Bernoulli **22**(4), 2016, 2548–2578
DOI: 10.3150/15-BEJ737

An α-stable limit theorem under sublinear expectation

ERHAN BAYRAKTAR* and ALEXANDER MUNK**

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA.
E-mail: *erhan@umich.edu; **amunk@umich.edu

For $\alpha \in (1, 2)$, we present a generalized central limit theorem for α-stable random variables under sublinear expectation. The foundation of our proof is an interior regularity estimate for partial integro-differential equations (PIDEs). A classical generalized central limit theorem is recovered as a special case, provided a mild but natural additional condition holds. Our approach contrasts with previous arguments for the result in the linear setting which have typically relied upon tools that are non-existent in the sublinear framework, for example, characteristic functions.

Keywords: generalized central limit theorem; partial-integro differential equations; stable distribution; sublinear expectation

1. Introduction

The purpose of this manuscript is to prove a generalized central limit theorem for α-stable random variables in the setting of sublinear expectation. Such a result complements the limit theorems for G-normal random variables due to Peng and others in this context and answers in the affirmative a question posed by Neufeld and Nutz in [22] (see below).

When working with a sublinear expectation, one is simultaneously considering a potentially uncountably infinite and non-dominated collection of probability measures. A construction of this kind is motivated by the study of pricing under volatility uncertainty. Needless to say, a variety of frequently called upon devices from the classical setting are unavailable. The complications encompass further issues as well: new behaviors are occasionally observed like those outlined in [4].

Analogues of significant theorems from classical probability and stochastic analysis are nevertheless moderately abundant. For instance, versions of the law of large numbers can be found in [26] and [28]; the martingale representation theorem is given in [37,38] and [31]; Girsanov's theorem is obtained in [23,39] and [9]; and a Donsker-type result is shown in [8]. To conduct investigations along these lines, standard proofs must often be re-imagined. For instance, Peng's proof of the central limit theorem under sublinear expectation in [26] resorts to interior regularity estimates for fully nonlinear parabolic partial differential equations (PDEs). His idea has since been extended to prove a number of variants of his original result, for example, see [12,20,28] and [40].

We will operate in the sublinear expectation framework unless explicitly indicated otherwise. The objects of our special attention here, the α-stable random variables for $\alpha \in (1, 2)$, were introduced in [22]. The authors pondered whether or not these could be the subject of a generalized central limit theorem. Classical generalized central limit theorems ordinarily come in one of three flavors:

(i) a statement indicating that a random variable has a non-empty domain of attraction if and only if it is α-stable such as Theorem 2.1.1 in [13],
(ii) a characterization theorem for the domain of attraction of an α-stable random variable such as Theorem 2.6.1 in [13], or
(iii) a characterization theorem for the domain of normal attraction for an α-stable random variable such as Theorem 2.6.7 in [13].

Recall that an i.i.d. sequence $(Y_i)_{i=1}^{\infty}$ of random variables is in the *domain of attraction* of a random variable X if there exist sequences of constants $(A_i)_{i=1}^{\infty}$ and $(B_i)_{i=1}^{\infty}$ so that

$$B_n \sum_{i=1}^{n} Y_i - A_n$$

converges in distribution to X as $n \to \infty$. $(Y_i)_{i=1}^{\infty}$ is in the *domain of normal attraction* of X if

$$B_n = \frac{1}{bn^{1/\alpha}}$$

for some $b > 0$.

We confine our search to the direction suggested by (iii) because of the particular importance classically of results of this type (cf. the central limit theorem). Our main findings are summarized in Theorem 3.1, which details sufficient conditions for membership in the domain of normal attraction of a given α-stable random variable. While the initial appearance of our distributional hypotheses is perhaps forbidding, in point of fact, our assumptions are manageable. This is illustrated by the discussion immediately following Theorem 3.1, as well as Examples 4.1 and 4.2.

Example 4.1 establishes that the α-stable random variables under consideration are in their own domain of normal attraction. Although one need not apply Theorem 3.1 for this purpose, the write-up serves a clarifying role and any credible result clearly must pass this litmus test.

Example 4.2 is more substantive. Setting aside a few mild "uniformity" conditions which arise due to the supremum, this example can be understood in an intuitive manner (see Section 4). This falls out of our analysis just below Theorem 3.1, where we describe the relationship between our work and the classical result noted in (iii) above. More specifically, Theorem 3.1 detects all classical random variables in this collection with mean zero and a cumulative distribution function (cdf) that satisfies a small differentiability requirement. An extra regularity condition on the cdf is unavoidable, as one must translate its form into properties that can be stated only in terms of expectation.

The strategy of our proof is to reduce demonstrating convergence in distribution to showing that a certain limit involving the solution to the backward version of our generating PIDE is zero. Upon breaking up our domain and summing the corresponding increments of the solution, regularity properties of this function are employed to argue that size of the terms being added together decay rapidly enough in the limit to furnish the desired conclusion. This general scheme is similar to that initiated in [26], except that the generating equation there is

$$\partial_t u - \frac{1}{2}\left(\overline{\sigma}^2(\partial_{xx}u)^+ - \underline{\sigma}^2(\partial_{xx}u)^-\right) = 0, \qquad (t,x) \in (0,\infty)\times\mathbb{R},$$
$$u(0,x) = \psi(x), \qquad x \in \mathbb{R}$$

for some $0 \le \underline{\sigma}^2 \le \overline{\sigma}^2$ and appropriate function ψ. Recall that this equation is known as the Barenblatt equation if $\underline{\sigma}^2 > 0$ and has been studied in [2] and [1], for instance. Ours is given by (2.1), a difference that leads to a few difficulties as reflected by the increased complexity of our hypotheses. To overcome these difficulties, we use the technology from [18,19] and [7].

The work in this paper offers a step toward understanding α-stability under sublinear expectation. The simple interpretation admitted by Example 4.2 is promising, as developing intuition in this environment is usually a tough undertaking for the reasons mentioned previously.

A brief overview of necessary background material can be found in Section 2. We prove our main result and discuss its connection to the classical case in Section 3. Examples highlighting the applications of our main result are contained in Section 4. We give some prerequisite material for the proof of the essential interior regularity estimate for our PIDE in Appendix A. The proof of this estimate is in Appendix B.

2. Background

We now offer a concise account of those aspects of sublinear expectations, α-stable random variables, and PIDEs which are required for the sequel.[1] References for more comprehensive treatments of these topics are also included for the convenience of the interested reader.

Definition 2.1. *Let $\mathcal{H}$ be a collection of real-valued functions on a set Ω. A sublinear expectation is an operator $\mathcal{E}:\mathcal{H}\longrightarrow\mathbb{R}$ which is*:

(i) *monotonic*: $\mathcal{E}[X]\le\mathcal{E}[Y]$ *if* $X\le Y$,
(ii) *constant-preserving*: $\mathcal{E}[c]=c$ *for any* $c\in\mathbb{R}$,
(iii) *sub-additive*: $\mathcal{E}[X+Y]\le\mathcal{E}[X]+\mathcal{E}[Y]$, *and*
(iv) *positive homogeneous*: $\mathcal{E}[\lambda X]=\lambda\mathcal{E}[X]$ *for* $\lambda\ge 0$.

The triple $(\Omega,\mathcal{H},\mathcal{E})$ *is called a sublinear expectation space*.

One views $\mathcal{H}$ as a space of random variables on Ω. Typically, it is assumed that $\mathcal{H}$

(i) is a linear space,
(ii) contains all constant functions, and
(iii) contains $\psi(X_1,X_2,\dots,X_n)$ for every $X_1,X_2,\dots,X_n\in\mathcal{H}$ and $\psi\in C_{\mathrm{b.Lip}}(\mathbb{R}^n)$, where $C_{\mathrm{b.Lip}}(\mathbb{R}^n)$ is the set of bounded Lipschitz functions on $\mathbb{R}^n$;

[1] Further information on PIDE interior regularity theory is contained in Appendix A.

however, we will expend little attention on either Ω or $\mathcal{H}$. Delicacy needs to be exercised while computing sublinear expectations. A rare instance when a classical technique can be justly employed is the following.

Lemma 2.2. *Consider two random variables $X, Y \in \mathcal{H}$ such that $\mathcal{E}[Y] = -\mathcal{E}[-Y]$. Then*

$$\mathcal{E}[X + \alpha Y] = \mathcal{E}[X] + \alpha \mathcal{E}[Y]$$

for all $\alpha \in \mathbb{R}$.

This result is notably useful in the case where $\mathcal{E}[Y] = \mathcal{E}[-Y] = 0$.

Definition 2.3. *A random variable $Y \in \mathcal{H}$ is said to be independent from a random variable $X \in \mathcal{H}$ if for all $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R}^2)$, we have*

$$\mathcal{E}\big[\psi(X, Y)\big] = \mathcal{E}\big[\mathcal{E}\big[\psi(x, Y)\big]_{x=X}\big].$$

Observe the deliberate wording. This choice is crucial, as independence can be asymmetric in our context. Note that this definition reduces to the traditional one if $\mathcal{E}$ is a classical expectation. The same is true for the next three concepts.

Definition 2.4. *Let X, Y and $(Y_n)_{n=1}^{\infty}$ be random variables, that is, X, Y and $(Y_n)_{n=1}^{\infty} \in \mathcal{H}$.*

(i) *X and Y are identically distributed, denoted $X \sim Y$, if*

$$\mathcal{E}\big[\psi(X)\big] = \mathcal{E}\big[\psi(Y)\big]$$

for all $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$.

(ii) *If X and Y are identically distributed and Y is independent from X, then Y is an independent copy of X.*

(iii) *$(Y_n)_{n=1}^{\infty}$ converges in distribution to Y, which we denote by $Y_n \xrightarrow{\mathrm{d}} Y$, if*

$$\lim_{n\to\infty} \mathcal{E}\big[\psi(Y_n)\big] = \mathcal{E}\big[\psi(Y)\big]$$

for all $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$.

Random variables need not be defined on the same space to have appropriate notions of (i) or (iii). In this case, the above definitions require the obvious notational modifications. Further details concerning general sublinear expectation spaces can be found in [27] or [30].

Definition 2.5. *Let $\alpha \in (0, 2]$. A random variable X is said to be (strictly) α-stable if for all a, $b \geq 0$,*

$$aX + bY$$

and

$$\left(a^\alpha + b^\alpha\right)^{1/\alpha} X$$

are identically distributed, *where* Y *is an independent copy of* X.

Three examples of α-stable random variables exist in the current literature. For $\alpha = 1$, there are the maximal random variables discussed in references such as [28,30] and [10]. When $\alpha = 2$, we have the G-normal random variables of Peng. Resources on this topic are plentiful and include [21,27,29,30] and [4]. If $\alpha \in (1, 2)$, we can consider X_1 for a nonlinear α-stable Lévy process $(X_t)_{t\geq 0}$ in the framework of [22]. Our focus shall be restricted to the last situation.

The construction of nonlinear Lévy processes in [22] extends that studied in [11,25,32] and [24] and is much more general than our present objectives demand. We limit our presentation to a few key ideas. Let

(i) $\alpha \in (1, 2)$;
(ii) $K_\pm$ be a bounded measurable subset of $\mathbb{R}_+$;
(iii) $F_{k_\pm}$ be the α-stable Lévy measure

$$F_{k_\pm}(dz) = (k_- \mathbf{1}_{(-\infty,0)} + k_+ \mathbf{1}_{(0,\infty)})(z)|z|^{-\alpha-1}\,dz$$

for all $k_\pm \in K_\pm$; and
(iv) $\Theta = \{(0, 0, F_{k_\pm}) : k_\pm \in K_\pm\}$.

One can then produce a process $(X_t)_{t\geq 0}$ which is a nonlinear Lévy process whose local characteristics are described by the set of Lévy triplets Θ. This means the following:

(i) $(X_t)_{t\geq 0}$ is a real-valued càdlàg process.
(ii) $X_0 = 0$.
(iii) $(X_t)_{t\geq 0}$ has stationary increments, that is, $X_t - X_s$ and X_{t-s} are identically distributed for all $0 \leq s \leq t$.
(iv) $(X_t)_{t\geq 0}$ has independent increments, that is, $X_t - X_s$ is independent from $(X_{s_1}, \ldots, X_{s_n})$ for all $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$.
(v) If $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$ and u is defined by

$$u(t, x) = \mathcal{E}\left[\psi(x + X_t)\right]$$

for all $(t, x) \in [0, \infty) \times \mathbb{R}$, then u is the unique[2] viscosity solution[3] of

$$\begin{aligned} \partial_t u(t, x) - \sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} \delta_z u(t, x) F_{k_\pm}(dz) \right\} &= 0, \qquad (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, x) &= \psi(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

[2]The uniqueness of a viscosity solution of (2.1) can be viewed as a special case of Theorem 2.5 in [22].

[3]We take the following definition from Section 2.2 of [22]. Let $C_b^{2,3}((0, \infty) \times \mathbb{R})$ denote the set of functions on $(0, \infty) \times \mathbb{R}$ having bounded continuous partial derivatives up to the second and third order in t and x, respectively. A bounded upper semicontinuous function u on $[0, \infty) \times \mathbb{R}$ is a *viscosity subsolution* of (2.1) if

$$u(0, \cdot) \leq \psi(\cdot)$$

Here, we use the notation

$$\delta_z u(t,x) := u(t,x+z) - u(t,x) - \partial_x u(t,x) z$$

since the right-hand side of this equation as well as similar expressions will frequently occur throughout the paper.

A critical feature of this setup is that if Θ is a singleton, $(X_t)_{t\geq 0}$ is a classical Lévy process with triplet Θ. That X_1 actually is an α-stable random variable is not immediately obvious. We give a brief argument in Example 4.1, but the core of this observation is a result from [22] (see Example 2.7).

Lemma 2.6. *For all $\beta > 0$ and $t \geq 0$, $X_{\beta t}$ and $\beta^{1/\alpha} X_t$ are identically distributed.*

The dynamic programming principle in Lemma 2.7 (see Lemma 5.1 in [22]) and the absolute value bound in Lemma 2.8 (see Lemma 5.2 in [22]) also play a central role when using our main result to check that X_1 is in its own domain of normal attraction.

Lemma 2.7. *For all $0 \leq s \leq t < \infty$ and $x \in \mathbb{R}$,*

$$u(t,x) = \mathcal{E}\big[u(t-s, x+X_s)\big].$$

Lemma 2.8. *We have that*

$$\mathcal{E}\big[|X_1|\big] < \infty.$$

The remaining essential ingredients for our purposes describe the regularity of u. The first result describes properties of u which are valid on the whole domain. It is a special case of Lemma 5.3 in [22].

Lemma 2.9. *The function u is uniformly bounded by $\|\psi\|_{L^\infty(\mathbb{R})}$ and jointly continuous. More precisely, $u(t,\cdot)$ is Lipschitz continuous with constant $\operatorname{Lip}(\psi)$, the Lipschitz constant of ψ, and*

and for any $(t,x) \in (0,\infty) \times \mathbb{R}$,

$$\partial_t \varphi(t,x) - \sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} \delta_z \varphi(t,x) F_{k_\pm}(dz) \right\} \leq 0$$

whenever $\varphi \in C_b^{2,3}((0,\infty) \times \mathbb{R})$ is such that

$$\varphi \geq u$$

on $(0,\infty) \times \mathbb{R}$ and

$$\varphi(t,x) = u(t,x).$$

To define a *viscosity supersolution* of (2.1), one reverses the inequalities and semicontinuity. A bounded continuous function is a *viscosity solution* of (2.1) if it is both a viscosity subsolution and supersolution. Viscosity solutions of the other PIDEs appearing in this paper, for example, see Lemma A.4, are defined similarly.

$u(\cdot, x)$ is locally $1/2$-Hölder continuous with a constant depending only on $\mathrm{Lip}(\psi)$ *and*

$$\sup_{k_\pm \in K_\pm} \left\{ \int_{\mathbb{R}} |z| \wedge |z|^2 F_{k_\pm}(dz) \right\} < \infty.$$

We will require even stronger regularity estimates for u. To obtain these, we must restrict our attention to the interior of the domain.

Proposition 2.10. *Suppose that for some λ, $\Lambda > 0$, we know $\lambda < k_\pm < \Lambda$ for all $k_\pm \in K_\pm$. For any $h > 0$,*

(i) *$\partial_t u$ and $\partial_x u$ exist and are bounded on $[h, h+1] \times \mathbb{R}$;*
(ii) *there are constants C, $\gamma > 0$ such that*

$$\left|\partial_t u(t_0, x) - \partial_t u(t_1, x)\right| \le C|t_0 - t_1|^{\gamma/\alpha},$$
$$\left|\partial_t u(t, x_0) - \partial_t u(t, x_1)\right| \le C|x_0 - x_1|^{\gamma}$$

for all (t_0, x), (t_1, x), (t, x_0), $(t, x_1) \in [h, h+1] \times \mathbb{R}$;
(iii) *u is a classical solution of (2.1) on $[h, h+1] \times \mathbb{R}$; and*
(iv) *if $K_\pm$ contains exactly one pair $\{k_\pm\}$, then $\partial^2_{xx} u$ exists and is bounded on $[h, h+1] \times \mathbb{R}$.*

The proof of Proposition 2.10 can be found in Appendix B.

3. Main result

To facilitate our discussion in the sequel, we now fix some notation. Compared with Section 2, we make only one alteration to our nonlinear α-stable Lévy process $(X_t)_{t\ge 0}$: additionally assume that $K_\pm$ is a subset of (λ, Λ) for some λ, $\Lambda > 0$. We will make use of this in conjunction with Proposition 2.10.

We also consider a sequence $(Y_i)_{i=1}^{\infty}$ of random variables on some sublinear expectation space. The only aspect of this space that we will invoke directly is the sublinear expectation itself, say $\mathcal{E}'$. Distinguishing between $\mathcal{E}$ and $\mathcal{E}'$ will be convenient for Example 4.2. We further specify that $(Y_i)_{i=1}^{\infty}$ is i.i.d. in the sense that Y_{i+1} is independent from $(Y_1, Y_2, \dots, Y_i)$ and $Y_{i+1} \sim Y_i$ for all $i \ge 1$. After proper normalization,

$$S_n := \sum_{i=1}^{n} Y_i$$

will be the sequence attracted to X_1.

Theorem 3.1. *Suppose that*

(i) $\mathcal{E}'[Y_1] = \mathcal{E}'[-Y_1] = 0$;
(ii) $\mathcal{E}'[|Y_1|] < \infty$; *and*

(iii) *for any* $0 < h < 1$ *and* $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$,

$$n\left|\mathcal{E}'\left[\delta_{B_nY_1}v(t,x)\right]-\left(\frac{1}{n}\right)\sup_{k_\pm\in K_\pm}\left\{\int_{\mathbb{R}}\delta_z v(t,x)F_{k_\pm}(dz)\right\}\right|\to 0 \tag{3.1}$$

uniformly on $[0,1]\times\mathbb{R}$ *as* $n\to\infty$, *where* v *is the unique viscosity solution of*

$$\begin{aligned}\partial_t v(t,x)+\sup_{k_\pm\in K_\pm}\left\{\int_{\mathbb{R}}\delta_z v(t,x)F_{k_\pm}(dz)\right\}&=0, \qquad (t,x)\in(-h,1+h)\times\mathbb{R}\\ v(1+h,x)&=\psi(x), \qquad x\in\mathbb{R}.\end{aligned} \tag{3.2}$$

Then

$$B_nS_n\xrightarrow{\mathrm{d}}X_1$$

as $n\to\infty$.

Admittedly, a cursory glance over our hypotheses leaves one with the impression that they are intractable. The opposite is true. Before presenting the proof of Theorem 3.1, let us demonstrate that when our attention is confined to the classical case, we are imposing only a mild and natural supplementary restriction on the attracted random variable. In addition to being a significant remark in itself, this work also underlies Example 4.2.

Assume that Θ is a singleton. Since $(X_t)_{t\geq 0}$ is the classical Lévy process with triplet $(0,0,F_{k_\pm})$, the characteristic function of X_1, denoted φ_{X_1}, is given by

$$\varphi_{X_1}(t)=\exp\left(k_-\int_{-\infty}^0\frac{\exp(itz)-1-itz}{|z|^{\alpha+1}}\,dz+k_+\int_0^\infty\frac{\exp(itz)-1-itz}{z^{\alpha+1}}\,dz\right)$$

for all $t\in\mathbb{R}$. In the case where Y_1 is a classical random variable with mean zero, Theorem 2.6.7 from [13] implies that

$$B_nS_n\xrightarrow{\mathrm{d}}X_1$$

as $n\to\infty$ if and only if the cdf of Y_1, denoted F_{Y_1}, has the form

$$F_{Y_1}(z)=\begin{cases}\left[b^\alpha(k_-/\alpha)+\beta_1(z)\right]\dfrac{1}{|z|^\alpha}, & z<0,\\ 1-\left[b^\alpha(k_+/\alpha)+\beta_2(z)\right]\dfrac{1}{z^\alpha}, & z>0,\end{cases}$$

for some functions β_1 and β_2 satisfying

$$\lim_{z\to-\infty}\beta_1(z)=\lim_{z\to\infty}\beta_2(z)=0.$$

As there is no appropriate counterpart of the cdf in the sublinear setting, we must recast this condition using expectation. To do so requires F_{Y_1} to possess further regularity properties. For convenience, say that after an extension, the β_i's are continuously differentiable on their respective closed half-lines. This is the lone extra requirement we shall need.

It follows that

$$\mathbb{E}\big[|Y_1|\big] < \infty$$

since

$$\begin{aligned}\int_0^\infty z\,dF_{Y_1}(z) &= -\int_0^1 \frac{\beta_2'(z)}{z^{\alpha-1}}\,dz + \int_0^1 \frac{b^\alpha k_+ + \alpha\beta_2(z)}{z^\alpha}\,dz + \beta_2(1)\\ &\quad + \int_1^\infty \frac{\beta_2(z)}{z^\alpha}\,dz + \int_1^\infty \frac{b^\alpha k_+}{z^\alpha}\,dz \\ &< \infty\end{aligned} \tag{3.3}$$

and similarly for the integral along the negative half-line. One could have cited Theorem 2.6.4 of [13] instead, but (3.3) will be helpful in Example 4.2. We also get

$$\begin{aligned}&n\left|\mathbb{E}\big[\delta_{B_nY_1}v(t,x)\big] - \left(\frac{1}{n}\right)\int_{\mathbb{R}} \delta_z v(t,x) F_{k_\pm}(dz)\right| \\ &\quad = \left(\frac{1}{b^\alpha}\right)\left|\int_{\mathbb{R}} \delta_z v(t,x)\left(\frac{\beta_1'(B_n^{-1}z)|B_n^{-1}z| + \alpha\beta_1(B_n^{-1}z)}{|z|^{\alpha+1}}\mathbf{1}_{(-\infty,0)}(z)\right.\right. \\ &\qquad \left.\left. + \frac{-\beta_2'(B_n^{-1}z)|B_n^{-1}z| + \alpha\beta_2(B_n^{-1}z)}{|z|^{\alpha+1}}\mathbf{1}_{(0,\infty)}(z)\right)dz\right|\end{aligned} \tag{3.4}$$

for all $(t,x) \in [0,1]\times\mathbb{R}$ and $n \geq 1$ by changing variables.

A careful application of elementary estimates shows that this last expression tends to zero uniformly on $[0,1]\times\mathbb{R}$ as $n \to \infty$. To see this, note that we can choose an upper bound, say M_1, for $|\partial_{xx}v|$, $|\partial_x v|$, and $|v|$ on $[0,1]\times\mathbb{R}$ by Lemma 2.9 and Proposition 2.10. Then using integration by parts and the dominated convergence theorem,

$$\begin{aligned}&\left|\int_1^\infty \delta_z v(t,x)\left(\frac{-\beta_2'(B_n^{-1}z)|B_n^{-1}z| + \alpha\beta_2(B_n^{-1}z)}{|z|^{\alpha+1}}\right)dz\right| \\ &\quad = \left|\delta_1 v(t,x)\beta_2\big(B_n^{-1}\big)\right. \\ &\qquad \left. + \int_1^\infty \frac{\beta_2(B_n^{-1}z)}{z^\alpha}\big[\partial_x v(t,x+z) - \partial_x v(t,x)\big]\,dz\right| \\ &\quad \leq 3M_1\big|\beta_2\big(B_n^{-1}\big)\big| + 2M_1\int_1^\infty \frac{|\beta_2(B_n^{-1}z)|}{z^\alpha}\,dz \\ &\quad \to 0\end{aligned} \tag{3.5}$$

as $n \to \infty$. The mean value theorem and a change of variables give

$$\begin{aligned}
&\left|\int_0^{B_n} \delta_z v(t,x)\left(\frac{-\beta_2'(B_n^{-1}z)|B_n^{-1}z| + \alpha\beta_2(B_n^{-1}z)}{|z|^{\alpha+1}}\right) dz\right| \\
&\quad\le \int_0^{B_n} M_1 \frac{|-\beta_2'(B_n^{-1}z)(B_n^{-1}z) + \alpha\beta_2(B_n^{-1}z)|}{z^{\alpha-1}}\, dz \\
&\quad= \left(\frac{M_1}{b^{2-\alpha}n^{2/\alpha-1}}\right)\int_0^1 \frac{|-\beta_2'(z)z + \alpha\beta_2(z)|}{z^{\alpha-1}}\, dz \\
&\quad\to 0
\end{aligned} \tag{3.6}$$

as $n \to \infty$. We have

$$\begin{aligned}
&\left|\int_{B_n}^{1} \delta_z v(t,x)\left(\frac{\alpha\beta_2(B_n^{-1}z)}{|z|^{\alpha+1}}\right) dz\right| \\
&\quad\le \int_{B_n}^{1} M_1 \frac{|\alpha\beta_2(B_n^{-1}z)|}{z^{\alpha-1}}\, dz \\
&\quad\le M_1\alpha \int_0^1 \frac{|\beta_2(B_n^{-1}z)|}{z^{\alpha-1}}\, dz \\
&\quad\to 0
\end{aligned} \tag{3.7}$$

as $n \to \infty$ by the mean value theorem and dominated convergence theorem. Finally,

$$\begin{aligned}
&\left|\int_{B_n}^{1} \delta_z v(t,x)\left(\frac{-\beta_2'(B_n^{-1}z)(B_n^{-1}z)}{|z|^{\alpha+1}}\right) dz\right| \\
&\quad= \Bigg|-\delta_1 v(t,x)\beta_2\big(B_n^{-1}\big) + \delta_{B_n} v(t,x)(B_n)^{-\alpha}\beta_2(1) \\
&\qquad + \int_{B_n}^{1}\big[\partial_x v(t,x+z) - \partial_x v(t,x)\big]\left(\frac{\beta_2(B_n^{-1}z)}{z^{\alpha}}\right) dz \\
&\qquad - \alpha\int_{B_n}^{1} \delta_z v(t,x)\left(\frac{\beta_2(B_n^{-1}z)}{z^{\alpha+1}}\right) dz\Bigg| \\
&\quad\le 3M_1\big|\beta_2\big(B_n^{-1}\big)\big| + M_1|\beta_2(1)|\left(\frac{1}{b^{2-\alpha}n^{2/\alpha-1}}\right) + \int_{B_n}^{1} M_1 \frac{|\beta_2(B_n^{-1}z)|}{z^{\alpha-1}}\, dz \\
&\qquad + \alpha\int_{B_n}^{1} M_1 \frac{|\beta_2(B_n^{-1}z)|}{z^{\alpha-1}}\, dz \\
&\quad\le 3M_1\big|\beta_2\big(B_n^{-1}\big)\big| + M_1|\beta_2(1)|\left(\frac{1}{b^{2-\alpha}n^{2/\alpha-1}}\right) + 2\alpha M_1\int_0^1 \frac{|\beta_2(B_n^{-1}z)|}{z^{\alpha-1}}\, dz \\
&\quad\to 0
\end{aligned} \tag{3.8}$$

as $n \to \infty$ by integration by parts, the dominated convergence theorem, and the mean value theorem. The integrals along the negative half-line are handled similarly.

Having established the connection between Theorem 3.1 and the classical case, we finally present its proof.

Proof of Theorem 3.1. We need to show that

$$\lim_{n\to\infty} \mathcal{E}'\big[\psi(B_n S_n)\big] = \mathcal{E}\big[\psi(X_1)\big] \tag{3.9}$$

for all $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$. Our initial step will be to reduce proving (3.9) to proving (3.12). The advantage of doing so is that we can then incorporate the regularity properties described in Lemma 2.9 and Proposition 2.10. These properties alone do much of the heavy lifting in the estimates at the heart of the argument, and our distributional assumptions do the rest.

Let $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$, and define u by

$$u(t,x) = \mathcal{E}\big[\psi(x + X_t)\big] \tag{3.10}$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}$. We know from Section 2 that u is the unique viscosity solution of (2.1).

It will be more convenient for our purposes to work with the backward equation. Since we will soon rely on the interior regularity results of Proposition 2.10, we also let $0 < h < 1$ and define v by

$$v(t,x) = u(1 + h - t, x) \tag{3.11}$$

for $(t,x) \in (-h, 1+h] \times \mathbb{R}$. Then v will be the unique viscosity solution of (3.2).

Observe that v inherits key regularity properties from u. At the moment, it is enough to note that for any $(t,x) \in (-h, 1+h] \times \mathbb{R}$, $v(\cdot, x)$ is 1/2-Hölder continuous with some constant K_1 and $v(t,\cdot)$ is Lipschitz continuous with constant $\mathrm{Lip}(\psi)$ by Lemma 2.9. Because the t-domain has length $1 + 2h$ and $0 < h < 1$, the 1/2-Hölder continuity is uniform, and we can assume that K_1 does not depend on h. It follows by (3.10) and (3.11) that

$$\begin{aligned}
&\limsup_{n\to\infty} \big|\mathcal{E}'\big[\psi(B_n S_n)\big] - \mathcal{E}\big[\psi(X_1)\big]\big| \\
&\quad \le \limsup_{n\to\infty} \Big(\big|\mathcal{E}'\big[\psi(B_n S_n)\big] - \mathcal{E}'\big[v(1, B_n S_n)\big]\big| + \big|\mathcal{E}'\big[v(1, B_n S_n)\big] - v(0,0)\big| \\
&\qquad + \big|v(0,0) - \mathcal{E}\big[\psi(X_1)\big]\big|\Big) \\
&\quad = \limsup_{n\to\infty} \Big(\big|\mathcal{E}'\big[v(1+h, B_n S_n)\big] - \mathcal{E}'\big[v(1, B_n S_n)\big]\big| + \big|\mathcal{E}'\big[v(1, B_n S_n)\big] - v(0,0)\big| \\
&\qquad + \big|v(0,0) - v(h,0)\big|\Big) \\
&\quad \le \limsup_{n\to\infty} \Big(\mathcal{E}'[K_1\sqrt{h}] + \big|\mathcal{E}'\big[v(1, B_n S_n)\big] - v(0,0)\big|\Big) + K_1\sqrt{h} \\
&\quad = 2K_1\sqrt{h} + \limsup_{n\to\infty} \big|\mathcal{E}'\big[v(1, B_n S_n)\big] - v(0,0)\big|.
\end{aligned}$$

As h is arbitrary, it is sufficient to show that

$$\lim_{n\to\infty} \mathcal{E}'\big[v(1, B_n S_n)\big] = v(0,0). \tag{3.12}$$

The required estimates are intricate, so we will give them in Lemma 3.2 below. □

Lemma 3.2. *In the setup of Theorem* 3.1,

$$\lim_{n\to\infty} \mathcal{E}'\big[v(1, B_n S_n)\big] = v(0,0).$$

Proof. For all $n \geq 3$,

$$\begin{aligned} &v(1, B_n S_n) - v(0,0) \\ &\quad = v(1, B_n S_n) - v\left(\frac{n-1}{n}, B_n S_n\right) + \sum_{i=2}^{n-1}\left[v\left(\frac{i}{n}, B_n S_{i+1}\right) - v\left(\frac{i-1}{n}, B_n S_i\right)\right] \\ &\qquad + v\left(\frac{1}{n}, B_n S_2\right) - v(0,0). \end{aligned} \tag{3.13}$$

Our analysis now becomes delicate. We would like to show that when we apply $\mathcal{E}'$ to (3.13) and let $n \to \infty$, the result goes to zero. Since the number of terms in this decomposition is growing with n, we must prove that our v-increments are decaying quite rapidly. The properties of v arising from Lemma 2.9 are only enough to manage the first and last terms. By the 1/2-Hölder continuity of $v(\cdot, x)$,

$$\mathcal{E}'\left[\left|v(1, B_n S_n) - v\left(\frac{n-1}{n}, B_n S_n\right)\right|\right] \leq \mathcal{E}'\left[K_1\sqrt{\frac{1}{n}}\right] = K_1\sqrt{\frac{1}{n}}. \tag{3.14}$$

If we also use the Lipschitz continuity of $v(t, \cdot)$ and the fact that Y_2 is independent from Y_1, we get

$$\begin{aligned} &\mathcal{E}'\left[\left|v\left(\frac{1}{n}, B_n S_2\right) - v(0,0)\right|\right] \\ &\quad \leq \mathcal{E}'\left[\left|v\left(\frac{1}{n}, B_n S_2\right) - v(0, B_n S_2)\right|\right] + \mathcal{E}'\big[\big|v(0, B_n S_2) - v(0,0)\big|\big] \\ &\quad \leq \mathcal{E}'\left[K_1\sqrt{\frac{1}{n}}\right] + \mathcal{E}'\big[\mathrm{Lip}(\psi) B_n |S_2|\big] \\ &\quad \leq K_1\sqrt{\frac{1}{n}} + 2\,\mathrm{Lip}(\psi) B_n \mathcal{E}'\big[|Y_1|\big]. \end{aligned} \tag{3.15}$$

We remark that although we only referred to $C_{\mathrm{b.Lip}}(\mathbb{R})$ in our definition of independence, our manipulations are still valid by Exercise 3.20 in [30].

Proposition 2.10 allows us to control the remaining terms. Again, this motivates our requirement that $K_{\pm} \subset (\lambda, \Lambda)$ for some $0 < \lambda < \Lambda$. We can find a constant $K_2 > 0$ such that $\partial_t v$ exists on $[0, 1] \times \mathbb{R}$ and

$$\begin{aligned} \left|\partial_t v(t_0, x) - \partial_t v(t_1, x)\right| &\le K_2 |t_0 - t_1|^{\gamma/\alpha}, \\ \left|\partial_t v(t, x_0) - \partial_t v(t, x_1)\right| &\le K_2 |x_0 - x_1|^{\gamma} \end{aligned} \tag{3.16}$$

for all (t_0, x), (t_1, x), (t, x_0), and $(t, x_1) \in [0, 1] \times \mathbb{R}$. We then break down the rest of (3.13) a bit further. If $2 \le i \le n - 1$,

$$\begin{aligned} & v\left(\frac{i}{n}, B_n S_{i+1}\right) - v\left(\frac{i-1}{n}, B_n S_i\right) \\ &\quad = v\left(\frac{i}{n}, B_n S_{i+1}\right) - v\left(\frac{i-1}{n}, B_n S_{i+1}\right) - \partial_t v\left(\frac{i-1}{n}, B_n S_i\right)\frac{1}{n} \\ &\qquad + \partial_t v\left(\frac{i-1}{n}, B_n S_i\right)\frac{1}{n} + v\left(\frac{i-1}{n}, B_n S_{i+1}\right) - v\left(\frac{i-1}{n}, B_n S_i\right). \end{aligned}$$

Let

$$C_i^n = v\left(\frac{i}{n}, B_n S_{i+1}\right) - v\left(\frac{i-1}{n}, B_n S_{i+1}\right) - \partial_t v\left(\frac{i-1}{n}, B_n S_i\right)\frac{1}{n}$$

and

$$D_i^n = \partial_t v\left(\frac{i-1}{n}, B_n S_i\right)\frac{1}{n} + v\left(\frac{i-1}{n}, B_n S_{i+1}\right) - v\left(\frac{i-1}{n}, B_n S_i\right).$$

We can establish an appropriate bound for the C_i^n's using (3.16):

$$\begin{aligned} \left|C_i^n\right| &= \left|\frac{1}{n}\int_0^1 \left[\partial_t v\left(\frac{i-1+\beta}{n}, B_n S_{i+1}\right) - \partial_t v\left(\frac{i-1}{n}, B_n S_{i+1}\right)\right] d\beta \right. \\ &\qquad \left. + \frac{1}{n}\left[\partial_t v\left(\frac{i-1}{n}, B_n S_{i+1}\right) - \partial_t v\left(\frac{i-1}{n}, B_n S_i\right)\right]\right| \\ &\le \frac{1}{n}\int_0^1 \left|\partial_t v\left(\frac{i-1+\beta}{n}, B_n S_{i+1}\right) - \partial_t v\left(\frac{i-1}{n}, B_n S_{i+1}\right)\right| d\beta \\ &\qquad + \frac{1}{n}\left|\partial_t v\left(\frac{i-1}{n}, B_n S_{i+1}\right) - \partial_t v\left(\frac{i-1}{n}, B_n S_i\right)\right| \\ &\le \frac{1}{n}\int_0^1 K_2 \left|\frac{\beta}{n}\right|^{\gamma/\alpha} d\beta + \frac{1}{n} K_2 B_n^{\gamma} |Y_{i+1}|^{\gamma} \\ &\le \frac{K_2}{n}\left[\left(\frac{1}{n}\right)^{\gamma/\alpha} + B_n^{\gamma} |Y_{i+1}|^{\gamma}\right]. \end{aligned}$$

Hence, for $2 \le i \le n-1$,

$$\mathcal{E}'\big[\big|C_i^n\big|\big] \le \frac{K_2}{n}\left[\left(\frac{1}{n}\right)^{\gamma/\alpha} + B_n^{\gamma}\mathcal{E}'\big[|Y_1|^{\gamma}\big]\right] \tag{3.17}$$

since Y_{i+1} and Y_1 are identically distributed. Note that hypothesis (ii) gives that

$$\mathcal{E}'\big[|Y_1|^{\gamma}\big] < \infty.$$

While we need (3.16) to bound the D_i^n's, we finally use (3.1), too. Let $\varepsilon > 0$. By (3.1), we can find N such that $n \ge N$ implies

$$n\left|\mathcal{E}'\big[\delta_{B_nY_1}v(t,x)\big] - \left(\frac{1}{n}\right)\sup_{k_\pm\in K_\pm}\left\{\int_{\mathbb{R}}\delta_z v(t,x)F_{k_\pm}(dz)\right\}\right| < \varepsilon$$

on $[0,1]\times\mathbb{R}$. Now

$$\begin{aligned}
&\mathcal{E}'\left[v\left(\frac{i-1}{n}, B_nx + B_nY_1\right)\right] - v\left(\frac{i-1}{n}, B_nx\right)\\
&\quad = \mathcal{E}'\left[\delta_{B_nY_1}v\left(\frac{i-1}{n}, B_nx\right)\right]
\end{aligned}$$

by (i), so for these n,

$$\begin{aligned}
&n\left|v\left(\frac{i-2}{n}, B_nx\right) - \mathcal{E}'\left[v\left(\frac{i-1}{n}, B_nx + B_nY_1\right)\right]\right|\\
&\quad = n\left|v\left(\frac{i-2}{n}, B_nx\right) - \mathcal{E}'\left[v\left(\frac{i-1}{n}, B_nx + B_nY_1\right)\right]\right.\\
&\qquad + v\left(\frac{i-1}{n}, B_nx\right) - v\left(\frac{i-1}{n}, B_nx\right)\\
&\qquad \left. + \left(\frac{1}{n}\right)\partial_t v\left(\frac{i-1}{n}, B_nx\right) + \left(\frac{1}{n}\right)\sup_{k_\pm\in K_\pm}\left\{\int_{\mathbb{R}}\delta_z v\left(\frac{i-1}{n}, B_nx\right)F_{k_\pm}(dz)\right\}\right|\\
&\quad \le \left|-\frac{v((i-2)/n, B_nx) - v((i-1)/n, B_nx)}{-1/n} + \partial_t v\left(\frac{i-1}{n}, B_nx\right)\right|\\
&\qquad + n\left|\mathcal{E}'\left[\delta_{B_nY_1}v\left(\frac{i-1}{n}, B_nx\right)\right]\right.\\
&\qquad \left. - \left(\frac{1}{n}\right)\sup_{k_\pm\in K_\pm}\left\{\int_{\mathbb{R}}\delta_z v\left(\frac{i-1}{n}, B_nx\right)F_{k_\pm}(dz)\right\}\right|\\
&\quad < \frac{K_2}{n^{\gamma/\alpha}} + \varepsilon
\end{aligned}$$

by the mean value theorem, (3.2), and (3.16). Then

$$\begin{aligned}
&\left|\partial_t v\left(\frac{i-1}{n}, B_n x\right)\frac{1}{n} + \mathcal{E}'\left[v\left(\frac{i-1}{n}, B_n x + B_n Y_{i+1}\right)\right] - v\left(\frac{i-1}{n}, B_n x\right)\right| \\
&\quad \le \frac{1}{n}\left|\partial_t v\left(\frac{i-1}{n}, B_n x\right) + \frac{v((i-2)/n, B_n x) - v((i-1)/n, B_n x)}{1/n}\right| \\
&\qquad + \left|\mathcal{E}'\left[v\left(\frac{i-1}{n}, B_n x + B_n Y_1\right)\right] - v\left(\frac{i-2}{n}, B_n x\right)\right| \\
&\quad < \frac{2K_2}{n^{1+\gamma/\alpha}} + \frac{\varepsilon}{n}
\end{aligned} \tag{3.18}$$

for $2 \le i \le n-1$, $x \in \mathbb{R}$, and $n \ge N$.

Since Y_{i+1} is independent from $(Y_1, \dots, Y_i)$, repeated application of (3.18) shows that for $n \ge N$,

$$\mathcal{E}'\left[\sum_{i=2}^{n-1} D_i^n\right] < (n-2)\left(\frac{2K_2}{n^{1+\gamma/\alpha}} + \frac{\varepsilon}{n}\right) < \frac{2K_2}{n^{\gamma/\alpha}} + \varepsilon \tag{3.19}$$

and

$$\mathcal{E}'\left[\sum_{i=2}^{n-1} D_i^n\right] > -(n-2)\left(\frac{2K_2}{n^{1+\gamma/\alpha}} + \frac{\varepsilon}{n}\right) > -\frac{2K_2}{n^{\gamma/\alpha}} - \varepsilon. \tag{3.20}$$

We only need to combine our bounds above and invoke hypothesis (ii) to finish the proof. By (3.14), (3.15), (3.17), (3.19) and (3.20),

$$\begin{aligned}
&\mathcal{E}'\left[v(1, B_n S_n)\right] - v(0,0) \\
&\quad = \mathcal{E}'\left[v(1, B_n S_n) - v\left(\frac{n-1}{n}, B_n S_n\right) + \sum_{i=2}^{n-1} C_i^n + \sum_{i=2}^{n-1} D_i^n + v\left(\frac{1}{n}, B_n S_2\right) - v(0,0)\right] \\
&\quad \le \mathcal{E}'\left[\left|v(1, B_n S_n) - v\left(\frac{n-1}{n}, B_n S_n\right)\right|\right] + \sum_{i=2}^{n-1} \mathcal{E}'\left[\left|C_i^n\right|\right] + \mathcal{E}'\left[\sum_{i=2}^{n-1} D_i^n\right] \\
&\qquad + \mathcal{E}'\left[\left|v\left(\frac{1}{n}, B_n S_2\right) - v(0,0)\right|\right] \\
&\quad < \left(K_1\sqrt{\frac{1}{n}}\right) + \left(K_2\left[\left(\frac{1}{n}\right)^{\gamma/\alpha} + B_n^\gamma \mathcal{E}'\left[|Y_1|^\gamma\right]\right]\right) + \left(\frac{2K_2}{n^{\gamma/\alpha}} + \varepsilon\right) \\
&\qquad + \left(K_1\sqrt{\frac{1}{n}} + 2\,\mathrm{Lip}(\psi) B_n \mathcal{E}'\left[|Y_1|\right]\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{E}'\big[v(1, B_n S_n)\big] - v(0,0) \\
&\quad > -\left(K_1\sqrt{\frac{1}{n}}\right) - \left(K_2\left[\left(\frac{1}{n}\right)^{\gamma/\alpha} + B_n^\gamma \mathcal{E}'\big[|Y_1|^\gamma\big]\right]\right) - \left(\frac{2K_2}{n^{\gamma/\alpha}} + \varepsilon\right) \\
&\qquad - \left(K_1\sqrt{\frac{1}{n}} + 2\,\mathrm{Lip}(\psi) B_n \mathcal{E}'\big[|Y_1|\big]\right)
\end{aligned}$$

for $n \geq N$. Since $\varepsilon > 0$ is arbitrary and $\lim_{n\to\infty} B_n = 0$, we have

$$\lim_{n\to\infty} \mathcal{E}'\big[v(1, B_n S_n)\big] = v(0,0). \qquad \square$$

4. Examples

Example 4.1. X_1 is in its own domain of normal attraction. While this follows directly from the α-stability of X_1, we will demonstrate this using Theorem 3.1 as well in order to unpack our main result.

Let $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$ and u be defined by

$$u(t,x) = \mathcal{E}\big[\psi(x + X_t)\big]$$

on $[0,\infty) \times \mathbb{R}$. If $\tilde{X}_1$ is an independent copy of X_1, then

$$\begin{aligned}
\mathcal{E}\big[\psi(aX_1 + b\tilde{X}_1)\big] &= \mathcal{E}\big[\mathcal{E}\big[\psi\big(ax + (b^\alpha)^{1/\alpha}\tilde{X}_1\big)\big]_{x=X_1}\big] \\
&= \mathcal{E}\big[u\big(b^\alpha, aX_1\big)\big] \\
&= u\big(a^\alpha + b^\alpha, 0\big) \\
&= \mathcal{E}\big[\psi\big((a^\alpha + b^\alpha)^{1/\alpha} X_1\big)\big]
\end{aligned}$$

for any $a, b \geq 0$ by Lemmas 2.6 and 2.7, that is, X_1 is α-stable. Exercise 3.20 in [30] implies that the same relation actually holds for a broader class of maps. In particular,

$$\begin{aligned}
2^{1/\alpha}\mathcal{E}[X_1] &= \mathcal{E}\big[\mathcal{E}[x + \tilde{X}_1]_{x=X_1}\big] \\
&= \mathcal{E}\big[X_1 + \mathcal{E}[X_1]\big] \\
&= 2\mathcal{E}[X_1],
\end{aligned}$$

so

$$\mathcal{E}[X_1] = 0.$$

It follows similarly that

$$\mathcal{E}[-X_1] = 0.$$

We know

$$\mathcal{E}\big[|X_1|\big] < \infty$$

from Lemma 2.8.

To check the final hypothesis, let $0 < h < 1$ and v be the unique viscosity solution of (3.2). Then for all $(t, x) \in [0, 1] \times \mathbb{R}$,

$$\begin{aligned}
& n\left|\mathcal{E}\big[\delta_{B_n X_1} v(t,x)\big] - \left(\frac{1}{n}\right) \sup_{k_\pm \in K_\pm} \left\{\int_{\mathbb{R}} \delta_z v(t,x) F_{k_\pm}(dz)\right\}\right| \\
&= n\left|\mathcal{E}\big[v(t, x + B_n X_1)\big] - v(t,x) + \left(\frac{1}{n}\right)\partial_t v(t,x)\right| \\
&= n\left|v\left(t - \frac{1}{n}, x\right) - v(t,x) + \left(\frac{1}{n}\right)\partial_t v(t,x)\right| \\
&= \left|\frac{v(t - 1/n, x) - v(t,x)}{1/n} + \partial_t v(t,x)\right| \\
&\le \frac{K_2}{n^{\gamma/\alpha}}
\end{aligned}$$

by (3.11), (3.16) and Lemma 2.7. Here, $b = 1$ or, equivalently,

$$B_n = \frac{1}{n^{1/\alpha}}.$$

Abusing notation, Theorem 3.1 shows that

$$B_n S_n \xrightarrow{\mathrm{d}} X_1$$

as $n \to \infty$.

Example 4.2. Up to some "uniformity" assumptions, this example has a straightforward interpretation.

Let the uncertainty subset of distributions (see [30]) of Y_1 be given by $\{\mathbb{P}_\theta : \theta \in \Theta\}$. If for all $\theta \in \Theta$, a classical random variable with distribution $\mathbb{P}_\theta$ is in the domain of normal attraction of a classical α-stable random variable with triplet θ, then Y_1 is in the domain of normal attraction of X_1.

Let b, $M > 0$ and f be a non-negative function on $\mathbb{N}$ tending to zero as $n \to \infty$. For each $k_\pm \in K_\pm$, let $W_{k_\pm}$ be a classical random variable such that

(i) $W_{k_\pm}$ has mean zero;
(ii) $W_{k_\pm}$ has a cdf $F_{W_{k_\pm}}$ of the form

$$F_{W_{k_\pm}}(z) = \begin{cases} \big[b^\alpha(k_-/\alpha) + \beta_{1,k_\pm}(z)\big]\dfrac{1}{|z|^\alpha}, & z < 0, \\ 1 - \big[b^\alpha(k_+/\alpha) + \beta_{2,k_\pm}(z)\big]\dfrac{1}{z^\alpha}, & z > 0, \end{cases} \tag{4.1}$$

for some continuously differentiable functions $\beta_{1,k_\pm}$ on $(-\infty, 0]$ and $\beta_{2,k_\pm}$ on $[0,\infty)$ with

$$\lim_{z\to-\infty}\beta_{1,k_\pm}(z) = \lim_{z\to\infty}\beta_{2,k_\pm}(z) = 0;$$

(iii) the following quantities are all less than M:

$$\left|\int_{-\infty}^{-1}\frac{\beta_{1,k_\pm}(z)}{(-z)^\alpha}\,dz\right|, \qquad \left|\int_{-1}^{0}\frac{\beta'_{1,k_\pm}(z)}{(-z)^{\alpha-1}}\,dz\right|, \qquad \int_{-1}^{0}\frac{|-\beta'_{1,k_\pm}(z)z+\alpha\beta_{1,k_\pm}(z)|}{(-z)^{\alpha-1}}\,dz,$$
$$\left|\int_{1}^{\infty}\frac{\beta_{2,k_\pm}(z)}{z^\alpha}\,dz\right|, \qquad \left|\int_{0}^{1}\frac{\beta'_{2,k_\pm}(z)}{z^{\alpha-1}}\,dz\right|, \qquad \int_{0}^{1}\frac{|-\beta'_{2,k_\pm}(z)z+\alpha\beta_{2,k_\pm}(z)|}{z^{\alpha-1}}\,dz;$$

and

(iv) the following quantities are less than $f(n)$ for all n:

$$\left|\beta_{2,k_\pm}\left(B_n^{-1}\right)\right|, \qquad \int_{1}^{\infty}\frac{|\beta_{2,k_\pm}(B_n^{-1}z)|}{z^\alpha}\,dz, \qquad \int_{0}^{1}\frac{|\beta_{2,k_\pm}(B_n^{-1}z)|}{z^{\alpha-1}}\,dz$$
$$\left|\beta_{1,k_\pm}\left(-B_n^{-1}\right)\right|, \qquad \int_{-\infty}^{-1}\frac{|\beta_{1,k_\pm}(B_n^{-1}z)|}{(-z)^\alpha}\,dz, \qquad \int_{-1}^{0}\frac{|\beta_{1,k_\pm}(B_n^{-1}z)|}{(-z)^{\alpha-1}}\,dz.$$

Note that by (ii) alone, the terms in (iii) are finite and the terms in (iv) approach zero as $n\to\infty$. In other words, the content of (iii) and (iv) is that uniform bounds and minimum rates of convergence exist.

Define an operator $\mathcal{E}'$ on a space $\mathcal{H}$ of suitable functions by

$$\mathcal{E}'[\varphi] = \sup_{k_\pm\in K_\pm}\int_{\mathbb{R}}\varphi(z)\,dF_{W_{k_\pm}}(z)$$

for all $\varphi\in\mathcal{H}$. The exact composition of $\mathcal{H}$ is irrelevant for our purposes here. Clearly, $(\mathbb{R},\mathcal{H},\mathcal{E}')$ is a sublinear expectation space.

Let Y_1 be the random variable on this space defined by

$$Y_1(x) = x$$

for all $x\in\mathbb{R}$. We will use Theorem 3.1 to show that

$$B_nS_n \xrightarrow{\text{d}} X_1$$

as $n\to\infty$. Most of the difficulties have already been addressed during our discussion of the classical case in Section 3.

Since each $W_{k_\pm}$ has mean zero,

$$\mathcal{E}'[Y_1] = \sup_{k_\pm\in K_\pm}\int_{\mathbb{R}} z\,dF_{W_{k_\pm}}(z) = 0$$

and

$$\mathcal{E}'[-Y_1] = \sup_{k_\pm \in K_\pm} \int_{\mathbb{R}} -z\, dF_{W_{k_\pm}}(z) = 0.$$

After recalling that $K_\pm \subset (\lambda, \Lambda)$, (iii) gives

$$\mathcal{E}'\big[|Y_1|\big] < \infty$$

using (3.3) and (4.1). Observe that we are solving (4.1) for the obvious expressions to obtain uniform bounds on the terms

$$\big|\beta_{2,k_\pm}(1)\big|, \qquad \big|\beta_{1,k_\pm}(-1)\big|, \qquad \left|\int_0^1 \frac{b^\alpha k_+ + \alpha\beta_{2,k_\pm}(z)}{z^\alpha}\, dz\right|$$

and

$$\left|\int_{-1}^0 \frac{b^\alpha k_- + \alpha\beta_{1,k_\pm}(z)}{(-z)^\alpha}\, dz\right|.$$

To check the remaining hypothesis, let $0 < h < 1$, $\psi \in C_{\mathrm{b.Lip}}(\mathbb{R})$, and v be the unique viscosity solution of (3.2). The techniques of (3.4) demonstrate that

$$\begin{aligned}
&n\left|\mathcal{E}'\big[\delta_{B_n Y_1} v(t,x)\big] - \left(\frac{1}{n}\right) \sup_{k_\pm \in K_\pm} \left\{\int_{\mathbb{R}} \delta_z v(t,x) F_{k_\pm}(dz)\right\}\right| \\
&\le \left(\frac{1}{b^\alpha}\right) \sup_{k_\pm \in K_\pm} \left|\int_{\mathbb{R}} \delta_z v(t,x) \left(\frac{\beta'_{1,k_\pm}(B_n^{-1}z)|B_n^{-1}z| + \alpha\beta_{1,k_\pm}(B_n^{-1}z)}{|z|^{\alpha+1}} \mathbf{1}_{(-\infty,0)}(z)\right.\right. \\
&\quad \left.\left. + \frac{-\beta'_{2,k_\pm}(B_n^{-1}z)|B_n^{-1}z| + \alpha\beta_{2,k_\pm}(B_n^{-1}z)}{|z|^{\alpha+1}} \mathbf{1}_{(0,\infty)}(z)\right) dz\right|
\end{aligned}$$

for $(t, x) \in [0, 1] \times \mathbb{R}$ and $n \ge 1$. Combining (3.5), (3.6), (3.7) and (3.8) with (iii) and (iv) proves that this last expression approaches zero in the required way.

Appendix A: Interior regularity theory background

Interior regularity theory for fully nonlinear integro-differential equations is rich and well developed. Before describing the results that we need for our proof, we provide a short discussion of the literature. Readers new to this field are encouraged to first consult [41] for an introduction.

Some results and methods from the interior regularity theory for PDEs can be imported to the non-local case after minor modifications. For other aspects of the theory, this is false. As described in Section 2 of [14], a Hölder estimate and the Harnack inequality appear together in the local setting; however, there are non-local equations for which a Hölder estimate holds in the absence of the Harnack inequality. A partial list of other ways that non-local results can significantly differ from their local counterparts can be found in [41].

Early work on the regularity of integro-differential equations focused on equations in divergence form. A survey of these results is contained in [15]. For equations in non-divergence form, [3] contains the first Harnack inequality and Hölder estimate. The equations studied in [3] are of the form

$$\int_{\mathbb{R}^d} \big[w(x+z) - w(x) - z\nabla w(x)\mathbf{1}_{B_1}(z)\big]k(x,z)\,dz = 0,$$

where k is a kernel such that

$$k(x,z) = k(x,-z) \tag{A.1}$$

and

$$\frac{\lambda_1}{|z|^{d+\alpha_1}} \le k(x,z) \le \frac{\Lambda_1}{|z|^{d+\alpha_1}} \tag{A.2}$$

for some constants $\lambda_1, \Lambda_1 > 0$ and $\alpha_1 \in (0,2)$. For a review of the extensions of this initial work, see [15].

The Hölder estimate in [3] blows up as $\alpha_1 \to 2$. Many other early estimates share this feature. The first paper to prove a Hölder estimate and Harnack inequality without this property is [5]. The equations are of the form

$$\inf_r \sup_s \left\{ \int_{\mathbb{R}^d} \big[w(x+z) - w(x) - z\nabla w(x)\mathbf{1}_{B_1}(z)\big]k^{rs}(z)\,dz \right\} = 0 \tag{A.3}$$

for kernels k^{rs} depending only on z and satisfying (A.1), (A.2) and an additional smoothness condition. More precisely, for some fixed positive constants ρ and C,

$$\int_{\mathbb{R}^d \setminus B_\rho} \frac{|k(z) - k(z-\varepsilon)|}{|\varepsilon|}\,dz \le C$$

whenever

$$|\varepsilon| < \frac{\rho}{2}.$$

The paper culminates in a $C^{1,\gamma}$ estimate for the solution of (A.3).

These findings have been extended in a number of ways. For instance, references such as [17,18,33,36] and [19] study equations with non-symmetric kernels, that is, kernels that do not satisfy (A.1). Other examples of recent work include [6,34] and [16].

We now collect the definitions and results from [18] and [19] that we need for our proof. These references describe properties of the solutions to a broad class of non-local fully nonlinear parabolic equations of the form

$$\partial_t w(t,x) - I w(t,x) = f(t).$$

Due to the general nature of these equations, [18] and [19] are quite technical. Since (2.1) is an easy case of the equations studied in these papers, we will simplify this material and present only the version that we need for our argument.

Notation A.1. Let

$$\mathfrak{C}_{\tau,r}(t,x) := (t-\tau,t] \times (x-r,x+r).$$

We write $\mathfrak{C}_{\tau,r}$ for the cylinder $\mathfrak{C}_{\tau,r}(0,0)$. For suitable functions w, let

$$\tilde{\delta}_z w(t,x) := w(t,x+z) - w(t,x) - \partial_x w(t,x)\mathbf{1}_{(-1,1)}(z)z;$$

$$\|w\|_{L^1(\nu)} := \int_{\mathbb{R}} |w(z)| \min\bigl(1,|z|^{-1-\alpha}\bigr)\,dz; \quad \text{and}$$

$$[w]_{C^{0,1}((t_0,t_1]\mapsto L^1(\nu))} := \sup_{(t-\tau,t]\subseteq(t_0,t_1]} \frac{\|w(t,\cdot)-w(t-\tau,\cdot)\|_{L^1(\nu)}}{\tau}.$$

We also let

$$b_{k_\pm} := (k_- - k_+)\int_1^\infty \frac{dz}{z^\alpha}$$

for all $k_\pm \in K_\pm$.

In the literature, one also works frequently with cylinders of the form

$$(t-\tau^\alpha,t] \times (x-r,x+r)$$

due to their convenient scaling properties. We introduce

$$\|\cdot\|_{L^1(\nu)}$$

and

$$[\cdot]_{C^{0,1}((t_0,t_1]\mapsto L^1(\nu))}$$

due to their role in upcoming Hölder estimates, namely, Lemmas A.4 and A.5. The symbols $\tilde{\delta}_z$ and $b_{k_\pm}$ facilitate the identification of (2.1) with the equations studied [18] and [19]. Observe that for all $k_\pm \in K_\pm$ and suitable functions w,

$$\int_{\mathbb{R}} \delta_z w(t,x) F_{k_\pm}(dz) = b_{k_\pm}\partial_x w(t,x) + \int_{\mathbb{R}} \tilde{\delta}_z w(t,x) F_{k_\pm}(dz). \tag{A.4}$$

Definition A.2. *Since $K_\pm \subset (\lambda,\Lambda)$, we can pick $\beta > 0$ such that*

$$\sup_{k_\pm\in K_\pm}\left\{\sup_{r\in(0,1)}\left\{r^{\alpha-1}\left|b_{k_\pm} + \int_{(-1,1)\setminus(-r,r)} zF_{k_\pm}(dz)\right|\right\}\right\} \le \beta.$$

Let $\mathcal{L}_0$ be the family of operators

$$w(t,x) \mapsto b\partial_x w(t,x) + \int_{\mathbb{R}} \tilde{\delta}_z w(t,x)\frac{k(z)}{|z|^{1+\alpha}}\,dz,$$

where k is a kernel and b is a constant such that $\lambda \le k \le \Lambda$ and

$$\sup_{r\in(0,1)} r^{\alpha-1}\left|b+\int_{(-1,1)\setminus(-r,r)} \frac{zk(z)}{|z|^{1+\alpha}}\,dz\right| \le \beta.$$

We say that an operator in $\mathcal{L}_0$ is in $\mathcal{L}_1$ if

$$\left|\partial_z k(z)\right| \le \frac{\Lambda}{|z|},$$

and an operator in $\mathcal{L}_1$ is in $\mathcal{L}_2$ if

$$\left|\partial^2_{zz} k(z)\right| \le \frac{\Lambda}{|z|^2}.$$

The stronger regularity requirements on the kernels (in $\mathcal{L}_2$, say, compared to those in $\mathcal{L}_0$) give rise to stronger regularity results. All of the operators

$$w(t,x) \mapsto b_{k_\pm}\partial_x w(t,x) + \int_{\mathbb{R}} \tilde{\delta}_z w(t,x) F_{k_\pm}(dz)$$

are in each of these families. As we will soon see in (B.1), we will be especially interested in the operator I defined by

$$Iw(t,x) = \inf_{k_\pm\in K_\pm}\left\{b_{k_\pm}\partial_x w(t,x) + \int_{\mathbb{R}} \tilde{\delta}_z w(t,x) F_{k_\pm}(dz)\right\}.$$

I is a specific case of an *extremal operator*.

Definition A.3. *For a collection of operators $\mathcal{L} \subseteq \mathcal{L}_0$, define the extremal operators $\mathcal{M}^+_{\mathcal{L}}$ and $\mathcal{M}^-_{\mathcal{L}}$ by*

$$\mathcal{M}^+_{\mathcal{L}} = \sup_{L\in\mathcal{L}} L \quad \textit{and} \quad \mathcal{M}^-_{\mathcal{L}} = \inf_{L\in\mathcal{L}} L.$$

I has a number of other key properties including the following.[4]

(i) $I0 = 0$.

(ii) I is *uniformly elliptic* with respect to $\mathcal{L}_j$, that is,

$$\mathcal{M}^-_{\mathcal{L}_j}(w_1 - w_2) \le Iw_1 - Iw_2 \le \mathcal{M}^+_{\mathcal{L}_j}(w_1 - w_2).$$

(iii) I is *translation invariant*, that is,

$$I\big(w(t_0+\cdot, x_0+\cdot)\big)(t,x) = (Iw)(t_0+t, x_0+x).$$

[4]Though we will not emphasize this point, we remark in passing that $Iw(t,x)$ is well-defined for any $w(t,\cdot) \in C^{1,1}(x) \cap L^1(\nu)$ (see Section 2 of [19]).

(i) is trivial. See Section 2 of [19] for (ii). Since I has constant coefficients, we get (iii). We highlight these classes of operators and properties of I for the convenience of the reader comparing the next three results to their original versions (see Theorem 2.3 in [19] for Lemma A.4; Theorems 1.1, 2.4, and 2.5 in [19] for Lemma A.5; and Theorem 3.3 in [18] for Lemma A.6).[5]

Lemma A.4. *Let w satisfy*

$$\partial_t w - M^+_{\mathcal{L}_0} w \leq 0,$$
$$\partial_t w - M^-_{\mathcal{L}_0} w \geq 0$$

in the viscosity sense on $\mathfrak{C}_{1,1}$. There is some $\gamma \in (0,1)$ and $C > 0$ depending only on λ, Λ, and β such that for every $(t_0, x_0), (t_1, x_1) \in \mathfrak{C}_{1/2,1/2}$,

$$\frac{|w(t_0,x_0) - w(t_1,x_1)|}{(|t_0 - t_1|^{1/\alpha} + |x_0 - x_1|)^\gamma} \leq C\|w\|_{L^1((-1,0]\mapsto L^1(\nu))}.$$

Lemma A.5. *Let w satisfy*

$$\partial_t w - Iw = 0$$

in the viscosity sense on $\mathfrak{C}_{1,1}$. There is some $\gamma \in (0,1)$ and $C > 0$ depending only on λ, Λ, and β such that for every $(t_0, x_0), (t_1, x_1) \in \mathfrak{C}_{1/2,1/2}$,

$$\left|\partial_x w(t_0,x_0)\right| + \frac{|\partial_x w(t_0,x_0) - \partial_x w(t_1,x_1)|}{(|t_0 - t_1|^{1/\alpha} + |x_0 - x_1|)^\gamma} \leq C\|w\|_{L^1((-1,0]\mapsto L^1(\nu))}$$

and

$$\left|\partial_t w(t_0,x_0)\right| + \frac{|\partial_t w(t_0,x_0) - \partial_t w(t_1,x_1)|}{(|t_0 - t_1|^{1/\alpha} + |x_0 - x_1|)^\gamma} \leq C[w]_{C^{0,1}((-1,0]\mapsto L^1(\nu))}.$$

We also have

$$\|w\|_{C^{\alpha+\gamma}(\mathfrak{C}_{1/2,1/2})} \leq C\big(\|w\|_{L^1((-1,0]\mapsto L^1(\nu))} + [w\mathbf{1}_{(-1,1)^c}]_{C^{0,1}((-1,0]\mapsto L^1(\nu))}\big).$$

Lemma A.6. *Let w_1, w_2 satisfy*

$$\partial_t w_i - Iw_i = 0$$

in the viscosity sense on some domain Ω. Then

$$\partial_t(w_1 - w_2) - M^+_{\mathcal{L}_0}(w_1 - w_2) \leq 0,$$
$$\partial_t(w_1 - w_2) - M^-_{\mathcal{L}_0}(w_1 - w_2) \geq 0$$

[5] A number of related results exist in the literature. We mention only a small sample. Theorem 12.1 in [5], Theorem 1.1 in [14] and Theorem 7.1 in [33] are C^γ estimates along the lines of Lemma A.4. Theorem 8.1 in [33], Theorem 13.1 in [5], Theorem 1.1 in [6] and Theorem 1.1 in [34] contain $C^{1,\gamma}$ or $C^{\alpha+\gamma}$ estimates similar to those in Lemma A.5. Like Lemma A.6, Theorem 5.9 in [5] and Lemma 3.2 in [35] investigate the difference of viscosity solutions.

also holds in the viscosity sense on Ω.

We will need one more result (for the original version, see Lemma 5.6 and the proof of Corollary 5.7 in [7]). It is the key to a standard technique from the literature allowing one to repeatedly apply an estimate such as Lemma A.4 in order to obtain a higher regularity estimate.

Lemma A.7. *Let* $0 < \beta_1 \le 1$, $0 < \beta_2 < 1$, $L > 0$, *and* $w \in L^\infty([-1, 1])$ *satisfy*

$$\|w\|_{L^\infty([-1,1])} \le L.$$

For $0 < |h_0| \le 1$, *define* w_{β_1,h_0} *by*

$$w_{\beta_1,h_0}(x) = \frac{w(x + h_0) - w(x)}{|h_0|^{\beta_1}}$$

for all $x \in I_{h_0}$, *where* $I_{h_0} = [-1, 1 - h_0]$ *if* $h_0 > 0$ *and* $I_{h_0} = [-1 - h_0, 1]$ *if* $h_0 < 0$. *Suppose that*

$$w_{\beta_1,h_0} \in C^{\beta_2}(I_{h_0})$$

and

$$\|w_{\beta_1,h_0}\|_{C^{\beta_2}(I_{h_0})} \le L$$

for any $0 < |h_0| \le 1$.

(i) *If* $\beta_1 + \beta_2 < 1$, *then*

$$w \in C^{\beta_1+\beta_2}\big([-1, 1]\big)$$

and

$$\|w\|_{C^{\beta_1+\beta_2}([-1,1])} \le CL.$$

(ii) *If* $\beta_1 + \beta_2 > 1$ *and* $\beta_1 \neq 1$, *then*

$$w \in C^{0,1}\big([-1, 1]\big)$$

and

$$\|w\|_{C^{0,1}([-1,1])} \le CL.$$

(iii) *If* $\beta_1 = 1$, *then* $w \in C^{1,\beta_2}([-1, 1])$ *and*

$$\|w\|_{C^{1,\beta_2}([-1,1])} \le CL.$$

In any of these cases, C *depends only on* $\beta_1 + \beta_2$.

We will often apply these results on different domains than we have listed above without comment. For instance, we might use Lemma A.5 on $\mathfrak{C}_{1,1}(t, x)$ or Lemma A.7 on an arbitrary closed interval. These "new" results are obtained merely by translating or rescaling, both standard routines in the literature. As an example of such an operation, notice that if w satisfies

$$\partial_t w - I w = 0$$

in the viscosity sense on $(t_1, t_2] \times \Omega$, then $\tilde{w}$ defined by

$$\tilde{w}(t, x) = w\big(r^\alpha t + t_0, rx + x_0\big)$$

satisfies

$$\partial_t \tilde{w} - I \tilde{w} = 0$$

in the viscosity sense on

$$\left(\frac{t_1 - t_0}{r^\alpha}, \frac{t_2 - t_0}{r^\alpha}\right] \times \frac{\Omega - x_0}{r}$$

(see Section 2.1.1 of [18]). Further information can be found in [18,19] and [7].

Appendix B: Proof of Proposition 2.10

In the hope of keeping the number of constants in our argument at a reasonable level, we will not issue a new subscript each time we introduce a new constant B below. Also, we will write $\bar{u}$ instead of $-u$. From (2.1) and (A.4), $\bar{u}$ is a viscosity solution of

$$\begin{aligned} \partial_t \bar{u}(t, x) - I\bar{u}(t, x) &= 0, \qquad (t, x) \in (0, \infty) \times \mathbb{R} \\ \bar{u}(0, x) &= -\psi(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{B.1}$$

It suffices to show that parts (i)–(iv) of Proposition 2.10 hold for $\bar{u}$ and (B.1).

The quantities

$$[\bar{u}]_{C^{0,1}((t_0,t_1] \mapsto L^1(\nu))}$$

play a crucial role in Lemma A.5, so our first goal will be to control them for t_0 greater than some positive number. We will do this by showing that $\bar{u}$ is uniformly Lipschitz as a function of time for times above some lower bound. Achieving a Lipschitz estimate can be done using a standard strategy. Specifically, we will begin by obtaining an initial $C^{\gamma/\alpha}$ estimate from Lemma A.4. Lemma A.6 will allow us to apply Lemma A.4 to get a $C^{\gamma/\alpha}$ estimate for the incremental quotients of $\bar{u}$. Then Lemma A.7 will give that $\bar{u}$ is $C^{2\gamma/\alpha}$ in time. We will repeat these steps to show that $\bar{u}$ is $C^{3\gamma/\alpha}$ in time, $C^{4\gamma/\alpha}$ in time, and so on until we conclude that $\bar{u}$ is Lipschitz in time.

Since

$$\mathcal{M}^-_{\mathcal{L}_0} w \le I w \le \mathcal{M}^+_{\mathcal{L}_0} w,$$

$\bar{u}$ satisfies

$$\partial_t \bar{u} - M^+_{\mathcal{L}_0} \bar{u} \le 0,$$

$$\partial_t \bar{u} - M^-_{\mathcal{L}_0} \bar{u} \ge 0$$

in the viscosity sense on $(0,\infty)\times\mathbb{R}$. For any $\bar{t}>1$,

$$\big\|\bar{u}(\bar{t}+\cdot,\cdot)\big\|_{L^1((-1,0]\mapsto L^1(\nu))}=\int_{-1}^{0}\int_{\mathbb{R}}\big|\bar{u}(\bar{t}+t,z)\big|\min\big(1,|z|^{-1-\alpha}\big)\,dz\,dt$$
$$\le\|\psi\|_{L^\infty(\mathbb{R})}\int_{-1}^{0}\int_{\mathbb{R}}\min\big(1,|z|^{-1-\alpha}\big)\,dz\,dt$$

by Lemma 2.9. Lemma A.4 implies that for some $B,\gamma>0$,

$$\frac{|\bar{u}(t_0,x_0)-\bar{u}(t_1,x_1)|}{(|t_0-t_1|^{1/\alpha}+|x_0-x_1|)^\gamma}\le B \tag{B.2}$$

for every $(t_0,x_0),(t_1,x_1)\in\mathfrak{C}_{1/2,1/2}(\bar{t},\bar{x})$ with $\bar{t}>1$.

For $0<|h_0|<1/2$, define $\bar{u}_{\gamma/\alpha,h_0}$ by

$$\bar{u}_{\gamma/\alpha,h_0}(t,x)=\frac{\bar{u}(t+h_0,x)-\bar{u}(t,x)}{|h_0|^{\gamma/\alpha}}$$

for all $(t,x)\in[1/2,\infty)\times\mathbb{R}$. Then

$$\|\bar{u}_{\gamma/\alpha,h_0}\|_{L^\infty((1,\infty)\times\mathbb{R})}\le B$$

by (B.2). Hence,

$$\big\|\bar{u}_{\gamma/\alpha,h_0}(\bar{t}+\cdot,\cdot)\big\|_{L^1((-1,0]\mapsto L^1(\nu))}\le B\int_{\mathbb{R}}\min\big(1,|z|^{-1-\alpha}\big)\,dz$$

for any $\bar{t}>2$.

Notice that

$$\partial_t\bar{u}(\cdot+h_0,\cdot)-I\bar{u}(\cdot+h_0,\cdot)=0$$

in the viscosity sense on $(1/2,\infty)\times\mathbb{R}$ because (B.1) has constant coefficients. Lemma A.6 implies that

$$\partial_t\bar{u}_{\gamma/\alpha,h_0}-M^+_{\mathcal{L}_0}\bar{u}_{\gamma/\alpha,h_0}\le 0,$$
$$\partial_t\bar{u}_{\gamma/\alpha,h_0}-M^-_{\mathcal{L}_0}\bar{u}_{\gamma/\alpha,h_0}\ge 0$$

in the viscosity sense on $(1/2,\infty)\times\mathbb{R}$. For some B,

$$\frac{|\bar{u}_{\gamma/\alpha,h_0}(t_0,x_0)-\bar{u}_{\gamma/\alpha,h_0}(t_1,x_1)|}{(|t_0-t_1|^{1/\alpha}+|x_0-x_1|)^\gamma}\le B$$

for every $(t_0,x_0),(t_1,x_1)\in\mathfrak{C}_{1/2,1/2}(\bar{t},\bar{x})$ with $\bar{t}>2$ by Lemma A.4.

Lemma A.7 shows that for a small r_1 (less than 1/4), we can find B such that

$$\bar{u}(\cdot,\bar{x})\in C^{2\gamma/\alpha}\big([\bar{t}-r_1,\bar{t}+r_1]\big)$$

and

$$\left\| \bar{u}(\cdot, \bar{x}) \right\|_{C^{2\gamma/\alpha}([\bar{t}-r_1, \bar{t}+r_1])} \leq B \tag{B.3}$$

for $\bar{t} > 2$.

Due to Lemma A.7, assume without loss of generality that α/γ is not an integer. Starting from the incremental quotient

$$\frac{\bar{u}(t + h_0, x) - \bar{u}(t, x)}{|h_0|^{2\gamma/\alpha}},$$

we can use these steps to produce a $C^{3\gamma/\alpha}$ estimate for $\bar{u}$ in time. By continuing to repeat this procedure, we will obtain a $C^{4\gamma/\alpha}$ estimate, a $C^{5\gamma/\alpha}$ estimate, and so on until we obtain a Lipschitz estimate for $\bar{u}$ in time. More precisely, we will find B and a small r_n such that

$$\bar{u}(\cdot, \bar{x}) \in C^{0,1}\big([\bar{t} - r_n, \bar{t} + r_n]\big)$$

and

$$\left\| \bar{u}(\cdot, \bar{x}) \right\|_{C^{0,1}([\bar{t}-r_n, \bar{t}+r_n])} \leq B$$

for $\bar{t} > \lceil \alpha/\gamma \rceil$.

For $t_0, t_1 > \lceil \alpha/\gamma \rceil$,

$$\begin{aligned}
\left| \bar{u}(t_0, x_0) - \bar{u}(t_1, x_0) \right| &\leq \left| \bar{u}(s_0, x_0) - \bar{u}(s_1, x_0) \right| + \cdots + \left| \bar{u}(s_{N-1}, x_0) - \bar{u}(s_N, x_0) \right| \\
&\leq B|s_0 - s_1| + \cdots + B|s_{N-1} - s_N| \\
&= B|t_0 - t_1|,
\end{aligned}$$

where $t_0 = s_0$, $t_1 = s_N$, and $s_i < s_{i+1} \leq s_i + 2r_n$. This indicates that

$$\bar{u}(\cdot, \bar{x}) \in C^{0,1}\big(\big(\lceil \alpha/\gamma \rceil, \infty\big)\big)$$

and

$$\left\| \bar{u}(\cdot, \bar{x}) \right\|_{C^{0,1}((\lceil \alpha/\gamma \rceil, \infty))} \leq B.$$

Then $t_0, t_1 > \lceil \alpha/\gamma \rceil$ implies

$$\begin{aligned}
[\bar{u}\mathbf{1}_{(-1,1)^c}]_{C^{0,1}((t_0,t_1] \mapsto L^1(\nu))} &\leq [\bar{u}]_{C^{0,1}((t_0,t_1] \mapsto L^1(\nu))} \\
&= \sup_{(t-\tau, t] \subseteq (t_0, t_1]} \frac{\|\bar{u}(t, \cdot) - \bar{u}(t-\tau, \cdot)\|_{L^1(\nu)}}{\tau} \\
&\leq B \int_{\mathbb{R}} \min\big(1, |z|^{-1-\alpha}\big)\, dz.
\end{aligned}$$

Lemma A.5 gives that for $\bar{t} > \lceil \alpha/\gamma \rceil$,

$$\left| \partial_x \bar{u}(t_0, x_0) \right| + \frac{|\partial_x \bar{u}(t_0, x_0) - \partial_x \bar{u}(t_1, x_1)|}{(|t_0 - t_1|^{1/\alpha} + |x_0 - x_1|)^{\gamma}} \leq B \tag{B.4}$$

and

$$\left|\partial_t \bar{u}(t_0, x_0)\right| + \frac{|\partial_t \bar{u}(t_0, x_0) - \partial_t \bar{u}(t_1, x_1)|}{(|t_0 - t_1|^{1/\alpha} + |x_0 - x_1|)^\gamma} \le B \tag{B.5}$$

for every $(t_0, x_0), (t_1, x_1) \in \mathfrak{C}_{1/2,1/2}(\bar{t}, \bar{x})$. It also shows that

$$\|\bar{u}\|_{C^{\alpha+\gamma}(\mathfrak{C}_{1/2,1/2}(\bar{t},\bar{x}))} \le B. \tag{B.6}$$

After suitably rescaling, we see that these inequalities actually hold for $\bar{t} > (1+h)/2$. Part (i) of Proposition 2.10 then follows from (B.4) and (B.5), while part (iii) follows from (B.6). From (B.5) and a simple covering argument, we know that as long as the distance between x_0 and x_1 is under some arbitrary bound, we can find B such that

$$\left|\partial_t \bar{u}(t, x_0) - \partial_t \bar{u}(t, x_1)\right| \le B|x_0 - x_1|^\gamma$$

for $t \in [h, h+1]$. Since $\partial_t \bar{u}$ is bounded on $[h, h+1] \times \mathbb{R}$, we can drop the distance constraint and get the second inequality in part (ii). A similar covering argument finishes the proof of the first inequality and yields part (ii) of Proposition 2.10.

It remains to prove part (iv). In this case, the equation for $\bar{u}$ is

$$\begin{aligned} \partial_t \bar{u}(t,x) - b_{k_\pm}\partial_x \bar{u}(t,x) - \int_{\mathbb{R}} \tilde{\delta}_z \bar{u}(t,x) F_{k_\pm}(dz) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R} \\ \bar{u}(0,x) = -\psi(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{B.7}$$

Since $\bar{u}$ is a classical solution of this equation on $[h, \infty) \times \mathbb{R}$, $\bar{u}(\cdot, \bar{x} + \cdot)$ also classically satisfies

$$\partial_t \bar{u}(\cdot, \bar{x}+\cdot) - b_{k_\pm}\partial_x \bar{u}(\cdot, \bar{x}+\cdot) - \int_{\mathbb{R}} \tilde{\delta}_z \bar{u}(\cdot, \bar{x}+\cdot) F_{k_\pm}(dz) = 0$$

on $[h, \infty) \times \mathbb{R}$. Then

$$\hat{u}_{h_0}(t,x) := \frac{\bar{u}(t, x+h_0) - \bar{u}(t,x)}{|h_0|}$$

is a classical solution of (B.7) on $[h, \infty) \times \mathbb{R}$ as well.

Lemma 2.9 implies that

$$\left\|\hat{u}_{h_0}(\bar{t}+\cdot, \cdot)\right\|_{L^1((-1,0]\mapsto L^1(\nu))} \le \operatorname{Lip}(\psi) \int_{\mathbb{R}} \min\left(1, |z|^{-1-\alpha}\right) dz$$

for $\bar{t} > 1$. By Lemma A.5, it follows that for some B,

$$\left|\partial_x \hat{u}_{h_0}(t_0, x_0)\right| + \frac{|\partial_x \hat{u}_{h_0}(t_0, x_0) - \partial_x \hat{u}_{h_0}(t_1, x_1)|}{(|t_0 - t_1|^{1/\alpha} + |x_0 - x_1|)^\gamma} \le B$$

for every $(t_0, x_0), (t_1, x_1) \in \mathfrak{C}_{1/2,1/2}(\bar{t}, \bar{x})$ with $\bar{t} > 1$. Rewriting this in terms of $\bar{u}$, we see that we have found a γ-Hölder estimate for

$$\frac{\partial_x \bar{u}(t_0, x+h_0) - \partial_x \bar{u}(t_0, x)}{|h_0|}.$$

By Lemma A.7, $\partial^2_{xx}\bar{u}$ exists and is bounded on $(1/2, \infty) \times \mathbb{R}$. By rescaling, we get that this actually holds on $[h, h+1] \times \mathbb{R}$.

Acknowledgements

We gratefully acknowledge Dennis Kriventsov of the University of Texas at Austin and Luis Silvestre of the University of Chicago for valuable conversations regarding interior regularity estimates for PIDEs. We also thank the anonymous referees for their helpful advice on improving the paper. This work is supported by the National Science Foundation under Grant DMS-0955463.

References

[1] Avellaneda, M., Levy, A. and Parás, A. (1995). Pricing and hedging derivative securities in markets with uncertain volatilities. *Appl. Math. Finance* **2** 73–88.

[2] Barenblatt, G.I. (1979). *Similarity, Self-Similarity, and Intermediate Asymptotics*. New York–London: Consultants Bureau [Plenum]. Translated from the Russian by Norman Stein, translation edited by Milton Van Dyke, with a foreword by Ya.B. Zel'dovich [Ja.B. Zel'dovič]. MR0556234

[3] Bass, R.F. and Levin, D.A. (2002). Harnack inequalities for jump processes. *Potential Anal.* **17** 375–388. MR1918242

[4] Bayraktar, E. and Munk, A. (2015). Comparing the G-normal distribution to its classical counterpart. *Commun. Stoch. Anal.* **9** 1–18.

[5] Caffarelli, L. and Silvestre, L. (2009). Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.* **62** 597–638. MR2494809

[6] Caffarelli, L. and Silvestre, L. (2011). The Evans–Krylov theorem for nonlocal fully nonlinear equations. *Ann. of Math.* (2) **174** 1163–1187. MR2831115

[7] Caffarelli, L.A. and Cabré, X. (1995). *Fully Nonlinear Elliptic Equations*. *American Mathematical Society Colloquium Publications* **43**. Providence, RI: Amer. Math. Soc. MR1351007

[8] Dolinsky, Y., Nutz, M. and Soner, H.M. (2012). Weak approximation of G-expectations. *Stochastic Process. Appl.* **122** 664–675. MR2868935

[9] Hu, M., Ji, S., Peng, S. and Song, Y. (2014). Comparison theorem, Feynman–Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. *Stochastic Process. Appl.* **124** 1170–1195. MR3138611

[10] Hu, M. and Li, X. (2014). Independence under the G-expectation framework. *J. Theoret. Probab.* **27** 1011–1020. MR3245996

[11] Hu, M. and Peng, S. (2009). G-Lévy processes under sublinear expectations. Preprint. Available at arXiv:0911.3533v1.

[12] Hu, Z.-C. and Zhou, L. (2015). Multi-dimensional central limit theorems and laws of large numbers under sublinear expectations. *Acta Math. Sin.* (*Engl. Ser.*) **31** 305–318. MR3297852

[13] Ibragimov, I.A. and Linnik, Y.V. (1971). *Independent and Stationary Sequences of Random Variables*. Groningen: Wolters-Noordhoff Publishing. With a supplementary chapter by I.A. Ibragimov and V.V. Petrov, translation from the Russian edited by J.F.C. Kingman. MR0322926

[14] Kassmann, M., Rang, M. and Schwab, R.W. (2014). Integro-differential equations with nonlinear directional dependence. *Indiana Univ. Math. J.* **63** 1467–1498. MR3283558

[15] Kassmann, M. and Schwab, R.W. (2014). Regularity results for nonlocal parabolic equations. *Riv. Math. Univ. Parma* (*N.S.*) **5** 183–212. MR3289601

[16] Kriventsov, D. (2013). $C^{1,\alpha}$ interior regularity for nonlinear nonlocal elliptic equations with rough kernels. *Comm*. *Partial Differential Equations* **38** 2081–2106. MR3169771

[17] Lara, H.C. and Dávila, G. (2012). Regularity for solutions of nonlocal, nonsymmetric equations. *Ann*. *Inst*. *H*. *Poincaré Anal*. *Non Linéaire* **29** 833–859. MR2995098

[18] Lara, H.C. and Dávila, G. (2014). Hölder estimates for non-local parabolic equations with critical drift. Preprint. Available at arXiv:1408.0676.

[19] Lara, H.C. and Dávila, G. (2014). $C^{\sigma+\alpha}$ estimates for concave, non-local parabolic equations with critical drift. Preprint. Available at arXiv:1408.5149.

[20] Li, M. and Shi, Y. (2010). A general central limit theorem under sublinear expectations. *Sci*. *China Math*. **53** 1989–1994. MR2679081

[21] Luo, P. and Jia, G. (2014). A note on characterizations of G-normal distribution. Preprint. Available at arXiv:1402.4631.

[22] Neufeld, A. and Nutz, M. (2015). Nonlinear Lévy processes and their characteristics. *Trans*. *Amer*. *Math*. *Soc*. To appear.

[23] Osuka, E. (2013). Girsanov's formula for G-Brownian motion. *Stochastic Process*. *Appl*. **123** 1301–1318. MR3016224

[24] Paczka, K. (2012). Itô calculus and jump diffusions for G-Lévy processes. Preprint. Available at arXiv:1211.2973.

[25] Paczka, K. (2014). G-martingale representation in the G-Lévy setting. Preprint. Available at arXiv:1404.2121.

[26] Peng, S. (2007). Law of large numbers and central limit theorem under nonlinear expectations. Preprint. Available at arXiv:math/0702358.

[27] Peng, S. (2007). G-Brownian motion and dynamic risk measure under volatility uncertainty. Preprint. Available at arXiv:0711.2834.

[28] Peng, S. (2008). A new central limit theorem under sublinear expectations. Preprint. Available at arXiv:0803.2656.

[29] Peng, S. (2009). Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations. *Sci*. *China Ser*. *A* **52** 1391–1411. MR2520583

[30] Peng, S. (2010). Nonlinear expectations and stochastic calculus under uncertainty. Preprint. Available at arXiv:1002.4546.

[31] Peng, S., Song, Y. and Zhang, J. (2014). A complete representation theorem for G-martingales. *Stochastics* **86** 609–631. MR3230070

[32] Ren, L. (2013). On representation theorem of sublinear expectation related to G-Lévy process and paths of G-Lévy process. *Statist*. *Probab*. *Lett*. **83** 1301–1310. MR3041277

[33] Schwab, R.W. and Silvestre, L. (2014). Regularity for parabolic integro-differential equations with very irregular kernels. Preprint. Available at arXiv:1412.3790.

[34] Serra, J. (2014). $C^{\sigma+\alpha}$ regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels. Preprint. Available at arXiv:1405.0930.

[35] Silvestre, L. (2011). On the differentiability of the solution to the Hamilton–Jacobi equation with critical fractional diffusion. *Adv*. *Math*. **226** 2020–2039. MR2737806

[36] Silvestre, L. (2014). Regularity estimates for parabolic integro-differential equations and applications. In *Proceedings of the International Congress of Mathematicians* (*Seoul*, 2014) (S.Y. Jang, Y.R. Kim, D.-W. Lee and I. Yie, eds.) 873–894. Seoul: Kyung Moon SA. Available at http://www.icm2014.org/en/vod/proceedings.

[37] Soner, H.M., Touzi, N. and Zhang, J. (2011). Martingale representation theorem for the G-expectation. *Stochastic Process*. *Appl*. **121** 265–287. MR2746175

[38] Song, Y. (2011). Some properties on G-evaluation and its applications to G-martingale decomposition. *Sci*. *China Math*. **54** 287–300. MR2771205

[39] Xu, J., Shang, H. and Zhang, B. (2011). A Girsanov type theorem under G-framework. *Stoch*. *Anal*. *Appl*. **29** 386–406. MR2795552

[40] Zhang, D. and Chen, Z. (2014). A weighted central limit theorem under sublinear expectations. *Comm*. *Statist*. *Theory Methods* **43** 566–577. MR3171055

[41] Nonlocal Equations Wiki. Available at https://www.ma.utexas.edu/mediawiki/index.php/Starting_page. Accessed 2015-01-25.

Received January 2015 and revised May 2015