\rho\big((A_1 \cdots A_m)^{(t)}\big)^{\frac{1}{m}} \\ &\leq \rho(A_1A_2 \cdots A_m).\end{aligned} \tag{3.6}$$

Proof. Since

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_m) = \rho(A_1 \circ A_2 \circ \cdots \circ A_m)^{1-\frac{t}{m}}\rho(A_1 \circ A_2 \circ \cdots \circ A_m)^{\frac{t}{m}},$$

the result follows by applying (3.3). □

By applying Theorems 2.1 and 3.2, we obtain the following result, which generalizes [4, Proposition 2.4] and generalizes and refines [13, Theorem 4].

Corollary 3.5. *Let $A_1, \dots, A_m$ be nonnegative matrices that define operators on $l^2(R)$ and $t \in [1, m]$. If we denote $S_i = A_i A_i^T$ and $T_i = S_i^{(t)} S_{i+1}^{(t)} \cdots S_m^{(t)} S_1^{(t)} S_2^{(t)} \cdots S_{i-1}^{(t)}$ for $i = 1, \dots, m$, then*

$$\begin{aligned}\|A_1 \circ A_2 \circ \cdots \circ A_m\|^2 &\le \rho(S_1 \circ S_2 \circ \cdots \circ S_m) \le \rho(T_1^{(\frac{1}{t})} \circ \cdots \circ T_m^{(\frac{1}{t})})^{\frac{1}{m}} \\ &\le \rho(S_1^{(t)} \cdots S_m^{(t)})^{\frac{1}{t}} \\ &\le \rho\big((S_1 \cdots S_m)^{(t)}\big)^{\frac{1}{t}} \le \rho(S_1 \cdots S_m).\end{aligned} \tag{3.7}$$

Proof. By Theorem 2.1, we have

$$\begin{aligned}&(A_1 \circ A_2 \circ \cdots \circ A_m)(A_1 \circ A_2 \circ \cdots \circ A_m)^T \\ &\quad = (A_1 \circ A_2 \circ \cdots \circ A_m)(A_1^T \circ A_2^T \circ \cdots \circ A_m^T) \\ &\quad \le (A_1 A_1^T) \circ (A_2 A_2^T) \circ \cdots \circ (A_m A_m^T) = S_1 \circ S_2 \circ \cdots \circ S_m,\end{aligned}$$

and so it follows by (2.2) and Theorem 2.1 that

$$\|A_1 \circ A_2 \circ \cdots \circ A_m\|^2 = \rho\big((A_1 \circ A_2 \circ \cdots \circ A_m)(A_1 \circ A_2 \circ \cdots \circ A_m)^T\big) \le \rho(S_1 \circ S_2 \circ \cdots \circ S_m),$$

which proves the first inequality (3.7). Now the result follows by applying (3.3). □

The following Cauchy–Schwarz-type inequality for the spectral radius of nonnegative $(n \times n)$-matrices was proved in [4, Proposition 2.6] using the trace description: if A, B are nonnegative $(n \times n)$-matrices, then

$$\rho(A \circ B) \le \rho(A \circ A)^{1/2} \rho(B \circ B)^{\frac{1}{2}}. \tag{3.8}$$

This result has already been implicitly known and also applied (see, e.g., the proof of [16, Theorem 3.7]). Moreover, an easy application of Corollary 2.2 gives the following infinite-dimensional generalization of (3.8) and its analogues for the operator norm and the numerical radius.

Theorem 3.6. *Given $L \in \mathcal{L}$, let $A_1, \dots, A_m$ be nonnegative matrices that define operators on L. Define functions $r, N : [1, \infty) \mapsto \mathbb{R}$ by*

$$r(t) = \big(\rho(A_1^{(t)}) \rho(A_2^{(t)}) \cdots \rho(A_m^{(t)})\big)^{1/t} \qquad \text{and}$$
$$N(t) = \big(\|A_1^{(t)}\| \|A_2^{(t)}\| \cdots \|A_m^{(t)}\|\big)^{\frac{1}{t}}.$$

Then the function r is decreasing on $[1, \infty)$, and $\rho(A_1 \circ A_2 \circ \cdots \circ A_m)$ is its lower bound on the interval $[1, m]$. Similarly, the function N is decreasing on $[1, \infty)$, and $\|A_1 \circ A_2 \circ \cdots \circ A_m\|$ is its lower bound on the interval $[1, m]$.

If, in addition, $L = l^2(R)$, then

$$w(A_1 \circ A_2 \circ \cdots \circ A_m) \le \big(w(A_1^{(m)}) w(A_2^{(m)}) \cdots w(A_m^{(m)})\big)^{\frac{1}{m}}. \tag{3.9}$$

If, in addition, $L = \mathbb{C}^n$ and $A_1, \dots, A_m$ are nonnegative $(n \times n)$-matrices, then the functions $t \mapsto r(t)$ and $t \mapsto N(t)$ are well-defined decreasing functions on $(0, \infty)$, with lower bounds on the interval $(0, m]$ equal to $\rho(A_1 \circ A_2 \circ \cdots \circ A_m)$ and $\|A_1 \circ A_2 \circ \cdots \circ A_m\|$, respectively.

Proof. The expression $\rho(A_i^{(t)})^{\frac{1}{t}}$ is decreasing in $t \in [1, \infty)$. Indeed, if $s \geq t > 0$, then the inequality (2.10) implies that

$$\rho(A_i^{(s)})^{\frac{1}{s}} = \rho\big((A_i^{(t)})^{(\frac{s}{t})}\big)^{\frac{1}{s}} \leq \rho(A_i^{(t)})^{\frac{1}{t}}.$$

So, it follows that the function r is decreasing.

If $1 \leq t \leq m$, then $\frac{m}{t} \geq 1$, and so we have, by (2.7),

$$r(t) \geq \rho\big((A_1^{(t)})^{(1/t)} \circ (A_2^{(t)})^{(1/t)} \circ \cdots \circ (A_m^{(t)})^{(1/t)}\big) = \rho(A_1 \circ A_2 \circ \cdots \circ A_m).$$

Therefore, on the interval $[1, m]$ the function r is bounded below by $\rho(A_1 \circ A_2 \circ \cdots \circ A_m)$.

In a similar manner, one can show the properties of the function N. Furthermore, the inequality (3.9) follows from the inequality (2.12).

In the case $L = \mathbb{C}^n$, the proof above remains correct, if we replace the intervals $[1, \infty)$ and $[1, m]$ with $(0, \infty)$ and $(0, m]$, respectively. □

Remark 3.7. In general, we do not have that $\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \leq r(t)$ for $t > m$. For example, in the case $m = 1$, take $A_1 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Then $\rho(A_1) = 2 > \rho(A_1^{(t)})^{\frac{1}{t}} = 2^{\frac{1}{t}}$ for $t > 1$. This matrix can also be used in the general case $m \geq 2$. Setting $A_k := A_1$ for $k = 2, \ldots, m$ we have $\rho(A_1 \circ A_2 \circ \cdots \circ A_m) = \rho(A_1) = 2 > (\rho(A_1^{(t)})\rho(A_2^{(t)}) \cdots \rho(A_m^{(t)}))^{\frac{1}{t}} = 2^{\frac{m}{t}}$ for $t > m$.

Note that the limit $\mu(A) := \lim_{k \to \infty} \rho(A^{(t)})^{\frac{1}{t}}$ plays (at least in the case of nonnegative $(n \times n)$-matrices) the role of the spectral radius in the algebraic system max algebra (see, e.g., [2], [14], [8], [7], [10], [9], [3], [11], and the references cited therein for various applications).

Remark 3.8. We can use an example from [5] to show that the product

$$\big(w(A_1^{(t)})w(A_2^{(t)}) \cdots w(A_m^{(t)})\big)^{\frac{1}{t}}$$

is not necessarily decreasing in t. Let $L = \mathbb{C}^2$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $A^{(t)} = A$ for all $t > 0$, $w(A) = \frac{1}{2}$, and so $w(A^{(t)}) = \frac{1}{2} > (\frac{1}{2})^t = w(A)^t$ for $t > 1$. Therefore, choose $A_1 = \cdots = A_m = A$ above.

The following result generalizes [13, Theorem 5].

Theorem 3.9. *Let $A_1, \ldots, A_m$ be nonnegative matrices that define operators on $l^2(R)$. If m is even, then*

$$\begin{aligned} &\|A_1 \circ A_2 \circ \cdots \circ A_m\|^2 \\ &\quad \leq \rho(A_1^T A_2 A_3^T A_4 \cdots A_{m-1}^T A_m)\rho(A_1 A_2^T A_3 A_4^T \cdots A_{m-1} A_m^T) \\ &\quad = \rho(A_1^T A_2 A_3^T A_4 \cdots A_{m-1}^T A_m)\rho(A_m A_{m-1}^T \cdots A_4 A_3^T A_2 A_1^T). \end{aligned} \tag{3.10}$$

If m is odd, then

$$\begin{aligned} &\|A_1 \circ A_2 \circ \cdots \circ A_m\|^2 \\ &\quad \leq \rho(A_1 A_2^T A_3 A_4^T \cdots A_{m-2} A_{m-1}^T A_m A_1^T A_2 A_3^T A_4 \cdots A_{m-2}^T A_{m-1} A_m^T). \end{aligned} \tag{3.11}$$

Proof. If m is even, we have, by (2.3),

$$\begin{aligned}
&\big((A_1 \circ A_2 \circ \cdots \circ A_m)^T (A_1 \circ A_2 \circ \cdots \circ A_m)\big)^{\frac{m}{2}} \\
&\quad = (A_1^T \circ A_2^T \circ \cdots \circ A_m^T)(A_2 \circ \cdots \circ A_m \circ A_1)(A_3^T \circ A_4^T \circ \cdots \circ A_m^T \circ A_1^T \circ A_2^T) \\
&\qquad \times (A_4 \circ \cdots \circ A_m \circ A_1 \circ A_2 \circ A_3) \cdots (A_{m-1}^T \circ A_m^T \circ A_1^T \circ \cdots \circ A_{m-2}^T) \\
&\qquad \times (A_m \circ A_1 \circ \cdots \circ A_{m-1}) \\
&\quad \le (A_1^T A_2 A_3^T A_4 \cdots A_{m-1}^T A_m) \circ (A_2^T A_3 A_4^T A_5 \cdots A_m^T A_1) \\
&\qquad \circ \cdots \circ (A_{m-1}^T A_m A_1^T A_2 \cdots A_{m-3}^T A_{m-2}) \circ (A_m^T A_1 A_2^T A_3 \cdots A_{m-2}^T A_{m-1}).
\end{aligned}$$

It follows by (2.5) that

$$\begin{aligned}
&\|A_1 \circ A_2 \circ \cdots \circ A_m\|^m \\
&\quad = \rho\big((A_1 \circ A_2 \circ \cdots \circ A_m)^T (A_1 \circ A_2 \circ \cdots \circ A_m)\big)^{\frac{m}{2}} \\
&\quad \le \rho\big((A_1^T A_2 A_3^T A_4 \cdots A_{m-1}^T A_m) \circ (A_2^T A_3 A_4^T A_5 \cdots A_m^T A_1) \\
&\qquad \circ \cdots \circ (A_{m-1}^T A_m A_1^T A_2 \cdots A_{m-3}^T A_{m-2}) \\
&\qquad \circ (A_m^T A_1 A_2^T A_3 \cdots A_{m-2}^T A_{m-1})\big) \\
&\quad \le \rho(A_1^T A_2 A_3^T A_4 \cdots A_{m-1}^T A_m) \rho(A_2^T A_3 A_4^T A_5 \cdots A_m^T A_1) \\
&\qquad \times \cdots \rho(A_{m-1}^T A_m A_1^T A_2 \cdots A_{m-3}^T A_{m-2}) \rho(A_m^T A_1 A_2^T A_3 \cdots A_{m-2}^T A_{m-1}) \\
&\quad = \rho^{\frac{m}{2}}(A_1^T A_2 A_3^T A_4 \cdots A_{m-1}^T A_m) \rho^{\frac{m}{2}}(A_1 A_2^T A_3 A_4^T \cdots A_{m-1} A_m^T),
\end{aligned} \tag{3.12}$$

which proves (3.10).

If m is odd, then we have, by (2.3),

$$\begin{aligned}
&\big((A_1 \circ A_2 \circ \cdots \circ A_m)^T (A_1 \circ A_2 \circ \cdots \circ A_m)\big)^m \\
&\quad = (A_1^T \circ A_2^T \circ \cdots \circ A_m^T)(A_2 \circ \cdots \circ A_m \circ A_1)(A_3^T \circ A_4^T \circ \cdots \circ A_m^T \circ A_1^T \circ A_2^T) \\
&\qquad \times (A_4 \circ \cdots \circ A_m \circ A_1 \circ A_2 \circ A_3) \cdots \\
&\qquad \times (A_{m-1} \circ A_m \circ A_1 \circ \cdots \circ A_{m-2})(A_m^T \circ A_1^T \circ \cdots \circ A_{m-1}^T) \\
&\qquad \times (A_1 \circ A_2 \circ \cdots \circ A_m)(A_2^T \circ \cdots \circ A_m^T \circ A_1^T)(A_3 \circ A_4 \circ \cdots \circ A_m \circ A_1 \circ A_2) \\
&\qquad \times \cdots (A_{m-1}^T \circ A_m^T \circ A_1^T \circ \cdots \circ A_{m-2}^T)(A_m \circ A_1 \circ \cdots \circ A_{m-1}) \\
&\quad \le (A_1^T A_2 A_3^T A_4 \cdots A_{m-1} A_m^T A_1 A_2^T A_3 A_4^T \cdots A_{m-1}^T A_m) \\
&\qquad \circ (A_2^T A_3 A_4^T \cdots A_{m-1}^T A_m A_1^T A_2 A_3^T A_4 \cdots A_{m-1} A_m^T A_1) \\
&\qquad \circ \cdots \circ (A_m^T A_1 A_2^T A_3 A_4^T \cdots A_{m-1}^T A_m A_1^T A_2 A_3^T A_4 \cdots A_{m-1}).
\end{aligned}$$

It follows by (2.5) that

$$\begin{aligned}
&\|A_1 \circ A_2 \circ \cdots \circ A_m\|^{2m} \\
&\quad \le \rho\big((A_1^T A_2 A_3^T A_4 \cdots A_{m-1} A_m^T A_1 A_2^T A_3 A_4^T \cdots A_{m-1}^T A_m) \\
&\qquad \circ \cdots \circ (A_2^T A_3 A_4^T \cdots A_{m-1}^T A_m A_1^T A_2 A_3^T A_4 \cdots A_{m-1} A_m^T A_1) \\
&\qquad \circ \cdots \circ (A_m^T A_1 A_2^T A_3 A_4^T \cdots A_{m-1}^T A_m A_1^T A_2 A_3^T A_4 \cdots A_{m-1})\big) \\
&\quad \le \rho^{\frac{m+1}{2}}(A_1^T A_2 A_3^T A_4 \cdots A_{m-1} A_m^T A_1 A_2^T A_3 A_4^T \cdots A_{m-1}^T A_m)
\end{aligned} \tag{3.13}$$

$$\times \rho^{\frac{m-1}{2}}(A_1A_2^TA_3A_4^T\cdots A_{m-1}^TA_mA_1^TA_2A_3^TA_4\cdots A_{m-1}A_m^T)$$
$$= \rho^m(A_1A_2^TA_3A_4^T\cdots A_{m-1}^TA_mA_1^TA_2A_3^TA_4\cdots A_{m-1}A_m^T),$$

which completes the proof. □

The following result follows from Theorem 3.9 and its proof. It generalizes and refines [13, Corollary 6] and [4, Corollary 2.3].

Corollary 3.10. *Let A, B, and C be nonnegative matrices that define operators on $l^2(R)$. Then*

$$\|A\circ B\| \le \rho^{\frac{1}{2}}\big((A^TB)\circ(B^TA)\big) \le \rho(A^TB) \tag{3.14}$$

and

$$\begin{aligned}&\|A\circ B\circ C\|\\ &\quad\le \rho^{\frac{1}{6}}\big((A^TBC^TAB^TC)\circ(B^TCA^TBC^TA)\circ(C^TAB^TCA^TB)\big)\\ &\quad\le \rho^{\frac{1}{2}}(AB^TCA^TBC^T).\end{aligned} \tag{3.15}$$

Proof. It follows by (3.12) that

$$\|A\circ B\| \le \rho^{\frac{1}{2}}\big((A^TB)\circ(B^TA)\big) \le \rho^{\frac{1}{2}}(A^TB)\rho^{\frac{1}{2}}(B^TA) = \rho(A^TB),$$

which proves (3.14).
Similarly, (3.15) follows from (3.13). □

The inequalities (3.15) yield the following lower bounds for the operator norm of the Jordan triple product ABA.

Corollary 3.11. *Let A and B be nonnegative matrices that define operators on $l^2(R)$. Then*

$$\begin{aligned}&\|A\circ B^T\circ A\|\\ &\quad\le \rho^{\frac{1}{6}}\big((A^TB^TA^TABA)\circ(BAA^TB^TA^TA)\circ(A^TABAA^TB^T)\big)\\ &\quad\le \|ABA\|.\end{aligned} \tag{3.16}$$

Proof. It follows by (3.15) that

$$\begin{aligned}&\|A\circ B^T\circ A\|\\ &\quad\le \rho^{\frac{1}{6}}\big((A^TB^TA^TABA)\circ(BAA^TB^TA^TA)\circ(A^TABAA^TB^T)\big)\\ &\quad\le \rho^{\frac{1}{2}}(ABAA^TB^TA^T) = \|ABA\|,\end{aligned} \tag{3.17}$$

which completes the proof. □

In contrast to (3.16), the inequality $\|A\circ B\circ A\| \le \|ABA\|$ is not valid in general, as the following example from [13] shows.

Example 3.12. If $A = \left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]$, then $\|A\circ B\circ A\| = 1 > 0 = \|ABA\|$.

Note that the inequalities (3.14) refine the well-known inequality $\|A \circ B\| \le \|A\|\|B\|$ and that we have

$$\rho(A \circ B) \le \|A \circ B\| \le \rho^{\frac{1}{2}}\big((A^T B) \circ (B^T A)\big) \le \rho(A^T B) \le \|A^T B\| \le \|A\|\|B\|.$$

Note also that $\|A \circ B\| \le \rho(AB)$ is not valid in general, as the matrices from Example 3.12 show (as has already been pointed out in [13]).

We conclude the article by combining the spectral mapping theorem for analytic functions and the inequality (3.2). To this end, let $\mathcal{A}_+$ denote the collection of all power series

$$f(z) = \sum_{j=0}^{\infty} \alpha_j z^j$$

having nonnegative coefficients $\alpha_j \ge 0$ $(j = 0, 1, \dots)$. Let R_f be the radius of convergence of $f \in \mathcal{A}_+$; that is, we have

$$\frac{1}{R_f} = \limsup_{j \to \infty} \alpha_j^{1/j}.$$

If A is an operator on a Banach space such that $\rho(A) < R_f$, then the operator $f(A)$ is defined by

$$f(A) = \sum_{j=0}^{\infty} \alpha_j A^j.$$

Theorem 3.13. *Given $L \in \mathcal{L}$, let $A_1, \dots, A_m$ be nonnegative matrices that define operators on L. If $f \in \mathcal{A}_+$ and $\rho(A_1 \cdots A_m) < R_f$, then*

$$\rho\big(f(A_1 \circ \cdots \circ A_m)\big) \le \rho\big(f(A_1 \cdots A_m)\big).$$

Proof. If $\rho(A_1 \cdots A_m) < R_f$, then it follows from the spectral mapping theorem and (3.2) that

$$\begin{aligned}\rho\big(f(A_1 \circ \cdots \circ A_m)\big) &= f\big(\rho(A_1 \circ \cdots \circ A_m)\big) \\ &\le f\big(\rho(A_1 \cdots A_m)\big) = \rho\big(f(A_1 \cdots A_m)\big),\end{aligned}$$

which completes the proof. □

Choosing the exponential series for $f \in \mathcal{A}_+$, we obtain the following corollary.

Corollary 3.14. *Given $L \in \mathcal{L}$, let $A_1, \dots, A_m$ be nonnegative matrices that define operators on L. Then*

$$\rho\big(\exp(A_1 \circ \cdots \circ A_m)\big) \le \rho\big(\exp(A_1 \cdots A_m)\big).$$

By applying the C. Neumann series for the resolvent

$$(\lambda I - A)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n, \quad |\lambda| > \rho(A),$$

we also obtain the following result.

Corollary 3.15. *Given $L \in \mathcal{L}$, let $A_1, \ldots, A_m$ be nonnegative matrices that define operators on L. If $\lambda > \rho(A_1 \cdots A_m)$, then*

$$\rho\big((\lambda I - A_1 \circ \cdots \circ A_m)^{-1}\big) \leq \rho\big((\lambda I - A_1 \cdots A_m)^{-1}\big).$$

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