

Lévy processes and stochastic integrals in the sense of generalized convolutions

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In this paper, we present a comprehensive theory of generalized and weak generalized convolutions, illustrate it by a large number of examples, and discuss the related infinitely divisible distributions. We consider Lévy and additive process with respect to generalized and weak generalized convolutions as certain Markov processes, and then study stochastic integrals with respect to such processes. We introduce the representability property of weak generalized convolutions. Under this property and the related weak summability, a stochastic integral with respect to random measures related to such convolutions is constructed.

Keywords: Lévy process; scale mixture; stochastic integral; symmetric stable distribution; weakly stable distribution

1. Introduction

Motivated by the seminal work of Kingman [13], K. Urbanik introduced and developed the theory of generalized convolutions in his fundamental papers [30,31,33,34]. Roughly speaking, a generalized convolution is a binary associative operation \star on probability measures such that the convolution of point-mass measures $\delta_x \star \delta_y$ can be a nondegenerate probability measure, while the usual convolution gives δ_{x+y} . The study of weakly stable distributions, initiated by Kucharczak and Urbanik (see [15,32]) and followed by a series of papers by Urbanik, Kucharczak, Panorska, and Vol'kovich (see, e.g., [14,16,22,36–38]), provided a new and rich class of weak generalized convolutions on \mathbb{R}_+ (called also \mathcal{B} -generalized convolutions). Misiewicz, Oleszkiewicz and Urbanik [20] gave full characterization of weakly stable distributions with nontrivial discrete part and proved some uniqueness properties of weakly stable distributions that will be used in this paper. For additional information on generalized convolutions and weakly stable laws, see [5–10, 17–19,21,23].

In this paper, we present a comprehensive theory of generalized and weak generalized convolutions and discuss the related classes of infinitely divisible distributions. We construct Lévy and additive processes with respect to such convolutions. Lévy process with respect to generalized convolutions form interesting subclasses of Markov processes, such as the class of Bessel

processes in the case of Kingman's convolution (see [29]), but in general, they are heavy tailed Markov processes (see Remark 4.5). Then we construct stochastic integrals of deterministic functions associated with such convolutions and the corresponding Lévy processes. We also introduce the weak summability property of generalized convolutions. If a convolution admits the weak summability, then the stochastic integration theory related to such convolutions becomes more explicit and concrete.

This paper is organized as follows. In Section 2, we give definitions and properties of generalized and weak generalized convolutions that will be used throughout this work. We also provide an extensive list of examples. In Section 3, we recall main results on infinite divisibility with respect to generalized and weak generalized convolutions. This information is crucial for further considerations. The main result of Section 4 states that under minimal assumptions on generalized convolutions an analog of processes with independent increments can be constructed. We follow and extend an approach of N. Van Thu [28]. In Section 5, we consider stochastic integral processes with respect to generalized convolutions. Section 6 is devoted to the property of weak generalized summation. In Section 7, we construct "independently scattered" random measures based on a weak generalized summation; these measures are used in Section 8 to construct Lévy and additive processes. Finally, in Section 9 we define stochastic integrals of deterministic functions with respect to such random measures and generalized convolutions.

Throughout this paper, the distribution of the random element X is denoted by $\mathcal{L}(X)$. If $\lambda = \mathcal{L}(X)$ and $a \in \mathbb{R}$, we denote the law of aX by $T_a\lambda$. $\mathcal{P}(\mathbb{E})$ denotes the family of all probability measures on the Borel σ -algebra $\mathcal{B}(\mathbb{E})$ of a Polish space \mathbb{E} . For short, we write $\mathcal{P}(\mathbb{R}) = \mathcal{P}$ and $\mathcal{P}(\mathbb{R}_+) = \mathcal{P}_+$. The set of all symmetric probability measures on \mathbb{R} is denoted by \mathcal{P}_s . If $\lambda \in \mathcal{P}$ and $\lambda = \mathcal{L}(\theta)$, then $|\lambda| \in \mathcal{P}_+$ is defined by $|\lambda| = \mathcal{L}(|\theta|)$. If $\mu = \mathcal{L}(X)$ and $\lambda = \mathcal{L}(\theta)$ are such that X and θ are independent, then by $\mu \circ \lambda$ we denote the distribution of $X\theta$.

2. Generalized convolutions

2.1. Urbanik's generalized convolutions

Urbanik [30] introduced a *generalized convolution* as a binary, symmetric, associative and commutative operation \diamond on \mathcal{P}_+ having the following properties:

- (i) $\lambda \diamond \delta_0 = \lambda$ for all $\lambda \in \mathcal{P}_+$;
- (ii) $(p\lambda_1 + (1-p)\lambda_2) \diamond \lambda = p(\lambda_1 \diamond \lambda) + (1-p)(\lambda_2 \diamond \lambda)$ for each $p \in [0, 1]$ and $\lambda, \lambda_1, \lambda_2 \in \mathcal{P}_+$;
- (iii) $T_a(\lambda_1 \diamond \lambda_2) = (T_a\lambda_1) \diamond (T_a\lambda_2)$ for all $a \geq 0$ and $\lambda_1, \lambda_2 \in \mathcal{P}_+$;
- (iv) if $\lambda_n \rightarrow \lambda$ and $\nu_n \rightarrow \nu$, then $(\lambda_n \diamond \nu_n) \rightarrow (\lambda \diamond \nu)$, where \rightarrow denotes the weak convergence;
- (v) there exists a sequence of positive numbers (c_n) such that $T_{c_n}\delta_1^{\diamond n}$ converges weakly to a measure $\nu \neq \delta_0$ (here $\lambda^{\diamond n} = \lambda \diamond \dots \diamond \lambda$ denotes the generalized convolution of n identical measures λ).

The property (v) is important. It states that for the generalized convolution a kind of limit theorem holds with a nontrivial limit measure. Another important property, which follows from (ii) and

(iv), is that for every $\lambda_1, \lambda_2 \in \mathcal{P}_+$ and a Borel set $A \subset \mathbb{R}_+$

$$\lambda_1 \diamond \lambda_2(A) = \int_0^\infty \int_0^\infty (\delta_x \diamond \delta_y)(A) \lambda_1(dx) \lambda_2(dy) \quad (2.1)$$

(see Lemma 2.7 for the proof of a related equality). In view of (2.1), in order to specify \diamond we only need to know $\delta_x \diamond \delta_y$ for all x, y . Actually, it is enough to know $\delta_z \diamond \delta_1$ for all $z \in [0, 1]$, because $\delta_x \diamond \delta_y = T_x(\delta_1 \diamond \delta_{y/x})$ for any $x > y$.

Examples

For details, see [1,3,8,13,16,30–36].

Example 2.0. The classical convolution ([30,35]) is evidently an example of generalized convolution. It will be denoted as usual by $*$:

$$\delta_a * \delta_b = \delta_{a+b}.$$

Example 2.1. Symmetric generalized convolution ([30,35]) on \mathcal{P}_+ is defined by

$$\delta_a *_s \delta_b = \frac{1}{2} \delta_{|a-b|} + \frac{1}{2} \delta_{a+b}.$$

The name *symmetric* comes from the fact that this convolution can be easily extended to a generalized convolution on \mathcal{P} taking values in the set of symmetric measures \mathcal{P}_s :

$$\delta_a *_s \delta_b = \frac{1}{4} \delta_{a-b} + \frac{1}{4} \delta_{-a+b} + \frac{1}{4} \delta_{-a-b} + \frac{1}{4} \delta_{a+b}.$$

Example 2.2. In a similar way another generalized convolution (called by Urbanik $(\alpha, 1)$ -convolution in [30,32]) can be defined for every $\alpha > 0$ by means of

$$\delta_a *_s, \alpha \delta_b = \frac{1}{2} \delta_{|a^\alpha - b^\alpha|^{1/\alpha}} + \frac{1}{2} \delta_{(a^\alpha + b^\alpha)^{1/\alpha}}.$$

Example 2.3. For every $p \in (0, \infty]$, the formula

$$\delta_a *_p \delta_b = \delta_c, \quad a, b \geq 0, \quad c = \|(a, b)\|_p = (a^p + b^p)^{1/p}$$

defines a generalized convolution $*_p$ (p -stable convolution) on \mathcal{P}_+ . For details, see [30,34].

Example 2.4. The Kendall convolution \diamond_α on \mathcal{P}_+ , $\alpha > 0$, is defined ([8]) by

$$\delta_x \diamond_\alpha \delta_1 = x^\alpha \pi_{2\alpha} + (1 - x^\alpha) \delta_1, \quad x \in [0, 1],$$

where $\pi_{2\alpha}$ is a Pareto measure with density $g_{2\alpha}(x) = 2\alpha x^{-2\alpha-1} \mathbf{1}_{[1, \infty)}(x)$.

Example 2.5. The Kingman convolution \otimes_{ω_s} on \mathcal{P}_+ , $s > -\frac{1}{2}$, is defined in [13] by

$$\delta_a \otimes_{\omega_s} \delta_b = \mathcal{L}(\sqrt{a^2 + b^2 + 2ab\theta_s}),$$

where θ_s is absolutely continuous with the density function

$$f_s(x) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)} (1-x^2)_+^{s-1/2}.$$

If $n := 2(s+1) \in \mathbb{N}$, $n > 1$, the variable θ_s can be interpreted as one dimensional projection of the random vector $\mathbf{U} = (U_1, \dots, U_n)$ having uniform distribution ω_n on the unit sphere $S_{n-1} \subset \mathbb{R}^n$. If $n = 1$ and $s = -\frac{1}{2}$, then θ_s has the discrete distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

Example 2.6. ∞ -convolution ([16,35]) is defined by

$$\delta_a \oslash \delta_b = \delta_{\max\{a,b\}}.$$

Example 2.7. A combination of Kingman convolution and $(\alpha, 1)$ convolution, called by Urbanik (α, β) -convolution in [30], for $0 < \alpha < \infty$, $0 < \beta < \infty$, is defined for $a, b > 0$ as

$$\delta_a \otimes_{\alpha,\beta} \delta_b = \mathcal{L}((a^{2\alpha} + b^{2\alpha} + 2a^\alpha b^\alpha \theta)^{1/2\alpha}),$$

where $\theta = \theta_{(\beta-2)/2}$ is a random variable with the density function

$$f_{(\beta-2)/2}(x) = \frac{\Gamma(\beta/2)}{\sqrt{\pi}\Gamma((\beta-1)/2)} (1-x^2)_+^{(\beta-3)/2}.$$

Example 2.8. A kind of generalization of Kendall convolution called the Kucharczak–Urbanik convolution ([1]) was obtained by the following definition for $\alpha > 0$ and $s \in [0, 1]$

$$\begin{aligned} \delta_s \diamond_{\alpha,n} \delta_1(dx) &= (1-s^\alpha)_+^n \delta_1(dx) \\ &+ \frac{\alpha(n+1)s^{\alpha(n+1)}}{x^{2\alpha n+1}} \sum_{k=1}^n \binom{n}{k} \binom{n}{k-1} \frac{(x^\alpha - s^\alpha)^{k-1} (x^\alpha - 1)_+^{n-k}}{s^{\alpha k}} dx. \end{aligned}$$

Example 2.9. The Kucharczak convolution $\@$, $\alpha \in (0, 1)$, is defined in [35] by

$$\delta_a \@ \delta_b(dx) = \frac{a^\alpha b^\alpha \sin(\pi\alpha)(2x - a - b)}{\pi(x - a - b)^\alpha (x - a)^\alpha (x - b)^\alpha} \mathbf{1}_{((a^\alpha + b^\alpha)^{1/\alpha}, \infty)}(x) dx.$$

Example 2.10. The Vol'kovich convolution $\triangle_{1,\beta}$ for $0 < \beta < \frac{1}{2}$ (see [36]) is given by

$$\delta_a \triangle_{1,\beta} \delta_b(dx) = \frac{2a^{2\beta} b^{2\beta}}{B(\beta, (1/2) - \beta)} [(x^2 - (a-b)^2)_+ ((a+b)^2 - x^2)_+]^{-\beta-1/2} dx.$$

Example 2.11. In [16] for $\alpha \in (0, 1)$, the authors considered the following measure:

$$\mu = (2 - 2^{-\alpha}) \sum_{n=0}^{\infty} 2^{-1-n(\alpha+1)} T_{2^n}(\pi_{\alpha}),$$

where π_{α} is the Pareto distribution with the density $\alpha x^{-\alpha-1} \mathbf{1}_{[1, \infty)}(x)$. They proved that for every pair $a, b > 0$ there exists a unique probability measure $\varrho(a, b) \in \mathcal{P}_+$ fulfilling the equality

$$T_a(\mu) \odot T_b(\mu) = \mu \circ \varrho(a, b).$$

Setting $\delta_a \nabla_{\alpha} \delta_b := \varrho(a, b)$ they obtained a generalized convolution. In a similar way, many other generalized convolutions can be constructed on the basis of known convolutions (see, e.g., [8]).

Example 2.12. We say that the distribution μ on \mathbb{R}^n is ℓ_1 -symmetric (sometimes the name ℓ_1 -pseudo-isotropic is used here) if the characteristic function of μ has the following form

$$\widehat{\mu}(\xi) = \varphi(\|\xi\|_1),$$

for some function φ , where $\|\xi\|_1 = |\xi_1| + \dots + |\xi_n|$. This means that the random vector \mathbf{X} is ℓ_1 -symmetric (ℓ_1 -pseudo-isotropic) if for every $\xi \in \mathbb{R}^n$ the following equation holds

$$\langle \xi, \mathbf{X} \rangle = \sum_{k=1}^n \xi_k X_k \stackrel{d}{=} \|\xi\|_1 \cdot X_1.$$

In 1983, Cambanis, Keener and Simons [3] described the set of extreme points of the family of ℓ_1 -symmetric distributions on \mathbb{R}^n . They proved that the random vector \mathbf{X} is ℓ_1 -pseudo-isotropic iff there exists a nonnegative random variable Θ such that

$$\mathbf{X} \stackrel{d}{=} \left(\frac{U_1}{\sqrt{D_1}}, \dots, \frac{U_n}{\sqrt{D_n}} \right) \cdot \Theta =: \mathbf{V} \cdot \Theta, \quad (2.2)$$

where $\mathbf{U}^n = (U_1, \dots, U_n)$ has uniform distribution on the unit sphere in \mathbb{R}^n , $\mathbf{D} = (D_1, \dots, D_n)$ has Dirichlet distribution with parameters $(\frac{1}{2}, \dots, \frac{1}{2})$, \mathbf{U}^n , \mathbf{D} and Θ are independent. This means that the set of extreme points for the set of ℓ_1 -pseudo-isotropic distributions on \mathbb{R}^n is equal to

$$\{T_a \mathcal{L}(\mathbf{V}) : a \geq 0\}.$$

Let $\varphi(\|\xi\|_1)$ be the characteristic function of \mathbf{V} , that is, $\varphi(\|\xi\|_1) = \mathbf{E} e^{i\langle \xi, \mathbf{V} \rangle}$. Then the characteristic function of $a\mathbf{V} + b\mathbf{V}'$, where \mathbf{V}' is an independent copy of a \mathbf{V} , is of the form

$$\Phi(\|\xi\|_1) = \varphi(a\|\xi\|_1) \varphi(b\|\xi\|_1),$$

thus it also depends only on $\|\xi\|_1$. By (2.2), there exists a random variable $\Theta = \Theta(a, b)$ independent of \mathbf{V} such that

$$a\mathbf{V} + b\mathbf{V}' \stackrel{d}{=} \mathbf{V}\Theta.$$

Now we obtain a generalized convolution ∇_{ℓ_1} setting

$$\delta_a \nabla_{\ell_1} \delta_b = \mathcal{L}(\Theta(a, b)).$$

Unfortunately, an explicit formula for $\mathcal{L}(\Theta(a, b))$ is unknown.

Remark 2.1. By Schoenberg's classical result (see [27]), we have that a random vector \mathbf{X} on \mathbb{R}^n is ℓ_2 -pseudo-isotropic (ℓ_2 -symmetric, rotationally invariant) iff $\mathbf{X} \stackrel{d}{=} \mathbf{U}\sqrt{\Theta}$ for some nonnegative variable Θ independent of \mathbf{U} . This leads to the family of Kingman's convolutions in special cases $n = 2(s + 1) \in \mathbb{N}$. The characterization (2.2) proven in [3] gives a general form for ℓ_1 -pseudo-isotropic and leads to the generalized convolution ∇_{ℓ_1} . In both cases the distributions of the extreme points of ℓ_i -pseudo-isotropic measures, $i = 1, 2$, that is, $\mathcal{L}(\mathbf{U})$ and $\mathcal{L}(\mathbf{V})$ are weakly stable. A full characterization of ℓ_α -symmetric distributions for $\alpha \notin \{1, 2\}$ is unknown. All we know is that only $\alpha \leq 2$ can be considered here.

A pair $(\mathcal{P}_+, \diamond)$ is called a *generalized convolution algebra*. A continuous mapping $h : \mathcal{P}_+ \rightarrow \mathbb{R}$ is called a *homomorphism* of $(\mathcal{P}_+, \diamond)$ if

- $\forall a \in [0, 1] \forall \lambda_1, \lambda_2 \in \mathcal{P}_+ \quad h(a\lambda_1 + (1-a)\lambda_2) = ah(\lambda_1) + (1-a)h(\lambda_2),$
- $\forall \lambda_1, \lambda_2 \in \mathcal{P}_+ \quad h(\lambda_1 \diamond \lambda_2) = h(\lambda_1)h(\lambda_2).$

Obviously, $h(\cdot) \equiv 0$ and $h(\cdot) \equiv 1$ are the trivial homomorphisms. A generalized convolution algebra $(\mathcal{P}_+, \diamond)$ is said to be *regular* if it admits a nontrivial homomorphism.

Definition 2.2. We say that a nontrivial generalized convolution algebra $(\mathcal{P}_+, \diamond)$ admits a *characteristic function* if there exists one-to-one correspondence $\lambda \leftrightarrow \Phi_\lambda$ between probability measures $\lambda \in \mathcal{P}_+$ and real valued functions Φ_λ on $[0, \infty)$ such that for $\lambda, \nu \in \mathcal{P}_+$

1. $\Phi_{p\lambda+q\nu} = p\Phi_\lambda + q\Phi_\nu$ for $p, q \geq 0, p + q = 1$;
2. $\Phi_{\lambda \diamond \nu} = \Phi_\lambda \cdot \Phi_\nu$;
3. $\Phi_{T_a \lambda}(t) = \Phi_\lambda(at)$;
4. the uniform convergence of Φ_{λ_n} on every bounded interval is equivalent to the weak convergence of λ_n .

The function Φ_λ is called the *characteristic function of the probability measure λ in the algebra $(\mathcal{P}_+, \diamond)$* or *$\diamond$ -generalized characteristic function of λ* .

It can be shown (see [33]) that Φ is uniquely determined up to a scale parameter.

The \diamond -generalized characteristic function in generalized convolution algebra plays the same role as the classical Laplace or Fourier transform for convolutions defined by addition of independent random elements. The following fact is crucial for further investigations, see [30] for the proof.

Proposition 2.3. A nontrivial generalized convolution algebra $(\mathcal{P}_+, \diamond)$ admits a characteristic function Φ if and only if it is regular. In this case

$$\Phi_\lambda(t) = h(T_t \lambda), \quad t \geq 0, \lambda \in \mathcal{P}_+,$$

where h is the nontrivial homomorphism of $(\mathcal{P}_+, \diamond)$. Moreover, the map $\lambda \mapsto \Phi_\lambda$ is an integral transform:

$$\Phi_\lambda(t) = \int_0^\infty \Omega(tx) \lambda(dx),$$

where $\Omega(t) := h(\delta_t)$. Ω is called the kernel of the \diamond -generalized characteristic function Φ .

It can be shown that for each nontrivial homomorphism h on a regular algebra $(\mathcal{P}_+, \diamond)$ there exists an open neighborhood of zero U such that

$$\forall x \in U \setminus \{0\}, \quad 0 < |h(\delta_x)| < 1.$$

This property implies that the \diamond -generalized characteristic function $\Phi_\lambda(\cdot)$ of the measure $\lambda \in \mathcal{P}_+$ has a very useful property: if $\Phi_\lambda(t_n) = 1$ for some $t_n \searrow 0$, then $\lambda = \delta_0$. One can find more about generalized convolutions in [8,12,15,16,30–34,36–38].

2.2. Weak generalized convolutions

Weak generalized convolutions were studied in [6,10,18–20,32,36]. They are derived from the concept of weakly stable probability measures.

Definition 2.4. The distribution μ of a random vector \mathbf{X} , taking values in a separable Banach space \mathbb{E} , is weakly stable if for every $a, b \in \mathbb{R}$ there exists a random variable θ independent of \mathbf{X} such that

$$a\mathbf{X}_1 + b\mathbf{X}_2 \stackrel{d}{=} \theta\mathbf{X}, \quad (*)$$

where $\mathbf{X}_1, \mathbf{X}_2$ are independent copies of \mathbf{X} and $\stackrel{d}{=}$ denotes equality in distribution.

If the condition $(*)$ holds only for nonnegative constants a, b , then we say that \mathbf{X} is \mathbb{R}_+ -weakly stable. It was shown in [20] that if a weakly stable measure μ has an atom, then either $\mu = \delta_0$ or $\mu = \frac{1}{2}\delta_{\mathbf{a}} + \frac{1}{2}\delta_{-\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{E}$. In both cases we shall call such measures trivial.

It was proved in [20] that the condition $(*)$ is equivalent to the following:

$$\forall \theta_1, \theta_2 \exists \theta \quad \theta_1\mathbf{X}_1 + \theta_2\mathbf{X}_2 \stackrel{d}{=} \theta\mathbf{X}, \quad (**)$$

where θ_1, θ_2 are random variables such that $\theta_1, \theta_2, \mathbf{X}_1, \mathbf{X}_2$ are independent and θ is independent of \mathbf{X} . Then $(**)$ can be written in the language of distributions in the following way:

$$(\mu \circ \lambda_1) * (\mu \circ \lambda_2) = \mu \circ \lambda,$$

where $\mathcal{L}(\theta_i) = \lambda_i$, $i = 1, 2$, $\mathcal{L}(\theta) = \lambda$. If the measure μ is nonsymmetric, then λ is uniquely determined from $\mu \circ \lambda$, but when μ is symmetric, then only the measure $|\lambda| = \mathcal{L}(|\theta|)$ (equivalently, $\frac{1}{2}(\lambda + T_{-1}\lambda)$) is uniquely determined (see [20]).

Having a weakly stable random vector \mathbf{X} with distribution μ , we are able to define a weak generalized convolution:

Definition 2.5. Let $\mu \in \mathcal{P}(\mathbb{R})$ be a nontrivial weakly stable measure, and let $\lambda_1, \lambda_2 \in \mathcal{P}$. If

$$(\mu \circ \lambda_1) * (\mu \circ \lambda_2) = \mu \circ \lambda,$$

then the weak generalized convolution (also called μ -weak generalized convolution) of the measures λ_1, λ_2 with respect to the measure μ (notation $\lambda_1 \otimes_\mu \lambda_2$) is defined as follows

$$\lambda_1 \otimes_\mu \lambda_2 = \begin{cases} \lambda & \text{if } \mu \text{ is not symmetric;} \\ |\lambda| & \text{if } \mu \text{ is symmetric.} \end{cases}$$

Sometimes it is more convenient to define $\lambda_1 \otimes_\mu \lambda_2 = \frac{1}{2}(\lambda + T_{-1}\lambda)$, when μ is symmetric. The pair $(\mathcal{P}, \otimes_\mu)$ is called a weak generalized convolution algebra.

The following lemma describes basic properties of weak generalized convolution.

Lemma 2.6. If the weakly stable measure $\mu \in \mathcal{P}(\mathbb{R})$ is not trivial, then for all $\lambda, \lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}$

- (1) $\lambda_1 \otimes_\mu \lambda_2$ is uniquely determined;
- (2) $\lambda_1 \otimes_\mu \lambda_2 = \lambda_2 \otimes_\mu \lambda_1$;
- (3) $(\lambda_1 \otimes_\mu \lambda_2) \otimes_\mu \lambda_3 = \lambda_1 \otimes_\mu (\lambda_2 \otimes_\mu \lambda_3)$;
- (4) $\lambda \otimes_\mu \delta_0 = \lambda$ ($\lambda \otimes_\mu \delta_0 = |\lambda|$ if μ is symmetric);
- (5) $(p\lambda_1 + (1-p)\lambda_2) \otimes_\mu \lambda = p(\lambda_1 \otimes_\mu \lambda) + (1-p)(\lambda_2 \otimes_\mu \lambda)$ for each $p \in [0, 1]$;
- (6) $T_a(\lambda_1 \otimes_\mu \lambda_2) = (T_a\lambda_1) \otimes_\mu (T_a\lambda_2)$;
- (7) if $\lambda_n \rightarrow \lambda$ and $v_n \rightarrow v$, then $\lambda_n \otimes_\mu v_n \rightarrow \lambda \otimes_\mu v$.

Proof. Property (1) follows from Theorems 3 and 4 in [20]. Properties (2)–(6) are simple consequences of the definition and the uniqueness property (1). To see (7) it is enough to notice that for independent random sequences Y, Y_1, Y_2, \dots and Z, Z_1, Z_2, \dots the following implications hold

$$Y_n \xrightarrow{d} Y, \quad Z_n \xrightarrow{d} Z \quad \Rightarrow \quad \begin{cases} Y_n \cdot Z_n \xrightarrow{d} Y \cdot Z; \\ Y_n + Z_n \xrightarrow{d} Y + Z, \end{cases}$$

where \xrightarrow{d} denotes convergence of distributions, and then use the uniqueness (1). □

Lemma 2.7. If μ is not symmetric, then for every $\lambda_1, \lambda_2 \in \mathcal{P}$ and $A \in \mathcal{B}(\mathbb{R})$ we have

$$\lambda_1 \otimes_\mu \lambda_2(A) = \int_{\mathbb{R}^2} (\delta_x \otimes_\mu \delta_y)(A) \lambda_1(dx) \lambda_2(dy). \quad (2.3)$$

If μ is symmetric, then this equality holds with \mathbb{R} replaced by \mathbb{R}_+ and $\lambda_1, \lambda_2 \in \mathcal{P}_+$.

Proof. If λ_1, λ_2 have finite supports, $\lambda_1 = \sum_{i=1}^m p_i \delta_{x_i}$, $\lambda_2 = \sum_{j=1}^n q_j \delta_{y_j}$, then by (2) and (5), $\lambda_1 \otimes_\mu \lambda_2 = \sum_{i,j} p_i q_j \delta_{x_i} \otimes_\mu \delta_{y_j}$. Hence for a bounded continuous function f on \mathbb{R} , we

have

$$\begin{aligned} \int_{\mathbb{R}} f(z)(\lambda_1 \otimes_{\mu} \lambda_2)(dz) &= \sum_{i,j} p_i q_j \int_{\mathbb{R}} f(z)(\delta_{x_i} \otimes_{\mu} \delta_{y_j})(dz) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(z)(\delta_x \otimes_{\mu} \delta_y)(dz) \lambda_1(dx) \lambda_2(dy). \end{aligned} \quad (2.4)$$

Let $\lambda_1, \lambda_2 \in \mathcal{P}$ be arbitrary. Choose $\lambda_{i,n} \in \mathcal{P}$ with finite supports such that $\lambda_{i,n} \rightarrow \lambda_i$ as $n \rightarrow \infty$, $i = 1, 2$. We have for any bounded continuous function f on \mathbb{R}

$$\begin{aligned} &\int_{\mathbb{R}^2} \int_{\mathbb{R}} f(z)(\delta_x \otimes_{\mu} \delta_y)(dz) \lambda_1(dx) \lambda_2(dy) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(z)(\delta_x \otimes_{\mu} \delta_y)(dz) \lambda_{1,n}(dx) \lambda_{2,n}(dy) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(z)(\lambda_{1,n} \otimes_{\mu} \lambda_{2,n})(dz) = \int_{\mathbb{R}} f(z)(\lambda_1 \otimes_{\mu} \lambda_2)(dz). \end{aligned}$$

The first equality holds because the map $(x, y) \mapsto \int f(z)(\delta_x \otimes_{\mu} \delta_y)(dz)$ is continuous by (7) and bounded, the second one follows from (2.4), and the third uses (7). We have shown

$$\int_{\mathbb{R}} f(z)(\lambda_1 \otimes_{\mu} \lambda_2)(dz) = \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(z)(\delta_x \otimes_{\mu} \delta_y)(dz) \lambda_1(dx) \lambda_2(dy)$$

for any bounded continuous function f . By a standard monotone class argument, we deduce that this equality holds for any $f = \mathbf{1}_A$, $A \in \mathcal{B}(\mathbb{R})$, which gives (2.3). The proof in the symmetric case of μ is similar. \square

Notice that for a weak generalized convolution the condition (v) of the Urbanik definition of generalized convolution does not have to be satisfied. In [6], we can find a wide description of properties of the generalized convolutions on \mathbb{R} without property (v). However it was shown in [20] that if the measure μ has a finite weak moment of order $\varepsilon > 0$, then there exists a measure λ such that $\mu \circ \lambda$ is symmetric α -stable for some (and then for every) $\alpha \leq \min\{\varepsilon, 2\}$. This means that $T_{c_n} \lambda^{\otimes \mu^n} = \lambda$ for a properly chosen sequence (c_n) , and the property (v) holds if we replace δ_1 by λ .

The weak generalized convolution is always regular with

$$\Omega(t) = h(\delta_t) := \widehat{\mu}(t) = \int_{\mathbb{R}} e^{itx} \mu(dx),$$

see Proposition 2.3.

Examples

Example 2.1a. Let θ be a random variable with distribution $\lambda_0 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ and let θ' be its independent copy. It is easy to check that for all $a, b \geq 0, a \neq b$

$$a\theta + b\theta' \equiv |a\theta + b\theta'| \cdot \frac{a\theta + b\theta'}{|a\theta + b\theta'|},$$

where the two factors on the right are independent and

$$\frac{a\theta + b\theta'}{|a\theta + b\theta'|} \stackrel{d}{=} \theta.$$

This shows that θ is weakly stable. Moreover, since

$$\mathcal{L}(|a\theta + b\theta'|) = \frac{1}{2}\delta_{|a-b|} + \frac{1}{2}\delta_{a+b},$$

we have that the symmetric generalized convolution is a weak generalized convolution and $*_s = \otimes_{\lambda_0}$.

Example 2.3a. Not for all $p > 0$, but for $p \in (0, 2]$ the convolution $*_p$ can be extended to a weak generalized convolution on \mathcal{P} taking values in \mathcal{P}_+ defined by γ_p -symmetric p -stable measure which is weakly stable since

$$a\Gamma_p + b\Gamma'_p \equiv \|(a, b)\|_p \Gamma''_p, \quad \text{where } \Gamma''_p := \frac{a}{\|(a, b)\|_p} \Gamma_p + \frac{b}{\|(a, b)\|_p} \Gamma_p,$$

where Γ_p, Γ'_p are independent with the distribution γ_p . Evidently, the first equality holds everywhere and, by the basic properties of stable variables, Γ''_p also has the distribution γ_p .

Example 2.4a. Not for all $\alpha > 0$, but for $\alpha \in (0, 1]$ the Kendall convolution \diamond_α can be extended to a weak generalized convolution on \mathcal{P} taking values in \mathcal{P}_s defined by the measure μ_α with the characteristic function $\widehat{\mu_\alpha}(t) = (1 - |t|^\alpha)_+$.

Example 2.5a. For $2(s+1) \in \mathbb{N}$ the Kingman convolution has the natural interpretation as a weak generalized convolution with respect to the weakly stable uniform distribution on the unit sphere $S_{2s+1} \subset \mathbb{R}^{2(s+1)}$. More precisely:

Let $\mathbf{U}_n, n \geq 2$, denotes the random vector with the uniform distribution ω_n on the unit sphere $S_{n-1} \subset \mathbb{R}^n$. It is known that if \mathbf{U}, \mathbf{U}' are independent copies of \mathbf{U}_n , then for each $a, b \in \mathbb{R}, ab \neq 0$, the random variables

$$\|a\mathbf{U} + b\mathbf{U}'\|_2 \quad \text{and} \quad \frac{a\mathbf{U} + b\mathbf{U}'}{\|a\mathbf{U} + b\mathbf{U}'\|_2}$$

are independent and the second one has the distribution ω_n . Since

$$a\mathbf{U} + b\mathbf{U}' \equiv \|a\mathbf{U} + b\mathbf{U}'\|_2 \frac{a\mathbf{U} + b\mathbf{U}'}{\|a\mathbf{U} + b\mathbf{U}'\|_2} \quad \text{a.e.}$$

this implies that ω_n is weakly stable and it defines the weakly stable convolution \otimes_{ω_n} on \mathcal{P} in the following way

$$\delta_a \otimes_{\omega_n} \delta_b = \mathcal{L}(\|a\mathbf{U} + b\mathbf{U}'\|_2).$$

For $2s = -1$, we simply have

$$\delta_a \otimes_{\omega_1} \delta_b = \frac{1}{2}\delta_{|a-b|} + \frac{1}{2}\delta_{a+b},$$

which is \ast_s convolution considered in Example 2.1.

Example 2.12a. By the result of Cambanis, Keener and Simons [3], the distribution of \mathbf{V} is weakly stable, and by our construction

$$\nabla_{\ell_1} = \otimes_{\mathcal{L}(\mathbf{V})}.$$

3. Infinite divisibility with respect to generalized convolutions

3.1. Infinite divisibility (decomposability) of measures on \mathbb{R}_+

It is natural to consider infinitely divisible measures with respect to generalized convolutions. Following Urbanik [30], sometimes we will call such measures *infinitely decomposable*.

Definition 3.1. A measure $\lambda \in \mathcal{P}_+$ is said to be *infinitely divisible with respect to the generalized convolution \diamond (\diamond -infinitely decomposable)* in the algebra $(\mathcal{P}_+, \diamond)$ if for every $n \in \mathbb{N}$ there exists a probability measure $\lambda_n \in \mathcal{P}_+$ such that $\lambda = \lambda_n^{\diamond n}$.

The proof of the following proposition can be found in [30].

Proposition 3.2. Let $\lambda \in \mathcal{P}_+$ be \diamond -infinitely divisible. There exists a collection of measures $\lambda^{\diamond r}$, $r \geq 0$ such that

- (i) $\lambda^{\diamond 0} = \delta_0$, $\lambda^{\diamond 1} = \lambda$;
- (ii) $\lambda^{\diamond r} \diamond \lambda^{\diamond s} = \lambda^{\diamond(r+s)}$, $r, s \geq 0$;
- (iii) $\lambda^{\diamond r_n} \rightarrow \delta_0$ if $r_n \searrow 0$.

Similarly as in the classical theory, one of the most important examples of \diamond -infinitely divisible distribution is given by

$$\text{Exp}_{\diamond}(a\lambda) \stackrel{\text{def}}{=} e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} \lambda^{\diamond k},$$

where $\lambda \in \mathcal{P}_+$ and $a > 0$. The measure $\text{Exp}_{\diamond}(a\lambda)$ is called a *generalized compound Poisson measure* or \diamond -*compound Poisson measure*. If $\lambda = \delta_1$, then it is called a *generalized Poisson measure*

or \diamond -Poisson measure. To see that $\text{Exp}_\diamond(a\lambda)$ is infinitely divisible with respect to \diamond it is sufficient to observe that

$$\left(\text{Exp}_\diamond\left(\frac{a}{n}\lambda\right)\right)^{\diamond n} = \text{Exp}_\diamond(a\lambda).$$

Another important example of a \diamond -infinitely divisible distribution gives the following:

Definition 3.3. Let $\lambda \in \mathcal{P}_+$. We say that λ is stable in the generalized convolution algebra $(\mathcal{P}_+, \diamond)$ if the following condition holds:

$$\forall a, b \geq 0 \exists c \geq 0 \quad T_a \lambda \diamond T_b \lambda = T_c \lambda.$$

Remark 3.4. A measure λ is stable in the generalized convolution algebra $(\mathcal{P}_+, \diamond)$ (or simply \diamond -stable) if and only if there exists a sequence of positive numbers (c_n) and $\eta \in \mathcal{P}_+$ such that

$$T_{c_n} \eta^{\diamond n} \rightarrow \lambda.$$

For details of the proof see Theorem 14 in [30].

In the formulation of the analog of the Lévy–Khintchine formula for a \diamond -infinitely divisible distribution we need the characteristic exponent $\varkappa(\diamond)$ for the generalized convolution \diamond defined in the following theorem of Urbanik [34]:

Theorem 3.5. For every generalized convolution \diamond on \mathcal{P}_+ there exists a constant $\varkappa(\diamond) \in (0, \infty]$ such that for every $p \in (0, \varkappa(\diamond)]$ there exists a measure $\sigma_p \in \mathcal{P}_+$ with the \diamond -generalized characteristic function

$$\Phi_{\sigma_p}(t) = \begin{cases} e^{-t^p} & \text{if } p < \infty; \\ \mathbf{1}_{[0,1]}(t) & \text{if } p = \infty. \end{cases}$$

Moreover, the set of all \diamond -stable measures coincides with the set

$$\{T_a(\sigma_p): a > 0, 0 < p \leq \varkappa(\diamond)\}.$$

In particular we have that

$$e^{-t^\varkappa} = \int_0^\infty \Omega(ts) \sigma_\varkappa(ds).$$

Let γ_{p+} , $p \in (0, 1)$, be the completely skewed to the right stable measure $\gamma_{p+} \sim S_p(\sigma, 1, 0)$ with $\sigma^p = 2^p \cos \frac{p\pi}{2}$ and the Laplace transform $e^{-2^p t^p}$ with the notation $S_p(\sigma, \beta, \mu)$ as in the representation 1.1.6 in [25] and let γ_{p+} be the distribution of θ_p . Since

$$\int_0^\infty e^{-t^\varkappa s/2} \gamma_{p+}(ds) = e^{-t^{p\varkappa}},$$

we see that for $s < \varkappa$ the measure $\sigma_s = T_{2^{1/\varkappa}}\sigma_{\varkappa} \circ \mathcal{L}(\theta_p^{1/\varkappa})$ for $p = \frac{s}{\varkappa}$ is absolutely continuous with respect to the Lebesgue measure. In many cases also the measure σ_{\varkappa} is absolutely continuous with respect to the Lebesgue measure. We denote by f_s the density function for the standard s -stable measure with respect to the generalized convolution \diamond .

It was proven (see Theorem 7 in [30]) by Urbanik that the characteristic exponent does not depend on the choice of nontrivial homomorphism and consequently on the choice of the \diamond -generalized characteristic function.

The examples given below illustrate the material of this section. Examples 3.0 and 3.3 belong to the classical theory of stable distributions. Formulas for densities in Examples 3.4 and 3.11 are new. Detailed calculations related to Examples 3.5–3.11 can mostly be found in [35]. The formula for the generalized characteristic function in Example 3.12 is new.

Examples

Example 3.0. As a nontrivial homomorphism in the case of usual convolution on \mathcal{P}_+ we can simply take $h(\lambda) = \int_0^\infty e^{-x\lambda} \lambda(dx)$, that is, the kernel of the transform can be given by $\Omega(t) = h(T_t \delta_1) = e^{-t} \mathbf{1}_{[0,\infty)}(t)$. Moreover $\varkappa(*) = 1$, $\sigma_1 = \delta_1$ and $\sigma_p = \gamma_{p+}$. In particular, for $p = \frac{1}{2}$ we have $\sigma = 1$ and the density of $\gamma_{1/2+}$ can be written in terms of elementary functions, namely

$$\gamma_{1/2+}(dx) = \frac{1}{\sqrt{2\pi}} x^{-3/2} \exp\left\{-\frac{1}{2x}\right\} dx.$$

It has been shown in [39], that

$$T_{1/2}\gamma_{1/3+}(dx) = \frac{1}{3\pi} x^{-3/2} K_{1/3}\left(\sqrt{\frac{4}{27x}}\right) dx,$$

where $K_{1/3}$ is the MacDonald function and

$$T_{1/2}\gamma_{2/3+}(dx) = \frac{1}{x\sqrt{3\pi}} W_{1/2,1/6}\left(\frac{4}{27x^2}\right) \exp\left\{-\frac{2}{27x^2}\right\} dx,$$

where $W_{p,q}$ is the Whittaker function.

Example 3.3. For the generalized convolution $*_p$ on \mathcal{P}_+ we have $\Omega(t) = e^{-t^p}$, $\varkappa(*_p) = p$ and $\sigma_s = \gamma_{\frac{s}{p}+}$ for $s < p$.

Example 3.4. For the generalized Kendall convolution \diamond_α on \mathcal{P}_+ we have $\Omega(t) = (1 - t^\alpha)_+$, $\varkappa(\diamond_\alpha) = \alpha$ and for $p \in (0, \alpha]$

$$f_p(x) = px^{-p-1} \left(1 - \frac{p}{\alpha} + \frac{p}{\alpha} x^{-p}\right) e^{-x^{-p}} \mathbf{1}_{(0,\infty)}(x).$$

For the same convolution considered as an operation on \mathcal{P}_s the functions Ω and f_p shall be symmetrized.

Example 3.5. For the generalized Kingman convolution, we have

$$\Omega(t) = \Gamma(s+1) \left(\frac{t}{2}\right)^s J_s(t),$$

where J_r is the Bessel function, $\varkappa(\otimes_{\omega_s}) = 2$,

$$f_1(x) = \frac{\Gamma(s+3/2)}{\sqrt{\pi}\Gamma(s+1)} \frac{x^s \mathbf{1}_{(0,\infty)}(x)}{(1+x)^{s+3/2}}, \quad f_2(x) = \frac{\mathbf{1}_{(0,\infty)}(x)}{2^{s+1}\Gamma(s+1)} x^s e^{-x/2},$$

and for $0 < p < 2$

$$f_p(x) = \frac{x^s}{2^{s+1}\Gamma(s+1)} \int_0^\infty y^{-s-1} \exp\left\{-\frac{x}{2y}\right\} \gamma_{p/2+}(dy).$$

Example 3.6. For the ∞ -convolution

$$\Omega(t) = \mathbf{1}_{[0,1]}(t)$$

and $\varkappa(\bigotimes) = \infty$, $\sigma_\infty = \delta_1$ and

$$f_p(x) = px^{-p-1} \exp\{-x^{-p}\} \mathbf{1}_{(0,\infty)}(x)$$

is the Weibull–Gnedenko distribution. It has been proven by Urbanik [34] that $\varkappa(\diamond) = \infty$ if and only if $\diamond = \bigotimes$.

Example 3.9. For the Kucharczak convolution $\@$, $\alpha \in (0, 1)$,

$$\Omega(t) = \Gamma(\alpha)^{-1} \Gamma(\alpha, t),$$

where $\Gamma(\alpha, t)$ is the incomplete Gamma function, $\varkappa(\@) = \alpha$ and

$$\sigma_p([0, x]) = x^{1-\alpha} \int_0^x (x-y)^{\alpha-1} \gamma_{p+}(dy).$$

Example 3.10. For the Vol’kovich convolution with $0 < \beta < \frac{1}{2}$ we have

$$\Omega(t) = \frac{2^{1-\beta} t^\beta}{\Gamma(\beta)} K_\beta(t),$$

where K_β is the MacDonald function and $\varkappa(\triangle_{1,\beta}) = 2\beta$.

Example 3.11. For the generalized convolution ∇_α , $\alpha \in (0, 1)$, under ∞ -convolution we have

$$\Omega(t) = (1 - 2^{(1+\alpha)[\log_2 t]} - (2 - 2^{-\alpha})(1 - 2^{\lfloor \log_2 t \rfloor})t^\alpha) \mathbf{1}_{[0,1]}(t),$$

where the square brackets denote the integer part and $\varkappa(\nabla_\alpha) = \alpha$. Moreover

$$\sigma_p([0, x)) = \frac{2^{1+\alpha}}{2^{1+\alpha} - 1} \left(1 + \frac{p}{\alpha x^p} \right) e^{-x^{-p}} - \frac{1}{2^{1+\alpha} - 1} \left(1 + \frac{p 2^p}{\alpha x^p} \right) e^{-2^p x^{-p}}.$$

Example 3.12. For the Cambanis, Keener and Simons convolution, we have

$$\Omega(t) = \mathbb{E} e^{itV_1} = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n-1)/2)} \int_1^\infty \Omega_n(ur^2) u^{-n/2} (u-1)^{(n-3)/2} du,$$

where $\Omega_n(r^2)$ is the characteristic function of the first coordinate of the vector \mathbf{U}^n and $\varkappa(\nabla_{\ell_1}) = 1$. The measure σ_p , $p \leq 1$, in this case is such that $\mathcal{L}(\mathbf{V}) \circ \sigma_p = \gamma_p$, for γ_p being the symmetric p -stable measure (abbreviation: SPS measure).

The following theorem (see Theorem 13 in [30]) gives the Lévy–Khintchine formula for \diamond -generalized characteristic function for a \diamond -infinitely divisible distribution.

Theorem 3.6. Let $(\mathcal{P}_+, \diamond)$ be a regular generalized convolution algebra. A function $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a \diamond -generalized characteristic function of a \diamond -infinitely divisible measure iff it has the following representation

$$\Phi(t) = \exp \left\{ -At^{\varkappa(\diamond)} + \int_0^\infty \frac{\Omega(tx) - 1}{v(x)} m(dx) \right\},$$

where m is a finite Borel measure on $[0, \infty)$,

$$v(x) = \begin{cases} 1 - \Omega(x) & \text{if } 0 \leq x \leq x_0, \\ 1 - \Omega(x_0) & \text{if } x \geq x_0 \end{cases}$$

and $x_0 > 0$ is such that $\Omega(x) < 1$ whenever $0 < x \leq x_0$.

3.2. Weak infinite divisibility

It is known that if a weakly stable measure μ is symmetric and such that $\int_{\mathbb{E}} |\langle \xi, \mathbf{x} \rangle|^\varepsilon \mu(d\mathbf{x}) < \infty$ for some $\varepsilon > 0$ and all continuous linear functionals $\xi \in \mathbb{E}^*$, then the weak generalized convolution \otimes_μ is a generalized convolution in the Urbanik sense (i.e., \otimes_μ has property (v)), see [10]. Consequently, the infinite divisibility with respect to such convolutions on \mathcal{P}_+ was already described in the previous subsection. Adding the information about weakly stable measures μ , that generate such convolutions, will make this description more detailed and concrete.

Definition 3.7. Let $\mu \in \mathcal{P}(\mathbb{E})$ be a weakly stable measure. We say that the measure λ is μ -weakly infinitely divisible if for every $n \in \mathbb{N}$ there exists a probability measure λ_n such that

$$\lambda = \lambda_n^{\otimes_{\mu} n} \equiv \lambda_n^n \stackrel{\text{def}}{=} \lambda_n \otimes_{\mu} \cdots \otimes_{\mu} \lambda_n, \quad (n\text{-times}),$$

where (for the uniqueness) $\lambda, \lambda_n \in \mathcal{P}_+$ if μ is \mathbb{R}_+ -weakly stable or if μ is symmetric, and $\lambda, \lambda_n \in \mathcal{P}$ if μ is weakly stable nonsymmetric.

Notice that if λ is μ -weakly infinitely divisible, then $\mu \circ \lambda$ is infinitely divisible in the classical sense. This information can be of some help in investigations, however we shall remember that the opposite implication does not hold. There are measures λ and weakly stable measures μ such that $\mu \circ \lambda$ is infinitely divisible and λ is not μ -weakly infinitely divisible. Counterexamples are known even for μ symmetric Gaussian and symmetric stable measures μ (see Example 2 in [10]). Special properties of infinitely divisible sub-stable distributions are discussed in [17, 25, 26].

It was proven in [10] that for every nontrivial weakly stable measure μ and μ -weakly infinitely divisible measure λ there exists a family of measures $\{\lambda^r : r \geq 0\}$ such that

- (1) $\lambda^0 = \delta_0, \lambda^1 = \lambda$;
- (2) $\lambda^r \otimes_{\mu} \lambda^s = \lambda^{r+s}, r, s \geq 0$;
- (3) $\lambda^r \rightarrow \delta_0$ if $r \rightarrow 0$.

The μ -weak compound Poisson measure for the μ -weak generalized convolution is defined exactly in the same way (see [10]) as the compound Poisson measure for generalized convolution:

$$\text{Exp}_{\otimes_{\mu}}(a\lambda) \stackrel{\text{def}}{=} e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} \lambda^{\otimes_{\mu} k},$$

where $\lambda \in \mathcal{P}$ and $a > 0$. Sometimes this measure is called μ -weak generalized exponent of the measure $a\lambda$. If $\lambda = \delta_1$, then it is called a μ -weak Poisson measure. In the case of μ -weak generalized convolution the following additional interesting property holds:

$$\mu \circ \text{Exp}_{\otimes_{\mu}}(a\lambda) = \exp(a(\mu \circ \lambda)),$$

that is, every μ -weak compound Poisson measure is a factor of some compound Poisson measure. In some cases, we get the explicit formulas for the generalized Poisson distribution.

Examples

Example 3.3a. Let $\mu = \gamma_p$, $p \in (0, 2]$ be symmetric p -stable distribution on \mathbb{R} with the characteristic function $e^{-A|r|^p}$, $A > 0$. Then the μ -weak Poisson measure is purely discrete with the distribution

$$\text{Exp}_{\otimes_{\mu}}(c\delta_1) = e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \delta_{k^{1/p}}.$$

To see this, it is enough to notice that if X_1, \dots, X_k are independent random variables with distribution γ_p , then $X_1 + \dots + X_k \stackrel{d}{=} k^{1/p} X_1$, thus

$$\delta_1^{\otimes_{\gamma_p} k} = \delta_{k^{1/p}}.$$

Example 3.4a_s. Consider the Kendall weak generalized convolution \diamond_α on \mathcal{P}_s with respect to the weakly stable measure μ_α with the characteristic function $\widehat{\mu}_\alpha(t) = (1 - |t|^\alpha)_+$, $\alpha \in (0, 1]$. It was shown in [9] that

$$(1 - |t|^\alpha)_+^k = \int_{\mathbb{R}} (1 - |ts|^\alpha)_+ \lambda_k(ds),$$

where $\lambda_0 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ and for $k \geq 1$ we have

$$\lambda_k(ds) = \frac{\alpha k(k-1)}{2} (1 - |s|^{-\alpha})^{k-2} |s|^{-(2\alpha+1)} \mathbf{1}_{(1,\infty)}(|s|) ds.$$

This means that $\delta_1^{\diamond_\alpha k} = \lambda_k$ for $k \geq 1$, thus $\mu_\alpha \circ \lambda_k = \mu_\alpha^{*k}$, and the μ_α -weak generalized exponent of $c\delta_1$ can be calculated as

$$\begin{aligned} \text{Exp}_{\diamond_\alpha}(c\delta_1)(ds) &= e^{-c}\delta_0(ds) + e^{-c}c\lambda_0(ds) \\ &\quad + e^{-c} \sum_{k=2}^{\infty} \frac{c^k}{k!} \frac{\alpha k(k-1)}{2} (1 - |s|^{-\alpha})^{k-2} |s|^{-(2\alpha+1)} \mathbf{1}_{(1,\infty)}(|s|) ds \\ &= e^{-c}(\delta_0 + c\lambda_0)(ds) + \frac{\alpha c^2}{2|s|^{(2\alpha+1)}} e^{-c|s|^{-\alpha}} \mathbf{1}_{(1,\infty)}(|s|) ds. \end{aligned}$$

Example 3.4a₊. Consider the same Kendall weak generalized convolution as an operator on \mathcal{P}_+ . Then, similarly as before for \mathcal{P}_s case, we obtain that $\exp(c\mu_\alpha) = \mu_\alpha \circ \text{Exp}_{\diamond_\alpha}(c\delta_1)$, where

$$\text{Exp}_{\diamond_\alpha}(c\delta_1)(du) = e^{-c}\delta_0(du) + ce^{-c}\delta_1(du) + \frac{c^2\alpha}{u^{2\alpha+1}} e^{-cu^{-\alpha}} \mathbf{1}_{(1,\infty)}(u) du.$$

Example 3.5a. For the technical reasons we consider here the special case of the Kingman weak generalized convolution $\otimes_{\omega_3} : \mathcal{P}_s \rightarrow \mathcal{P}_s$. Since the generalized convolutions defined by ω_3 and by its one-dimensional projection $\omega_{3,1}$ are the same and $\omega_{3,1}(du) = \frac{1}{2}\mathbf{1}_{[-1,1]}(u) du$, the calculations are simpler than in the general case.

For any $c > 0$ we need to calculate $\lambda = \text{Exp}_{\otimes_{\omega_{3,1}}}(\frac{1}{2}c\delta_1 + \frac{1}{2}c\delta_{-1})$ because in \mathcal{P}_s the role of δ_1 is played by the measure $\lambda_0 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Since $\widehat{\omega_{3,1}}(r) = \frac{\sin r}{r}$ and $\omega_{3,1} \circ \lambda = \exp(c\omega_{3,1})$ then

$$\widehat{\omega_{3,1} \circ \lambda}(r) = e^{-c(1 - \sin r/r)}.$$

On the other hand, we can write

$$\widehat{\omega_{3,1} \circ \lambda}(r) = \int_{\mathbb{R}} \widehat{\lambda}(rs) \omega_{3,1}(ds) = \int_{-1}^1 \frac{1}{2} \widehat{\lambda}(rs) ds = \frac{1}{2r} \int_{-r}^r \widehat{\lambda}(s) ds.$$

Thus,

$$\int_{-r}^r \widehat{\lambda}(s) ds = 2re^{-c(1 - \sin r/r)}.$$

From the last equation it follows that

$$\widehat{\lambda}(r) = \frac{d}{dr} \left(r e^{-c(1-\sin r/r)} \right) = e^{-c(1-\sin r/r)} \left(1 - c \frac{\sin r}{r} + c \cos r \right).$$

This implies that

$$\text{Exp}_{\otimes_{\omega_{3,1}}} (c\lambda_0) = \exp(c\omega_{3,1}) * (\delta_0 - c\omega_{3,1} + c\lambda_0).$$

These examples show that the μ -weak Poisson measure does not need to be discrete, although it is a linear combination of μ -weak generalized convolutions of the Dirac measure δ_1 .

Definition 3.8. Let $\mu \in \mathcal{P}$ be a nontrivial weakly stable measure. A measure $\lambda \in \mathcal{P} \setminus \{\delta_0\}$ is μ -weakly stable if there exists a sequence of positive numbers (c_n) and a measure $\nu \in \mathcal{P}$ such that

$$T_{c_n} \nu^{\otimes_{\mu} n} \rightarrow \lambda.$$

We denote by $\mathcal{S}(\mu)$ the set of all μ -weakly stable measures. Let

$$\mathcal{S}_p(\mu) = \{ \lambda \in \mathcal{P} \setminus \{\delta_0\} : T_a \lambda \otimes_{\mu} T_b \lambda = T_{g_p(a,b)} \lambda \},$$

where $g_p(a, b) = (|a|^p + |b|^p)^{1/p}$. Measures λ in $\mathcal{S}_p(\mu)$ will be referred to as μ -weakly p -stable. For every symmetric weakly stable measure μ there exists a parameter $\varkappa = \varkappa(\mu)$ called the characteristic exponent, such that

$$\varkappa(\mu) = \sup \{ p \in (0, 2] : \mathcal{S}_p(\mu) \neq \emptyset \}.$$

In our convention the supremum over the empty set equals zero. The parameter \varkappa is related to the symmetric p -stable measure γ_p in the usual sense. Note that $\varkappa(\mu) \leq 2$ for every weakly stable measure μ while the corresponding characteristic exponent $\varkappa(\diamond)$ of the Urbanik type generalized convolution \diamond can take any value from the positive half-line including infinity. It was proven in [10] that $\varkappa(\mu)$ has the following characterization.

Theorem 3.9. For every weakly stable distribution μ and $\mathcal{M}(\mu) = \{ \mu \circ \lambda : \lambda \in \mathcal{P} \}$ we have

$$\begin{aligned} \varkappa(\mu) &= \sup \left\{ p \in [0, 2] : \int_{\mathbb{R}} |x|^p \mu(dx) < \infty \right\} \\ &= \sup \{ p \in [0, 2] : \gamma_p \in \mathcal{M}(\mu) \}. \end{aligned}$$

The next theorem gives us the analogue of the Lévy–Khintchine representation for infinitely divisible distributions in the sense of weak generalized convolution on \mathcal{P}_s . Here $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

Theorem 3.10. Assume that μ is a nontrivial symmetric weakly stable measure on \mathbb{R} with \otimes_{μ} acting on \mathcal{P}_s and $\varkappa(\mu) > 0$. A measure $\lambda \in \mathcal{P}_s$ is μ -weakly infinitely divisible if and only if there

exists $A \geq 0$ and a symmetric σ -finite measure ν on \mathbb{R}_0 such that $\nu([-a, a]^c) < \infty$ for each $a > 0$,

$$\int_0^\infty \mu([-s, s]^c) \nu(ds) < \infty$$

and

$$\int_{\mathbb{R}} e^{itx} (\mu \circ \lambda)(dx) = \exp \left\{ -A|t|^{\varkappa(\mu)} - \int_{\mathbb{R}_0} (1 - \widehat{\mu}(ts)) \nu(ds) \right\}.$$

For details of the proof see [10]. The parameter A and the measure ν we call the *scale parameter* and μ -weak generalized Lévy measure respectively. Below we present some examples of μ -weakly stable distributions. Since we consider symmetric measures, it is enough to restrict the corresponding spectral measure ν to the positive half-line.

Examples

Example 3.4b. Consider the Kendall weak generalized convolution \diamond_α , $\alpha \in (0, 1]$, on \mathcal{P}_+ defined by the measure $\mu_\alpha \in \mathcal{P}_s$ with the characteristic function $\widehat{\mu}_\alpha(t) = (1 - |t|^\alpha)_+$ and the characteristic exponent $\varkappa(\mu_\alpha) = \alpha$. We know (for details see, e.g., [10]) that for every $p \leq \alpha$ there exists a probability measure $\nu_{\alpha,p} \in \mathcal{P}_+$ such that $\gamma_p = \mu_\alpha \circ \nu_{\alpha,p}$. The density of $\nu_{\alpha,p}$ (which is μ_α -weakly p -stable) for $p < \alpha$ is given by

$$g_{\alpha,p}(s) = p\alpha^{-1}((\alpha - p)s^{-p-1} + ps^{-2p-1})e^{-s^{-p}}\mathbf{1}_{(0,\infty)}(s).$$

In the same paper [10] it was shown that

$$\exp\{-|t|^p\} = \exp\left\{-\int_0^\infty (1 - (1 - |ts|^\alpha)_+) \frac{p(\alpha - p)}{\alpha s^{p+1}} ds\right\}.$$

Thus the Lévy measure for symmetric p -stable measure with the characteristic function $\exp\{-|t|^p\}$ can be written as $\mu_\alpha \circ \lambda_p$, where $\lambda_p(ds) = p(\alpha - p)\alpha^{-1}s^{-p-1}\mathbf{1}_{(0,\infty)}(s)ds$. For $p = \alpha$ such a measure λ_α does not exist, but we have that

$$\exp\{-|t|^\alpha\} = \lim_{p \nearrow \alpha} \exp\left\{-\int_0^\infty (1 - \widehat{\mu}_\alpha(ts)) \frac{p(\alpha - p)}{\alpha s^{p+1}} ds\right\}.$$

Example 3.5b. Consider the weakly stable Kingman distributions

$$\omega_{s,1}(dx) = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma(s+1/2)}(1-x^2)^{s-1/2}\mathbf{1}_{(-1,1)}(x)dx,$$

$s > -\frac{1}{2}$, with the characteristic exponent $\varkappa(\omega_{s,1}) = 2$. There exists a probability measure $\nu_{s,2}$ such that $\omega_{s,1} \circ \nu_{s,2} = N(0, 1)$, where the density of $\nu_{s,2}$ is given by

$$f_{s,2}(x) = \frac{1}{2^s\Gamma(s+1)}x^{2s+1}e^{-x^2/2}\mathbf{1}_{(0,\infty)}(x).$$

If by λ_p we denote the distribution of the random variable $\sqrt{\Theta}$, where Θ is the positive $p/2$ -stable random variable with the Laplace transform $\exp\{-(2t)^{p/2}\}$, then $\omega_{s,1} \circ \nu_{s,2} \circ \lambda_p = N(0, 1) \circ \lambda_p$ is symmetric p -stable. For $p < 2$ the spectral measure for γ_p is a scale mixture of $\omega_{s,1}$ since for a suitable constant $K > 0$

$$|t|^p = \int_0^\infty (1 - \widehat{\omega_{s,1}}(tr)) \frac{K}{r^{p+1}} dr.$$

4. Lévy and additive processes with respect to generalized and weak generalized convolutions

In this section, we consider an analog of a process with independent increments, when the usual convolution is replaced by a generalized one. To see that this is a natural generalization, consider the usual process with independent increments $X = \{X_t : t \geq 0\}$. X is also a Markov process with transition probabilities $P_{s,t}(x, \cdot) = \delta_x * \lambda_{s,t}$, where probability measures $\lambda_{s,t} = \mathcal{L}(X_t - X_s)$ satisfy an obvious consistency condition: $\lambda_{s,t} * \lambda_{t,u} = \lambda_{s,u}$, $s < t < u$. Conversely, given a family of distributions $\{\lambda_{s,t}\}$ satisfying the above consistency condition, there is a Markov process X with transition probabilities $P_{s,t}(x, \cdot) = \delta_x * \lambda_{s,t}$. Due to the consistency condition, the increments of X are independent and determined by $\lambda_{s,t}$. Therefore, the existence of a process with independent increments follows from a standard construction of a Markov process with given transition probabilities (see, e.g., Theorems 9.7 and 10.4 in [26]).

This approach was also applied by Nguyen Van Thu [29] in the context of generalized convolutions, and for Kingman's convolutions in particular, to relate generalized Lévy processes to Bessel processes.

We will use this approach to define and construct additive processes for generalized and weak generalized convolutions. We will identify properties of convolutions that are needed for this construction to go through, which indicates possible extensions beyond the types of convolutions considered in this paper. The consistency condition stated above naturally extends to the case of generalized convolutions as follows

$$\lambda_{s,t} \diamond \lambda_{t,u} = \lambda_{s,u} \quad \forall s < t < u. \quad (4.1)$$

It turns out that, given (4.1) and properties of generalized convolutions,

$$P_{s,t}(x, \cdot) := \delta_x \diamond \lambda_{s,t}(\cdot), \quad s < t, x \in \mathbb{R}_+, \quad (4.2)$$

satisfy the Chapman–Kolmogorov equations (see Theorem 4.2 below), hence generalized additive process can be well-defined.

Definition 4.1. $X = \{X_t : t \geq 0\}$ is said to be a \diamond -additive process (associated with $\{\lambda_{s,t}\}$ satisfying (4.1)) if X is a Markov process with transition probabilities given by (4.2). If $\lambda_{s,t} = \lambda^{\diamond(t-s)}$ for some \diamond -infinitely decomposable measure and all $0 \leq s < t$, then X is called a \diamond -Lévy process generated by λ . The definition of \otimes_μ -additive and \otimes_μ -Lévy processes is analogous, we replace \diamond in the above by \otimes_μ .

The next theorem is stated in a greater generality to show that only minimal conditions on convolutions are needed for the existence of generalized additive processes.

Theorem 4.2. *Let \mathbb{E} be a Polish space. Let \star be a binary associative operation on $\mathcal{P}(\mathbb{E})$ such that the map $\mathbb{E}^2 \ni (x, y) \mapsto \delta_x \star \delta_y(A) \in [0, 1]$ is measurable for each $A \in \mathcal{B}(\mathbb{E})$, and for every $\lambda_1, \lambda_2 \in \mathcal{P}(\mathbb{E})$*

$$\lambda_1 \star \lambda_2(A) = \int_{\mathbb{E}^2} (\delta_x \star \delta_y)(A) \lambda_1(dx) \lambda_2(dy). \quad (4.3)$$

Given a family $\{\lambda_{s,t} : 0 \leq s < t\} \subset \mathcal{P}(\mathbb{E})$ such that

$$\lambda_{s,u} = \lambda_{s,t} \star \lambda_{t,u}, \quad s < t < u,$$

the probability kernels $P_{s,t}(x, \cdot) := \delta_x \star \lambda_{s,t}(\cdot)$ on $\mathbb{E} \times \mathcal{B}$ satisfy the Chapman–Kolmogorov equations, that is, for every $0 < s < t < u$, $x \in \mathbb{E}$ and $A \in \mathcal{B}(\mathbb{E})$,

$$P_{s,u}(x, A) = \int_{\mathbb{E}} P_{s,t}(x, dy) P_{t,u}(y, A). \quad (4.4)$$

Consequently, for any $\mu_0 \in \mathcal{P}(\mathbb{E})$, there exists a Markov process $X = \{X_t : t \geq 0\}$ in \mathbb{E} such that $\mathcal{L}(X_0) = \mu_0$ and, for all $t > s$, $x \in \mathbb{E}$,

$$\mathbf{P}(X_t \in (\cdot) | X_s = x) = \delta_x \star \lambda_{s,t}(\cdot). \quad (4.5)$$

Proof. Let s, t, u, x and A be as in (4.4). We have

$$\begin{aligned} P_{s,u}(x, A) &= \delta_x \star \lambda_{s,u}(A) = \delta_x \star (\lambda_{s,t} \star \lambda_{t,u})(A) \\ &= (\delta_x \star \lambda_{s,t}) \star \lambda_{t,u}(A) \\ &= \int_{\mathbb{E}^2} (\delta_y \star \delta_z)(A) (\delta_x \star \lambda_{s,t})(dy) \lambda_{t,u}(dz) \\ &= \int_{\mathbb{E}^3} (\delta_x \star \lambda_{s,t})(dy) (\delta_w \star \delta_z)(A) \delta_y(dw) \lambda_{t,u}(dz) \\ &= \int_{\mathbb{E}} (\delta_x \star \lambda_{s,t})(dy) (\delta_y \star \lambda_{t,u})(A) \\ &= \int_{\mathbb{E}} P_{s,t}(x, dy) P_{t,u}(y, A), \end{aligned}$$

where the third equality uses the associativity of \star ; we also applied (4.1)–(4.3). The existence of the process X with desired properties follows now from (4.4) by Kolmogorov’s extension theorem. \square

Remark 4.3. Given a probability kernel $\mathbb{E}^2 \ni (x, y) \mapsto \rho_{x,y} \in \mathcal{P}(\mathbb{E})$, one can define a “convolution” on $\mathcal{P}(\mathbb{E})$ setting $\delta_x \star \delta_y := \rho_{x,y}$, and then extending \star to arbitrary measures by

(4.3). If \mathbb{E} is also a semigroup (not-necessarily commutative), then it is natural to assume that $\rho_{x,0} = \rho_{0,x} = \delta_x$. If $(\delta_x \star \delta_y) \star \delta_z = \delta_x \star (\delta_y \star \delta_z)$ for all $x, y, z \in \mathbb{E}$, then \star is associative on $\mathcal{P}(\mathbb{E})$. In this way, new classes of Markov processes, which are Lévy processes relative to such convolutions, can be defined.

Theorem 4.4. *Let \star denote either a generalized convolution \diamond or a weak generalized convolution \otimes_μ . Then for any consistent family of probability measures $\{\lambda_{s,t} : 0 \leq s < t\}$ there exists a \star -additive process $X = \{X_t : t \geq 0\}$ generated by this family and starting from 0. If $\lim_{t \downarrow s} \lambda_{s,t} = \delta_0$ for every $s \geq 0$ [$\lim_{s \uparrow t} \lambda_{s,t} = \delta_0$ for every $t > 0$, resp.], then X is right [left, resp.] continuous in probability. Any \star -Lévy process is continuous in probability.*

Proof. Suppose that $\lambda_{s,t} \rightarrow \delta_0$ as $t \downarrow s$. For every $\varepsilon > 0$, by (4.5) we have

$$\begin{aligned} \mathbf{P}(|X_t - X_s| > \varepsilon) &= \int \mathbf{P}(|X_t - x| > \varepsilon | X_s = x) \mathcal{L}(X_s)(dx) \\ &= \int \delta_x \star \lambda_{s,t}(\{y : |y - x| > \varepsilon\}) \mathcal{L}(X_s)(dx) \\ &\rightarrow \int \delta_x \star \delta_0(\{y : |y - x| > \varepsilon\}) \mathcal{L}(X_s)(dx) = 0 \end{aligned}$$

as $t \downarrow s$. Similarly we treat continuity from the left. Now, if X is a Lévy process, then the continuity of $\lambda_{s,t} = \lambda^{\star(t-s)}$ follows from Proposition 3.2 and the beginning of Section 3.2. \square

Remark 4.5. The \diamond -Lévy processes are Markov processes in classical sense. By Theorem 2.6 in [28], it follows that if \diamond is a generalized convolution on \mathbb{R}_+ , or a weak generalized convolution with $\varkappa(\diamond) > 0$, then each \diamond -Lévy processes has strong Markov property, the Feller property, it is continuous in probability and has càdlàg trajectories. Consequently, for each such process starting from a fixed (nonrandom) point the Blumenthal's 0–1 law holds (see, e.g., Proposition 40.4 in [26]).

Moreover, \diamond -Lévy processes have heavy-tailed distributions in each of the examples considered in this paper, provided $\varkappa(\diamond) < 2$ and \diamond is not the maximum or stable convolution. To see this it is enough to notice that in these cases for all $x, y \in \mathbb{R} \setminus \{0\}$ the measure $\delta_x \diamond \delta_y$ has infinite p -moment for $p > \varkappa(\diamond)$. Such processes provide interesting new models for the study of heavy-tail phenomena and possible long range dependence.

5. Stochastic integral processes with respect to \diamond -Lévy processes

For λ being \diamond -infinitely decomposable probability measure with the \diamond -generalized characteristic function

$$\Phi_\lambda(t) = \exp \left\{ -At^{\varkappa(\diamond)} - \int_0^\infty \frac{1 - \Omega(tx)}{\nu(x)} m(dx) \right\},$$

let \mathcal{A}_λ be the class of nonnegative functions f on the positive half-line which are nonnegative, measurable, bounded on compact intervals, and such that for every $t, u > 0$

$$\int_0^t f(x)^{\mathcal{Z}(\diamond)} dx < \infty, \quad \int_0^\infty \int_0^t \frac{1 - \Omega(uf(x)s)}{v(s)} dx m(ds) < \infty.$$

By $\{X_t: t \geq 0\}$ we denote the \diamond -additive process based on λ defined in the previous section. We want to define a stochastic process

$$Y_t = \diamond \int_0^t f(s) dX_s, \quad t \geq 0$$

as a Markov process with the transition probabilities $P_{s,t}^f(x, \cdot) = \delta_x \diamond P_{s,t}^f(0, \cdot)$ defined by the \diamond -generalized characteristic function of $P_{s,t}^f(0, \cdot)$:

$$\Psi(f, s, t, u) = \exp \left\{ -Au^{\mathcal{Z}(\diamond)} \int_s^t f(x)^{\mathcal{Z}(\diamond)} dx - \int_0^\infty \int_s^t \frac{1 - \Omega(uf(x)s)}{v(s)} dx m(ds) \right\}.$$

In view of the previous section the construction will be completed when we prove the following:

Lemma 5.1. *For each $f \in \mathcal{A}_\lambda$ and every $s, t \geq 0$, $s < t$ the function $\Psi(f, s, t, \cdot)$ is a \diamond -generalized characteristic function of a \diamond -infinitely decomposable measure $P_{s,t}^f$.*

Proof. Assume first that f is a simple function, which means that $f(x) = \sum_{k=1}^n a_k \mathbf{1}_{B_k}(x)$, where $B_j \cap B_k = \emptyset$ for $j \neq k$ and $\bigcup_{k=1}^n B_k = [s, t]$. We define the following measure

$$P_{s,t}^f := T_{a_1} \lambda^{\diamond \ell(B_1)} \diamond \dots \diamond T_{a_n} \lambda^{\diamond \ell(B_n)},$$

where ℓ is the Lebesgue measure on the positive half-line. We see that

$$\begin{aligned} \Phi_{P_{s,t}^f}(u) &= \int_0^\infty \Omega(ux) P_{s,t}^f(dx) \\ &= \exp \left\{ -Au^{\mathcal{Z}(\diamond)} \sum_{k=1}^n a_k^{\mathcal{Z}(\diamond)} \ell(B_k) - \sum_{k=1}^n \ell(B_k) \int_0^\infty \frac{1 - \Omega(ua_k x)}{v(x)} m(dx) \right\} \\ &= \exp \left\{ -Au^{\mathcal{Z}(\diamond)} \int_s^t f(r)^{\mathcal{Z}(\diamond)} dr - \int_s^t \int_0^\infty \frac{1 - \Omega(uf(r)x)}{v(x)} m(dx) dr \right\}. \end{aligned}$$

Now, if $f \in \mathcal{A}_\lambda$, then there exists a sequence of simple functions f_n monotonically increasing to f in each point $r \in [s, t]$. By the Lebesgue dominated convergence theorem we have that

$$\lim_{n \rightarrow \infty} \Phi_{P_{s,t}^{f_n}}(u) = \exp \left\{ -Au^{\mathcal{Z}(\diamond)} \int_s^t f(r)^{\mathcal{Z}(\diamond)} dr + \int_s^t \int_0^\infty \frac{\Omega(uf(r)x) - 1}{v(x)} m(dx) dr \right\}.$$

Since the sequence of continuous functions converging to a continuous function is converging uniformly on every compact interval, by the definition of \diamond -generalized characteristic function there exists a probability measure $P_{s,t}^f(0, \cdot)$ such that $P_{s,t}^{f_n} \rightarrow P_{s,t}^f$ weakly if $n \rightarrow \infty$ and

$$\int_0^\infty \Omega(ux) P_{s,t}^f(dx) = \Psi(f, s, t, u).$$

Infinite decomposability follows from the fact that

$$\Psi(f, s, t, u) = \Psi_{A,m}(f, s, t, u) = \Psi_{A/n, m/n}^n(f, s, t, u).$$

It can also be derived from the following property:

$$P_{s,t}^f \diamond P_{t,u}^f = P_{s,u}^f, \quad s < t < u. \quad \square$$

From this lemma, we conclude the following theorem.

Theorem 5.2. *Let λ be \diamond -infinitely decomposable and let $X = \{X_t : t \geq 0\}$ be the corresponding \diamond -Lévy process associated with λ . For given $f \in A_\lambda$, there exists nonhomogenous Markov process $Y = \{Y_t : t \geq 0\}$ with transition probabilities $\delta_x \diamond P_{s,t}^f$, where $P_{s,t}$ are transition probabilities of X . The process Y is a \diamond -additive process which is denoted by*

$$Y_t = \diamond \int_0^t f(s) dX_s, \quad t \geq 0.$$

6. Weak generalized summation

Naturally, one would like to describe a generalized convolution in terms of an operation on independent random variables. To this aim, one can consider a weak generalized summation $X \oplus Y$ of nonnegative random variables, where \oplus is a binary operation on \mathbb{R}_+ . It turns out that this method is very restrictive, only convolutions described in Examples 2.3 and 2.6 can be realized this way. Indeed, if we assume that for all $a, b, c \geq 0$, $a \oplus b = b \oplus a$, $a \oplus 0 = a$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ and $c(a \oplus b) = (ca) \oplus (cb)$, together with an assumption on continuity, then by Bohnenblust's theorem (see [2]), for some $\alpha \in (0, \infty]$,

$$a \oplus b = \begin{cases} (a^\alpha + b^\alpha)^{1/\alpha} & \text{if } \alpha < \infty, \\ \max\{a, b\} & \text{if } \alpha = \infty. \end{cases}$$

The problem of describing a generalized convolution in the language of random variables seems to be difficult. However, weak generalized convolutions open some new possibilities in this direction.

Recall that the random vector \mathbf{X} and its distribution μ is weakly stable if for all random variables θ_1, θ_2 and $\mathbf{X}_1, \mathbf{X}_2$ independent copies of \mathbf{X} such that $\theta_1, \theta_2, \mathbf{X}_1, \mathbf{X}_2$ are independent there

exists a random variable θ independent of \mathbf{X} such that

$$\mathbf{X}_1\theta_1 + \mathbf{X}_2\theta_2 \stackrel{d}{=} \mathbf{X}\theta. \quad (**)$$

Until now, we were satisfied by defining the weak generalized convolution based on this property:

$$\mathcal{L}(\theta_1) \otimes_{\mu} \mathcal{L}(\theta_2) = \mathcal{L}(\theta).$$

Now we want to use the original property in defining weak generalized addition which involves all random elements appearing in (**).

Lemma 6.1. *Let μ be a nontrivial weakly stable distribution. Suppose that \mathbf{X}, \mathbf{X}_1 and \mathbf{X}_2 are i.i.d. with distribution μ . Then for all nonnegative random variables θ_1, θ_2 such that $\theta_1, \theta_2, \mathbf{X}_1, \mathbf{X}_2$ are independent there exist random elements $\mathcal{X}, \Theta, \mathcal{X} \stackrel{d}{=} \mathbf{X}$, such that*

$$\theta_1\mathbf{X}_1 + \theta_2\mathbf{X}_2 = \mathcal{X} \cdot \Theta \quad a.e.$$

Proof. Let $\theta_1, \theta_2, \mathbf{X}_1, \mathbf{X}_2$ be as assumed in the lemma, with random elements taking values in a separable Banach space \mathbb{E} . By weak stability of \mathbf{X} we have that there exists independent random variable Θ independent of \mathbf{X} such that

$$\theta_1\mathbf{X}_1 + \theta_2\mathbf{X}_2 \stackrel{d}{=} \mathbf{X}\Theta.$$

Corollary 5.11 in [11] states that for each two Borel spaces S and T , a measurable mapping $f: T \rightarrow S$ and some random elements ξ in S and η in T with $\xi \stackrel{d}{=} f(\eta)$ there exists a random element $\tilde{\eta} \stackrel{d}{=} \eta$ in T with $\xi = f(\tilde{\eta})$ a.e. We see that it is enough to apply this corollary for $\xi = \theta_1\mathbf{X}_1 + \theta_2\mathbf{X}_2$, $\eta = (\mathbf{X}, \Theta)$ and $f: \mathbb{E} \times [0, \infty) \mapsto \mathbb{E}$ given by $f(\mathbf{x}, s) = \mathbf{x}s$ to obtain existence of $\tilde{\eta} = (\mathcal{X}, \Theta)$ such that

$$\theta_1\mathbf{X}_1 + \theta_2\mathbf{X}_2 = \mathcal{X}\Theta \quad a.e. \quad \square$$

In the following definition by \mathbb{K} we understand one of the sets \mathbb{R} or $\mathbb{R}_+ = [0, \infty)$. If μ is symmetric, then we can take $\mathcal{P}(\mathbb{K}) = \mathcal{P}_s$ as well as $\mathcal{P}(\mathbb{K}) = \mathcal{P}_+$.

Definition 6.2. *Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a rich enough probability space, $\mu \in \mathcal{P}(\mathbb{E})$ be a nontrivial weakly stable distribution, and let $s, t \in \mathbb{K}$.*

The weak generalized convolution algebra $(\mathcal{P}(\mathbb{K}), \otimes_{\mu})$ is representable (or the weak generalized convolution \otimes_{μ} is representable) if there exist measurable functions

$$\Theta: (\mathbb{K} \times \mathbb{E})^2 \rightarrow \mathbb{K} \quad \text{and} \quad \mathcal{X}: (\mathbb{K} \times \mathbb{E})^2 \rightarrow \mathbb{E}$$

such that for every choice of i.i.d. vectors $(\mathbf{X}_i)_{i \in \mathbb{N}}$ with distribution μ and all $i \neq j$, $i, j \in \mathbb{N}$, the following conditions hold

$$(1) \quad \Theta(s, \mathbf{X}_i; t, \mathbf{X}_j) = \Theta(t, \mathbf{X}_j; s, \mathbf{X}_i) \text{ and } \mathcal{X}(s, \mathbf{X}_i; t, \mathbf{X}_j) = \mathcal{X}(t, \mathbf{X}_j; s, \mathbf{X}_i);$$

- (2) $\mathcal{X}(s, \mathbf{X}_i; t, \mathbf{X}_j) \stackrel{d}{=} \mathbf{X}_1$;
- (3) $\mathcal{L}(\Theta(s, \mathbf{X}_i; t, \mathbf{X}_j)) = \delta_s \otimes_\mu \delta_t$;
- (4) $\Theta(s, \mathbf{X}_i; t, \mathbf{X}_j)$ and $\mathcal{X}(s, \mathbf{X}_i; t, \mathbf{X}_j)$ are independent;
- (5) $s\mathbf{X}_i + t\mathbf{X}_j = \Theta(s, \mathbf{X}_i; t, \mathbf{X}_j)\mathcal{X}(s, \mathbf{X}_i; t, \mathbf{X}_j)$ a.e.;
- (6) $\Theta(\Theta(s, \mathbf{X}_i; t, \mathbf{X}_j), \mathcal{X}(s, \mathbf{X}_i; t, \mathbf{X}_j); u, \mathbf{X}_k) = \Theta(s, \mathbf{X}_i; \Theta(t, \mathbf{X}_j; u, \mathbf{X}_k), \mathcal{X}(t, \mathbf{X}_j; u, \mathbf{X}_k))$ a.e. and $\mathcal{X}(\Theta(s, \mathbf{X}_i; t, \mathbf{X}_j), \mathcal{X}(s, \mathbf{X}_i; t, \mathbf{X}_j); u, \mathbf{X}_k) = \mathcal{X}(s, \mathbf{X}_i; \Theta(t, \mathbf{X}_j; u, \mathbf{X}_k), \mathcal{X}(t, \mathbf{X}_j; u, \mathbf{X}_k))$ a.e.;
- (7) If $\sum_{i=1}^{\infty} s_i \mathbf{X}_i$ converges a.e. for some choice of $s_i \in \mathbb{K}, i \in \mathbb{N}$, then $S_n \rightarrow S$ a.e. and $\mathcal{X}_n \rightarrow \mathbf{X}$ a.e., where

$$\begin{aligned} S_1 &= s_1, & S_{n+1} &= \Theta(S_n, \mathcal{X}_n; s_{n+1}, \mathbf{X}_{n+1}) && \text{a.e.;} \\ \mathcal{X}_1 &= \mathbf{X}_1, & \mathcal{X}_{n+1} &= \mathcal{X}(S_n, \mathcal{X}_n; s_{n+1}, \mathbf{X}_{n+1}) && \text{a.e.} \end{aligned}$$

Lemma 6.3. Assume that the weak generalized convolution algebra $(\mathcal{P}(\mathbb{K}), \otimes_\mu)$ is representable. If θ_1, θ_2 are independent with distributions λ_1, λ_2 respectively, and they are independent of $\mathbf{X}_1, \mathbf{X}_2$, then the random elements $\Theta(\theta_1, \mathbf{X}_1; \theta_2, \mathbf{X}_2)$ and $\mathcal{X}(\theta_1, \mathbf{X}_1; \theta_2, \mathbf{X}_2)$ are independent.

Proof. In fact, the result follows from properties (2) and (4) of Definition 6.2 by the following arguments:

$$\begin{aligned} & \mathbf{P}\{\mathcal{X}(\theta_1, \mathbf{X}_1; \theta_2, \mathbf{X}_2) \in B, \Theta(\theta_1, \mathbf{X}_1; \theta_2, \mathbf{X}_2) \in A\} \\ &= \int_{\mathbb{K}} \int_{\mathbb{K}} \mathbf{P}\{\mathcal{X}(s, \mathbf{X}_1; t, \mathbf{X}_2) \in B, \Theta(s, \mathbf{X}_1; t, \mathbf{X}_2) \in A\} \lambda_1(ds) \lambda_2(dt) \\ &\stackrel{(4)}{=} \int_{\mathbb{K}} \int_{\mathbb{K}} \mathbf{P}\{\mathcal{X}(s, \mathbf{X}_1; t, \mathbf{X}_2) \in B\} \mathbf{P}\{\Theta(s, \mathbf{X}_1; t, \mathbf{X}_2) \in A\} \lambda_1(ds) \lambda_2(dt) \\ &\stackrel{(2)}{=} \mu(B) \int_{\mathbb{K}} \int_{\mathbb{K}} \mathbf{P}\{\Theta(s, \mathbf{X}_1; t, \mathbf{X}_2) \in A\} \lambda_1(ds) \lambda_2(dt) \\ &= \mathbf{P}\{\mathcal{X}(\theta_1, \mathbf{X}_1; \theta_2, \mathbf{X}_2) \in B\} \mathbf{P}\{\Theta(\theta_1, \mathbf{X}_1; \theta_2, \mathbf{X}_2) \in A\}. \end{aligned} \quad \square$$

The following are examples of weak generalized convolutions that are representable.

Examples

Example 6.1. The symmetric convolution as the convolution on \mathcal{P}_+ is representable and we have

$$\begin{aligned} \Theta: (\mathbb{R}_+ \times \mathbb{R})^2 &\rightarrow \mathbb{R}_+, & \Theta(s, x; t, y) &= |sx + ty|, \\ \mathcal{X}: (\mathbb{R}_+ \times \mathbb{R})^2 &\rightarrow \mathbb{R}_+, & \mathcal{X}(s, x; t, y) &= \frac{sx + ty}{|sx + ty|} = \text{sign}(sx + ty). \end{aligned}$$

Example 6.3. For $p \in (0, 2]$ the weak generalized convolution algebra $(\mathcal{P}_+, *_p)$ generated by symmetric p -stable, weakly stable distribution γ_p is evidently representable:

$$\begin{aligned}\Theta: (\mathbb{R}_+ \times \mathbb{R})^2 &\rightarrow \mathbb{R}_+, & \Theta(s, x; t, y) &= \|(s, t)\|_p, \\ \mathcal{X}: (\mathbb{R}_+ \times \mathbb{R})^2 &\rightarrow \mathbb{R}_+, & \mathcal{X}(s, x; t, y) &= \frac{s}{\|(s, t)\|_p} x + \frac{t}{\|(s, t)\|_p} y.\end{aligned}$$

Example 6.5. The weak generalized convolution algebra $(\mathcal{P}_+, \otimes_{\omega_n})$ is representable. The corresponding functions are the following

$$\begin{aligned}\Theta: (\mathbb{R}_+ \times \mathbb{R}^n)^2 &\rightarrow \mathbb{R}_+, & \Theta(s, \mathbf{x}; t, \mathbf{y}) &= \|s\mathbf{x} + t\mathbf{y}\|_2; \\ \mathcal{X}: (\mathbb{R}_+ \times \mathbb{R}^n)^2 &\rightarrow \mathbb{R}, & \mathcal{X}(s, \mathbf{x}; t, \mathbf{y}) &= \frac{s\mathbf{x} + t\mathbf{y}}{\|s\mathbf{x} + t\mathbf{y}\|_2}.\end{aligned}$$

If a nontrivial weak generalized convolution \otimes_μ is representable and this will not lead to misunderstanding, we use the notation

$$\Theta(\theta_1, \mathbf{X}_1, \theta_2, \mathbf{X}_2) = \theta_1 \oplus_\mu \theta_2.$$

In most of the cases, we shall however write

$$\Theta(\theta_1, \mathbf{X}_1, \theta_2, \mathbf{X}_2) = (\theta_1 | \mathbf{X}_1) \oplus_\mu (\theta_2 | \mathbf{X}_2)$$

and

$$(\theta_1 | \mathbf{X}_1) \oplus_\mu (\theta_2 | \mathbf{X}_2) \oplus_\mu \cdots \oplus_\mu (\theta_n | \mathbf{X}_n) =: \sum_{i \leq n}^{\oplus_\mu} (\theta_i | \mathbf{X}_i).$$

To see the advantage of introducing representability for the weak generalized convolution consider examples constructed as follows:

Let \mathbf{X} with distribution μ be weakly stable and such that the weak generalized convolution \otimes_μ is representable. As in Section 4, for any distribution λ there exists a Markov process $\{S_n: n \in \mathbb{N}_0\}$ with the transition probabilities

$$P_{n,k}(x, \cdot) = \delta_x \otimes_\mu \lambda^{\otimes_\mu(k-n)}.$$

The existence of the process $\{S_n: n \in \mathbb{N}_0\}$ follows from a kind of existence theorem. Using the representability we can do it more explicitly:

Let $\theta_i, i \in \mathbb{N}$, be a sequence of i.i.d. random variables with distribution λ and $\mathbf{X}_i, i \in \mathbb{N}$ be a sequence of i.i.d. vectors with distribution μ . Now we define

$$S_n := \sum_{i \leq n}^{\oplus_\mu} (\theta_i | \mathbf{X}_i), \quad \mathbf{Z}_n := \sum_{i \leq n} \mathbf{X}_i \theta_i = S_n \mathcal{X}_n,$$

where $\mathcal{X}_1 = \mathbf{X}_1$, $\mathcal{X}_{n+1} = \mathcal{X}(\theta_{n+1}, \mathbf{X}_{n+1}; S_n, \mathcal{X}_n)$. We see that the sequence $\{S_n \mathcal{X}_n: n \in \mathbb{N}\}$ is a classical independent increments homogenous random walk with the step distribution

$\mu \circ \lambda$. Considering simultaneously both processes $\{(S_n, \mathbf{Z}_n): n \in \mathbb{N}\}$ or even all three processes $\{(S_n, \mathcal{X}_n, \mathbf{Z}_n): n \in \mathbb{N}\}$ we obtain more information than considering them separately.

Examples

Example 6.5a. In the case of $\mu = \omega_d$ uniform distribution on the unit sphere $S_{d-1} \subset \mathbb{R}^d$ and λ with the density function

$$f_{d-1,2}(r) = \frac{1}{2^{d-1}\Gamma(d)} r^{2d-1} e^{-r^2/2} \mathbf{1}_{(0,\infty)}(r)$$

we see that $Z_n, n \in \mathbb{N}$, is the classical Wiener process describing the position of the particle in \mathbb{R}^d observed in discrete times, $S_n, n \in \mathbb{N}$, describes the actual distance of the particle from the origin, and the stationary process $\mathcal{X}_n, n \in \mathbb{N}$ describes the projection of the actual position of the particle on the unit sphere in \mathbb{R}^d .

Example 6.3a. Another interesting example is connected with the symmetric α -stable Lévy motion, where symmetry means in fact spherical symmetry of the distribution of increments. To see this notice first that every zero mean Gaussian random vector \mathbf{X} is weakly stable and defines the representable weak generalized convolution $*_2$:

$$a\mathbf{X} + b\mathbf{X}' = \sqrt{a^2 + b^2} \left(\frac{a}{\sqrt{a^2 + b^2}} \mathbf{X} + \frac{b}{\sqrt{a^2 + b^2}} \mathbf{X}' \right).$$

Thus we have $\Theta(a, \mathbf{X}; b, \mathbf{X}') = \|(a, b)\|_2$, and

$$\mathcal{X}(a, \mathbf{X}; b, \mathbf{X}') = \left(\frac{a}{\|(a, b)\|_2} \mathbf{X} + \frac{b}{\|(a, b)\|_2} \mathbf{X}' \right).$$

We see that for the sequence $\mathbf{X}_n, n \in \mathbb{N}$, of i.i.d. random vectors with rotationally invariant Gaussian distribution and $\theta_i, i \in \mathbb{N}$, i.i.d. sequence of random variables such that θ_i^2 has $\frac{\alpha}{2}$ -stable distribution with the Laplace transform $e^{-t^{\alpha/2}}$ the sequence $Z_n = S_n \mathcal{X}_n$ consists of variables with symmetric α -stable distribution.

Consequently the sequences S_n, \mathcal{X}_n appearing in the condition (7) of Definition 6.2 are such that

$$S_n := \left(\sum_{i=1}^n \theta_i^2 \right)^{1/2} \quad \text{is a square root of a positive } \frac{\alpha}{2}\text{-stable process,}$$

$$Z_n := \sum_{i \leq n} X_i \theta_i = S_n \mathcal{X}_n \quad \text{is S}\alpha\text{S rotationally invariant Lévy process.}$$

Lemma 6.4. The sequence $\sum_{1 \leq i \leq n}^{\oplus \mu} (\theta_i | \mathbf{X}_i)$ converges a.e. if and only if the sequence $\sum_{1 \leq i \leq n} \theta_i \mathbf{X}_i$ converges a.e.

Proof. Assume that the sequence $\sum_{1 \leq i \leq n}^{\oplus \mu} (\theta_i | \mathbf{X}_i)$ converges a.e. (and in particular weakly) to a random variable θ . Since

$$\left(\sum_{i \leq n}^{\oplus \mu} (\theta_i | \mathbf{X}_i) \right) \mathcal{X}_n = \sum_{i=1}^n \theta_i \mathbf{X}_i \quad \text{a.e.}$$

we see that the right-hand side of this equality converges weakly to a random variable with distribution $\mathcal{L}(\theta) \circ \mu$. Since the summands $\theta_i \mathbf{X}_i$ are independent, the Lévy's equivalence theorem implies that $\sum_{1 \leq i \leq n} \theta_i \mathbf{X}_i$ converges a.e.

The opposite implication is a direct consequence of the property (7) of representable weak generalized convolution. \square

7. Random measures with weak generalized summation

Let $(\mathbb{S}, \mathcal{E})$ be a measurable space equipped with a σ -finite measure ϱ . We define

$$\mathcal{E}_0 = \{A \in \mathcal{E} : \varrho(A) < \infty\}.$$

By $L^0(\Omega, \mathbb{E})$ we denote the space of all random elements on Ω taking values in a separable Banach space \mathbb{E} .

Definition 7.1. Let $\mu \in \mathcal{P}(\mathbb{E})$ be a nontrivial weakly stable measure with representable convolution \otimes_μ and let $\lambda \in \mathcal{P}$ be μ -weakly infinitely divisible measure. The set function

$$\mathbf{M}_{\varrho, \lambda, \mu} = \mathbf{M} : \mathcal{E}_0 \rightarrow L^0(\Omega; \mathbb{R}) \times L^0(\Omega; \mathbb{E})$$

is called the μ -weak generalized random measure on a measurable space $(\mathbb{S}, \mathcal{E})$ with the control measure ϱ if the following conditions hold:

- (1) $\mathbf{M}(\emptyset) = (0, \bar{0})$ a.e.,
- (2) $\mathbf{M}(A) = (\mathbf{M}_\mu(A), \mathbf{M}_\mu(A)\mathcal{Y}(A))$, where $\mathbf{M}_\mu(A)$ has the distribution $\lambda^{\otimes_{\mu} \varrho(A)}$, $\mathcal{Y}(A)$ has the distribution μ , $\mathbf{M}_\mu(A)$ and $\mathcal{Y}(A)$ are independent for every set $A \in \mathcal{E}_0$,
- (3) if the sets $A_1, A_2, \dots, A_n \in \mathcal{E}_0$ are disjoint, then the random vectors $\mathbf{M}(A_1), \mathbf{M}(A_2), \dots, \mathbf{M}(A_n)$ are independent,
- (4) if sets $A_1, A_2, \dots \in \mathcal{E}_0$ are disjoint and $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{E}_0$, then

$$\mathbf{M}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \left(\mathbf{M}_\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right), \sum_{i \in \mathbb{N}} \mathbf{M}_\mu(A_i)\mathcal{Y}(A_i)\right) \quad \text{a.e.}$$

For simplicity, we use the following notation

$$\mathbf{M}_\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}}^{\oplus \mu} \mathbf{M}_\mu(A_i),$$

when the sets (A_i) are disjoint. Notice that this means that on the second coordinate we have a random measure in the classical sense, that is, the set function

$$\{\mathbf{M}_*(A) = \mathbf{M}_\mu(A) \mathcal{V}(A) : A \in \mathcal{E}_0\}$$

is a classical independently scattered random measure and $\mathbf{M}_*(A)$ has the distribution $(\mu \circ \lambda)^{*Q(A)}$.

The existence of the μ -weak generalized random measure can be derived from the Kolmogorov extension theorem by showing the consistency conditions for finite-dimensional distributions of the process $\{\mathbf{M}(A) : A \in \mathcal{E}_0\}$. Instead of checking this directly we show that finite-dimensional distributions of this process can be represented as distributions of random vectors built with a collection of independent two-dimensional random vectors and their \oplus_μ -sums.

Let $A_1, \dots, A_n \in \mathcal{E}_0$. Then there exist disjoint sets $B_1, \dots, B_N \in \mathcal{E}_0$ and sets $I_1, \dots, I_n \subseteq \{1, \dots, N\}$ such that

$$A_i = \bigcup_{j \in I_i} B_j, \quad i = 1, \dots, n.$$

Consequently $Q(A_i) = \sum_{j \in I_i} Q(B_j)$. For each choice of $B_1, \dots, B_N \in \mathcal{E}_0$ we can choose independent random variables $\theta_1, \dots, \theta_N$ with distributions $\lambda^{\otimes_{\mu} Q(B_1)}, \dots, \lambda^{\otimes_{\mu} Q(B_N)}$ respectively and a sequence of i.i.d. random vectors $\mathbf{X}_1, \dots, \mathbf{X}_N$ with distribution μ such that

$$\Theta(A_i) = \sum_{j \in I_i}^{\oplus_\mu} (\theta_j | \mathbf{X}_j) \quad \text{a.e.}$$

The random variables $\Theta(A_i)$ are well defined in view of representability of the weak generalized convolution \otimes_μ . By the same argument, similarly as in the condition (6) in Definition 6.2 we have uniquely, up to equality almost everywhere, defined vectors $\mathbf{X}(A_i)$, $i = 1, \dots, n$ such that $\Theta(A_i)$ and $\mathbf{X}(A_i)$ are independent and

$$\sum_{j \in I_i} \mathbf{X}_j \theta_j = \mathbf{X}(A_i) \Theta(A_i) \quad \text{a.e.}$$

Now it is easy to see that the random vector

$$((\Theta(A_1), \mathbf{X}(A_1) \Theta(A_1)), \dots, (\Theta(A_n), \mathbf{X}(A_n) \Theta(A_n)))$$

has the distribution desired for $(\mathbf{M}(A_1), \dots, \mathbf{M}(A_n))$ and the consistency conditions in the Kolmogorov extension theorem are evidently satisfied.

8. Lévy processes with respect to weak generalized summation

In this section, we assume that $\mathbb{S} = [0, \infty)$, $\mathcal{E} = \mathcal{B}([0, \infty))$ and Q is a σ -finite measure on $[0, \infty)$, finite on compact sets.

Definition 8.1. Let μ be a weakly stable measure on \mathbb{E} with representable generalized convolution \otimes_μ and let $\lambda \in \mathcal{P}$ be μ -weakly infinitely divisible. If $\mathbf{M}_{\varrho, \lambda, \mu}$ is the μ -weak generalized random measure on $([0, \infty), \mathcal{E})$ with the control measure ϱ , then the stochastic process

$$\{Z_{\varrho, \lambda, \mu}(t) := \mathbf{M}_\mu([0, t]): t \geq 0\}$$

is μ -weakly additive that is, has μ -weakly independent increments, and the additive in classical sense process

$$\{\mathbf{Y}_{\varrho, \lambda, \mu}(t) := \mathbf{M}_\mu([0, t])\mathcal{V}([0, t]): t \geq 0\}$$

is said to be associated with $\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\}$.

If the measures ϱ, λ, μ are fixed and this does not cause a misunderstanding, then we use simplified notation

$$\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\} = \{Z_t: t \geq 0\}, \quad \{\mathbf{Y}_{\varrho, \lambda, \mu}(t): t \geq 0\} = \{\mathbf{Y}_t: t \geq 0\}.$$

The finite dimensional distributions of μ -weakly additive process are uniquely determined by the measure ϱ and the distribution $\mathcal{L}(Z_1) = \lambda^{\otimes_\mu \varrho([0, 1])}$.

Remark 8.2. Notice that if we have two independent μ -weakly additive processes $\{Z_{\varrho_1, \lambda_1, \mu}(t): t \geq 0\}$ and $\{Z_{\varrho_2, \lambda_2, \mu}(t): t \geq 0\}$, then their \oplus_μ -sum

$$\{Z_t: t \geq 0\} = \{Z_{\varrho_1, \lambda_1, \mu}(t) \oplus_\mu Z_{\varrho_2, \lambda_2, \mu}(t): t \geq 0\}$$

is also μ -weakly additive in the following two cases:

- (1) if there exists a constant $a > 0$ such that $\lambda_1^{\otimes_\mu a} = \lambda_2$, and then

$$Z_t = Z_{\varrho_1 + a\varrho_2, \lambda_1, \mu}(t),$$

- (2) if there exists $c > 0$ such that $\varrho_2 = c\varrho_1$, and then

$$Z_t = Z_{\varrho_1, \lambda_1 \otimes_\mu \lambda_2^{\otimes_\mu c}, \mu}(t).$$

We want to consider a μ -weakly additive process $\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\}$ as a process with independent increments, but these increments shall not be defined as a usual difference of random variables. Thus, for every $0 \leq s \leq t$ we define the increment between Z_s and Z_t to be the random variable $Z_{s,t} = \mathbf{M}_\mu([s, t])$. By the assumption, $Z_{s,t}$ is independent of Z_s and

$$Z_s \oplus_\mu Z_{s,t} = Z_t \quad \text{a.e.}$$

Definition 8.3. A μ -weakly additive stochastic process

$$\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\}$$

is μ -weak Lévy process in law if the control measure ϱ for the corresponding μ -weak generalized random measure $\mathbf{M}_{\varrho, \lambda, \mu}$ is proportional to the Lebesgue measure.

It is easy to see that the stochastic process $\{\mathbf{Y}_t: t \geq 0\}$ associated with the μ -weak Lévy process $\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\}$ is a Lévy process in law in the classical sense.

A Lévy process in law is an additive process with stationary increments, that is continuous in probability. Since the control measure ϱ in the definition of the μ -weak Lévy process in law is proportional to the Lebesgue measure, stationarity of increments is evident. The next proposition implies that our process is also continuous in probability.

Proposition 8.4. *Let μ be a nontrivial weakly stable measure and let λ be μ -weakly infinitely divisible. If the measure ϱ on $[0, \infty)$ does not have any atoms, then both μ -weakly additive process $\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\}$ and the process $\{\mathbf{Y}_t: t \geq 0\}$ associated with $\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\}$ are continuous in probability.*

Proof. Since $\varrho([s, t)) \rightarrow 0$ for $t \searrow s$,

$$\mathcal{L}(Z_{s,t}) = \lambda^{\otimes_{\mu} \varrho([s,t))} \rightarrow \delta_0,$$

which implies continuity in probability for the process $\{Z_{\varrho, \lambda, \mu}(t): t \geq 0\}$. Consequently, we have also

$$\mathcal{L}(Y_t - Y_s) = \mu \circ \lambda^{\otimes_{\mu} \varrho([s,t))} \rightarrow \mu \circ \delta_0 = \delta_0 \quad \text{for } t \searrow s. \quad \square$$

Definition 8.5. *Let $\mu \in \mathcal{P}(\mathbb{E})$ be a nontrivial weakly stable measure and let ℓ be the Lebesgue measure on $[0, \infty)$. The μ -weak Lévy process*

$$\{N_{\mu}(t): t \geq 0\} \stackrel{\text{def}}{=} \{Z_{\ell, \lambda, \mu}(t): t \geq 0\}$$

is μ -weak Poisson processes with the intensity $c > 0$ if $\lambda = \text{Exp}_{\otimes_{\mu}}(c\delta_1)$.

Examples

Example 8.3. Let $\mu = \gamma_p$, $p \in (0, 2]$ be a symmetric p -stable distribution on \mathbb{R} with the characteristic function $e^{-A|r|^p}$, $A > 0$. Then the μ -weak Poisson process $\{N_{\gamma_p}(t): t \geq 0\}$ is purely discrete with the distribution

$$\mathcal{L}(N_{\gamma_p}(t)) = \text{Exp}_{\otimes_{\gamma_p}}(ct\delta_1) = e^{-ct} \sum_{k=0}^{\infty} \frac{(ct)^k}{k!} \delta_{k^{1/p}}.$$

It is easy to notice that the stochastic process $\{Y_t: t \geq 0\}$ associated with $\{N_{\gamma_p}(t): t \geq 0\}$ is such that

$$\mathbb{E} e^{irY_t} = \widehat{\exp(ct\gamma_p)}(r) = \exp\{-ct(1 - \widehat{\gamma_p}(r))\} = \exp\{-ct(1 - e^{-A|r|^p})\}.$$

Example 8.4. Consider the Kendall weak generalized convolution $\diamond_{\alpha}: \mathcal{P}_s \rightarrow \mathcal{P}_s$ defined by the weakly stable distribution μ_{α} on \mathbb{R} with the characteristic function $\widehat{\mu_{\alpha}}(t) = (1 - |t|^{\alpha})_+$, $\alpha \in$

$(0, 1]$. By Example 3.4a_s we know that the distribution of $N_{\mu_\alpha}(t)$ is given by

$$\text{Exp}_{\otimes_{\mu_\alpha}}(ct\delta_1)(ds) = e^{-ct}(\delta_0 + ct\lambda_0)(ds) + \frac{\alpha(ct)^2}{2|s|^{2\alpha+1}} e^{-ct|s|^{-\alpha}} \mathbf{1}_{(1,\infty)}(|s|) ds.$$

The Lévy stochastic process in law $\{Y_t: t \geq 0\}$ associated with the μ_α -weak Poisson process $\{N_{\mu_\alpha}(t): t \geq 0\}$ is such that

$$\mathbb{E}e^{irY_t} = \exp\{-ct(1 - \widehat{\mu_\alpha}(r))\} = e^{-ct|r|^\alpha} \mathbf{1}_{[-1,1]}(r) + e^{-ct} \mathbf{1}_{[-1,1]^c}(r).$$

This means that

$$\mathcal{L}(Y_t)(ds) = e^{-ct}\delta_0(ds) + (1 - e^{-ct})f_\alpha(s) ds,$$

where

$$f_\alpha(s) = \frac{1}{\pi} \frac{1}{1 - e^{-ct}} \int_0^1 \cos(sr) (e^{-ct|r|^\alpha} - e^{-ct}) dr.$$

For $\alpha = 1$ we obtain

$$f_1(s) = \frac{ct}{\pi(e^{ct} - 1)} \frac{se^{ct} - ct \sin(s) - s \cos(s)}{s((ct)^2 + s^2)}.$$

Example 8.5. Consider $\{N_{\omega_{3,1}}(t): t \geq 0\}$, the $\omega_{3,1}$ -weak Poisson process with the intensity $c > 0$. In this construction, we assume that $\otimes_{\omega_{3,1}}: \mathcal{P}_s \rightarrow \mathcal{P}_s$.

The distribution of $N_{\omega_{3,1}}(t)$ we obtain substituting c by ct in the formula obtained in Example 3.5a, thus

$$\mathcal{L}(N_{\omega_{3,1}}(t)) = \exp(ct\omega_{3,1}) * (\delta_0 - ct\omega_{3,1} + ct\lambda_0).$$

The Lévy process in law $\{Y_t: t \geq 0\}$ associated with $\{N_{\omega_{3,1}}(t): t \geq 0\}$ is such that

$$\mathcal{L}(Y_t) = \exp(ct\omega_{3,1}) = e^{-ct} \sum_{n=0}^{\infty} \frac{(ct)^n}{n!} \omega_{3,1}^{*n}.$$

Since $\omega_{3,1}$ is the uniform distribution on $[-1, 1]$, $\omega_{3,1}^{*n}$ are also well known and, for example, in [12] we can find that $\omega_{3,1}^{*n}$ has the following density function:

$$f^{(n)}(x) = \begin{cases} \sum_{i=0}^k (-1)^i \binom{n}{i} \frac{(x+n-2i)^{n-1}}{(n-1)!2^n}, & x \in [-n+2k, -n+2(k+1)), \\ 0, & k = 0, \dots, n-1, \\ & \text{otherwise.} \end{cases}$$

Remark 8.6. The idea of a stochastic process associated with another process suggests a natural connections with the idea of subordinated processes described in Feller's monograph [4]. The

construction there was the following: We start with two independent stochastic processes $\{X_t \in \mathbb{R}: t \geq 0\}$ and $\{\mathbb{T}(t) \in [0, \infty): t \geq 0\}$, $\mathbb{T}(t)$ increasing, and we define

$$\{X_{\mathbb{T}(t)}: t \geq 0\}.$$

The process $\{X_{\mathbb{T}(t)}: t \geq 0\}$ is subordinated to the process $\{X_t: t \geq 0\}$ by $\{\mathbb{T}(t): t \geq 0\}$. This construction is rich enough to cover many cases.

One of the best known subordinated processes is the sub-stable independent increments process. It is based on a strictly stable process $\{X_t: t \geq 0\}$ with independent stationary increments. This means that

$$X_t \stackrel{d}{=} t^{1/\alpha} X_1, \quad X_{t+s} - X_t \stackrel{d}{=} s^{1/\alpha} X_1, \quad X_t \perp (X_{t+s} - X_t).$$

The corresponding time stochastic process $\{\mathbb{T}(t): t \geq 0\}$ takes values in the positive half-line, has independent increments and the Laplace transform $\mathbb{E}e^{-r\mathbb{T}_t} = \exp\{-tr^\beta\}$ for some $\beta < 1$. Then $\{X_{\mathbb{T}(t)}: t \geq 0\}$ is an $(\alpha\beta)$ -stable stochastic process with independent increments.

The same process can be obtained by our construction as associated with the μ -weakly additive process

$$\{Z_{\varrho, \lambda, \gamma_\alpha}(t) = \mathbb{T}(t)^{1/\alpha}: t \geq 0\},$$

where

$$\lambda = \mathcal{L}(\mathbb{T}(1)^{1/\alpha}), \quad \mu = \gamma_\alpha = \mathcal{L}(X_1), \quad \varrho = \ell.$$

In this case $Z_{s,t} = M_\mu([s, t]) = (\mathbb{T}(t) - \mathbb{T}(s))^{1/\alpha}$ and $Y_t - Y_s \stackrel{d}{=} Z_{s,t} X_1$. Thus the associated process $\{Y_t: t \geq 0\}$ was obtained by some operation on the space, not by randomizing the time, as $\{X_{\mathbb{T}(t)}: t \geq 0\}$; however they are stochastically equivalent. In the case $\alpha = 2$ and $\{X_t: t \geq 0\}$ being multidimensional Brownian motion we again obtain rotationally invariant independent increment symmetric 2β -stable stochastic process.

9. Weak stochastic integrals

In this section, we give a construction of a stochastic integral using the weak generalized summation. We assume that the considered nontrivial weakly stable measure μ belongs to \mathcal{P} (instead of $\mu \in \mathcal{P}(\mathbb{E})$ for the sake of simplicity) and that the weak generalized convolution \otimes_μ is representable. Let λ be μ -weakly infinitely divisible, $\mathbf{M}_{\varrho, \lambda, \mu}$ be μ -weak generalized random measure for some σ -finite measure ϱ on $(\mathbb{S}, \mathcal{E})$, and let $\mathcal{E}_0 = \{A \in \mathcal{E}: \varrho(A) < \infty\}$.

The representability property of \otimes_μ allows us to construct a stochastic integral as in the case of the usual convolution (see, e.g., Rajput and Rosiński [24]). We will only outline this construction. For a simple function

$$f(x) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}(x),$$

where $A_1, \dots, A_n \in \mathcal{E}_0$ are disjoint sets and $a_1, \dots, a_n \in \mathbb{R}$, put

$$I_{\varrho, \lambda, \mu}(f) = \int_{\mathbb{S}} f(x) \mathbf{M}_{\mu}(dx) \stackrel{\text{def}}{=} \sum_{i \leq n}^{\oplus \mu} a_i \mathbf{M}_{\mu}(A_i).$$

Lemma 9.1. Assume that we have two representations for the simple function f , that is,

$$f(x) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}(x) \quad \text{and} \quad f(x) = \sum_{i=1}^m b_i \mathbf{1}_{B_i}(x),$$

such that $A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{E}_0$ and $A_i \cap A_j = \emptyset$, $B_i \cap B_j = \emptyset$ for $i \neq j$. Then

$$\sum_{i \leq n}^{\oplus \mu} a_i \mathbf{M}_{\mu}(A_i) = \sum_{j \leq m}^{\oplus \mu} b_j \mathbf{M}_{\mu}(B_j) \quad \text{a.e.}$$

Proof. There exists a family of disjoint sets $C_1, \dots, C_N \in \mathcal{E}_0$ such that for every $i \leq n$ and $j \leq m$ there exists $I_i = \{k_{1,i}, \dots, k_{n_i,i}\} \subset \{1, \dots, N\}$ and $J_j = \{\ell_{1,j}, \dots, \ell_{m_j,j}\} \subset \{1, \dots, N\}$ such that

$$\bigcup_{k \in I_i} C_k = A_i, \quad \bigcup_{\ell \in J_j} C_{\ell} = B_j, \quad i \leq n, j \leq m.$$

Of course $I_k \cap I_i = \emptyset$ and $J_k \cap J_i = \emptyset$ for $k \neq i$. Thanks to representability of \otimes_{μ} it makes sense to consider generalized sums, thus by our construction

$$\mathbf{M}_{\mu}(A_i) = \sum_{k \in I_i}^{\oplus \mu} \mathbf{M}_{\mu}(C_k) \quad \text{a.e.} \quad \text{and} \quad \mathbf{M}_{\mu}(B_j) = \sum_{\ell \in J_j}^{\oplus \mu} \mathbf{M}_{\mu}(C_{\ell}) \quad \text{a.e.}$$

Put $c_k := a_i = b_j$ if $C_k \subset A_i \cap B_j$. Now we see that the following equalities hold almost everywhere

$$\begin{aligned} \sum_{i \leq n}^{\oplus \mu} a_i \mathbf{M}_{\mu}(A_i) &= \sum_{i \leq n}^{\oplus \mu} a_i \sum_{k \in I_i}^{\oplus \mu} \mathbf{M}_{\mu}(C_k) = \sum_{i \leq n}^{\oplus \mu} \sum_{k \in I_i}^{\oplus \mu} a_i \mathbf{M}_{\mu}(C_k) \\ &= \sum_{i \leq n}^{\oplus \mu} \sum_{k \in I_i}^{\oplus \mu} c_k \mathbf{M}_{\mu}(C_k) = \sum_{k \leq N}^{\oplus \mu} c_k \mathbf{M}_{\mu}(C_k) \\ &= \sum_{j \leq m}^{\oplus \mu} \sum_{\ell \in J_j}^{\oplus \mu} c_{\ell} \mathbf{M}_{\mu}(C_{\ell}) = \sum_{j \leq m}^{\oplus \mu} b_j \sum_{\ell \in J_j}^{\oplus \mu} \mathbf{M}_{\mu}(C_{\ell}) \\ &= \sum_{j \leq m}^{\oplus \mu} b_j \mathbf{M}_{\mu}(B_j). \end{aligned}$$

□

Remark 9.2. Let $\mu = \mathcal{L}(\mathbf{X}) \in \mathcal{P}(\mathbb{E})$ be a nontrivial weakly stable measure and let $\lambda = \text{Exp}_{\otimes_{\mu}}(\delta_1)$. The μ -weak generalized random measure \mathbf{M}_{μ} consists of the variables with μ -weak

Poisson distribution and $\mathcal{L}(\mathbf{M}_\mu(A)) = \text{Exp}_{\otimes_\mu}(\varrho(A)\delta_1)$ for $A \in \mathcal{E}_0$. Since the \otimes_μ -generalized characteristic function of the measure $\text{Exp}_{\otimes_\mu}(a\delta_1)$ is equal to the classical characteristic function of $\text{Exp}_{\otimes_\mu}(a\delta_1) \circ \mu = \exp(a\delta_1 \circ \mu) = \exp(a\mu)$ then

$$\begin{aligned} \mathbf{E} \exp\{i\langle \mathbf{t}, a\mathbf{M}_\mu(A)\mathbf{X} \rangle\} &= \exp\{-(1 - \widehat{\mu}(a\mathbf{t}))\varrho(A)\} \\ &= \exp\left\{-\int_{\mathbb{S}} (1 - \widehat{\mu}(a\mathbf{1}_A(x)\mathbf{t}))\varrho(\mathrm{d}x)\right\}. \end{aligned}$$

If $f(x) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}(x)$, for disjoint $A_1, \dots, A_n \in \mathcal{E}_0$, then

$$\begin{aligned} \mathbf{E} \exp\{i\langle \mathbf{t}, I_{\varrho, \lambda, \mu}(f)\mathbf{X} \rangle\} &= \prod_{k=1}^n \mathbf{E} \exp\{i\langle \mathbf{t}, a_k \mathbf{M}_\mu(A_k)\mathbf{X} \rangle\} \\ &= \exp\left\{-\sum_{k=1}^n \int_{\mathbb{S}} (1 - \widehat{\mu}(a_k \mathbf{1}_{A_k}(x)\mathbf{t}))\varrho(\mathrm{d}x)\right\} \\ &= \exp\left\{-\int_{\mathbb{S}} (1 - \widehat{\mu}(f(x)\mathbf{t}))\varrho(\mathrm{d}x)\right\} \\ &= \exp\left\{-\int_{\mathbb{S}} (1 - \widehat{\mu}(s\mathbf{t}))\varrho_f(\mathrm{d}s)\right\}, \end{aligned}$$

where $\varrho_f(A) = \varrho(f^{-1}(A)) = \varrho(\{x \in \mathbf{E} : f(x) \in A\})$, $A \in \mathcal{E}_0$. This means that

$$\mathcal{L}(I_{\varrho, \lambda, \mu}(f)) = \text{Exp}_{\otimes_\mu}(\varrho_f).$$

Proposition 9.3. Assume that the weakly stable measure $\mu = \mathcal{L}(X)$ on \mathbb{R} is nontrivial and symmetric with the characteristic exponent \varkappa . Let λ be μ -weakly infinitely divisible with the scale parameter $A \geq 0$ and the μ -weak generalized Lévy measure ν . Let $f : \mathbb{S} \mapsto \mathbb{R}$ be a measurable function such that

$$\int_{\mathbb{S}} |f(x)|^{\varkappa} \varrho(\mathrm{d}x) < \infty \quad \text{and} \quad \int_{\mathbb{R}} \int_{\mathbb{S}} |1 - \widehat{\mu}(f(x)ts)| \varrho(\mathrm{d}x) \nu(\mathrm{d}s) < \infty.$$

Then the stochastic integral $I_{\varrho, \lambda, \mu}(f)$ exists as the limit in probability of stochastic integrals of simple functions. Moreover, the \otimes_μ -generalized characteristic function of $I_{\varrho, \lambda, \mu}(f)$ is of the form

$$\begin{aligned} \mathbf{E} \exp\{it I_{\varrho, \lambda, \mu}(f)X\} \\ = \exp\left\{-A|t|^{\varkappa} \int_{\mathbb{S}} |f(x)|^{\varkappa} \varrho(\mathrm{d}x) - \int_{\mathbb{R}} \int_{\mathbb{S}} (1 - \widehat{\mu}(f(x)ts)) \varrho(\mathrm{d}x) \nu(\mathrm{d}s)\right\}. \end{aligned}$$

Proof. It is enough to prove this for simple function $f = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ for disjoint sets A_1, \dots, A_n . Notice that the generalized characteristic function for λ is the following

$$\widehat{\lambda \circ \mu}(t) = \exp \left\{ -A|t|^{\mathcal{Z}(\mu)} - \int_{\mathbb{R}} (1 - \widehat{\mu}(ts)) \nu(ds) \right\}.$$

Since

$$\begin{aligned} (T_{a_1} \lambda^{\varrho(A_1)} \otimes_{\mu} \dots \otimes_{\mu} T_{a_n} \lambda^{\varrho(A_n)}) \circ \mu &= T_{a_1} \lambda^{\varrho(A_1)} \circ \mu * \dots * T_{a_n} \lambda^{\varrho(A_n)} \circ \mu \\ &= T_{a_1} (\lambda \circ \mu)^{* \varrho(A_1)} * \dots * T_{a_n} (\lambda \circ \mu)^{* \varrho(A_n)}, \end{aligned}$$

where, for the simplicity, we write $\lambda^{\varrho(A_i)}$ instead of $\lambda^{\otimes_{\mu} \varrho(A_i)}$, we have

$$\begin{aligned} &\mathbf{E} \exp \{it I_{\varrho, \lambda, \mu}(f) X\} \\ &= \prod_{i=1}^n \exp \left\{ -A|ta_i|^{\mathcal{Z}(\mu)} \varrho(A_i) - \varrho(A_i) \int_{\mathbb{R}} (1 - \widehat{\mu}(a_i ts)) \nu(ds) \right\} \\ &= \exp \left\{ -A \sum_{i=1}^n |ta_i|^{\mathcal{Z}(\mu)} \varrho(A_i) - \int_{\mathbb{R}} \sum_{i=1}^n (1 - \widehat{\mu}(a_i ts) \varrho(A_i)) \nu(ds) \right\} \\ &= \exp \left\{ -A|t|^{\mathcal{Z}} \int_{\mathbb{S}} |f(x)|^{\mathcal{Z}} \varrho(dx) - \int_{\mathbb{R}} \int_{\mathbb{S}} (1 - \widehat{\mu}(f(x)ts)) \varrho(dx) \nu(ds) \right\}. \end{aligned}$$

This ends the proof. \square

Remark 9.4. The Proposition 9.3 states in particular that the random variable $I_{\varrho, \lambda, \mu}(f)$ is μ -weakly infinitely divisible with the scale parameter

$$A' = A \int_{\mathbb{S}} |f(x)|^{\mathcal{Z}(\mu)} \varrho(dx),$$

and the μ -weak generalized Lévy measure $\varrho_f \circ \nu$, where for $A \in \mathcal{E}_0$ $\varrho_f(A) = \varrho(f^{-1}(A))$.

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