Asymptotic total variation tests for copulas

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We propose a new platform of goodness-of-fit tests for copulas, based on empirical copula processes and nonparametric bootstrap counterparts. The standard Kolmogorov–Smirnov type test for copulas that takes the supremum of the empirical copula process indexed by orthants is extended by test statistics based on the empirical copula process indexed by families of L_n disjoint boxes, with L_n slowly tending to infinity. Although the underlying empirical process does not converge, the critical values of our new test statistics can be consistently estimated by nonparametric bootstrap techniques, under simple or composite null assumptions. We implemented a particular example of these tests and our simulations confirm that the power of the new procedure is oftentimes higher than the power of the standard Kolmogorov–Smirnov or the Cramér–von Mises tests for copulas.

Keywords: bootstrap; copula; empirical copula process; goodness-of-fit test; weak convergence

1. Introduction

This paper introduces new powerful goodness-of-fit (GOF) tests for copulas in $[0, 1]^d$, $d \geq 2$, based on the empirical copula process

$$\mathbb{Z}_n(\mathbf{u}) = \sqrt{n}(\mathbb{C}_n - C)(\mathbf{u}), \qquad \mathbf{u} = (u_1, \ldots, u_d) \in [0, 1]^d, \qquad (1.1)$$

given a sample of n independent random vectors $\mathbf{X}_i = (X_{i1}, \ldots, X_{id}) \in \mathbb{R}^d$, $i = 1, \ldots, n$, from a common distribution function H. Let C be the associated copula function, as given by Sklar's theorem [30]. Here, $\mathbb{C}_n$ is the usual empirical copula, as introduced by Deheuvels [7]: denoting by $\mathbb{H}_n$ the joint c.d.f. of the sample $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$, $\mathbb{F}_{n,j}$ the jth empirical c.d.f. associated to $(X_{1j}, \ldots, X_{nj})$, $j = 1, \ldots, d$, and $\mathbb{F}_{n,j}^-$ its empirical quantile function, we have

$$\mathbb{C}_n(\mathbf{u}) = \mathbb{H}_n\big(\mathbb{F}_{n,1}^-(u_1), \ldots, \mathbb{F}_{n,d}^-(u_d)\big)$$

by definition, for every $\mathbf{u} = (u_1, \ldots, u_d) \in [0, 1]^d$. The Kolmogorov–Smirnov (KS) test statistic for testing of the null hypothesis $\mathrm{H}_0 : C = C_0$ is

$$\mathrm{KS}_n = \sup_{\mathbf{u} \in [0,1]^d} \big|\sqrt{n}(\mathbb{C}_n - C_0)(\mathbf{u})\big|. \qquad (1.2)$$

The Cramér–von Mises statistic (CvM) is

$$\mathrm{CM}_n = \int \left\{ \sqrt{n} (\mathbb{C}_n - C_0)(\mathbf{u}) \right\}^2 \mathrm{d}\mathbb{C}_n(\mathbf{u}). \tag{1.3}$$

It is well known (see, for instance, [11]) that $\mathbb{Z}_n$ and its bootstrap counterpart $\mathbb{Z}_n^*$, defined in (2.4) below, both converge weakly to the same tight Gaussian process in $\ell^\infty([0,1]^d)$ under the null hypothesis. Therefore, we can compute the α-upper points of KS_n and CM_n via the bootstrap. To the best of our knowledge, all the proposed GOF tests rely on simulation-based procedures to calculate their corresponding p-values, with the notable exception of the distribution-free test statistics of Fermanian [9]. The latter idea has been further developed by Scaillet [26] and Fermanian and Wegkamp [12]. A parametric bootstrap has been proposed [14] to tackle composite null hypotheses, while Rémillard and Scaillet [25] advocate the use of the multiplier central limit theorem to build an alternative bootstrap empirical copula process. Bücher and Dette [5] give a survey and a comparison of various bootstrap methods.

The goal of this paper is to develop more powerful tests than the KS test (1.2) and CvM test (1.3) for simple and composite null hypotheses. The next section offers a class of such tests. For instance, in the case of a null simple hypothesis $\mathrm{H}_0 : C = C_0$, we propose the test that rejects H_0 for large values of the test statistic

$$\mathbb{T}_n := \sup_{B_1, \ldots, B_{L_n}} \sum_{k=1}^{L_n} \left| \mathbb{Z}_n(B_k) \right|. \tag{1.4}$$

The supremum is taken over all disjoint boxes $B_1, \ldots, B_{L_n} \subset [0,1]^d$ of the form $\prod_{j=1}^d (a_j, b_j]$, using the convention

$$\mathbb{Z}_n \big((a_1, b_1] \times \cdots \times (a_d, b_d] \big) = \Delta_{a_1, b_1}^1 \Delta_{a_2, b_2}^2 \cdots \Delta_{a_d, b_d}^d \mathbb{Z}_n(\mathbf{u}) \tag{1.5}$$

for any arbitrary point $\mathbf{u} \in [0,1]^d$ and for all $0 \le a_j < b_j \le 1$, $j = 1, \ldots, d$. Here, we have used the usual operators Δ^j defined for every function f by

$$\left(\Delta_{a,b}^j f \right)(\mathbf{u}) = f(u_1, \ldots, u_{j-1}, b, u_{j+1}, \ldots, u_d) - f(u_1, \ldots, u_{j-1}, a, u_{j+1}, \ldots, u_d)$$

for all $\mathbf{u} \in [0,1]^d$, and all real numbers a and b.

We will also consider the related statistics

$$\widetilde{\mathbb{T}}_n = \max_{B_1, \ldots, B_{L_n}} \sum_{i=1}^{L_n} \left| \mathbb{Z}_n(B_i) \right|, \tag{1.6}$$

with the maximum taken over all disjoint rectangles $B_1, \ldots, B_{L_n}$ of the form $B = \prod_{j=1}^d (a_j, b_j]$ with a_j, b_j belonging to a grid $\{ n^{-1/d}, 2n^{-1/d}, \ldots, \lfloor n^{1/d} \rfloor n^{-1/d} \}$. Asymptotically, $\widetilde{\mathbb{T}}_n$ and $\mathbb{T}_n$ are the same (see Proposition 13 in Section 5), but $\widetilde{\mathbb{T}}_n$ is computationally much more tractable.

Now, if $L_n = L$ for all n, the collection of boxes is sufficiently small that we can still appeal to the weak convergence of $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$ in conjunction with the continuous mapping theorem, to obtain α-upper points of the test statistic $\mathbb{T}_n$ via the bootstrap. Taking $L_n = +\infty$ for all n, or equivalently, if we consider all families of disjoint boxes in $[0, 1]^d$ (possibly partitions), the statistic $\mathbb{T}_n$ is equal to the total variation distance $\mathrm{TV}(\mathbb{Z}_n)$ of $\mathbb{Z}_n$. The resulting test is not statistically meaningful as $\mathrm{TV}(\mathbb{Z}_n)$ is maximal, to wit, $\mathrm{TV}(\mathbb{Z}_n) = n^{1/2} \to +\infty$. The problem is to find a rich collection that quickly detects departure from the null, but still yields a consistent test. The main novelty of our approach is the fact that we let L_n, the number of boxes, slowly tend to ∞ in that $L_n \sim (\log n)^\gamma$, $0 < \gamma < 1$. While in this case the process $\mathbb{Z}_n$ no longer converges, Theorem 1 in Section 2 states that we can still consistently estimate the distribution of the process $\mathbb{Z}_n$ by the bootstrap. We refer to our procedure as the Asymptotic Total Variation (ATV) test. The considered families of boxes are finer and finer, presumably improving the power of the test, while for each n large enough, we still have a consistent test in that we control the type 1 error. A key observation is that under the null hypothesis $\mathrm{H}_0 : C = C_0$, we have $\mathbb{T}_n \le L_n \sup_B |\mathbb{Z}_n(B)| = \mathrm{O}_p(L_n)$, while under the alternative $\mathrm{H}_A : C = C_1$ for some fixed $C_1 \ne C_0$, $\mathbb{T}_n$ is much larger since the bias is at least of order $\mathrm{O}(n^{1/2})$.

Theorem 1 extends the surprising result obtained by Radulović [23] for empirical processes indexed by sums of indicator functions of VC-graph classes (see Theorem 14 in the Appendix). We require very mild conditions on the copula function C. This is one of the few notable exceptions known to us in the literature where the bootstrap "works", that is, the conditional bootstrap distribution consistently estimates the distribution of the test statistic, while the distribution of the statistic itself does not converge. For other instances of this phenomenon, we refer to [4] and, more recently, [22–24].

Section 3 considers the more general hypothesis that the underlying copula C belongs to some parametric copula family $\{C_\theta, \theta \in \Theta \subset \mathbb{R}^p\}$. Given a sufficiently regular estimator $\widehat{\theta}$ and its bootstrap counterpart $\widehat{\theta}^*$, we adjust our statistic (1.4) and its nonparametric bootstrap counterpart to obtain a consistent level α test (Theorem 4). Again, the result is established under very mild regularity conditions on the copula C_θ and the estimators $\widehat{\theta}$ and $\widehat{\theta}^*$. Incidentally, we introduce a new bootstrap procedure under composite null hypotheses, an alternative to the usual parametric bootstrap or the multiplier CLT.

Section 4 then reports a small numerical study where we show that, in complex but realistic situations, our test (1.4) is superior to the Kolmogorov–Smirnov and the Cramér–von Mises tests. We also comment on a possible inadequacy in the way the copula GOF tests are commonly evaluated. Finally, the proofs are collected in Section 5. The Appendix contains some technical results from [27] and [23] and a description of the implementation of the proposed tests.

2. The asymptotic total variation test

Notation. *Let H be the distribution function of the random vector $\mathbf{X}$ with marginals $F_1, \ldots, F_d$. We will assume throughout the paper that H is continuous. Let $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be independent copies of $\mathbf{X}$. We denote the generalized inverse of a distribution function F by F^-. For instance,*

$F_j^-(u) = \inf\{x \mid F_j(x) \geq u\}$. *The empirical counterparts of H and any F_j are, respectively,*

$$\mathbb{H}_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{\mathbf{X}_i \leq \mathbf{x}\}, \qquad \mathbf{x} \in \mathbb{R}^d,$$

$$\mathbb{F}_{n,j}(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{X_{i,j} \leq x\}, \qquad x \in \mathbb{R}, j = 1,\ldots,d.$$

The copula function of $\mathbf{X}$ *is* $C(\mathbf{u}) = H(F_1^-(u_1),\ldots,F_d^-(u_d))$, $\mathbf{u} = (u_1,\ldots,u_d) \in [0,1]^d$, *and its empirical estimate is* $\mathbb{C}_n(\mathbf{u}) = \mathbb{H}_n(\mathbb{F}_{n,1}^-(u_1),\ldots,\mathbb{F}_{n,d}^-(u_d))$. *The empirical copula process* $\mathbb{Z}_n(\mathbf{u}) = \sqrt{n}(\mathbb{C}_n - C)(\mathbf{u})$ *is already defined in* (1.1). *We define $\mathcal{F}_n$ as the class of functions*

$$f(\mathbf{x}) = \sum_{k=1}^{L_n} c_k \mathbf{1}\{\mathbf{x} \in B_k\}, \tag{2.1}$$

with $c_k \in \{-1,+1\}$ *and disjoint boxes B_k of the form* $\prod_{j=1}^{d}(a_j,b_j]$ *in the unit cube* $[0,1]^d$, *for all* $1 \leq k \leq L_n$. *We let*

$$\mathbb{Z}_n(f) = \sum_{k=1}^{L_n} c_k \mathbb{Z}_n(B_k)$$

and observe that

$$\mathbb{T}_n = \sup_{f \in \mathcal{F}_n} \left|\mathbb{Z}_n(f)\right| = \sup_{B_1,\ldots,B_{L_n}} \sum_{k=1}^{L_n} \left|\mathbb{Z}_n(B_k)\right|,$$

where the supremum is taken over all disjoint boxes $B_1,\ldots,B_{L_n}$ of the unit square $[0,1]^d$.

If $L_n = L$ for all n, then $\mathcal{F}_n = \mathcal{F}$ and $\mathbb{Z}_n$ converges in $\ell^\infty(\mathcal{F})$ to a Gaussian process under regularity conditions on C; see, for instance, [11] *and* [27]. *As a consequence of the continuous mapping theorem, $\mathbb{T}_n$ trivially converges weakly as well. However, if $L_n \to \infty$, as $n \to \infty$, this is no longer true as the process $\mathbb{Z}_n$ does not converge weakly.*

The main point of this paper is to show that, provided $L_n = (\log n)^\gamma$ for some $0 < \gamma < 1$, the distribution of $\mathbb{T}_n$ can be estimated by the bootstrap. The bootstrap counterparts of the above processes are defined as follows. Let the bootstrap sample $(\mathbf{X}_1^*,\ldots,\mathbf{X}_n^*)$ be obtained by sampling with replacement from $\mathbf{X}_1,\ldots,\mathbf{X}_n$. We write

$$\mathbb{H}_n^*(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{\mathbf{X}_i^* \leq \mathbf{x}\}, \qquad \mathbf{x} \in \mathbb{R}^d \tag{2.2}$$

for the empirical c.d.f. based on the bootstrap, with marginals

$$\mathbb{F}_{n,j}^*(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{X_{i,j}^* \leq x\}, \qquad x \in \mathbb{R}, j = 1,\ldots,d. \tag{2.3}$$

We denote its associated empirical copula function by $\mathbb{C}_n^*$. The bootstrap empirical copula process is

$$\mathbb{Z}_n^* = \sqrt{n}\big(\mathbb{C}_n^* - \mathbb{C}_n\big) = \sqrt{n}\big\{\mathbb{H}_n^*\big(\mathbb{F}_{n,1}^{*-}, \ldots, \mathbb{F}_{n,d}^{*-}\big) - \mathbb{H}_n\big(\mathbb{F}_{n,1}^{-}, \ldots, \mathbb{F}_{n,d}^{-}\big)\big\}. \qquad (2.4)$$

Assumptions. *We will assume the following set of assumptions*:

(C1) *For any $j = 1, \ldots, d$, for all $\mathbf{u} \in [0, 1]^d$ with $0 < u_j < 1$, the first-order partial derivative $C_j(\mathbf{u}) = \partial C(\mathbf{u})/\partial u_j$ exists and is of bounded variation ([16], e.g.). Moreover, it satisfies, for some $r > 0$, $\beta \geq 0$ and $K < \infty$,*

$$\big|C_j(\mathbf{u}) - C_j(\mathbf{v})\big| \leq K\big(u_j^{-\beta}(1 - u_j)^{-\beta} + v_j^{-\beta}(1 - v_j)^{-\beta}\big) \sum_{l=1}^{d} |u_l - v_l|^r$$

for all $\mathbf{u}, \mathbf{v} \in [0, 1]^d$, $0 < u_j, v_j < 1$. As in [27], we extend the domain of each C_j to the whole $[0, 1]^d$ by setting

$$C_j(\mathbf{u}) := \begin{cases} \displaystyle\limsup_{h \downarrow 0} \frac{C(\mathbf{u} + h\mathbf{e}_j)}{h} & \text{if } \mathbf{u} \in [0, 1]^d, u_j = 0; \\ \displaystyle\limsup_{h \downarrow 0} \frac{C(\mathbf{u}) - C(\mathbf{u} - h\mathbf{e}_j)}{h} & \text{if } \mathbf{u} \in [0, 1]^d, u_j = 1. \end{cases}$$

Here, $\mathbf{e}_j$ is the jth coordinate vector in $\mathbb{R}^d$.

(C2) *The number L_n is of order $(\log n)^\gamma$ for some $0 < \gamma < 1$.*

Remark. We know that continuity of the partial derivatives of C on $(0, 1)^d$ is required for weak convergence; see [11] and [27]. The requirement that the partial derivatives are of bounded variation is natural since we compute the supremum of $\mathbb{Z}_n$ over increasingly finer families of boxes in $[0, 1]^d$. The process $\mathbb{Z}_n(\mathbf{u})$ is asymptotically equivalent to $\alpha_n(\mathbf{u}) - \sum_{j=1}^{d} C_j(\mathbf{u})\alpha_{n,j}(u_j)$ with $\alpha_n(\mathbf{u}) = \sqrt{n}(\mathbb{H}_n - H)(\mathbf{u})$ and $\alpha_{n,j}(u_j) = \sqrt{n}(\mathbb{F}_{n,j} - F)(u_j)$ (see Proposition 10).

Remark. The additional requirement (C1) is weaker than imposing a Hölder condition on the derivatives. Segers [27] imposes a slightly stronger condition on the second-order partial derivatives of C (corresponding to $r = 1$ and $\beta = 1$) to obtain an almost sure representation of the empirical copula process.

As a counterexample, consider the bivariate Archimedean copula C whose generator is given by $\psi : (0, 1] \to \mathbb{R}^+$, $\psi(t) := \exp(t^{-\theta}) - e$ for some $\theta > 0$. This copula (see display (4.2.20) in [21]) is

$$C(u_1, u_2) = \big[\ln\big(\exp\big(u_1^{-\theta}\big) + \exp\big(u_2^{-\theta}\big) - e\big)\big]^{-1/\theta}$$

for any $\mathbf{u} \in [0, 1]^2$. It can be checked easily that, when $u \to 0$, the copula density $C_{12}(u, u)$ behaves like $u^{-\theta-1}$. Therefore, C cannot fulfill Condition 4.1 in [27]. Nonetheless, by the mean value theorem and simple calculations, we can prove that

$$\big|C_1(\mathbf{u}) - C_1(\mathbf{v})\big| \leq K\big(\min(u_1, v_1)\big)^{-2\theta-2}|u_1 - v_1| + K\big(\min(u_1, v_1)\big)^{-\theta-1}|u_2 - v_2|.$$

Moreover, C_1 is of bounded variation: introduce the cross-derivative function $\partial_{12}^2 C_1 : (u_1, u_2) \mapsto \partial^3 C(u_1, u_2)/\partial^2 u_1 \partial u_2$. This function exists on $(0, 1)^2$ and is integrable. Indeed, after lengthy calculations, it can be proved that, for every $\mathbf{u} \in (0, 1)^2$,

$$\left|\partial_{12}^2 C_1(u_1, u_2)\right| \leq K u_1^{-2\theta-2} u_2^{-\theta-1} \min(u_1, u_2)^{3\theta+1} \frac{\exp(2u_1^{-\theta} + u_2^{-\theta})}{(\exp(u_1^{-\theta}) + \exp(u_2^{-\theta}) - e)^3}$$

for some positive constant K. Integration of the latter upper bound with respect to u_2 yields the integrable function $u_1 \mapsto u_1^{\theta-1}$ and Tonelli's theorem implies that the total variation $\int |dC_1| = \int |\partial_{12}^2 C_1(u_1, u_2)| \, du_1 \, du_2$ of C_1 is finite. The same reasoning applies to C_2, and we conclude that condition (C1) is fulfilled with this copula family.

The second assumption (C2) allows for sub-logarithmic rate in the sample size for the number of boxes considered. In practice, even this fairly slow rate yields much better tests, see our simulations in Section 4. And we have not observed any significant differences empirically between choosing $\gamma = 1$ and γ close to one.

Our first result states that the processes $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$ are close in the bounded Lipschitz distance that characterizes weak convergence. Formally, we show that

$$\mathbb{E}\left[\sup_h \left|\mathbb{E}[h(\mathbb{Z}_n)] - \mathbb{E}^*[h(\mathbb{Z}_n^*)]\right|\right] \tag{2.5}$$

is asymptotically negligible. Here, $\mathbb{E}^*$ is the conditional expectation with respect to the bootstrap sample and the supremum in (2.5) is taken over $\mathrm{BL}_1 = \mathrm{BL}_1(\ell^\infty(\mathcal{F}_n))$, the class of all uniformly bounded, Lipschitz functionals $h : \ell^\infty(\mathcal{F}_n) \to \mathbb{R}$ with Lipschitz constant 1, that is,

$$\sup_{x \in \ell^\infty(\mathcal{F}_n)} \left|h(x)\right| \leq 1 \tag{2.6}$$

and, for all $x, y \in \ell^\infty(\mathcal{F}_n)$,

$$\left|h(x) - h(y)\right| \leq \sup_{f \in \mathcal{F}_n} \left|x(f) - y(f)\right|. \tag{2.7}$$

Theorem 1. *Let* $\mathbb{Z}_n = \{\mathbb{Z}_n(f), f \in \mathcal{F}_n\}$ *and* $\mathbb{Z}_n^* = \{\mathbb{Z}_n^*(f), f \in \mathcal{F}_n\}$ *with* $\mathcal{F}_n$ *as defined in* (2.1) *above. Under conditions* (C1) *and* (C2), *we have*

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{h \in \mathrm{BL}_1} \left|\mathbb{E}[h(\mathbb{Z}_n)] - \mathbb{E}^*[h(\mathbb{Z}_n^*)]\right|\right] = 0. \tag{2.8}$$

Corollary 2. *Consider any sequence of Lipschitz functionals* $\phi_n : \ell^\infty(\mathcal{F}_n) \to \mathbb{R}$ *with Lipschitz constants* 1. *Under conditions* (C1) *and* (C2), *we have*

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_g \left|\mathbb{E}[g(\phi_n(\mathbb{Z}_n))] - \mathbb{E}^*[g(\phi_n(\mathbb{Z}_n^*))]\right|\right] = 0.$$

The supremum is taken over all uniformly bounded Lipschitz functions $g : \mathbb{R} \to \mathbb{R}$ with $\sup_x |g(x)| \leq 1$ and $|g(x) - g(y)| \leq |x - y|$.

Corollary 2 follows directly from Theorem 1 since, for a fixed Lipschitz function ϕ, the set of compositions $g \circ \phi$ above is a class of uniformly bounded Lipschitz functions (with the same Lipschitz constant). In particular, since the mapping $\phi_n(X) = \sup_{f \in \mathcal{F}_n} |X(f)|$ is Lipschitz (with the Lipschitz constant 1), Corollary 2 implies that we can approximate the distribution of the statistic $\mathbb{T}_n$ by the conditional (bootstrap) distribution of

$$\mathbb{T}_n^* = \sup_{f \in \mathcal{F}_n} \left| \mathbb{Z}_n^*(f) \right| = \sup_{B_1, \dots, B_{L_n}} \sum_{k=1}^{L_n} \left| \mathbb{Z}_n^*(B_k) \right|. \tag{2.9}$$

Corollary 3. *Under conditions* (C1) *and* (C2), *we have*

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_g \left| \mathbb{E}\left[g(\mathbb{T}_n) \right] - \mathbb{E}^*\left[g\left(\mathbb{T}_n^* \right) \right] \right| \right] = 0. \tag{2.10}$$

The supremum is taken over all uniformly bounded Lipschitz functions $g : \mathbb{R} \to \mathbb{R}$ with $\sup_x |g(x)| \leq 1$ and $|g(x) - g(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

We may replace with impunity in Theorem 1 and its corollaries, the Lipschitz constant 1 by an arbitrary but fixed (independent on n) Lipschitz constant K.

Actually, $\mathbb{T}_n$ and $\widetilde{\mathbb{T}}_n$ are just two examples of many potentially useful asymptotic variation type statistics. We mention two other possible statistics:

- *Generalized χ^2 statistics.* Form an equidistant grid i/p, $i = 0, \dots, p = \lfloor L_n^{1/d} \rfloor + 1$ on each axis of $[0, 1]^d$, and use the $(p+1)^d$ points of the resulting equidistant grid on $[0, 1]^d$ as the corners of p^d disjoint boxes B_i. We define the statistic $\sum_i |\mathbb{Z}_n(B_i)|^2$, which, for fixed L_n, reduces to a nonnormalized χ^2 statistics, in the same spirit as in [8]. Here, since the statistic as a function of $\mathbb{Z}_n$ is Lipschitz on $\ell^\infty(\mathcal{F}_n)$, $L_n \to \infty$ is allowed. However, we suspect that the full power of Theorem 1 is not needed, since Radulović [24] proved a result similar to Theorem 1 via a more direct approach, in the noncopula, i.i.d. setting under a weaker restriction on the partition size.

- *Generalized Kuiper statistics.* We start with the usual Kuiper statistics

$$K_1 = \mathbb{Z}_n(B_1) = \sup_B \left| \mathbb{Z}_n(B) \right|,$$

where supremum is taken over all boxes $B \subseteq [0, 1]^d$, and achieved at B_1. Then we define recursively, given boxes $B_1, \dots, B_m$ with $m < L_n$,

$$K_{m+1} = \mathbb{Z}_n(B_{m+1}) = \sup_{B \cap B_j = \varnothing, j = 1, \dots, m} \left| \mathbb{Z}_n(B) \right|.$$

The supremum is taken over all boxes B that are disjoint with $B_1, \dots, B_m$, and we denote by B_{m+1} for the box at which supremum is achieved. The resulting sum $\sum_{j=1}^{L_n} K_j$ of statis-

tics K_j, based on disjoint boxes B_j, is a Lipschitz functional of $\mathbb{Z}_n$ and Corollary 2 applies to this statistic as well.

The performance and the actual implementation of these additional statistics will not be discussed here, but we will report on them elsewhere. This paper offers a numerical study only as a proof of principle and for this purpose we used the straightforward statistic $\widetilde{\mathbb{T}}_n$ and optimization scheme (pure random search) to demonstrate the applicability of Theorem 1. Nevertheless, even this conservative approach resulted in a superior performance.

Remark. We may approximate the α-upper point of the statistic $\mathbb{T}_n$ by that of the bootstrap counterpart $\mathbb{T}_n^*$. Unlike the classical bootstrap situation that assumes a continuous limiting distribution function, the bootstrap quantile approximation can be used as follows. Let $\varepsilon > 0$ be arbitrary (independent of n) and define the Lipschitz function

$$g_{t,\varepsilon}(x) = \mathbf{1}\{x \le t\} + \frac{t + \varepsilon - x}{\varepsilon}\mathbf{1}\{t < x \le t + \varepsilon\}.$$

We have, for $\delta_n := \sup_h |\mathbb{E}[h(\mathbb{T}_n)] - \mathbb{E}^*[h(\mathbb{T}_n^*)]|$ with the supremum taken over all $h \in \mathrm{BL}_1$, uniformly in $t \in \mathbb{R}$,

$$\mathbb{P}\{\mathbb{T}_n \le t\} = \mathbb{E}^*\big[g_{t,\varepsilon}\big(\mathbb{T}_n^*\big)\big] + \mathbb{E}\big[g_{t,\varepsilon}\big(\mathbb{T}_n\big)\big] - \mathbb{E}^*\big[g_{t,\varepsilon}\big(\mathbb{T}_n^*\big)\big]$$
$$\le \mathbb{P}^*\big\{\mathbb{T}_n^* \le t + \varepsilon\big\} + \delta_n/\varepsilon,$$

since $g_{t,\varepsilon}$ has Lipschitz constant $1/\varepsilon$. Note that this value is different of one, but is not a problem to apply our theoretical results, as explained above. A similar computation shows that $\mathbb{P}^*\{\mathbb{T}_n^* \le t - \varepsilon\} - \delta_n/\varepsilon \le \mathbb{P}\{\mathbb{T}_n \le t\}$, so that, uniformly in t, and each $\varepsilon > 0$

$$\mathbb{P}^*\big\{\mathbb{T}_n^* \le t - \varepsilon\big\} - \delta_n/\varepsilon \le \mathbb{P}\{\mathbb{T}_n \le t\} \le \mathbb{P}^*\big\{\mathbb{T}_n^* \le t + \varepsilon\big\} + \delta_n/\varepsilon \qquad (2.11)$$

and in the same way we may prove

$$\mathbb{P}\{\mathbb{T}_n \le t - \varepsilon\} - \delta_n/\varepsilon \le \mathbb{P}^*\big\{\mathbb{T}_n^* \le t\big\} \le \mathbb{P}\{\mathbb{T}_n \le t + \varepsilon\} + \delta_n/\varepsilon, \qquad (2.12)$$

uniformly in t, and each $\varepsilon > 0$. For instance, if t^* is the bootstrap 95% critical value of $\mathbb{T}_n^*$, it is prudent to reject the null for values of $\mathbb{T}_n$ larger than $t^* + \varepsilon$.

Remark. The test for $\mathrm{H}_0 : C = C_0$ based on the critical regions $\{\mathbb{T}_n > c\}$ is consistent. Indeed, under the null, since $\mathbb{T}_n \le L_n \sup_B |\mathbb{Z}_n(B)|$, we have $L_n^{-1}\mathbb{T}_n$ is bounded in probability, while under the alternative hypothesis, $\mathrm{H}_A : C = C_1$ for a fixed $C_1 \ne C_0$, we have that $\mathbb{T}_n \ge \sqrt{n}|C_0(B) - C_1(B)| - |\mathbb{Z}_n(B)|$, so that $n^{-1/2}\mathbb{T}_n \ge \frac{1}{2}|C_0(B) - C_1(B)|$, with probability tending to one, for any box B where C_0 and C_1 differ. Such a box exists under the alternative and the increasing sequence $\mathcal{F}_n$ likely contains at least one such box for relatively small n. The improved power of our test statistic is confirmed in our simulation study.

3. Parametric hypothesis

In this section, we consider the problem of testing if the underlying copula C belongs to a parametric family $\mathcal{C} := \{C_\theta, \theta \in \Theta\}$. That is, the null hypothesis states that $C = C_{\theta_0}$ for some $\theta_0 \in \Theta$. Here $\Theta \subset \mathbb{R}^p$, equipped with the Euclidean norm $\|\cdot\|_2$. Suppose that we have a consistent estimator $\widehat{\theta} = \widehat{\theta}(\mathbb{H}_n)$ of θ_0.

Replacing C_0 by $C_{\widehat{\theta}}$ in the definition of the test statistic $\mathbb{T}_n$, we consider the process

$$\mathbb{Y}_n = \sqrt{n}(\mathbb{C}_n - C_{\widehat{\theta}}) = \mathbb{Z}_n - \sqrt{n}(C_{\widehat{\theta}} - C), \tag{3.1}$$

and its bootstrap version

$$\mathbb{Y}_n^* = \mathbb{Z}_n^* - \sqrt{n}(C_{\widehat{\theta}^*} - C_{\widehat{\theta}}), \tag{3.2}$$

based on the *nonparametric* bootstrap estimate $\widehat{\theta}^* = \widehat{\theta}(\mathbb{H}_n^*)$, obtained after resampling with replacement from the original sample. Note that

$$\mathbb{Y}_n^* = \sqrt{n}\big(\mathbb{C}_n^* - C_{\widehat{\theta}^*}\big) - \sqrt{n}(\mathbb{C}_n - C_{\widehat{\theta}}) \neq \sqrt{n}\big(\mathbb{C}_n^* - C_{\widehat{\theta}^*}\big). \tag{3.3}$$

The process $\sqrt{n}(\mathbb{C}_n^* - C_{\widehat{\theta}^*})$, while perhaps a natural candidate, does not yield a consistent estimate of the distribution of $\mathbb{Y}_n$. Indeed, the "distance" between $\mathbb{Y}_n$ and the latter process will be of the order of $\mathbb{Z}_n^*$, thus asymptotically tight. On the other hand, the distance between $\mathbb{Y}_n$ and $\mathbb{Y}_n^*$ will be of the same order of magnitude as the distance between $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$, that tends to zero (see the proof of Theorem 1).

We stress that our approach does not involve the *parametric* bootstrap, as studied by Genest and Rémillard [14], to estimate the limiting law of copula-based statistics. In other words, we calculate $\widehat{\theta}^*$ after resampling from the empirical distribution $\mathbb{H}_n$, and not from the law given by the parametric copula $C_{\widehat{\theta}}$.

We impose some regularity on our parameter estimate $\widehat{\theta}$.

(C3) There exists a $\psi : \mathbb{R}^d \mapsto \mathbb{R}^p$ with $\int \|\psi\|_2^4 \, dH < \infty$ such that

$$\widehat{\theta} - \theta_0 = \int \psi \, d(\mathbb{H}_n - H) + \varepsilon_n \quad \text{and} \quad \widehat{\theta}^* - \widehat{\theta} = \int \psi \, d\big(\mathbb{H}_n^* - \mathbb{H}_n\big) + \varepsilon_n^*,$$

under the null hypothesis, with $\|\varepsilon_n\|_2 = o_p(n^{-1/2}/L_n)$ and $\|\varepsilon_n^*\|_2 = o_{p^*}(n^{-1/2}/L_n)$ in probability.

Note that the estimators satisfying (C3) are closely related to the estimators in the class $\mathcal{R}$ of regular estimators, as defined by Genest and Rémillard [14].

Example (Estimators based on the inversion of Kendall's tau). As an example, we verify condition (C3) for estimators based on the inversion of Kendall's tau in the bivariate case ($d = 2$). Let $\theta = g(\tau)$ for some twice differentiable function g and Kendall's $\tau := 4\mathbb{E}[C_\theta(U, V)] - 1$, with the expectation taken over $(U, V) \sim C_\theta$. Kendall's τ is estimated empirically by

$$\widehat{\tau}_n := \frac{4}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{1}\big\{(Y_j - Y_i)(X_j - X_i) > 0\big\} - 1.$$

Then $U_n := \widehat{\tau}_n + 1$ is a U-statistic of order 2 for the kernel

$$h\big((x_1, y_1); (x_2, y_2)\big) = 2 \cdot \mathbf{1}\big\{(y_2 - y_1)(x_2 - x_1) > 0\big\}.$$

The projection of $U_n - \mathbb{E}[U_n]$ onto the space of all statistics of the form $\sum_{i=1}^{n} g_i(X_i, Y_i)$, for arbitrary measurable functions g_i with $\mathbb{E}[g_i^2(X, Y)] < \infty$, is

$$\widehat{U}_n = \sum_{i=1}^{n} \mathbb{E}\big[U_n - \mathbb{E}[U_n] | X_i, Y_i\big] = \frac{2}{n} \sum_{i=1}^{n} \big\{\psi(X_i, Y_i) - \mathbb{E}\big[\psi(X_i, Y_i)\big]\big\}$$

with

$$\psi(x, y) = P(X < x, Y < y) + P(X > x, Y > y).$$

By Hájek's projection principle,

$$\mathrm{Var}(U_n - \widehat{U}_n) = \mathrm{Var}(U_n) - \mathrm{Var}(\widehat{U}_n).$$

From the proof of Theorem 12.3 in [31], due to [17],

$$\mathrm{Var}(U_n) - \mathrm{Var}(\widehat{U}_n) = \frac{4(n-2)}{n(n-1)}\zeta_1 + \frac{2}{n(n-1)}\zeta_2 - \frac{4}{n}\zeta_1 = \frac{2\zeta_2 - 4\zeta_1}{n(n-1)}$$

with $\zeta_1 = \mathrm{Cov}(h(X, Y_1), h(X, Y_2))$ for X independent of Y_1 and Y_2, and with the same distribution as X_1, and $\zeta_2 = \mathrm{Var}(h(X_1, Y_1))$. Thus, the difference is $\mathrm{Var}(U_n) - \mathrm{Var}(\widehat{U}_n)$ is of order $O(1/n^2)$. Consequently, $U_n - \mathbb{E}[U_n] = \widehat{U}_n + R_n$ with $R_n = O_p(1/n)$ so that

$$\widehat{\tau}_n - \tau = U_n - \mathbb{E}[U_n] = \widehat{U}_n + R_n = \frac{2}{n} \sum_{i=1}^{n} \big\{\psi(X_i, Y_i) - \mathbb{E}\big[\psi(X_i, Y_i)\big]\big\} + O_p(1/n).$$

Hence, if g is twice continuously differentiable in the neighborhood of τ, a limited expansion ensures that $\widehat{\theta} := g(\widehat{\tau}_n)$ satisfies the first part of (C3). The second (bootstrap) part of (C3) follows from the same reasoning: We set $\widehat{\tau}_n^* := U_n^* - 1$ with

$$U_n^* = \frac{4}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbf{1}\big\{(Y_j^* - Y_i^*)(X_j^* - X_i^*) > 0\big\}$$

and for

$$\widehat{U}_n^* = \sum_{i=1}^{n} \mathbb{E}^*\big[U_n^* - \mathbb{E}^*[U_n^*] | X_i^*, Y_i^*\big]$$

we can show that

$$\mathrm{Var}^*\big(U_n^* - \widehat{U}_n^*\big) = \mathrm{Var}^*\big(U_n^*\big) - \mathrm{Var}^*\big(\widehat{U}_n^*\big)$$

is of order $O(1/n)$ almost surely, using the same arguments as above, keeping in mind that the empirical counterparts of ζ_1 and ζ_2 are bounded everywhere. Moreover, for

$$\psi_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{X_i < x, Y_i < y\} + \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{X_i > x, Y_i > y\},$$

we find

$$\widehat{U}_n^* = \sum_{i=1}^{n} \mathbb{E}^*\big[U_n^* - \mathbb{E}^*[U_n^*]|X_i^*, Y_i^*\big]$$

$$= \frac{2}{n} \sum_{i=1}^{n} \psi_n(X_i^*, Y_i^*) - \mathbb{E}^*\big[\psi_n(X_i^*, Y_i^*)\big]$$

$$= \frac{2}{n} \sum_{i=1}^{n} \{\psi(X_i^*, Y_i^*) - \mathbb{E}^*[\psi(X_i^*, Y_i^*)]\}$$

$$+ \frac{2}{n} \sum_{i=1}^{n} \{(\psi_n - \psi)(X_i^*, Y_i^*) - \mathbb{E}^*[(\psi_n - \psi)(X_i^*, Y_i^*)]\}.$$

The second term on the right is of order $O_{p^*}(1/n)$ as its variance equals

$$\frac{4}{n} \operatorname{Var}^*\big((\psi_n - \psi)(X_1^*, Y_1^*)\big) \le \frac{4}{n} \sum_{i=1}^{n} (\psi_n - \psi)^2 (X_i, Y_i) = O_{p^*}(1/n^2),$$

by the reasoning in [3], page 1202. This implies

$$\widehat{\tau}_n^* - \widehat{\tau}_n = U_n^* - \mathbb{E}^*[U_n^*] = \frac{2}{n} \sum_{i=1}^{n} \{\psi(X_i^*, Y_i^*) - \mathbb{E}^*[\psi(X_i^*, Y_i^*)]\} + O_{p^*}(1/n).$$

Again, for a g that is twice continuously differentiable in the neighborhood of τ, a limited expansion ensures that $\widehat{\theta}^*$ satisfies the second part of (C3).

Moreover, we need more regularity concerning $\theta \mapsto C_\theta$ itself.

(C4) For every $(s, t) \in [0, 1]^d$, the function $\theta \mapsto C_\theta(\mathbf{u})$ has continuous partial derivatives $\dot{C}_\theta(\mathbf{u}) = (\partial/\partial\theta)C_\theta(\mathbf{u})$ that satisfy a Hölder condition with Hölder exponent $\nu > 0$ locally: there exists a constant $K < \infty$ such that

$$\sup_{\mathbf{u}} \big\|\dot{C}_\theta(\mathbf{u}) - \dot{C}_{\theta_0}(\mathbf{u})\big\|_2 \le K \|\theta - \theta_0\|_2^\nu$$

for every θ in a neighborhood of θ_0. Moreover, $\dot{C}_{\theta_0}$ is of bounded variation.

The regularity condition (C4) is satisfied for most of standard copula families. Simple calculations show that it is the case for the Gaussian-, Clayton- and the Frank-copula families in particular. Although copula partial derivatives with respect to their arguments often exhibit discontinuities or nonexistence near their boundaries, justifying conditions such as (C1) (see [27]), the derivatives $\partial C_\theta(x, y)/\partial \theta$ with respect to the copula parameter θ behave a lot more regularly.

Theorem 4. *Let* $\mathbb{Y}_n = \{\mathbb{Y}_n(f), f \in \mathcal{F}_n\}$ *and* $\mathbb{Y}_n^* = \{\mathbb{Y}_n^*(f), f \in \mathcal{F}_n\}$ *with* $\mathcal{F}_n$ *in* (2.1) *as defined above. Assume that conditions* (C1), (C2), (C3) *and* (C4) *hold. Then, under the null hypothesis* $\mathrm{H}_0 : C = C_\theta, \theta \in \Theta,$ *we have*

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{h \in \mathrm{BL}_1} \left| \mathbb{E}\big[h(\mathbb{Y}_n)\big] - \mathbb{E}^*\big[h\big(\mathbb{Y}_n^*\big)\big] \right| \right] = 0. \tag{3.4}$$

This result implies that the distribution of the test statistic

$$\widehat{\mathbb{T}}_n = \sup_{f \in \mathcal{F}_n} \left| \mathbb{Y}_n(f) \right| = \sup_{B_1, \dots, B_{L_n}} \sum_{k=1}^{L_n} \left| \mathbb{Y}_n(B_k) \right| \tag{3.5}$$

can be "bootstrapped" by the distribution of

$$\widehat{\mathbb{T}}_n^* = \sup_{f \in \mathcal{F}_n} \left| \mathbb{Y}_n^*(f) \right| = \sup_{B_1, \dots, B_{L_n}} \sum_{k=1}^{L_n} \left| \mathbb{Y}_n^*(B_k) \right|. \tag{3.6}$$

Corollary 5. *Assume that conditions* (C1), (C2), (C3) *and* (C4) *hold. Then, under the null hypothesis* $\mathrm{H}_0 : C = C_\theta, \theta \in \Theta,$

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_g \left| \mathbb{E}\big[g(\widehat{\mathbb{T}}_n)\big] - \mathbb{E}^*\big[g\big(\widehat{\mathbb{T}}_n^*\big)\big] \right| \right] = 0, \tag{3.7}$$

with the supremum taken over all Lipschitz functions $g : \mathbb{R} \to [-1, 1]$ *with Lipschitz constant* 1.

Often, (C3) can be replaced by

(C3′) There exists a $\psi : \mathbb{R}^d \mapsto \mathbb{R}^p$ with $\int \|\psi\|_2^4 \, dC < \infty$ such that

$$\widehat{\theta} - \theta_0 = \frac{1}{n} \sum_{i=1}^{n} \left\{ \psi\big(\mathbb{F}_{n,1}(X_{i,1}), \dots, \mathbb{F}_{n,d}(X_{i,d})\big) - \mathbb{E}\big[\psi\big(F_1(X_{i,1}), \dots, F_d(X_{i,d})\big)\big] \right\} + \varepsilon_n,$$

$$\widehat{\theta}^* - \widehat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \psi\big(\mathbb{F}_{n,1}^*\big(X_{i,1}^*\big), \dots, \mathbb{F}_{n,d}^*\big(X_{i,d}^*\big)\big) - \psi\big(\mathbb{F}_{n,1}(X_{i,1}), \dots, \mathbb{F}_{n,d}(X_{i,d})\big) \right\} + \varepsilon_n^*,$$

under the null hypothesis, with $\|\varepsilon_n\|_2 = \mathrm{o}_p(n^{-1/2}/L_n)$ and $\|\varepsilon_n^*\|_2 = \mathrm{o}_{p^*}(n^{-1/2}/L_n)$ in probability.

This is a consequence of the following result.

Proposition 6. *Assume* (C1) *holds. Any estimator* $\widehat{\theta}$ *satisfying* (C3′), *satisfies* (C3).

Copula parameters are typically estimated through pseudo-observations or ranks, without any assumption on the marginal distributions. For this reason, the copula estimators that satisfy (C3′) are relevant. They are very closely related to the estimators in the class $\mathcal{R}_1$ of Genest and Rémillard [14]. In particular, the maximum pseudo-likelihood estimator that maximizes the pseudo log-likelihood function $\int \log c_\theta \, \mathrm{d}\mathbb{C}_n$ over $\theta \in \Theta$ (see, for instance, [13] or [28]) satisfies (C3′) under suitable regularity conditions on the copula density c_θ.

Since the bootstrapped copula process $\mathbb{Y}_n^*$ is new, it is noteworthy to stress that it provides a valuable alternative to the usual parametric bootstrap. Now, assume $L_n = L$ is a constant, to retrieve the standard framework.

Corollary 7. *Assume that conditions* (C1), (C3) *and* (C4) *hold. Then the process* $\{\mathbb{Y}_n(\mathbf{u}), \mathbf{u} \in [0, 1]^d\}$ *tends weakly toward a Gaussian process in* $\ell^\infty([0, 1]^d)$. *Moreover, the bootstrapped process* $\{\mathbb{Y}_n^*(\mathbf{u}), \mathbf{u} \in [0, 1]^d\}$ *converges weakly to the same Gaussian process in probability in* $\ell^\infty([0, 1]^d)$.

4. Applications and numerical studies

We present a limited numerical study, serving as a proof of principle rather than the final word on this subject. The evaluation of GOF tests in copula settings is a complex problem and only partial answers can be found in literature; see the surveys of Berg [2], Genest *et al.* [15] and, more recently, Fermanian [10]. Here, we restrict ourselves to the bivariate case. A full-scale numerical analysis is beyond the scope of this paper.

We have implemented $\widetilde{\mathbb{T}}_n$, a computationally simpler version of $\mathbb{T}_n$; see Appendix C for the algorithm. In the case of a composite null hypothesis, we have implemented a simplified version of $\widehat{\mathbb{T}}_n$ in the same way, by restricting the boxes B to be of the form $B = \prod_{i=1}^d (a_i, b_i]$ with $a_i, b_i \in \{n^{-1/d}, 2n^{-1/d}, \ldots\} \subset [0, 1]$. Since the distance between $\mathbb{T}_n$ and $\widetilde{\mathbb{T}}_n$ tends to zero in probability (as a result of Lemma 9 and Proposition 10 in Section 5), the weak convergence results are valid with $\widetilde{\mathbb{T}}_n$ instead of $\mathbb{T}_n$ or $\widehat{\mathbb{T}}_n$. Moreover, the reasoning to approximate p-values by bootstrap still applies.

4.1. Heuristics

For two copula densities c_0 and c_1, we define the *difference* sets A^+ and A^- as

$$A^+ = \big\{(s, t) : c_0(s, t) > c_1(s, t)\big\} \quad \text{and} \quad A^- = \big\{(s, t) : c_0(s, t) < c_1(s, t)\big\}.$$

The proposed test statistics are designed to sample L_n boxes in order to maximize the difference between the "true" and postulated copulas. In situations where the geometry of the difference sets A^+ and A^- is complex, statistics such as $\widetilde{\mathbb{T}}_n$ can "pick out" disjoint subregions of A^+ and A^-, and one could expect superior performance consequently. However, sometimes just a single well
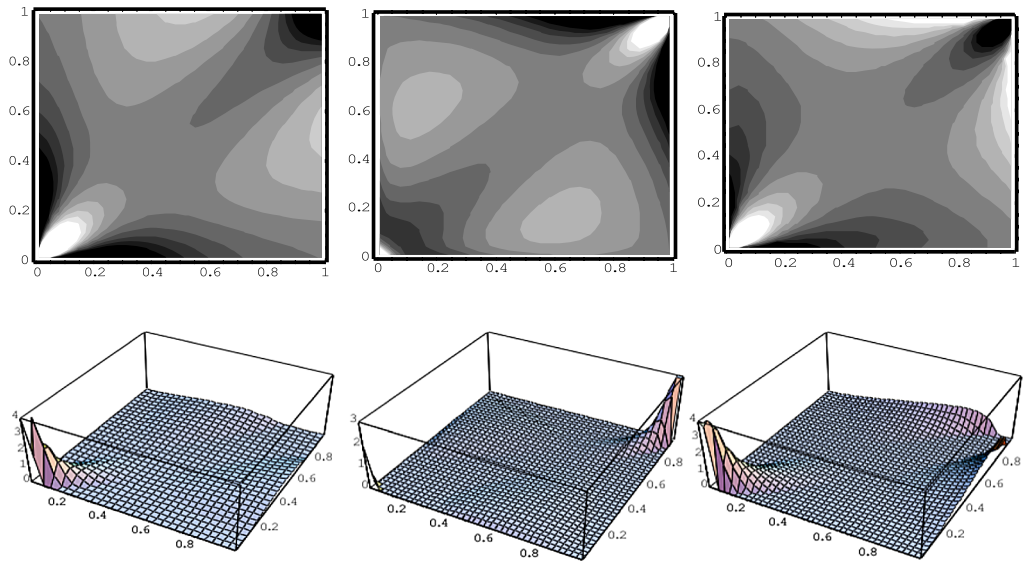
Figure 1. *Common comparisons*. Copula density differences, through contour plots and 3D plots of synthetic data: Clayton − Frank (left), Gumbel − Frank (center), Clayton − Gumbel (right). Their Kendall's tau is 0.4.

placed box can pick essentially all the mass of sets A^+ or A^-, while the remaining $L_n - 1$ boxes are just collecting noise and consequently diminish the power of the statistic $\mathbb{T}_n$.

Most common scenarios encountered in the literature compare Frank, Clayton, Gumbel, and Gauss copulas with each other, after controlling for some dependence indicator (typically Kendall's tau): see, for instance, [2,14] and [15]. However, all these pairings produce trivial difference sets A^+ and A^-, as revealed in the contour plots and 3D plots of $c_0 - c_1$ of Figure 1. We see that nearly all the mass difference between copula densities c_0 and c_1 is concentrated in a single spot, located in either the lower left or upper right corner. Here Kendall's $\tau = 0.4$, but we observed similar plots for different values of τ. Therefore, these common simulation scenarios are tailored toward many standard GOF tests such as KS and CvM tests. We are not aware of any argument that justifies such specific types of pairing, except for analytical tractability. Figures 2 and 3, however, paint a very different scenario with more elaborate difference sets A^+ and A^- that appear in real life situations. How often and to what extent this complex situation is encountered in reality is largely an open empirical issue.

In this study, the copula densities c_1 were estimated by kernel density estimators based on the following data:

- The bivariate *ARCH*-like process $(X_1, Y_1), \ldots, (X_n, Y_n)$, with $n = 10^6$, was generated as follows: First, we created independent $Z_i \sim N(0, 1)$ and $W_i = Z_i(1 + 0.6W_{i-1}^2)^{1/2}$, with $W_0 = 0$. Second, we set $(X_i, Y_i) := (W_{100i}, W_{100i+1})$, creating nearly independent couples (of strongly dependent observations). Such models are commonly used in empirical finance, for instance.
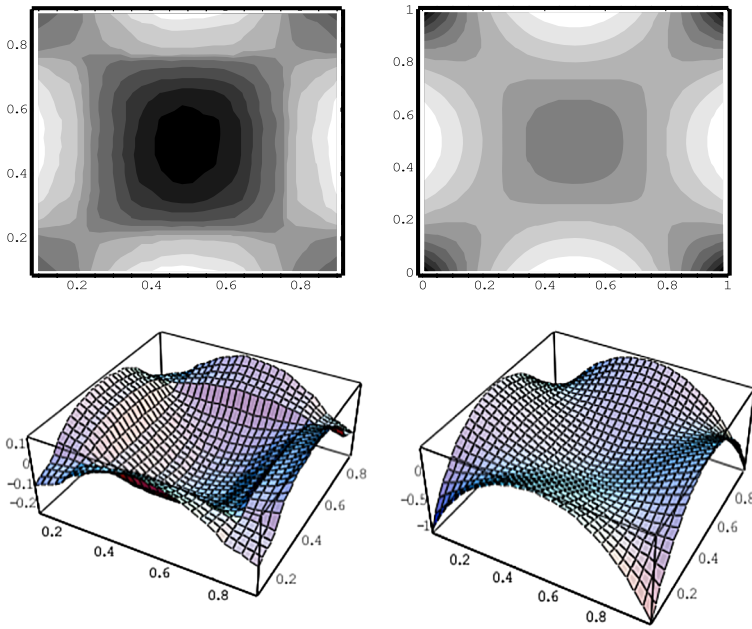
Figure 2. *Complex relation (synthetic data).* Copula density differences, through contour plots and 3D plots: ARCH (left) and mixture copula (right), compared to the independence copula.

- The *Mixture Copula* data $(X_1, Y_1), \ldots, (X_n, Y_n)$, with $n = 10^6$, are generated from the mixture $c_1(s,t) = \frac{1}{2}c_F(s,t) + \frac{1}{2}c_F(1-s,t)$ for the Frank copula c_F with Kendall's $\tau = 0.4$. Therefore, this copula has asymmetrical features, contrary to most copulas that are tested in the literature. Obviously, other asymmetrical copulas could be built, following [19] for instance.
- The *Euro–Dollar* data $(X_1, Y_1), \ldots, (X_n, Y_n)$, with $n = 1800$, are quoted currency exchange values. X is the daily percentage change of the Euro against the US dollar, while Y corresponds to the daily change of the Canadian dollar against the US dollar.
- The *Silver–Gold* data $(X_1, Y_1), \ldots, (X_n, Y_n)$, with $n = 5000$, presents the log ratio of the average daily price of silver and gold futures, respectively. For instance, $X_i = \log(S_{i+1}/S_i)$ based on the average price S_i of silver in US dollars on day i.

We compared *Mixture copula* and *ARCH* with the independence copula, for which $c_0(s,t) = 1$ (see Figure 2). In the case of real data (*Euro–Dollar* and *Silver–Gold*), we chose the Frank copula density with parameters $\tau = 2.6$ and $\tau = 3.4$, respectively, for c_0 (see Figure 3). The latter parameters were chosen after minimizing the (estimated) L_1-distance between c_0 and c_1. The difference sets are easily depicted by dark and bright sections of the contour plots, and the 3D plots clearly indicate that the mass difference between copula densities c_0 and c_1 is not concentrated in a single spot.
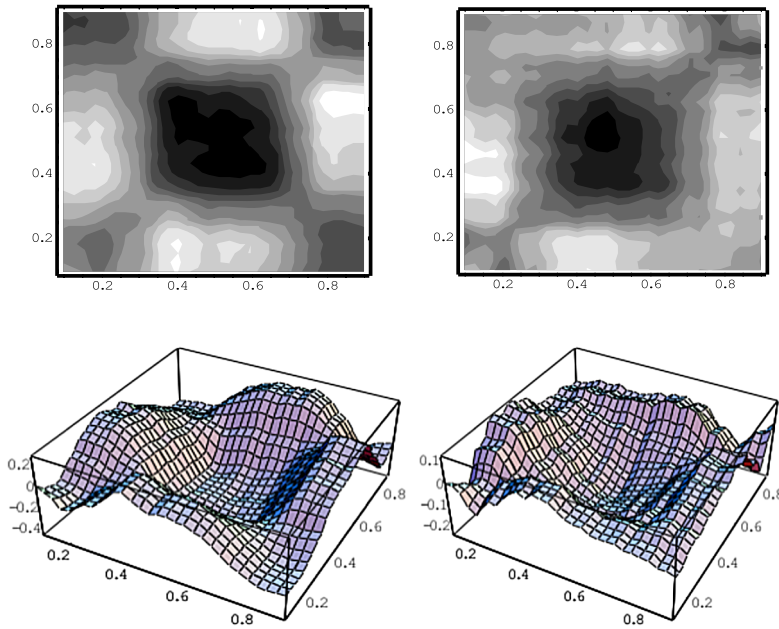
Figure 3. *Complex relation* (*actual data*). Copula density differences, through contour plots and 3D plots: Euro − Dollar (left) and Silver − Gold (right), compared to Frank copulas (with Kendall's tau equal to 2.6 and 3.4, resp.).

4.2. GOF tests in practice

We generated the data sets *ARCH* and *Mixture Copula* as described above. For each data set, we run two sets of simulations:

- (ARCH-S and Mixture-S) Test the simple null hypothesis $C_0(s, t) = st$ using the methodology of Section 2.
- (ARCH-C and Mixture-C) Test the composite null hypothesis that C_0 is a Frank copula using the procedure described in Section 3.

In both cases, the null hypothesis is wrong and should be rejected.

In our simulations, the number of boxes is $L_n = \lfloor \ln^{0.95}(n) \rfloor - 2$. We approximated the p-values of all the statistics we consider via the bootstrap procedures introduced in Sections 2 and 3. For each approximation, we used 1000 bootstrap samples. For the second set of simulations (ARCH-C and Mixture-C), we computed the parameters $\widehat{\theta}$ and $\widehat{\theta}^*$ by the usual pseudo-maximum likelihood procedure. Each procedure is repeated 100 times. We report the percentage of times that the computed p-value is below $\alpha = 0.05$.

Our limited numerical study confirms the above assessment. Table 1 shows that the ATV test outperforms largely the KS and CvM tests in the case of complex pairing, while Table 2 confirms that the ATV test is inferior in case of the commonly used pairings of Figure 1.

Table 1. *Complex pairing*, related to Figure 2: relative frequencies of rejected null hypotheses under $\alpha = 0.05$

Type	n	ARCH-S	ARCH-C	Mixture-S	Mixture-C
ATV	400	**75%**	**80%**	**41%**	**25%**
KS	400	6%	4%	8%	12%
CvM	400	25%	50%	6%	15%
ATV	800	**100%**	**99%**	**94%**	**98%**
KS	800	32%	50%	20%	25%
CvM	800	50%	92%	31%	84%

In Table 2, for each pair of copulas, say Clayton–Frank, we generated n observations from the first copula (Clayton), and we tested the null hypothesis that the second copula (Frank) is the true underlying copula. In this simple scenario, the sophistication of $\mathbb{T}_n$ is a disadvantage compared to simpler usual test statistics. The former test looks for discrepancies everywhere in the unit hypercube (at the price of noise), while the simpler KS and CvM tests pick up easily the right boxes (by chance, in our opinion).

Table 3 shows that the significance level of the ATV test is below 0.05. The data were simulated from the null hypothesis. In all tables, Kendall's $\tau = 0.4$.

5. Proofs

Throughout the proofs, we assume without loss of generality that $F_j = I$ for every $j = 1, \ldots, d$ (uniform marginal distributions). This implies that $H = C$. This is justified by the following lemma.

Lemma 8. *Let F_j, $j = 1, \ldots, d$ be continuous distribution functions. Denote by $\widetilde{H}$ the c.d.f. of $(F_1(X_1), \ldots, F_d(X_d))$ and by $\widetilde{C}$ its associated copula. The empirical copula associated to the sample $(F_1(X_{i1}), \ldots, F_d(X_{id}))$, $i = 1, \ldots, n$, is denoted by $\widetilde{\mathbb{C}}_n$. We have*

$$C(\mathbf{u}) = \widetilde{C}(\mathbf{u}) = \widetilde{H}(\mathbf{u}) \qquad \text{for all } \mathbf{u} \in [0, 1]^d.$$

Table 2. *Trivial pairing*, related to Figure 1: relative frequencies of rejected null hypotheses under $\alpha = 0.05$

Type	n	Clayton $-$ Frank	Gumbel $-$ Frank	Clayton $-$ Gumbel
ATV	400	42%	26%	88%
KS	400	58%	25%	90%
CvM	400	**84%**	**47%**	**95%**
ATV	800	92%	58%	94%
KS	800	98%	53%	98%
CvM	800	**100%**	**73%**	**100%**

Table 3. *Errors of the first kind*: relative frequencies of rejected null hypotheses under $\alpha = 0.05$

Type	n	Clayton − Clayton	Gumbel − Gumbel	Frank − Frank
ATV	400	3%	2%	2%
KS	400	4%	5%	4%
CvM	400	4%	5%	4%
ATV	800	2%	4%	3%
KS	800	3%	3%	5%
CvM	800	5%	3%	6%

Moreover,

$$\mathbb{C}_n\left(\frac{i_1}{n}, \dots, \frac{i_d}{n}\right) = \widetilde{\mathbb{C}}_n\left(\frac{i_1}{n}, \dots, \frac{i_d}{n}\right) \qquad \text{for } i_1, \dots, i_d \in \{0, 1, \dots, n\}.$$

Proof. This is a straightforward extension of Lemma 1 in [11]. $\qquad\qquad\square$

Since the letter C is reserved for the copula function, we use the letters K, K_0, K_1, etc. in the sequel to denote generic constants, and we write $\|\mathbf{s}\|_\infty = \max_{1 \le j \le d} |s_j|$ of $\mathbf{s} = (s_1, \dots, s_d) \in [0, 1]^d$.

5.1. Proof of preliminary results

In general, note that, for each $f \in \mathcal{F}_n$ defined in (2.1), we can write

$$\mathbb{Z}_n(f) = \sum_{k=1}^{L_n} c_k \mathbb{Z}_n(B_k) = \sum_{l=1}^{2^d L_n} \sigma_l \mathbb{Z}_n(\mathbf{s}_l)$$

and

$$\mathbb{Z}_n^*(f) = \sum_{l=1}^{2^d L_n} \sigma_l \mathbb{Z}_n^*(\mathbf{s}_l)$$

for some $\sigma_l \in \{-1, +1\}$ and $\mathbf{s}_l \in [0, 1]^d$, using formula (1.5). Let $\alpha_n(\mathbf{u}) := \sqrt{n}(\mathbb{H}_n - H)(\mathbf{u}) = \sqrt{n}(\mathbb{H}_n(\mathbf{u}) - \mathbf{u})$ be the ordinary uniform empirical process in $[0, 1]^d$, and let its oscillation modulus be defined as

$$\mathbb{M}_n(\delta) := \sup\left\{\left|\alpha_n(\mathbf{s}) - \alpha_n(\mathbf{s}')\right| : \left\|\mathbf{s} - \mathbf{s}'\right\|_\infty \le \delta; \mathbf{s}, \mathbf{s}' \in [0, 1]^d\right\} \tag{5.1}$$

for any $\delta > 0$.

Lemma 9. *Let $(\delta_n)_{n \geq 0}$ be a sequence of positive real numbers such that $n\delta_n / \log n \to \infty$. Then we have*

$$\mathbb{M}_n(\delta_n) = \mathrm{O}\big(\delta_n^{1/2}(\log n)^{1/2}\big) \qquad \text{almost surely.}$$

Proof. We apply Proposition 15 with $\lambda_n = K_0 \delta_n^{1/2}(\log n)^{1/2}$ for some constant $K_0 > 0$. Since $n^{-1/2}\lambda_n / \delta_n = K_0 (\log n / (n\delta_n))^{1/2}$ tends to zero, this inequality can be rewritten

$$\mathbb{P}\big\{\mathbb{M}_n(\delta_n) > \lambda_n\big\} \leq \frac{K_1}{\delta_n} \exp\left(-\frac{K_2 \psi(1)\lambda_n^2}{\delta_n}\right) = K_1 n \exp\big(-K_2 K_0^2 \psi(1) \log n\big)$$

for some constants K_1, K_2 and n sufficiently large. When K_0 is sufficiently large, we check that

$$\mathbb{P}\big\{\mathbb{M}_n(\delta_n) > \lambda_n\big\} \leq \frac{K_3}{n^2}$$

for some constant K_3. Invoke the Borel–Cantelli lemma to conclude the proof. $\qquad\square$

In addition, let $\alpha_{n,j}(u) = \sqrt{n}(\mathbb{F}_{n,j} - F_j)(u) = \sqrt{n}(\mathbb{F}_{n,j}(u) - u)$ be the ordinary uniform (marginal) empirical process in $[0, 1]$, and we define

$$\widetilde{\mathbb{Z}}_n(\mathbf{s}) = \alpha_n(\mathbf{s}) - \sum_{j=1}^{d} C_j(\mathbf{s})\alpha_{n,j}(s_j). \tag{5.2}$$

Proposition 10. *Under conditions* (C1) *and* (C2), *we have*

$$\lim_{n \to \infty} \sup_{h \in \mathrm{BL}_1} \big|\mathbb{E}\big[h(\mathbb{Z}_n)\big] - \mathbb{E}\big[h(\widetilde{\mathbb{Z}}_n)\big]\big| = 0.$$

Proof. First, we observe that

$$\sup_{h \in \mathrm{BL}_1} \big|\mathbb{E}\big[h(\mathbb{Z}_n) - h(\widetilde{\mathbb{Z}}_n)\big]\big| \leq \delta + 2\mathbb{P}\Big\{ \sup_{f \in \mathcal{F}_n} \big|\mathbb{Z}_n(f) - \widetilde{\mathbb{Z}}_n(f)\big| > \delta\Big\}.$$

The latter inequality holds for any $\delta > 0$, and uses the fact that $|h|$ is bounded by 1 and has Lipschitz constant 1. It remains to show that

$$\sup_{f \in \mathcal{F}_n} \big|\mathbb{Z}_n(f) - \widetilde{\mathbb{Z}}_n(f)\big| \to 0, \tag{5.3}$$

in probability, as $n \to \infty$. The remainder of the proof generalizes Proposition 4.2 of [27]. Now, we note that

$$\sup_{f \in \mathcal{F}_n} \big|\mathbb{Z}_n(f) - \widetilde{\mathbb{Z}}_n(f)\big| \leq 2^d L_n \sup_{\mathbf{s} \in [0,1]^d} \big|\mathbb{Z}_n(\mathbf{s}) - \widetilde{\mathbb{Z}}_n(\mathbf{s})\big| \leq 2^d L_n(I + II)$$

with

$$I = \sup_{\mathbf{s} \in [0,1]^d} \left| \alpha_n \big(\mathbb{F}_{n,1}^- s_1, \ldots, \mathbb{F}_{n,d}^- s_d \big) - \alpha_n(\mathbf{s}) \right|,$$

$$II = \sup_{\mathbf{s} \in [0,1]^d} \left| \sqrt{n} \big[C \big(\mathbb{F}_{n,1}^- s_1, \ldots, \mathbb{F}_{n,d}^- s_d \big) - C(\mathbf{s}) \big] + \sum_{j=1}^d C_j(\mathbf{s}) \alpha_{n,j}(s_j) \right|.$$

The first term, I, can be bounded as follows. Set $\beta_{n,j}(s) = \sqrt{n}(\mathbb{F}_{n,j}^- s - s)$, $j = 1, \ldots, d$. By the Chung–Smirnov LIL, we have

$$\max_{1 \leq j \leq d} \sup_{0 \leq s \leq 1} \left| \beta_{n,j}(s) \right| = \mathrm{O}\big((\log \log n)^{1/2} \big) \qquad \text{almost surely.}$$

Using Lemma 9 with $\delta = n^{-1/2} (\log \log n)^{1/2}$, we get

$$\sup_{\|\mathbf{s} - \mathbf{s}'\|_\infty < \delta} \left| \alpha_n(\mathbf{s}) - \alpha_n(\mathbf{s}') \right| = \mathrm{O}\big(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/4} \big),$$

almost surely. This implies that $I = \mathrm{O}(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/4})$, almost surely.

For the second term, we get by the mean value theorem that

$$II = \sup_{\mathbf{s} \in [0,1]^d} \left| \sqrt{n} \big[C \big(\mathbb{F}_{n,1}^- s_1, \ldots, \mathbb{F}_{n,d}^- s_d \big) - C(\mathbf{s}) \big] + \sum_{j=1}^d C_j(\mathbf{s}) \alpha_{n,j}(s_j) \right|$$

$$\leq \sup_{\mathbf{s} \in [0,1]^d} \left| \sum_{j=1}^d C_j(\mathbf{s}_n) \beta_{nj}(s_j) + \sum_{j=1}^d C_j(\mathbf{s}) \alpha_{n,j}(s_j) \right|,$$

where $\mathbf{s}_n$ is a vector in $[0,1]^d$ s.t. $\|\mathbf{s}_n - \mathbf{s}\|_\infty \leq n^{-1/2} \max_{1 \leq j \leq d} |\beta_{n,j}(s_j)|$. Since $|C_j| \leq 1$ for every $j = 1, \ldots, d$ (because copulas are Lipschitz with Lipschitz constant 1), we deduce

$$II \leq \sup_{\mathbf{s} \in [0,1]^d} \sum_{j=1}^d \left| \beta_{nj}(s_j) + \alpha_{n,j}(s_j) \right| + \sup_{\mathbf{s} \in [0,1]^d} \sum_{j=1}^d \left| \big[C_j(\mathbf{s}_n) - C_j(\mathbf{s}) \big] \alpha_{n,j}(s_j) \right|$$

$$\leq IIa + IIb.$$

The Bahadur–Kiefer theorem ([29], page 585) states that

$$\max_{1 \leq j \leq d} \sup_{0 \leq s \leq 1} \left| \beta_{n,j}(s) + \alpha_{n,j}(s_j) \right| = \mathrm{O}\big(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/4} \big) \qquad \text{almost surely.}$$

Then $IIa = \mathrm{O}(n^{-1/4} (\log n)^{1/2} (\log \log n)^{1/4})$ almost surely.

Concerning IIb, we consider a positive sequence (ε_n), $\varepsilon_n \to 0$, that will be specified later independently of any $\mathbf{s} = (s_1, \ldots, s_d) \in [0,1]^d$. For any index $j = 1, \ldots, d$ and any $\mathbf{s} \in [0,1]^d$, we will distinguish the two cases: $s_j \in [\varepsilon_n, 1 - \varepsilon_n]$ and the opposite.

If $s_j \in [\varepsilon_n, 1 - \varepsilon_n]$ then

$$s_{nj} = s_j \left(1 + \frac{s_{nj} - s_j}{s_j} \right) \geq s_j \left(1 - \frac{|s_{nj} - s_j|}{\varepsilon_n} \right) \geq \frac{s_j}{2}$$

and

$$1 - s_{nj} \geq (1 - s_j) \left(1 - \frac{|s_{nj} - s_j|}{\varepsilon_n} \right) \geq \frac{1 - s_j}{2},$$

almost surely and for n sufficiently large, for all $\varepsilon_n \to 0$ and $n\varepsilon_n^2 / \log n \to \infty$. Corollary 2 in [20] implies that

$$\max_{1 \leq j \leq d} \sup_{0 \leq s_j \leq 1} \left| s_j^{-1/2} (1 - s_j)^{-1/2} \alpha_{n,j}(s_j) \right| \leq K (\log n)^{1/2} \log \log n,$$

almost surely, for some constant $K > 0$.

In this case, using condition (C1), we deduce

$$\left| C_j(\mathbf{s}_n) - C_j(\mathbf{s}) \right| \left| \alpha_{n,j}(s_j) \right| \leq K_0 \|\mathbf{s}_n - \mathbf{s}\|^r \left\{ s_j^{-\beta}(1 - s_j)^{-\beta} + s_{nj}^{-\beta}(1 - s_{nj})^{-\beta} \right\} \left| \alpha_{n,j}(s_j) \right|$$

$$\leq K_1 \|\mathbf{s}_n - \mathbf{s}\|^r s_j^{1/2 - \beta}(1 - s_j)^{1/2 - \beta} (\log n)^{1/2} \log \log n$$

$$\leq K_2 n^{-r/2} (\log \log n)^{r/2} \max\left(\varepsilon_n^{1/2 - \beta}, 1 \right) (\log n)^{1/2} \log \log n,$$

almost surely, for some constants $K_0, K_1, K_2 > 0$ and every j.

If $s_j \notin [\varepsilon_n, 1 - \varepsilon_n]$, then

$$\left| C_j(\mathbf{s}_n) - C_j(\mathbf{s}) \right| \left| \alpha_{n,j}(s_j) \right| \leq 2 \left| \alpha_{n,j}(s_j) \right|$$

$$\leq 2 \varepsilon_n^{1/2} s_j^{-1/2} (1 - s_j)^{-1/2} \left| \alpha_{n,j}(s_j) \right|$$

$$\leq K \varepsilon_n^{1/2} (\log n)^{1/2} \log \log n \qquad \text{almost surely},$$

see Corollary 2 in [20].

Combining all these bounds entails then

$$IIb \leq K_3 \left[n^{-r/2} (\log \log n)^{r/2} \max\left(\varepsilon_n^{1/2 - \beta}, 1 \right) + \varepsilon_n^{1/2} \right] (\log n)^{1/2} \log \log n,$$

with $K_3 > 0$. We now specify the choice of $\varepsilon_n = n^{-p}$, with p depending on β and r only. If $2\beta > 2r + 1$, we take $0 < p < r/(2\beta - 1)$. If $\beta < 1/2$, set $p = 1/4$. Otherwise, take $p = \min(1/4, r/(4\beta - 2))$, for instance. In each case, these choices ensure that $IIb = \mathrm{O}(n^{-q})$ almost surely, for some $q > 0$.

Since $L_n = \mathrm{O}((\log n)^\gamma)$ by assumption (C2), we obtain $L_n(I + II) \to 0$ almost surely, as $n \to \infty$, and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we turn our attention to the bootstrap counterparts. We define $\alpha_n^*(\mathbf{s}) = \sqrt{n}(\mathbb{H}_n^* - \mathbb{H}_n)(\mathbf{s})$ as the ordinary bootstrap empirical process in $[0, 1]^d$. We prove the following exponential inequality for the oscillation modulus

$$\mathbb{M}_n^*(\delta) = \sup_{\|\mathbf{s}-\mathbf{s}'\|_\infty < \delta} \left| \alpha_n^*(\mathbf{s}) - \alpha_n^*(\mathbf{s}') \right|.$$

Lemma 11. *For all bounded sequences δ_n such that $n\delta_n/\log(n) \to \infty$ as $n \to \infty$,*

$$\mathbb{M}_n^*(\delta_n) = \mathrm{O}\left(\delta_n^{1/2}(\log n)^{1/2}\right) \qquad \textit{almost surely.} \tag{5.4}$$

Note that the sequence (δ_n) may be constant.

Proof of Lemma 11. Since α_n^* is a step function, we find that

$$\sup_{\|\mathbf{s}-\mathbf{s}'\|_\infty < \delta_n} \left| \alpha_n^*(\mathbf{s}) - \alpha_n^*(\mathbf{s}') \right| = \max \left| \alpha_n^*(X_{i_1,1}, \ldots, X_{i_d,d}) - \alpha_n^*(X_{i'_1,1}, \ldots, X_{i'_d,d}) \right|,$$

with the maximum taken over all $|X_{i_j,j} - X_{i'_j,j}| < \delta_n$, $j = 1, \ldots, d$, $i_1, i'_1, \ldots, i_d, i'_d \in \{1, \ldots, n\}$. For any $\mathbf{i} := (i_1, \ldots, i_d)$ and $\mathbf{i}' = (i'_1, \ldots, i'_d)$ in $\{1, \ldots, n\}^d$, we rewrite

$$\left| \alpha_n^*(X_{i_1,1}, \ldots, X_{i_d,d}) - \alpha_n^*(X_{i'_1,1}, \ldots, X_{i'_d,d}) \right| = n^{-1/2} \sum_{k=1}^n \left\{ V_{k,\mathbf{i},\mathbf{i}'} - \mathbb{E}^*[V_{k,\mathbf{i},\mathbf{i}'}] \right\},$$

as a sum of bounded independent random variables with

$$V_{k,\mathbf{i},\mathbf{i}'} := \mathbf{1}\left\{ X_{k,j}^* \le X_{i_j,j}, j = 1, \ldots, d \right\} - \mathbf{1}\left\{ X_{k,j}^* \le X_{i'_j,j}, j = 1, \ldots, d \right\},$$

conditionally on the sample $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$. Moreover, a simple calculation and Lemma 9 yield

$$\mathrm{Var}^*(V_{k,\mathbf{i},\mathbf{i}'}) \le \sum_{j=1}^d \mathbb{P}^* \left\{ \min(X_{i_j,j}, X_{i'_j,j}) \le X_{k,j}^* \le \max(X_{i_j,j}, X_{i'_j,j}) \right\}$$

$$\le \sum_{j=1}^d \sup_{s_j} \left[\mathbb{F}_{n,j}(s_j + \delta_n) - \mathbb{F}_{n,j}(s_j) \right]$$

$$\le d\delta_n + dn^{-1/2}\mathbb{M}_n(\delta_n)$$

$$\le d \max\left(\delta_n, \mathbb{M}_n(\delta_n)/\sqrt{n} \right)$$

$$\le K \max(\delta_n, \sqrt{\delta_n \log n}/\sqrt{n}) = K\delta_n$$

for n large enough, for almost all realizations and for some constant $K > 0$. Hence, by the union bound and Bernstein's exponential inequality for bounded random variables, we have, for some

constant K_0,

$$\mathbb{P}^*\left\{ \max_{\substack{\mathbf{i},\mathbf{i}'\in\{1,\dots,n\}^d \\ |X_{i_j,j}-X_{i'_j,j}|<\delta_n,\forall j}} \left| \alpha_n^*(X_{i_1,1},\dots,X_{i_d,d}) - \alpha_n^*(X_{i'_1,1},\dots,X_{i'_d,d}) \right| > x \right\}$$

$$\leq 2n^{2d}\exp\left(-K_0\left(\sqrt{n}x \wedge x^2\delta_n^{-1}\right)\right)$$

for all samples $(\mathbf{X}_1,\dots,\mathbf{X}_n)$. By integrating the previous inequality over $\mathbb{P}$, we get the same inequality, but replacing $\mathbb{P}^*$ by $\mathbb{P}$. Set $x = K_1\delta_n^{1/2}(\log n)^{1/2}$ and take a constant K_1 sufficiently large to obtain

$$\sum_{n=1}^{+\infty}\mathbb{P}\left\{\mathbb{M}_n^*(\delta_n) > K_1\delta_n^{1/2}(\log n)^{1/2}\right\} < +\infty.$$

Apply the Borel–Cantelli lemma to conclude the proof. □

Analogous to the approximation of the process $\mathbb{Z}_n$ by $\widetilde{\mathbb{Z}}_n$ before, we introduce a simpler process $\widetilde{\mathbb{Z}}_n^*$ to approximate $\mathbb{Z}_n^*$. Set

$$\widetilde{\mathbb{Z}}_n^*(\mathbf{s}) = \sqrt{n}\big(\mathbb{H}_n^* - \mathbb{H}_n\big)(\mathbf{s}) - \sum_{j=1}^{d} C_j(\mathbf{s})\sqrt{n}\big(\mathbb{F}_{n,j}^* - \mathbb{F}_{n,j}\big)(s_j). \tag{5.5}$$

Proposition 12. *Under conditions* (C1) *and* (C2), *we have*

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{h\in\mathrm{BL}_1}\big|\mathbb{E}^*\big[h\big(\mathbb{Z}_n^*\big) - h\big(\widetilde{\mathbb{Z}}_n^*\big)\big]\big|\Big] = 0.$$

Proof. First, we notice that, for any $\eta > 0$,

$$\mathbb{E}\Big[\sup_{h\in\mathrm{BL}_1}\big|\mathbb{E}^*\big[h\big(\mathbb{Z}_n^*\big) - h\big(\widetilde{\mathbb{Z}}_n^*\big)\big]\big|\Big] \leq \eta + 2\mathbb{E}\Big[\mathbb{P}^*\Big\{\sup_{f\in\mathcal{F}_n}\big|\mathbb{Z}_n^*(f) - \widetilde{\mathbb{Z}}_n^*(f)\big| \geq \eta\Big\}\Big]$$

$$\leq \eta + 2\mathbb{E}\Big[\mathbb{P}^*\Big\{\sup_{\mathbf{s}} 2^d L_n\big|\mathbb{Z}_n^*(\mathbf{s}) - \widetilde{\mathbb{Z}}_n^*(\mathbf{s})\big| \geq \eta\Big\}\Big].$$

Some straightforward adding and subtracting yields $\mathbb{Z}_n^*(\mathbf{s}) = \overline{\mathbb{Z}}_n^*(\mathbf{s}) + R_n^*(\mathbf{s})$ with

$$\overline{\mathbb{Z}}_n^*(\mathbf{s}) = \sqrt{n}\big\{\mathbb{H}_n^*(\mathbf{s}) - \mathbb{H}_n(\mathbf{s})\big\} - \sqrt{n}\big\{C\big(\mathbb{F}_{n,1}^* s_1,\dots,\mathbb{F}_{n,d}^* s_d\big) - C\big(\mathbb{F}_{n,1} s_1,\dots,\mathbb{F}_{n,d} s_d\big)\big\}$$

and $R_n^*(\mathbf{s}) = R_{n,1}^*(\mathbf{s}) + R_{n,2}^*(\mathbf{s}) + R_{n,3}^*(\mathbf{s}) + R_{n,4}^*(\mathbf{s})$ with

$$R_{n,1}^*(\mathbf{s}) = \alpha_n^*\big(\mathbb{F}_{n,1}^{*-} s_1,\dots,\mathbb{F}_{n,d}^{*-} s_d\big) - \alpha_n^*\big(\mathbb{F}_{n,1}^{-} s_1,\dots,\mathbb{F}_{n,d}^{-} s_d\big),$$

$$R_{n,2}^*(\mathbf{s}) = \alpha_n^*\big(\mathbb{F}_{n,1}^{-} s_1,\dots,\mathbb{F}_{n,d}^{-} s_d\big) - \alpha_n^*(\mathbf{s}),$$

$$R_{n,3}^*(\mathbf{s}) = \alpha_n\big(\mathbb{F}_{n,1}^{*-} s_1,\dots,\mathbb{F}_{n,d}^{*-} s_d\big) - \alpha_n\big(\mathbb{F}_{n,1}^{-} s_1,\dots,\mathbb{F}_{n,d}^{-} s_d\big),$$

$$R_{n,4}^*(\mathbf{s}) = \sqrt{n}\Big\{C\big(\mathbb{F}_{n,1}^{*-}s_1,\ldots,\mathbb{F}_{n,d}^{*-}s_d\big) - C\big(\mathbb{F}_{n,1}^{-}s_1,\ldots,\mathbb{F}_{n,d}^{-}s_d\big)\Big\}$$
$$+ \sqrt{n}\Big\{C\big(\mathbb{F}_{n,1}^{*}s_1,\ldots,\mathbb{F}_{n,d}^{*}s_d\big) - C\big(\mathbb{F}_{n,1}s_1,\ldots,\mathbb{F}_{n,d}s_d\big)\Big\}.$$

Let $\alpha_{n,j}^*(s) = \sqrt{n}(\mathbb{F}_{n,j}^* - \mathbb{F}_{n,j})(s)$ and $\beta_{n,j}^*(s) = \sqrt{n}(\mathbb{F}_{n,j}^{-*} - \mathbb{F}_{n,j}^{-})(s)$ be the bootstrap versions of the empirical processes $\alpha_{n,j}(s)$ and $\beta_{n,j}(s)$, respectively. Both converge to the same weak limit as

$$\sup_{0\le s_j\le 1} \big|\beta_{n,j}^*(s_j) + \alpha_{n,j}^*(s_j)\big| = \mathrm{O}\big(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}\big) \qquad \text{almost surely,}$$

see displays $(2.10')$ and $(2.12')$ in Theorem 2.1 of [6]. It remains to show that

$$\mathbb{P}^*\Big\{L_n \sup_{\mathbf{s}}\big|R_n^*(\mathbf{s})\big| > \eta\Big\} \to 0 \qquad \text{for all } \eta > 0,$$

conditionally given all sequences $(\mathbf{X}_1,\ldots,\mathbf{X}_n) \in \Omega_n$ for some sequence of events $\Omega_n \subset \mathbb{R}^{d\times n}$ with $\lim_{n\to\infty}\mathbb{P}(\Omega_n) = 1$.

Let $\delta_n = n^{-1/4}$. (Other choices are possible as well.) We have

$$\limsup_{n\to\infty}\mathbb{P}^*\big\{L_n\big\|R_{n,1}^*\big\|_\infty \ge \eta\big\} \le \limsup_{n\to\infty}\mathbb{P}^*\big\{L_n\mathbb{M}_n^*(\delta_n) \ge \eta\big\}$$
$$+ \limsup_{n\to\infty}\mathbb{P}^*\Big\{\max_j\big\|\beta_{n,j}^*\big\|_\infty \ge \sqrt{n}\delta_n\Big\} = 0,$$

by Lemma 11. Next, on the event $\max_j \|\beta_{n,j}\|_\infty \le \sqrt{n}\delta_n$ (that holds almost surely by the law of iterated logarithm),

$$\limsup_{n\to\infty}\mathbb{P}^*\big\{L_n\big\|R_{n,2}^*\big\|_\infty \ge \eta\big\} \le \limsup_{n\to\infty}\mathbb{P}^*\big\{L_n\mathbb{M}_n^*(\delta_n) \ge \eta\big\} = 0,$$

by Lemma 11. On the event $L_n\mathbb{M}_n(\delta_n) < \eta$ (that holds almost surely by Lemma 9), we have

$$\limsup_{n\to\infty}\mathbb{P}^*\big\{L_n\big\|R_{n,3}^*\big\|_\infty \ge \eta\big\} \le \limsup_{n\to\infty}\mathbb{P}^*\Big\{\max_j\big\|\beta_{n,j}^*\big\|_\infty > \sqrt{n}\delta_n\Big\} = 0,$$

by the weak convergence of $\beta_{n,j}^*$. Finally, for some s_j^* between $\mathbb{F}_{n,j}^{-*}(s_j)$ and $\mathbb{F}_{n,j}^{-}(s_j)$, and s_j^{**} between $\mathbb{F}_{n,j}^*(s_j)$ and $\mathbb{F}_{n,j}(s_j)$, we have

$$\big|R_{n,4}^*(\mathbf{s})\big| = \left|\sum_{j=1}^d C_j\big(s_j^*\big)\beta_{n,j}^*(s_j) + C_j\big(s_j^{**}\big)\alpha_{n,j}^*(s_j)\right|$$
$$\le \sum_{j=1}^d\big|\beta_{n,j}^*(s_j) + \alpha_{n,j}^*(s_j)\big| + \sum_{j=1}^d\big|\alpha_{n,j}^*(s_j)\big|\big|C_j\big(s_j^*\big) - C_j\big(s_j^{**}\big)\big|.$$

The first term is of order $\mathrm{O}(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4})$, uniformly in s_j. For the second term, we argue as in the proof of Proposition 10. First, we observe that $|s_j^{**} - s_j^*| \le |s_j^* - s_j| + |s_j^{**} - s_j|$

is of order $O_{p*}(n^{-1/2})$. Second, since the class $\mathbf{1}\{x \le t\}t^{-b}(1-t)^{-b}$ is a P-Donsker class for the uniform probability measure P on $[0,1]$, for all $0 \le b < 1/2$, see [32], Example 2.11.15 (page 214), the weak convergence of the bootstrap empirical process ([32], Theorem 3.6.1, page 347) implies that

$$\sup_{0<s<1} \left|\alpha_{n,j}^*(s)\right|/\left(s^b(1-s)^b\right) = O_{p*}(1).$$

Consequently, as in the proof of Proposition 10, we find that, for some constant $K < \infty$,

$$\sup_{\varepsilon_n \le s_j \le 1-\varepsilon_n} \left|\alpha_{n,j}^*(s_j)\right|\left|C_j\left(s_j^*\right) - C_j\left(s_j^{**}\right)\right|$$

$$\le K\left|s_j^{**} - s_j^*\right|^r s_j^{b-\beta}(1-s_j)^{b-\beta} \sup_{s_j}\left|\alpha_{n,j}^*(s_j)\right|/\left(s^b(1-s)^b\right)$$

which is of order $O_{p*}(1) \cdot \max(n^{-r/2}\max(1,\varepsilon_n^{b-\beta})$. On the other side,

$$\sup_{s_j \notin [\varepsilon,1-\varepsilon_n]} \left|\alpha_{n,j}^*(s_j)\right|\left|C_j\left(s_j^*\right) - C_j\left(s_j^{**}\right)\right| \le 2 \sup_{s_j \notin [\varepsilon,1-\varepsilon_n]} \left|\alpha_{n,j}^*(s_j)\right|$$

$$\le 2\varepsilon_n^b \sup_{s_j}\left|\alpha_{n,j}^*(s_j)\right|/\left(s^b(1-s)^b\right),$$

which is of order $O_{p*}(\varepsilon_n^b)$. Combining both bounds yields $\sup_{\mathbf{s}}|R_{n,4}(\mathbf{s})| = O_{p*}(\varepsilon_n^b + n^{-r/2} \times \max(1,\varepsilon_n^{b-\beta}))$. Taking $\varepsilon_n = n^{-p}$ with p depending on b, β and r, we get that $\lim_{n\to\infty} \mathbb{P}^*\{L_n \times \sup_{\mathbf{s}}|R_{n,4}^*(\mathbf{s})| \ge \eta\} = 0$ for all $\eta > 0$, conditionally on all sequences $(\mathbf{X}_1,\ldots,\mathbf{X}_n) \in \Omega_n$ for some sequence of events Ω_n with $\lim_n \mathbb{P}(\Omega_n) = 1$. This completes our proof. $\square$

Proposition 13. *Under conditions* (C1) *and* (C2), *we have* $|\widetilde{\mathbb{T}}_n - \mathbb{T}_n| = o_p(1)$.

Proof. From (5.3), it follows that

$$|\widetilde{\mathbb{T}}_n - \mathbb{T}_n| \le \left|\sup_{B_j} \sum_{j=1}^{L_n}\left|\widetilde{\mathbb{Z}}_n(B_j)\right| - \sup_{B_j'} \sum_{j=1}^{L_n}\left|\widetilde{\mathbb{Z}}_n\left(B_j'\right)\right|\right| + o_p(1). \tag{5.6}$$

The first supremum is taken over all disjoint boxes $B_1,\ldots,B_{L_n}$ of the form $B_j = \prod_{k=1}^d(a_{j,k}, b_{j,k}] \subset [0,1]^d$, while the second supremum is taken over all disjoint boxes $B_1',\ldots,B_{L_n}'$ of the form $B_j' = \prod_{k=1}^d(a_{j,k}', b_{j,k}']$, with $a_{j,k}', b_{j,k}'$ are chosen on the grid $\mathcal{I}_n = \{n^{-1/d}, 2n^{-1/d},\ldots, \lfloor n^{1/d}\rfloor n^{-1/d}\}$. Obviously,

$$\sup_{B_j} \sum_{j=1}^{L_n}\left|\widetilde{\mathbb{Z}}_n(B_j)\right| \ge \sup_{B_j'} \sum_{j=1}^{L_n}\left|\widetilde{\mathbb{Z}}_n\left(B_j'\right)\right|.$$

Conversely, for each set of disjoint boxes $B_j = \prod_{k=1}^d(a_{j,k}, b_{j,k}]$, $j = 1,\ldots,L_n$, we can construct an "approximate" set of disjoint boxes $B_j' = \prod_{k=1}^d(a_{j,k}', b_{j,k}']$ with $a_{j,k}', b_{j,k}' \in \mathcal{I}_n$, $j = 1,\ldots,L_n$,

by taking B'_j as the largest box of the latter form but inside B_j. If such a B'_j does not exist, that is, for some index k, we have $|b_{j,k} - a_{j,k}| < n^{-1/d}$ – we set $B'_j = \varnothing$. Recall $\widetilde{\mathbb{Z}}_n$ in (5.2), the fact that $\|C_j\|_\infty \leq 1$ for all $1 \leq j \leq d$, and Lemma 9, to prove that

$$\left|\widetilde{\mathbb{Z}}_n(B_j) - \widetilde{\mathbb{Z}}_n\left(B'_j\right)\right| \leq Kn^{-1/(2d)}(\log n)^{1/2},$$

almost surely, for some finite constant K. Consequently, we have a.e.

$$\sup_{B_j}\sum_{j=1}^{L_n}\left|\widetilde{\mathbb{Z}}_n(B_j)\right| \leq \sup_{B'_j}\sum_{j=1}^{L_n}\left|\widetilde{\mathbb{Z}}_n\left(B'_j\right)\right| + KL_n n^{-1/(2d)}(\log n)^{-1/2}$$

$$= \sup_{B'_j}\sum_{j=1}^{L_n}\left|\widetilde{\mathbb{Z}}_n\left(B'_j\right)\right| + \mathrm{o}_p(1).$$

Here, $\sup_{B_j}$ and $\sup_{B'_j}$ are taken as in (5.6) above. This proves the result. $\qquad\square$

5.2. Proof of Theorem 1

By triangle inequality, we have

$$\mathbb{E}\left[\sup_{h\in\mathrm{BL}_1}\left|\mathbb{E}\big[h(\mathbb{Z}_n)\big] - \mathbb{E}^*\big[h\big(\mathbb{Z}_n^*\big)\big]\right|\right] \leq \sup_{h\in\mathrm{BL}_1}\left|\mathbb{E}\big[h(\mathbb{Z}_n) - h(\widetilde{\mathbb{Z}}_n)\big]\right|$$

$$+ \mathbb{E}\left[\sup_{h\in\mathrm{BL}_1}\left|\mathbb{E}\big[h(\widetilde{\mathbb{Z}}_n)\big] - \mathbb{E}^*\big[h\big(\widetilde{\mathbb{Z}}_n^*\big)\big]\right|\right]$$

$$+ \mathbb{E}\left[\sup_{h\in\mathrm{BL}_1}\left|\mathbb{E}^*\big[h\big(\widetilde{\mathbb{Z}}_n^*\big)\big] - h\big(\mathbb{Z}_n^*\big)\big]\right|\right].$$

In view of Proposition 10 and Proposition 12, it remains to show that the second term on the right is asymptotically negligible. We recall that

$$\widetilde{\mathbb{Z}}_n(f) = \sum_{k=1}^{2^d L_n}\sigma_k\widetilde{\mathbb{Z}}_n(\mathbf{s}_k) = \sum_{k=1}^{2^d L_n}\sigma_k\int f_k(\mathbf{x})\,\mathrm{d}\alpha_n(\mathbf{x})$$

for

$$f_k(\mathbf{x}) = \mathbf{1}\{\mathbf{x} \leq \mathbf{s}_k\} - \sum_{j=1}^{d}C_j(\mathbf{s}_k)\mathbf{1}\{x_j \leq s_{k,j}\}.$$

Now, let $h_f(\mathbf{x}) = \sum_{k=1}^{2^d L_n}\sigma_k f_k(\mathbf{x})$ so that

$$\widetilde{\mathbb{Z}}_n(f) = \int h_f\,\mathrm{d}\alpha_n, \tag{5.7}$$

and we can derive in the same way

$$\widetilde{\mathbb{Z}}_n^*(f) = \int h_f \, d\alpha_n^*. \tag{5.8}$$

We now apply Theorem 3 in [23], stated as Theorem 14 in the Appendix for convenience. We need to verify that

- the $d + 1$ classes

$$\mathcal{G}_k^a = \left\{ \mathbf{1}\{\mathbf{x} \le \mathbf{s}_k\}, \mathbf{s}_k \in [0, 1]^d \right\},$$

$$\mathcal{G}_k^{(j)} = \left\{ C_j(\mathbf{s}_k) \mathbf{1}\{x \le s_{k,j}\}, \mathbf{s}_k \in [0, 1]^d \right\}, \qquad j = 1, \ldots, d,$$

  have VC-indices V_k^a and $V_k^{(j)}$, respectively, with $\sum_{k=1}^{2^d L_n} (V_k^a + \sum_{j=1}^d V_k^{(j)}) \le K (\log n)^\gamma$ for some finite constant K and some $0 < \gamma < 1$;
- the class $\mathcal{H}_n = \{h_f : f \in \mathcal{F}_n\}$ has an envelope $H(\mathbf{x})$ with $\mathbb{E}[H^4(\mathbf{X})] < \infty$.

First, we verify the VC property. The class $\mathcal{G}_k^a$ is VC with VC-dimension $V_k^a = d + 1$ ([32], page 135), while the class $\mathcal{G}_k^{(j)}$ is a subclass of the class of functions $c\mathbf{1}\{a \le x \le b\}$ with $a, b \in \mathbb{R}$ and $c > 0$. This class has a VC index 3: see [32], Problem 20, page 153. Consequently,

$$\sum_{k=1}^{2^d L_n} \left(V_k^a + \sum_{j=1}^d V_k^{(j)} \right) \le (4d + 1) 2^d L_n \le K (\log n)^\gamma$$

for some $K < \infty$.

It remains to verify the envelope condition. We will show that $h_f(\mathbf{x})$ has envelope $1 + d + \sum_{j=1}^d \mathrm{TV}(C_j)$. Writing

$$g_{\mathbf{x}}(\mathbf{s}) = \mathbf{1}\{\mathbf{x} \le \mathbf{s}\} - \sum_{j=1}^d C_j(\mathbf{s}) \mathbf{1}\{x_j \le s_j\},$$

we see that

$$h_f(\mathbf{x}) = \sum_{k=1}^{L_n} c_k g_{\mathbf{x}}(B_k)$$

for $c_k = \pm 1$ and the operation $\phi(B_k)$ defined in (1.5) for any function $\phi : \mathbb{R}^d \to \mathbb{R}$. Furthermore, writing

$$\gamma_{\mathbf{x}}(\mathbf{s}) = \mathbf{1}\{\mathbf{x} \le \mathbf{s}\}, \qquad \zeta_x^{(j)}(\mathbf{s}) = C_j(\mathbf{s}) \mathbf{1}\{x \le s_j\}, \qquad j = 1, \ldots, d,$$

we have

$$\left| h_f(\mathbf{x}) \right| \le \sum_{k=1}^{L_n} \left| \gamma_{\mathbf{x}}(B_k) \right| + \sum_{j=1}^d \sum_{k=1}^{L_n} \left| \zeta_{x_j}^{(j)}(B_k) \right| \le 1 + \sum_{j=1}^d \sum_{k=1}^{L_n} \left| \zeta_{x_j}^{(j)}(B_k) \right|$$

since the boxes $B_1, \ldots, B_{L_n}$ are disjoint. Since each B_k is of the form $\prod_{j=1}^{d}(s_{k,j}^1, s_{k,j}^2]$, there is a (fine enough) lattice partition Π of $[0,1]^d$ with the property that each B_k can be written as a union of (disjoint) elements A_{k_j}, with $A_{k_j} \in \Pi$. A little reflexion shows that, for each $1 \leq j \leq d$,

$$\sum_{k=1}^{L_n} \left| \zeta_{x_j}^{(j)}(B_k) \right| \leq \sum_{A \in \Pi} \left| \zeta_{x_j}^{(j)}(A) \right|$$

and, moreover, for $A_m = \prod_{j=1}^{d}(s_{m,j}^1, s_{m,j}^2] \in \Pi$, $A_{m,-j} = \prod_{l \neq j}(s_{m,l}^1, s_{m,l}^2]$ and

$$C_j(\mathbf{s}_{-j}|t) := C_j(s_1, \ldots, s_{j-1}, t, s_{j+1}, \ldots, s_d)$$

for every $\mathbf{s}_{-j} \in [0,1]^{d-1}$ and every $t \in [0,1]$, a little algebra gives the identity

$$\zeta_{x_j}^{(j)}(A_m) = \mathbf{1}\{x_j \leq s_{m,j}^2\} C_j(A_m) + \mathbf{1}\{s_{m,j}^1 < x_j \leq s_{m,j}^2\} C_j\left(A_{m,-j}|s_{m,j}^1\right).$$

Since

$$C_j(\mathbf{s}_{-j}|s_j) = \mathbb{P}\{\mathbf{X}_{-j} \leq \mathbf{s}_{-j}|X_j = s_j\},$$

we obtain

$$\sum_{k=1}^{L_n} \left| \zeta_x^{(j)}(B_k) \right| \leq \sum_{A_m \in \Pi} \left| \zeta_x^{(j)}(A_m) \right|$$

$$\leq \sum_{A_m \in \Pi} \left| C_j(A_m) \right| + \sum_{A_m \in \Pi} \mathbf{1}\{s_{m,j}^1 < x_j \leq s_{m,j}^2\} \mathbb{P}\{\mathbf{X}_{-j} \in A_{m,-j}|X_j = s_{m,j}^1\}$$

$$\leq \mathrm{TV}(C_j) + \sum_{A_m \in \Pi} \mathbf{1}\{s_{m,j}^1 < x_j \leq s_{m,j}^2\} \mathbb{P}\{\mathbf{X}_{-j} \in A_{m,-j}|X_j = s_{m,j}^1\}.$$

Let $A_{m(\mathbf{x})} \in \Pi$ with $\mathbf{x} \in A_{m(\mathbf{x})}$ and $s_j^1 < x \leq s_j^2$ with $(s_j^1, s_j^2]$ be the projection of $A_{m(\mathbf{x})}$ on the jth axis of the lattice. Then the last term on the right of the previous display can be bounded as follows:

$$\sum_{A_m \in \Pi} \mathbf{1}\{s_{m,j}^1 < x_j \leq s_{m,j}^2\} \mathbb{P}\{\mathbf{X}_{-j} \in A_{m,-j}|X_j = s_{m,j}^1\}$$

$$\leq \sum_{A_m \in \Pi, s_{m,j}^1 = s_j^1, s_{m,j}^2 = s_j^2} \mathbb{P}\{\mathbf{X}_{-j} \in A_{m,-j}|X_j = s_j^1\}$$

$$\leq 1$$

since the boxes $A_m \in \Pi$ and, therefore, $A_{m,-j}$ are disjoint. We have shown that the class $\mathcal{H}_n$ has envelope $1 + d + \sum_{j=1}^{d} \mathrm{TV}(C_j)$.

We can now apply Theorem 14 to conclude that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{h\in BL_1} \left|\mathbb{E}\left[h(\widetilde{\mathbb{Z}}_n)\right] - \mathbb{E}^*\left[h(\widetilde{\mathbb{Z}}_n^*)\right]\right|\right] = 0,$$

and the proof is complete.

5.3. Proof of Theorem 4

We proceed as in the proof of Theorem 1. We write $\widehat{C} = C_{\widehat{\theta}}$ and $\widehat{C}^* = C_{\widehat{\theta}^*}$. Recall that

$$\mathbb{Y}_n = \mathbb{Z}_n - \sqrt{n}(\widehat{C} - C).$$

We may replace $\mathbb{Z}_n$ by $\widetilde{\mathbb{Z}}_n$ with impunity since

$$\sup_{h\in BL_1} \left|\mathbb{E}\left[h(\mathbb{Y}_n) - h\left(\widetilde{\mathbb{Z}}_n - \sqrt{n}(\widehat{C} - C)\right)\right]\right|$$

$$\leq \delta + 2\mathbb{P}\left\{\sup_{f\in\mathcal{F}_n} \left|\mathbb{Y}_n(f) - \widetilde{\mathbb{Z}}_n(f) + \sqrt{n}(\widehat{C} - C)(f)\right| \geq \delta\right\}$$

$$= \delta + 2\mathbb{P}\left\{\sup_{f\in\mathcal{F}_n} \left|\mathbb{Z}_n(f) - \widetilde{\mathbb{Z}}_n(f)\right| \geq \delta\right\}$$

$$\to \delta \qquad \text{as } n \to \infty$$

for every $\delta > 0$, as in the proof of Proposition 10. Next, by the mean value theorem and assumptions (C3) and (C4), we have

$$\sqrt{n}(\widehat{C} - C)(\mathbf{s}) = \sqrt{n}(\widehat{\theta} - \theta_0)'\dot{C}_{\theta_0}(\mathbf{s}) + \sqrt{n}(\widehat{\theta} - \theta_0)'\left\{\dot{C}_{\widetilde{\theta}}(\mathbf{s}) - \dot{C}_{\theta_0}(\mathbf{s})\right\}$$

$$\text{for some } \widetilde{\theta} \text{ between } \widehat{\theta} \text{ and } \theta_0$$

$$= \left(\int \psi \, d\alpha_n + n^{1/2}\varepsilon_n\right)' \dot{C}_{\theta_0}(\mathbf{s}) + \sqrt{n}(\widehat{\theta} - \theta_0)'\left\{\dot{C}_{\widetilde{\theta}}(\mathbf{s}) - \dot{C}_{\theta_0}(\mathbf{s})\right\}$$

$$= \left(\int \psi \, d\alpha_n\right)' \dot{C}_{\theta_0}(\mathbf{s}) + R_n(\mathbf{s})$$

for some remainder term R_n that satisfies

$$\left|R_n(\mathbf{s})\right| \leq n^{1/2}\|\varepsilon_n\|_2 \|\dot{C}_{\theta_0}(\mathbf{s})\|_2 + Kn^{1/2}\|\widehat{\theta} - \theta_0\|_2^{1+\nu}$$

$$= O_p\left(n^{1/2}\|\varepsilon_n\|_2 + n^{-\nu/2}\right)$$

$$= o_p(1/L_n).$$

This bound holds uniformly in $\mathbf{s}$. Consequently, for

$$\widetilde{\mathbb{Y}}_n(f) = \sum_{k=1}^{2^d L} \sigma_k \widetilde{\mathbb{Y}}_n(\mathbf{s}_k)$$

based on

$$\widetilde{\mathbb{Y}}_n(\mathbf{s}) = \widetilde{\mathbb{Z}}_n(\mathbf{s}) - \left(\int \psi \, d\alpha_n \right)' \dot{C}_{\theta_0}(\mathbf{s}),$$

we have

$$\sup_{h \in \mathrm{BL}_1} \left| \mathbb{E}\big[h\big(\widetilde{\mathbb{Z}}_n - \sqrt{n}(\widehat{C} - C) \big) \big] - \mathbb{E}\big[h(\widetilde{\mathbb{Y}}_n) \big] \right| = \sup_{h \in \mathrm{BL}_1} \left| \mathbb{E}\big[h(\widetilde{\mathbb{Y}}_n - R_n) \big] - \mathbb{E}\big[h(\widetilde{\mathbb{Y}}_n) \big] \right|.$$

Since

$$\sup_f \left| R_n(f) \right| \le 2^d L_n \sup_{\mathbf{s}} \left| R_n(\mathbf{s}) \right| \to 0$$

in probability, we get $\sup_h |\mathbb{E}[h(\widetilde{\mathbb{Z}}_n - \sqrt{n}(\widehat{C} - C))] - \mathbb{E}[h(\widetilde{\mathbb{Y}}_n)]| \to 0$, as $n \to \infty$. We conclude that

$$\limsup_{n \to \infty} \sup_{h \in \mathrm{BL}_1} \left| \mathbb{E}\big[h(\mathbb{Y}_n) \big] - \mathbb{E}\big[h(\widetilde{\mathbb{Y}}_n) \big] \right| = 0.$$

For the bootstrap counterpart, we can argue in the same way. Using the expansion

$$\sqrt{n}\big(\widehat{C}^* - \widehat{C} \big)(\mathbf{s}) = \left(\int \psi \, d\alpha_n^* \right)' \dot{C}_{\theta_0}(\mathbf{s}) + R_n^*(\mathbf{s})$$

for some remainder term R_n^* that satisfies

$$\sup_{\mathbf{s}} \left| R_n^*(\mathbf{s}) \right| \le K_0 n^{1/2} \|\varepsilon_n^*\|_2 + K_1 n^{1/2} \|\widehat{\theta} - \theta_0\|_2^{1+\nu} + K_2 n^{1/2} \|\widehat{\theta}^* - \widehat{\theta}\|_2^{1+\nu}$$

for some finite constants K_0, K_1 and K_2. We check that the processes $\mathbb{Y}_n^*$ and $\widetilde{\mathbb{Y}}_n^*$ are close with $\widetilde{\mathbb{Y}}_n^*$ based on

$$\widetilde{\mathbb{Y}}_n^*(\mathbf{s}) = \widetilde{\mathbb{Z}}_n^*(\mathbf{s}) - \left(\int \psi \, d\alpha_n^* \right)' \dot{C}_{\theta_0}(\mathbf{s}).$$

Note that $\widetilde{\mathbb{Y}}_n(f) = \sum_k \sigma_k \widetilde{\mathbb{Y}}_n(\mathbf{s}_k) = \int (\sum_k \sigma_k g_k) \, d\alpha_n$ with

$$g_k(\mathbf{x}) = \mathbf{1}\{\mathbf{x} \le \mathbf{s}_k\} - \sum_{j=1}^{d} C_j(\mathbf{s}_k) \mathbf{1}\{x \le s_{k,j}\} - \big(\psi(\mathbf{x})\big)' \dot{C}_{\theta_0}(\mathbf{s}_k).$$

As in the proof of Theorem 1, it remains to verify the two conditions of Theorem 14. Since the only difference with the proof of Theorem 1 is the addition of the term $(\psi(\mathbf{x}))'\dot{C}_{\theta_0}(\mathbf{s}_k)$, we concentrate on the class of functions $(\psi(\mathbf{x}))'\dot{C}_{\theta_0}(\mathbf{s}_k)$. Since it is a subclass of $c'\psi(\mathbf{x})$ with $c \in \mathbb{R}^p$, its VC dimension trivially is equal to p. Moreover, it is not hard to see from the proof of Theorem 1 that

$$\left| \sum_{k=1}^{2^d L_n} \sigma_k g_k(\mathbf{x}) \right| \le 1 + d + \sum_{j=1}^{d} \mathrm{TV}(C_j) + \left\| \psi(\mathbf{x}) \right\| \mathrm{TV}(\dot{C}_{\theta_0}).$$

Since $\mathbb{E}[\|\psi(\mathbf{X})\|_2^4] < \infty$, the conditions of Theorem 14 are met, and we conclude that

$$\mathbb{E}\left[\sup_{h \in \mathrm{BL}_1} \left| \mathbb{E}\left[h(\widetilde{\mathbb{Y}}_n) \right] - \mathbb{E}^*\left[h(\widetilde{\mathbb{Y}}_n^*) \right] \right| \right] \to 0$$

as $n \to \infty$.

5.4. Proof of Proposition 6

From the proofs of Proposition 10 and Proposition 12, we see that

$$\sup_{\mathbf{u} \in [0,1]^d} \left| \mathbb{Z}_n(\mathbf{u}) - \widetilde{\mathbb{Z}}_n(\mathbf{u}) \right| = \mathrm{O}_p(n^{-\mu}) \quad \text{and}$$

$$\sup_{\mathbf{u} \in [0,1]^d} \left| \mathbb{Z}_n^*(\mathbf{u}) - \widetilde{\mathbb{Z}}_n^*(\mathbf{u}) \right| = \mathrm{O}_{p^*}(n^{-\mu})$$

almost surely, for some $\mu > 0$. The result follows after integration by parts.

5.5. Proof of Corollary 7

By the delta-method, $\{\widetilde{\mathbb{Y}}_n(\mathbf{s}), \mathbf{s} \in [0,1]^d\}$ converges toward a Gaussian process in $\ell^\infty([0,1]^d)$. The proof of Theorem 4 shows that $\limsup_{n \to \infty} \sup_{h \in \mathrm{BL}_1} |\mathbb{E}[h(\mathbb{Y}_n) - h(\widetilde{\mathbb{Y}}_n)]| = 0$. Hence, the process $\mathbb{Y}_n$ converges weakly to the same weak limit as $\widetilde{\mathbb{Y}}_n$. This proves the first claim. The second part of the corollary is a straightforward consequence of Theorem 4 and the triangle inequality.

Appendix A

Let $X_1, \ldots, X_n$ be independent random variables with probability measure P. Let $\mathbb{P}_n$ be the empirical probability measure, putting mass $1/n$ at each observation, and let $\mathbb{P}_n^*$ be the nonparametric bootstrap measure based on n independent observations from $\mathbb{P}_n$. We index the empirical process $\sqrt{n}(\mathbb{P}_n - P)$ and its bootstrap counterpart $\sqrt{n}(\mathbb{P}_n^* - \mathbb{P}_n)$ by functions f that belong to a sequence of classes $\mathcal{F}_n$.

Theorem 14. *Let d_n be an integer sequence and, for each $1 \leq i \leq d_n$, let $\mathcal{G}_{i,n}$ be a VC class of functions with VC index $V_{i,n}$ and*

$$\sum_{i=1}^{d_n} V_{i,n} \leq K (\log n)^\gamma$$

for some $K < \infty$ and $0 < \gamma < 1$. Set

$$\mathcal{F}_n = \left\{ f = \sum_{i=1}^{d_n} g_i : g_i \in \mathcal{G}_{i,n} \right\},$$

and suppose that there exists an envelope function $F \geq \sup_{f \in \mathcal{F}_n} |f|$, independent of n, with $\mathbb{E}[F^4(X)] < \infty$. Then

$$\limsup_{n \to \infty} \mathbb{E}\left[\sup_{h \in BL_1} \left| \mathbb{E}\left[h\left(\sqrt{n}(\mathbb{P}_n - P)\right)\right] - \mathbb{E}^*\left[h\left(\sqrt{n}\left(\mathbb{P}_n^* - \mathbb{P}_n\right)\right)\right] \right| \right] = 0.$$

Proof. See Theorem 3 in [23]. □

Appendix B

Set $\mathbb{M}_n(\delta)$ as in (5.1) for $\delta \geq 0$, and define

$$\psi(x) = 2x^{-2}\left\{(1+x)\log(1+x) - x\right\}, \qquad x \in (-1, 0) \cup (0, \infty)$$

and $\psi(-1) = 2$ and $\psi(0) = 1$. This function is continuous and decreasing.

Proposition 15. *There exist constants K_1 and K_2 such that*

$$\mathbb{P}\left\{\mathbb{M}_n(a) \geq \lambda\right\} \leq \frac{K_1}{a} \exp\left\{-\frac{K_2\lambda^2}{a}\psi\left(\frac{\lambda}{\sqrt{n}a}\right)\right\} \tag{B.1}$$

for all $a \in (0, 1/2]$ and all $\lambda \in [0, \infty)$.

Proof. See Proposition A.1 of [27]. □

Appendix C

We present a stochastic optimization algorithm that approximates $\widetilde{\mathbb{T}}_n$. The algorithm is based on Pure Random Search and easily implementable.

Step 1. *Compute and store, for all* $i_j \in \{0, \ldots, \lfloor n^{1/d} \rfloor\}$,

$$F(i_1, \ldots, i_d) := \mathbb{Z}_n\left(\frac{i_1}{n^{1/d}}, \ldots, \frac{i_d}{n^{1/d}}\right).$$

Step 2. (a) *Compute and store, for all* $B_i = \prod_{j=1}^{d}(a_{i,j}n^{-1/d}, b_{i,j}n^{-1/d}]$, *with* $a_{i,j}, b_{i,j} \in \{0, \ldots, \lfloor n^{1/d} \rfloor\}$ *and* $a_{i,j} < b_{i,j}$,

$$G(B_i) := \Delta^1_{a_{i,1}, b_{i,1}} \Delta^2_{a_{i,2}, b_{i,2}} \cdots \Delta^d_{a_{i,d}, b_{i,d}} F.$$

(b) *Rank the* B_i *according to* $G(B_1) \geq \cdots \geq G(B_m)$. *We suggest* $m = n$ *as the default value.*

Step 3. (a) *Sample without replacement* $(A_1, \ldots, A_{L_n}) \in \mathcal{B} = \{B_1, \ldots, B_m\}$.
(b) *Compute, for* $\mathbf{A} = (A_1, \ldots, A_{L_n})$ *of part 3(a),*

$$\mathcal{T}(\mathbf{A}) = G(A_1) + G(A_2)\mathbf{1}_{\{A_1 \cap A_2 = \varnothing\}} + \cdots + G(A_{L_n})\mathbf{1}_{\{A_1 \cap \cdots \cap A_{L_n} = \varnothing\}}.$$

(c) *Repeat parts 3(a) and 3(b) K times. We suggest* $K = 10^4$ *as the default value.*

Step 4. *Find* $\mathcal{T}(\mathbf{A}^o) = \max_{\mathbf{A}} \mathcal{T}(\mathbf{A})$ *with the maximum taken over the obtained list* $\mathbf{A}_1, \ldots, \mathbf{A}_K$ *in Step 3, and use this to approximate* $\widetilde{\mathbb{T}}_n$.

Remark (Computational cost). Step 1 requires n computations. We would like to caution that Step 1, although negligible if coded in C++ or Fortran, tends to be very slow if performed using more elaborate programming languages like R or Mathematica. Step 2 requires less than n^2 summations. Step 3, the verification whether L_n rectangles overlap, requires at most $L_n^2/2$ verifications, each in turn requiring $2d$ operations. Thus, we need at most dKL_n^2 operations in Step 3.

For a typical (larger) case $n = 800$, $d = 2$, and $L_n = 4$, the number of computations needed for Step 2 and Step 3 is bounded by 10^6. Since an ATV test typically requires 10^3 bootstrap samples, the total number of summations needed is of the order 10^9. A typical desktop computer (using C++ or Fortran code) needed less than 5 seconds.

Remark (Improvements). We took $m = n$ and $K = 10^4$. Smaller values for m and K would speed up the computation, while larger values would offer more guarantees that we find the true optimum. We experimented with $m = 10n$, $m = 100n$, $K = 10^5$ and $K = 10^6$, but we did not observe any significant improvements.

It is possible to enhance the proposed algorithm by including an additional step, which would concentrate on local search. Implementation of more sophisticated algorithms such as the Accelerated Random Search algorithm [1] would allow us to quickly search the neighborhod of $\mathbf{A}^o$. We experimented with this approach, and although it produced slightly larger values for statistic $\widetilde{\mathbb{T}}_n$, the overall performance did not significantly change. We suspect that such an additional step would be more valuable in dimensions $d > 2$. For a good review of optimization schemes

relevant to this scenario, we refer to the paper by Hvattum and Glover [18], where the authors describe eight optimization schemes and contrasts their performance on numerous test functions in higher dimensions (up to dimension 64).

Acknowledgements

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