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Asymptotic mean stationarity and absolute continuity of point process distributions

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This paper relates – for point processes Φ on $\mathbb{R}$ – two types of asymptotic mean stationarity (AMS) properties and several absolute continuity results for the common probability measures emerging from point process theory. It is proven that Φ is AMS under the time-shifts if and only if it is AMS under the event-shifts. The consequences for the accompanying two types of ergodic theorem are considered. Furthermore, the AMS properties are equivalent or closely related to several absolute continuity results. Thus, the class of AMS point processes is characterized in several ways. Many results from stationary point process theory are generalized for AMS point processes. To obtain these results, we first use Campbell's equation to rewrite the well-known Palm relationship for general nonstationary point processes into expressions which resemble results from stationary point process theory.

Keywords: point process; Palm distributions; stationarity; nonstationarity; asymptotic mean stationarity; absolute continuity; Radon–Nikodym approach; inversion formulae

1. Introduction

Point process theory on $\mathbb{R}$ utilizes two types of shifts (event-shifts and time-shifts) and several closely related probability measures. Each type of shifts brings its own ergodic theorem and – by taking expectations – its own concept 'asymptotic mean stationarity' (AMS). This paper develops a theory of AMS point processes on $\mathbb{R}$. The basic result is that the two types of AMS are equivalent, thus extending a classical result of Kaplan [8] about equivalence of event-stationarity and time-stationarity. Furthermore, the paper extends classical ergodic theorems, including Birkhoff's theorem, to the AMS setting. It relates AMS to absolute continuity (AC) properties for probability measures welling up from point process theory and it generalizes results that are well known for stationary point processes.

The general theory of AMS probability measures, with an underlying shift transformation (say) T, was mainly developed during the period 1945–1985. Dowker [2] proved that, for invertible and nonsingular T, the ergodic theorem holds if and only if AMS is valid. Rechard [13] and finally Gray and Kieffer [5] derived similar results under weaker assumptions for T. AMS is frequently used to generalize stationarity, for instance, in information theory. See also Faigle and Schönhuth [3].

In point process theory, it is the presence of the *two* types of shifts and *two* types of stationarity which offers new possibilities. Furthermore, in point process theory we can employ the close relationships between the following probability distributions:

- the distribution P of the point process,

- the distributions P_n; that is, P as experienced *at* the nth occurrence, $n \in \mathbb{Z}$,
- the distribution P^* arising from P_0 by shifting the origin to a completely random position between 0 and the first positive occurrence,
- the Palm distributions P^x; that is: P given an occurrence at x, $x \in \mathbb{R}$,
- the shifted Palm distributions $P^{0,x}$; that is, P as experienced *at* an occurrence in x, $x \in \mathbb{R}$.

Some researchers noted the close connection between (Birkhoff's) ergodic results and the AMS concepts in point process theory. Daley and Vere-Jones ([1]; Chapter 13) give overviews of authors and results; see also Sigman ([15]; Chapter 2). They use coupling results to study AMS, as in Thorisson [17]. However, to the best knowledge of the author of the present paper, a precise and rather complete study of AMS for point processes on $\mathbb{R}$ has not yet been performed.

Starting point of the paper is the theorem which states that for P the concepts 'event-asymptotic mean stationarity' (EAMS) and 'time-asymptotic mean stationarity' (TAMS) are equivalent. This result and relationships between the (above mentioned) distributions are used to link AMS to several AC properties. Thus, the class of AMS point processes is characterized in many ways and well-known results from stationary point process theory are generalized.

In the remainder of the (current) Section 1, basic notations and definitions are introduced. In Section 2, we summarize important results of stationary point process theory. Especially the so-called Radon–Nikodym (RN) approach – typical for this research – is explained. Many of the results will be generalized later under (weaker) AMS conditions. In Section 3, some well-known formulae from *nonstationary* point process theory on $\mathbb{R}$ are rewritten into formulae resembling results from Section 2, into forms useful for later sections. General definitions of P^x, $P^{0,x}$, P_n and P^* are given. In Section 4, the concepts TAMS and EAMS are defined and their equivalence is proven. Also the relationship with accompanying ergodic results is considered. Section 5 is about the equivalence of AMS to AC properties for $\{P_n\}$ and P^*, and to a weak AC property for $\{P^{0,x}\}$. Results from Section 2 are generalized by using results from Section 3. Sections 6 and 7 are about AC properties for $\{P^{0,x}\}$ and P – respectively, with respect to an event-stationary and a time-stationary distribution –, both stronger than AMS. Again, results from Section 2 are generalized. In Section 8, AC properties for P and $\{P^{0,x}\}$ are related and the relationships between the RN-derivatives are considered.

Basic notations

In the present research, $\mathbb{R}$ denotes the set of real numbers and $\mathrm{Bor}(\mathbb{R})$ the set of Borel-sets of $\mathbb{R}$. For $k \in \mathbb{Z}$, the set $\mathbb{R}_k$ is defined as the positive half-line $(0, \infty)$ if $k > 0$ and as the nonpositive half-line $(-\infty, 0]$ if $k \leq 0$. The notations $:=$ and $\Leftrightarrow_{\text{def}}$ both mean *is by definition*. Furthermore, a.e. means *almost everywhere* and w.r.t. means *with respect to*.

Although many results in this paper can be generalized to more general (like marked) point processes, we will only consider point processes on $\mathbb{R}$. A *point process* is a measurable mapping Φ from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to the set E of all integer-valued measures φ on $\mathbb{R}$ for which $\varphi(B) < \infty$ for all bounded $B \in \mathrm{Bor}(\mathbb{R})$. E is endowed with the σ-field $\mathcal{E}$ generated by the sets $\{\varphi \in E\colon \varphi(B) = k\}$, for $k \in \mathbb{Z}$ and sets $B \in \mathrm{Bor}(\mathbb{R})$. We also define

$$M := \left\{\varphi \in E\colon \varphi(\mathbb{R}) > 0; \varphi\{y\} \leq 1 \text{ for all } y \in \mathbb{R}\right\},$$

and add the σ-field $\mathcal{M} := M \cap \mathcal{E}$. We denote the (probability) distribution of Φ by P. Reversely, probability distributions on $(M, \mathcal{M})$ are called *point process distributions*. We will only allow single occurrences; we assume that $P(M) = 1$.

The atoms (called *points*, *events*, *occurrences*, *arrivals*) of $\varphi \in M$ are denoted by $T_n(\varphi)$ under the convention that

$$\cdots < T_{-1}(\varphi) < T_0(\varphi) \leq 0 < T_1(\varphi) < T_2(\varphi) < \cdots,$$

provided that they are finite. Occasionally, we will also write $T(n)$ instead of T_n. We write $\alpha_n(\varphi) := T_{n+1}(\varphi) - T_n(\varphi)$, $n \in \mathbb{Z}$, for the *interval lengths* between finite occurrences. Sets in $\mathcal{M}$ will be called *eventualities*. Eventualities like the set $\{\varphi \in M\colon\ \alpha_n(\varphi) < 3\}$ will shortly be written as $[\alpha_n(\varphi) < 3]$ or even $[\alpha_n < 3]$. Some other subsets of M with natural σ-fields:

$$\begin{aligned}
&F_n := \bigl\{\varphi \in M\colon\ \bigl|T_n(\varphi)\bigr| < \infty\bigr\} \quad \text{and} \quad \mathcal{F}_n := F_n \cap \mathcal{M}, \qquad n \in \mathbb{Z},\\
&M_x := \bigl\{\varphi \in M\colon\ \varphi\{x\} = 1\bigr\} \quad \text{and} \quad \mathcal{M}_x := M_x \cap \mathcal{M}, \qquad x \in \mathbb{R},\\
&M^\infty := \bigl\{\varphi \in M\colon\ \varphi(-\infty, 0] = \varphi(0, \infty) = \infty\bigr\} \quad \text{and} \quad \mathcal{M}^\infty := M^\infty \cap \mathcal{M},\\
&M^0 := \bigl\{\varphi \in M^\infty\colon\ \varphi\{0\} = 1\bigr\} \quad \text{and} \quad \mathcal{M}^0 := M^0 \cap \mathcal{M}.
\end{aligned}$$

The family $\{\theta_y\colon\ y \in \mathbb{R}\}$ of *time-shifts* $\theta_y : E \to E$ defined by $\theta_y(\varphi) := \theta_y\varphi := \varphi(y + \cdot)$ is important. The same holds for the family $\{\eta_n\colon\ n \in \mathbb{Z}\}$ of *event-shifts* $\eta_n : F_n \to E$ with $\eta_n(\varphi) := \eta_n\varphi := \varphi(T_n(\varphi) + \cdot)$. Note that $\theta_y \circ \theta_x = \theta_{y+x}$ for all $y, x \in \mathbb{R}$ and that $\theta_y\varphi$ has occurrences in $T_k(\varphi) - y$ (if finite) for $k \in \mathbb{Z}$. Also note that $\eta_n \circ \eta_k = \eta_{n+k}$ for all $n, k \in \mathbb{Z}$, that $\eta_n\varphi$ has occurrences in $T_k(\varphi) - T_n(\varphi)$ (if finite), and that $\eta_m = (\eta_1)^m$ for all positive $m \in \mathbb{Z}$. Regarding these shifts, the following notations are adopted:

$$\begin{aligned}
&\theta_y^{-1}A := \{\varphi \in E\colon\ \theta_y\varphi \in A\}, \qquad y \in \mathbb{R} \text{ and } A \in \mathcal{E},\\
&\eta_n^{-1}A := \{\varphi \in F_n\colon\ \eta_n\varphi \in A\}, \qquad n \in \mathbb{Z} \text{ and } A \in \mathcal{E},\\
&\mathcal{I}' := \bigl\{A \in \mathcal{M}^\infty\colon\ \theta_y^{-1}A = A \text{ for all } y \in \mathbb{R}\bigr\} \quad \text{and} \quad \mathcal{I} := \bigl\{A \in \mathcal{M}^\infty\colon\ \eta_1^{-1}A = A\bigr\}.
\end{aligned}$$

In Nieuwenhuis ([11]; Lemma 2), it was proved that the invariant σ-fields $\mathcal{I}'$ and $\mathcal{I}$ coincide. As a consequence, it holds for all $\mathcal{I}$-measurable functions $f : M^\infty \to \mathbb{R}$ that:

$$f \circ \theta_y = f \quad \text{and} \quad f \circ \eta_n = f \qquad \text{for all } y \in \mathbb{R} \text{ and } n \in \mathbb{Z}. \tag{1.1}$$

For $A \in \mathcal{M}$ and $B \in \mathrm{Bor}(\mathbb{R})$, we define: an *$A$-occurrence* is an arrival time T_n for which the eventuality $[\eta_n\varphi \in A]$ occurs, $N(B)$ is the number of occurrences in B and $N_A(B)$ the number of the A-occurrences in B. That is:

$$N(B) := \sum_{n\in\mathbb{Z}} 1_{[T(n)\in B]} \quad \text{and} \quad N_A(B) := \sum_{n\in\mathbb{Z}} (1_{[T(n)\in B]} 1_A \circ \eta_n). \tag{1.2}$$

Expectation under $\mathbb{P}$ is denoted by $\mathbb{E}$, expectation under P by E. For measurable functions $f : M \to \mathbb{R}$, we use $\mathbb{E}f(\Phi)$, Ef, $Ef(\Phi)$ and even $Ef(\varphi)$ to denote the expectation of $f(\Phi)$.

From Section 4 onwards, we will use the notations P_{ts} and P_{es} to respectively denote an event-stationary (ES) and a time-stationary (TS) distribution on $(M, \mathcal{M})$. Furthermore, AC means 'absolute continuity'. For two probability measures P and Q on $(M, \mathcal{M})$, the notation $P \ll Q$ denotes that P is AC with respect to (w.r.t.) Q. We also say that Q *dominates* P and denote a Radon–Nikodym derivative as RN. A discrete-time stochastic process $\{X_n \colon n \in \mathbb{Z}\}$ on M is called *Q-stationary* or *stationary under Q* if it holds for each positive integer m that:

$$(X_1, \ldots, X_m) =_d (X_{k+1}, \ldots, X_{k+m}) \qquad \text{under } Q \text{ (for all } k \in \mathbb{Z}). \tag{1.3}$$

2. Stationary point processes

We offer a brief but coherent overview of stationary point process theory, enclosing only results that will be used or generalized later; our notations originate from Franken *et al.* [4]. The second half of this section is less known; it reflects the special approach in the present paper.

A point process Φ (and also its distribution P) is called *time-stationary* (shortly TS) if $P\theta_y^{-1}(A) := P(\theta_y^{-1}A) = P(A)$ for all $y \in \mathbb{R}$ and $A \in \mathcal{M}$. It is called *event-stationary* (ES) if $P(M^\infty) = 1$ and it holds for all $A \in \mathcal{M}^\infty$ that $P\eta_1^{-1}(A) := P(\eta_1^{-1}A) = P(A)$, and hence that $P(\eta_n^{-1}A) = P(A)$ for all $n \in \mathbb{Z}$. For TS distributions P with $P(M) = 1$, we have $P(M^\infty) = 1$ and $P(M^0) = 0$; ES distributions P satisfy $P(M^\infty) = 1$ and $P(M^0) = 1$. If Φ is TS, we call $\lambda := E(N(0,1])$ the *intensity* of Φ and P; we will always implicitly assume that $\lambda < \infty$.

Suppose that P is TS and $y \geq 0$. Then, $E(N(0,y]) = \lambda y$. For all $x > 0$ the definition below yields one probability measure P^0 on $(M^\infty, \mathcal{M}^\infty)$, the *Palm distribution* (PD) of Φ and P:

$$P^0(A) := \frac{1}{\lambda x} E\bigl(N_A(0,x]\bigr) = \frac{1}{\lambda x} E\left(\sum_{i=1}^{N(0,x]} 1_A \circ \eta_i\right) \qquad \text{for } A \in \mathcal{M}^\infty. \tag{2.1}$$

Informally, P^0 is the conditional distribution of the point process if there is an occurrence in the origin. We denote P^0-expectation by E^0. This PD has the following properties:

$$P^0\bigl(M^0\bigr) = 1, \qquad P^0\eta_n^{-1} = P^0 \qquad \text{for all } n \in \mathbb{Z}, \tag{2.2}$$

$$\lambda = \frac{1}{E^0(\alpha_0)} = E\left(\frac{1}{\alpha_0}\right), \tag{2.3}$$

$$P^0(A) = \frac{E(N_A(0,x])}{E(N(0,x])} \qquad \text{for all } x > 0 \text{ and } A \in \mathcal{M}^\infty. \tag{2.4}$$

Hence, the PD of a TS distribution P is ES and the sequence $\{\alpha_n\}$ is stationary under P^0. With $\lambda_A := E(N_A(0,1])$, the *intensity of the A-occurrences*, it follows that:

$$P^0(A) = \lambda_A / \lambda. \tag{2.5}$$

Compared to (2.1), the following so-called *inversion formulae* work the other way round:

$$\begin{aligned} P(A) &= \lambda \int_{\mathbb{R}_k} P^0\big[\varphi(-x+\cdot) \in A \text{ and } T_{-k}(\varphi) \le -x < T_{-k+1}(\varphi)\big]\,\mathrm{d}x \\ &= \lambda E^0\bigg(\int_{-T(-k+1)}^{-T(-k)} 1_A \circ \theta_{-x}\,\mathrm{d}x\bigg) = \lambda E^0\bigg(\int_{T(-k)}^{T(-k+1)} 1_A \circ \theta_y\,\mathrm{d}y\bigg). \end{aligned} \tag{2.6}$$

Here, $A \in \mathcal{M}^\infty$ and $k \in \mathbb{Z}$. It is allowed to replace $\mathbb{R}_k$ by $\mathbb{R}$. See Slivnyak [16] and Kaplan [8] for the one-to-one correspondence described in (2.4) and (2.6).

In Nieuwenhuis ([10]; Theorem 8.1) it was proved that, for TS distributions P and all $n \in \mathbb{Z}$, the *intermediate distribution* $P_n := P\eta_n^{-1}$ is equivalent (i.e., AC in two directions) to the PD P^0:

$$P_n \ll P^0 \quad \text{and} \quad P_n(A) = \lambda E^0(\alpha_{-n} 1_A), \tag{2.7a}$$

$$P^0 \ll P_n \quad \text{and} \quad P^0(A) = \frac{1}{\lambda} E_n\bigg(\frac{1}{\alpha_{-n}} 1_A\bigg) = \frac{1}{\lambda} E\bigg(\frac{1}{\alpha_0} 1_A \circ \eta_n\bigg), \qquad A \in \mathcal{M}^\infty. \tag{2.7b}$$

(We write E_n for P_n-expectation.) See also Ryll–Nardzewski [14] and Thorisson [17] for similar approaches. Results (2.7a), (2.7b), which reflect the so-called *Radon–Nikodym approach*, offer the opportunity to jump easily between P, P^0 and related distributions and are very important for this paper. We illustrate their use and derive some frequently used results.

Since $P(A) = \lambda E^0(\int_0^{\alpha_0} 1_A \circ \theta_y\,\mathrm{d}y)$, it follows from (2.7b) that $P(A)$ can be written otherwise as a P-expectation:

$$P(A) = E_0\bigg(\frac{1}{\alpha_0}\int_0^{\alpha_0} 1_A \circ \theta_y\,\mathrm{d}y\bigg) = E\bigg(\frac{1}{\alpha_0}\int_{T_0}^{T_1} 1_A \circ \theta_y\,\mathrm{d}y\bigg). \tag{2.8a}$$

By (2.8a) we obtain, for all functions $g : \mathbb{R} \to \mathbb{R}$ with $E|g(T_1)| < \infty$:

$$E\big(g(T_1)|(\alpha_n)_{n\in\mathbb{Z}}\big) = \frac{1}{\alpha_0}\int_0^{\alpha_0} g(x)\,\mathrm{d}x \qquad P\text{-a.s.} \tag{2.8b}$$

Hence: conditionally on α_0, the distribution of T_1 under P is uniform$(0, \alpha_0)$. Note that $h := \frac{1}{\alpha_0}\int_{T_0}^{T_1} 1_A \circ \theta_y\,\mathrm{d}y$ satisfies $h \circ \eta_0 = h$ on M^∞. We obtain by (2.8a), for P-integrable functions f and $g : M^\infty \to \mathbb{R}$:

$$E\bigg(f \cdot \frac{1}{\alpha_0}\int_{T(0)}^{T(1)} g \circ \theta_y\,\mathrm{d}y\bigg) = E\bigg(g \cdot \frac{1}{\alpha_0}\int_{T(0)}^{T(1)} f \circ \theta_y\,\mathrm{d}y\bigg). \tag{2.8c}$$

Set $\overline{\alpha} = E^0(\alpha_0|\mathcal{I})$ and $\overline{N} = E(N(0,1]|\mathcal{I})$. By (2.7a) and (2.7b), Birkhoff's ergodic results

$$\frac{1}{n}\sum_{i=1}^n \alpha_i \to \overline{\alpha} \qquad \text{as } n \to \infty \ P^0\text{-a.s.} \quad \text{and} \quad \frac{1}{x}N(0,x] \to \overline{N} \qquad \text{as } x \to \infty \ P\text{-a.s.}$$

are also valid P-a.s. and P^0-a.s., respectively. Furthermore, it can be proved that:

$$\overline{N} = \frac{1}{\overline{\alpha}} = E\left(\frac{1}{\alpha_0}\Big|\mathcal{I}\right) \qquad P^0\text{-a.s. and } P\text{-a.s.} \tag{2.9}$$

A TS point process is called *ergodic* if $P(C) = 0$ or 1 (or, equivalently, $P^0(C) = 0$ or 1) for all $C \in \mathcal{I}$. It is called *pseudo-ergodic* if $P^0[\lambda\overline{\alpha} = 1] = 1$; see also Nieuwenhuis [12].

Note that, for all $x \in \mathbb{R}$, $A \in \mathcal{M}^\infty$ and functions $f : M^\infty \to \mathbb{R}$ with $f = f \circ \eta_0$ on M^∞:

$$\begin{aligned} E^0(1_A \cdot f \circ \theta_{-x}) &= \sum_{k\in\mathbb{Z}} E^0(1_A \cdot f \circ \eta_k \cdot 1_{[T_k \le -x < T_{k+1}]}) \\ &= E^0\big(f \cdot N_A[x, x+\alpha_0)\big) \qquad (2.10a) \\ &= \frac{1}{\lambda} E\big(f \cdot N_A[x+T_0, x+T_1)/\alpha_0\big), \qquad (2.10b) \end{aligned}$$

$$P^0(A) = E\left(N_A[x+T_0, x+T_1)\frac{1}{\lambda\alpha_0}\right) \quad \text{and} \quad E\left(N[x+T_0, x+T_1)\frac{1}{\alpha_0}\right) = \lambda. \tag{2.10c}$$

In coming sections, ES distributions and TS distributions are usually denoted as P_{es} and P_{ts} (and the accompanying expectation operators as E_{es} and E_{ts}). The ES Palm distribution associated with a TS distribution P_{ts}, is denoted by P_{ts}^0. So, the relationships between P_{ts} and P_{ts}^0 are the same as the relationships between P and P^0 described in (2.1)–(2.10).

3. Non-stationary point processes

In this section, we consider, for *general* point processes on $\mathbb{R}$, the PDs $\{P^x\}$ and their shifted versions $\{P^{0,x}\}$. Furthermore, we carefully define the distributions P_n and P^* informally mentioned in Section 1. Campbell's equation is used to obtain inversion formulae (3.6) and (3.7) which resemble and generalize (2.6) and (2.7a). We generalize (2.5) for the case that $P = P^*$ and characterize the class of the TS point process distributions.

We assume that the point process Φ satisfies $P(M) = 1$, and that the *intensity measure* ν on $\mathrm{Bor}(\mathbb{R})$ with $\nu(B) := E(N(B))$ for $B \in \mathrm{Bor}(\mathbb{R})$ exists and is locally finite. Below, for $A \in \mathcal{M}$, also the locally finite measures ν_A and μ_A play important roles:

$$\nu_A(B) := E\big(N(B)1_A\big) \quad \text{and} \quad \mu_A(B) := E\big(N_A(B)\big); \qquad B \in \mathrm{Bor}(\mathbb{R}).$$

Palm distributions

For $A \in \mathcal{M}$, ν_A is dominated by ν. An RN-derivative is denoted by $x \to P^x(A)$, so:

$$\nu_A(B) = \int_B P^x(A)\,\mathrm{d}\nu(x); \qquad B \in \mathrm{Bor}(\mathbb{R}). \tag{3.1}$$

A basic result in Palm theory now is that $\{P^x(A)\colon\ x \in \mathbb{R} \text{ and } A \in \mathcal{M}\}$ can be chosen such that the function $x \to P^x(A)$ is measurable for all $A \in \mathcal{M}$, P^x is a probability measure on $\mathcal{M}$ for all $x \in \mathbb{R}$, and

$$\int_M \int_{-\infty}^{\infty} f(x,\varphi)\,\mathrm{d}\varphi(x)\,\mathrm{d}P(\varphi) = \int_{-\infty}^{\infty} \int_M f(x,\varphi)\,\mathrm{d}P^x(\varphi)\,\mathrm{d}\nu(x) \tag{3.2}$$

for all $\mathrm{Bor}(\mathbb{R}) \times \mathcal{M}$-measurable functions f on $\mathbb{R} \times M$ that are either nonnegative or satisfy $\mathbb{E}[\int_{-\infty}^{\infty} f(x,\Phi)\,\mathrm{d}\Phi(x)] < \infty$. Thus, the family $\{P^x\}$ of probability distributions turns out to be uniquely defined by (3.2) apart from a Borel-set in $\mathbb{R}$ with ν-measure 0. See Matthes [9]. See also Jagers [6] and Kallenberg [7]. In the sequel, we will assume that the family $\{P^x\}$ is chosen this way. Note that $f(x,\varphi) = 1_{B\times A}(x,\varphi)$, $x \in \mathbb{R}$ and $\varphi \in M$, returns (3.1). The probability measures in $\{P^x\colon\ x \in \mathbb{R}\}$ are called *Palm distributions* (PDs) of P. It can be proved that $P^x(M_x) = 1$ for ν-a.e. $x \in \mathbb{R}$. By letting A in (3.1) shrink to $\{x\}$, we obtain the intuitive meaning for $P^x(A)$ as the probability that $\Phi \in A$ under the condition that $\Phi\{x\} = 1$.

Shifted Palm distributions

We are especially interested in $\{P^{0,x}\}$, the family of shifted PDs defined by $P^{0,x} := P^x\theta_x^{-1}$. Note that $P^{0,x}$ satisfies $P^{0,x}(M_0) = 1$, and that, in queuing terms, it can be considered as the probability measure that under P is experienced by a customer arriving at time x. For time-stationary P we have $P^{0,x} = P^0$ for ν-a.e. $x \in \mathbb{R}$, where P^0 is the event-stationary PD of P in (2.1). Note that the choice $f(x,\varphi) = 1_A(\theta_x\varphi)1_B(x)$ in (3.2) yields that:

$$\mu_A(B) = \int_B P^{0,x}(A)\,\mathrm{d}\nu(x); \qquad B \in \mathrm{Bor}(\mathbb{R}) \text{ and } A \in \mathcal{M}. \tag{3.3}$$

Hence, for all $A \in \mathcal{M}$, the function $x \to P^{0,x}(A)$ is just an RN-derivative of μ_A with respect to ν. If ν is AC with respect to Leb with *intensity* $\lambda(\cdot)$, then μ_A is also AC with respect to Leb. If $x \to \lambda_A(x)$ denotes an accompanying RN-derivative (the *intensity* of the point process of the A-occurrences), it then follows for all $A \in \mathcal{M}$ that

$$P^{0,x}(A) = \lambda_A(x)/\lambda(x) \qquad \text{for } \nu\text{-a.e. } x \in \mathbb{R}; \tag{3.4}$$

cf. (2.5). However, it cannot be concluded that for ν-a.e. $x \in \mathbb{R}$ the shifted PDs satisfy $P^{0,x}(A) = \lambda_A(x)/\lambda(x)$ for all $A \in \mathcal{M}$; (3.4) not even necessarily defines a probability measure for ν-a.e. $x \in \mathbb{R}$. By letting B in (3.3) shrink to $\{x\}$ we obtain the intuitive meaning for $P^{0,x}(A)$ as the probability that $\theta_x\Phi \in A$ under the condition that $\Phi\{x\} = 1$.

Intermediate probability measures

For $n \in \mathbb{Z}$ with $P(F_n) > 0$, we define the *intermediate probability measure* P_n of P as a conditional probability distribution:

$$P_n(A) := P\big([\eta_n\varphi \in A] | F_n\big), \qquad A \in \mathcal{M}. \tag{3.5}$$

We investigate the relationships between P, $\{P_n\}$ and $\{P^{0,x}\}$. Set $I_x := (0, x]$ for $x > 0$ and $I_x := (x, 0]$ for $x \le 0$, and choose f in (3.2) as:

$$f(x, \varphi) = 1_A(\varphi) 1_{\{|k|\}}\big(\varphi(I_x)\big) 1_{\mathbb{R}_k}(x), \qquad A \in \mathcal{M} \text{ and } k \in \mathbb{Z}.$$

We obtain:

$$P(A \cap F_k) = \int_{\mathbb{R}_k} P^x\big(A \cap \big[\varphi(I_x) = |k|\big]\big)\,\mathrm{d}\nu(x), \tag{3.6a}$$

$$P(A \cap F_k) = \int_{\mathbb{R}_k} P^x\big(A \cap [T_k = x]\big)\,\mathrm{d}\nu(x), \tag{3.6b}$$

$$P(A \cap F_k) = \int_{\mathbb{R}_k} P^{0,x}\big([\theta_{-x}\varphi \in A] \cap [T_{-k} \le -x < T_{-k+1}]\big)\,\mathrm{d}\nu(x). \tag{3.6c}$$

Compare with (2.6). It follows that, for all $A \in \mathcal{M}$ and $k \in \mathbb{Z}$ with $P(F_k) > 0$:

$$P_k(A) = \frac{1}{P(F_k)} \int_{\mathbb{R}_k} P^{0,x}\big(A \cap [T_{-k} \le -x < T_{-k+1}]\big)\,\mathrm{d}\nu(x). \tag{3.7}$$

Note that it is allowed to replace $\mathbb{R}_k$ by $\mathbb{R}$ in (3.6b), (3.6c) and (3.7), and that (3.7) generalizes (2.7a). Substitution of $A \cap [T_k \in B]$ for A in (3.6c) yields

$$P\big(A \cap [T_k \in B]\big) = \int_B P^{0,x}\big([\theta_{-x}\varphi \in A] \cap [T_{-k} \le -x < T_{-k+1}]\big)\,\mathrm{d}\nu(x) \tag{3.8}$$

for all $k \in \mathbb{Z}$, $B \in \mathrm{Bor}(\mathbb{R})$ and $A \in \mathcal{M}$. By taking $\sum_{k \in \mathbb{Z}}$, the left-hand side becomes equal to $\nu_A(B)$ and we get (3.1) back. When A in (3.8) is replaced by $[\eta_k\varphi \in A]$, we obtain:

$$P\big([\eta_k\varphi \in A] \cap [T_k\varphi \in B]\big) = \int_B P^{0,x}\big(A \cap [T_{-k} \le -x < T_{-k+1}]\big)\,\mathrm{d}\nu(x). \tag{3.9}$$

Note that we get (3.3) back by taking $\sum_{k \in \mathbb{Z}}$. The choice $A = M_0$ in (3.9) ensures that, if $P(F_k)$ is larger than 0, the conditional distribution $P([T_k \in \cdot] | F_k)$ of T_k is AC with respect to ν with RN-derivative $\gamma(x) := P^{0,x}[T_{-k} \le -x < T_{-k+1}]/P(F_k)$. So:

$$P\big([T_k \in \cdot] | F_k\big) \ll \nu, \qquad \gamma(x) = P^x[T_k = x]/P(F_k) \qquad \text{for } \nu\text{-a.e. } x \in \mathbb{R}. \tag{3.10}$$

The distribution P^*

For P such that $P(M^\infty) = 1$, we define the distribution P^* as follows:

$$P^*(A) := E\bigg(\frac{1}{\alpha_0} \int_{T(0)}^{T(1)} 1_A \circ \theta_y\,\mathrm{d}y\bigg) = E_0\bigg(\frac{1}{\alpha_0} \int_0^{\alpha_0} 1_A \circ \theta_y\,\mathrm{d}y\bigg) \qquad \text{for } A \in \mathcal{M}^\infty. \tag{3.11}$$

By (2.8a) and (2.7a), time-stationary point processes (with finite intensity) satisfy:

(a) $P = P^*$ and (3.12a)

(b) there exists an ES point process distribution P_{es} such that:

$$P_0 \ll P_{es} \quad \text{and} \quad \mathrm{d}P_0/\mathrm{d}P_{es} = \lambda\alpha_0 \qquad \text{with } \lambda = 1/E_{es}(\alpha_0) \in (0,\infty). \tag{3.12b}$$

Reversely, if P satisfies (3.12a), (3.12b) then, for $A \in \mathcal{M}^\infty$:

$$P(A) = P^*(A) = \lambda E_{es}\left(\int_0^{\alpha_0} 1_A \circ \theta_y \,\mathrm{d}y\right) =: P_{ts}(A).$$

Note that P_{ts} (and hence P) is just the TS distribution such that the accompanying PD is P_{es}; see (2.6). Hence, (3.12a), (3.12b) characterize the class of TS point process distributions.

For $A \in \mathcal{M}^\infty$ and $B \in \mathrm{Bor}(\mathbb{R})$, set $\mu_A^*(B) := E^*(N_A(B))$ and $\nu^*(B) := E^*(N(B))$. Here, E^* refers to P^*-expectation.

Theorem 3.1. *Suppose that P is a point process distribution with $P(M^\infty) = 1$. Then*:

(1) *the intermediate distributions of P^* and P coincide*;
(2) *under P^*, the conditional distribution of T_1 given $(\alpha_n)_{n\in\mathbb{Z}}$, is* uniform$(0,\alpha_0)$;
(3) *for $A \in \mathcal{M}^\infty$ it holds that $\mu_A^* \ll \mathrm{Leb}$ and $\nu^* \ll \mathrm{Leb}$, with intensity functions*

$$\lambda_A^*(x) = E\left(\frac{1}{\alpha_0} N_A[x + T_0, x + T_1)\right) \quad \textit{and} \quad \lambda^*(x) = E\left(\frac{1}{\alpha_0} N[x + T_0, x + T_1)\right);$$

(4) *the shifted PDs of P^* satisfy $P^{*0,x}(A) = \lambda_A^*(x)/\lambda^*(x)$ for ν^*-a.e. $x \in \mathbb{R}$ and $A \in \mathcal{M}^\infty$.*

Proof. Part (1) is immediate. Part (2) holds since for all eventualities A in the σ-field generated by $(\alpha_n)_{n\in\mathbb{Z}}$ and all functions $g : \mathbb{R} \to R$ with $E^*|g(T_1)| < \infty$ we have:

$$\begin{aligned}
E^*\big(1_A E^*\big(g(T_1)|(\alpha_n)_{n\in\mathbb{Z}}\big)\big) &= E^*\big(1_A g(T_1)\big) = E\left(1_A \frac{1}{\alpha_0}\int_{T(0)}^{T(1)} g(T_1 \circ \theta_y)\,\mathrm{d}y\right) \\
&= E_0\left(1_A \frac{1}{\alpha_0}\int_0^{\alpha_0} g(x)\,\mathrm{d}x\right) = E^*\left(1_A \frac{1}{\alpha_0}\int_0^{\alpha_0} g(x)\,\mathrm{d}x\right).
\end{aligned}$$

For (3), note that for $A \in \mathcal{M}^\infty$ and $B \in \mathrm{Bor}(\mathbb{R})$ we have that $\mu_A^*(B)$ equals:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} P^*\big[T_k(\varphi) \in B \text{ and } \eta_k\varphi \in A\big] \\
&\quad = \sum_{k\in\mathbb{Z}} E\left(\frac{1}{\alpha_0}\int_{T(k)-T(1)}^{T(k)-T(0)} 1_B(y)\,\mathrm{d}y\, 1_{[\eta_k\varphi\in A]}\right) \\
&\quad = \sum_{k\in\mathbb{Z}} E\left(\frac{1}{\alpha_0}\int_B 1_{[y+T(0)\le T(k)<y+T(1)]}\,\mathrm{d}y\, 1_{[\eta_k\varphi\in A]}\right) = \int_B \lambda_A^*(x)\,\mathrm{d}x.
\end{aligned}$$

Hence, $\mu_A^* \ll \mathrm{Leb}$. The choice $A = M^\infty$ yields $\nu^* \ll \mathrm{Leb}$. For (4), first note that it holds for ν^*-a.e. $x \in \mathbb{R}$ that $Q^{0,x}(A) := \lambda_A^*(x)/\lambda^*(x)$ defines a probability measure on $(M^\infty, \mathcal{M}^\infty)$. Replacing P, $\{P^x\}$ and ν by P^*, $\{Q^{0,x}\theta_{-x}^{-1}\}$ and ν^* in (3.2) yields, for both sides:

$$E\left(\frac{1}{\alpha_0(\varphi)} \sum_{k\in\mathbb{Z}} \int_{T_0\varphi}^{T_1\varphi} f(T_k\varphi - y, \theta_y\varphi)\,\mathrm{d}y\right).$$

So, (4) follows from the uniqueness of the family of PDs. $\square$

It follows from Theorem 3.1(1), (2) that P^* arises from P_0 by shifting the origin to an arbitrary position in the interval $(0, \alpha_0)$. By (2.10c) and Theorem 3.1(4), we get a generalization of (2.5); cf. (2.10c):

$$P = P^* \quad \Rightarrow \quad P^{0,x}(A) = \lambda_A(x)/\lambda(x) \qquad \text{for } \nu\text{-a.e. } x \in \mathbb{R} \text{ and all } A \in \mathcal{M}^\infty. \tag{3.13}$$

4. Asymptotic mean stationarity

A point process (as well as its distribution P) is called *time-asymptotic(ally) mean stationary* (shortly TAMS) if a probability distribution P_{ts} on $(M, \mathcal{M})$ exists such that:

$$\frac{1}{x}\int_0^x P[\theta_y\varphi \in A]\,\mathrm{d}y \to P_{ts}(A) \qquad \text{as } x \to \infty,\ \text{for all } A \in \mathcal{M}. \tag{4.1}$$

Note that P_{ts} is indeed TS. We write "P is TAMS(P_{ts})" and call P_{ts} the *time-stationary limit distribution* of P. A point process (and its distribution P) with $P(M^\infty) = 1$ is called *event-asymptotic(ally) mean stationary* (shortly EAMS) if a probability distribution P_{es} on $(M^\infty, \mathcal{M}^\infty)$ exists such that:

$$\frac{1}{n}\sum_{i=1}^n P[\eta_i\varphi \in A] \to P_{es}(A) \qquad \text{as } n \to \infty,\ \text{for all } A \in \mathcal{M}^\infty. \tag{4.2}$$

We write "P is EAMS(P_{es})" and call P_{es} the *event-stationary limit distribution* of P. Note that P_{es} is ES and that, for P being EAMS, it is only its behavior on $\mathcal{M}^0$ which matters. Sigman [15] refers to the AMS concepts as time and event asymptotic stationarity; Daley and Vere-Jones [1] use $(C, 1)$-asymptotic stationarity and event-stationarity, respectively.

Note that TS point processes are TAMS and ES point processes are EAMS. However, we will see that the class of TAMS (EAMS) point processes is – considerably – larger than the class of TS (ES) point processes. As an example: with P_{ts} the distribution of the TS Poisson point process with intensity λ_{ts}, the definition $P(A) := E_{ts}(1_A \cdot N(0,1])/\lambda_{ts}$ for $A \in \mathcal{M}^\infty$ yields a point process distribution P which is absolutely continuous with respect to P_{ts} and hence (see (4.8b) below) is TAMS. However, P is *not* TS since, by Jensen's inequality:

$$E\big(N(0,1]\big) = E_{ts}\big[\big(N(0,1]\big)^2\big]/\lambda_{ts} > \lambda_{ts} = E\big(N(1,2]\big).$$

The following characterizations of EAMS and TAMS will be used frequently.

Theorem 4.1. *Let Φ be a point process with distribution P for which $P(M^\infty) = 1$, and let P_{es} and P_{ts}, respectively, be an ES and a TS point process distribution. Then*:

$$\begin{aligned} &P \text{ is } \mathrm{EAMS}(P_{es}) \quad \Leftrightarrow \quad P = P_{es} \quad \text{on } \mathcal{I}; \\ &P \text{ is } \mathrm{TAMS}(P_{ts}) \quad \Leftrightarrow \quad P = P_{ts} \quad \text{on } \mathcal{I}. \end{aligned} \tag{4.3}$$

Proof. The implications '$\Rightarrow$' follow from (1.1). For the implications '$\Leftarrow$', suppose respectively that $P = P_{es}$ on $\mathcal{I}$ and $P = P_{ts}$ on $\mathcal{I}$. Note that, for each $A \in \mathcal{M}^\infty$, the eventualities

$$\left[\frac{1}{n}\sum_{i=1}^{n} 1_A \circ \eta_i \to E_{es}(1_A|\mathcal{I})\right] \quad \text{and} \quad \left[\frac{1}{x}\int_0^x 1_A \circ \theta_y \, \mathrm{d}y \to E_{ts}(1_A|\mathcal{I})\right]$$

are elements of $\mathcal{I}$ which (by Birkhoff's ergodic theorem) have probability 1 under respectively, P_{es} and P_{ts}, and hence they both have P-probability 1. Take P-expectations. □

The next theorem roughly states that P is EAMS iff P is TAMS; see also Daley and Vere-Jones ([1]; Theorem 13.4.VI) for a similar (but different) theorem. If P is $\mathrm{EAMS}(P_{es})$, then $\lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i/n$ equals $\overline{\alpha} := E_{es}(\alpha_0|\mathcal{I})$; this holds P_{es}-a.s. and (by Theorem 4.1) also P-a.s. If P is $\mathrm{TAMS}(P_{ts})$, then $\lim_{x\to\infty} N(0,x]/x$ equals $\overline{N} := E_{ts}(N(0,1]|\mathcal{I})$, P_{ts}-a.s. and also P-a.s.

Theorem 4.2. *Let Φ be a point process with distribution P for which $P(M^\infty) = 1$. Then*:

$$\begin{gathered} P \text{ is } \mathrm{EAMS}(P_{es}) \quad \text{and} \quad P_{es}[0 < \overline{\alpha} < \infty] = 1 \\ \Leftrightarrow \\ P \text{ is } \mathrm{TAMS}(P_{ts}) \quad \text{and} \quad P_{ts}[0 < \overline{N} < \infty] = 1. \end{gathered}$$

These distributions P_{es} and P_{ts} are related as follows:

$$\begin{aligned} &P_{ts}(A) = E_{es}\left(\frac{1}{\overline{\alpha}}\int_0^{\alpha_0} 1_A \circ \theta_y \, \mathrm{d}y\right) \quad \text{and} \\ &P_{es}(A) = E_{ts}\left(\frac{1}{\alpha_0}\frac{1}{\overline{N}} 1_A \circ \eta_0\right) \qquad \text{for } A \in \mathcal{M}^\infty, \end{aligned} \tag{4.4}$$

$$\overline{N} = \frac{1}{\overline{\alpha}} = E_{ts}\left(\frac{1}{\alpha_0}\middle|\mathcal{I}\right) \qquad P_{es}\text{-a.s., } P_{ts}\text{-a.s., and } P\text{-a.s.} \tag{4.5}$$

$$P_{es} \text{ is the event-stationary PD of } P_{ts} \quad \Leftrightarrow \quad P_{es} \text{ is pseudo-ergodic.}$$

Proof.

Proof of '$\Rightarrow$'. By Birkhoff's ergodic theorem and the left-hand side of (4.3), we obtain that, for all P_{es}-integrable functions $f : M^\infty \to \mathbb{R}$, the following convergence holds not only P_{es}-a.s. but also P-a.s.:

$$\frac{1}{n}\sum_{i=1}^{n} f \circ \eta_i \to E_{es}(f|\mathcal{I}) \qquad \text{as } n \to \infty.$$

The choices $f = \alpha_0$ and $f = \int_{T_0}^{T_1} 1_A \circ \theta_y \, \mathrm{d}y$ (with $A \in \mathcal{M}^\infty$) respectively yield that, P-a.s.:

$$\frac{1}{n} T_n \to \overline{\alpha} \quad \text{and} \quad \frac{1}{n} \int_{T(0)}^{T(n)} 1_A \circ \theta_y \, \mathrm{d}y \to E_{es}\left(\int_0^{\alpha_0} 1_A \circ \theta_y \, \mathrm{d}y \middle| \mathcal{I} \right) \qquad \text{as } n \to \infty.$$

After replacing n by $N(0, x]$ and using that $P[\overline{\alpha} > 0] = 1$, we obtain that it holds P-a.s. that:

$$\frac{N(0,x]}{x} \to \frac{1}{\overline{\alpha}} \quad \text{and}$$

$$\frac{1}{N(0,x]} \int_0^x 1_A \circ \theta_y \, \mathrm{d}y \to E_{es}\left(\int_0^{\alpha_0} 1_A \circ \theta_y \, \mathrm{d}y \middle| \mathcal{I} \right) \qquad \text{as } x \to \infty;$$

$$\frac{N(0,x]}{x} \frac{1}{N(0,x]} \int_0^x 1_A \circ \theta_y \, \mathrm{d}y \to \frac{1}{\overline{\alpha}} E_{es}\left(\int_0^{\alpha_0} 1_A \circ \theta_y \, \mathrm{d}y \middle| \mathcal{I} \right) \qquad \text{as } x \to \infty.$$

By taking P-expectations we get, again from (4.3):

$$\frac{1}{x} \int_0^x P[\theta_y \varphi \in A] \, \mathrm{d}y \to P_{ts}(A) := E_{es}\left(\frac{1}{\overline{\alpha}} \int_0^{\alpha_0} 1_A \circ \theta_y \, \mathrm{d}y \right) \qquad \text{as } x \to \infty.$$

So, P is TAMS(P_{ts}). Note that $P = P_{es} = P_{ts}$ on $\mathcal{I}$. Hence, $\overline{N} := E_{ts}(N(0,1]|\mathcal{I})$ and $1/\overline{\alpha}$ are both the P_{es}-, P- and P_{ts}-a.s. limit of $N(0,x]/x$. So:

$$\overline{N} = \frac{1}{\overline{\alpha}} \qquad P_{ts}\text{-a.s., } P\text{-a.s. and } P_{es}\text{-a.s.} \quad \text{and} \quad P_{ts}[0 < \overline{N} < \infty] = 1.$$

Proof of '$\Leftarrow$'. By Birkhoff's ergodic theorem and the right-hand side of (4.3), we obtain that, for all P_{ts}-integrable functions $f : M^\infty \to \mathbb{R}$, the following convergence not only holds P_{ts}-a.s. but also P-a.s.:

$$\frac{1}{x} \int_0^x f \circ \theta_y \, \mathrm{d}y \to E_{ts}(f|\mathcal{I}) \qquad \text{as } x \to \infty.$$

After replacing x by T_{n+1} and choosing $f = 1_A \circ \eta_0 / \alpha_0$ with $A \in \mathcal{M}^\infty$, we get P_{ts}-a.s. and P-a.s.:

$$\frac{1}{T_{n+1}} \int_{T_1}^{T_{n+1}} 1_A \circ \eta_0 \circ \theta_y \frac{1}{\alpha_0 \circ \theta_y} \, \mathrm{d}y \to E_{ts}\left(1_A \circ \eta_0 \frac{1}{\alpha_0} \middle| \mathcal{I} \right) \qquad \text{as } n \to \infty. \tag{4.6}$$

If, for $\varphi \in M^\infty$, y is such that $T_i(\varphi) \leq y < T_{i+1}(\varphi)$, then:

$$\alpha_0 \circ \theta_y(\varphi) = \alpha_i(\varphi) \quad \text{and} \quad 1_A \circ \eta_0 \circ \theta_y(\varphi) = 1_A \circ \eta_i(\varphi).$$

Hence, the left-hand side of (4.6) is equal to:

$$\frac{1}{T_{n+1}} \sum_{i=1}^{n} \int_{T_i}^{T_{i+1}} 1_A \circ \eta_i \frac{1}{\alpha_i} \, \mathrm{d}y = \frac{1}{T_{n+1}} \sum_{i=1}^{n} 1_A \circ \eta_i.$$

Note that $N(0,x]/x \to \overline{N}$ holds P_{ts}- and P-a.s. Replacing x by T_{n+1} yields that $T_{n+1}/n \to 1/\overline{N}$ also holds P_{ts}- and P-a.s. By (4.6) we obtain, P_{ts}- and P-a.s.:

$$\frac{1}{n}\sum_{i=1}^{n} 1_A \circ \eta_i = \frac{T_{n+1}}{n}\frac{1}{T_{n+1}}\sum_{i=1}^{n} 1_A \circ \eta_i \to \frac{1}{\overline{N}} E_{ts}\left(1_A \circ \eta_0 \frac{1}{\alpha_0}\Big|\mathcal{I}\right) \qquad \text{as } n \to \infty.$$

Since $P = P_{ts}$ on $\mathcal{I}$, we obtain by taking P-expectation:

$$\frac{1}{n}\sum_{i=1}^{n} P[\eta_i\varphi \in A] \to P_{es}(A) := E_{ts}\left(\frac{1}{\alpha_0}\frac{1}{\overline{N}} 1_A \circ \eta_0\right) \qquad \text{as } n \to \infty.$$

Especially, $P = P_{es}$ on $\mathcal{I}$. For $C \in \mathcal{I}$ we have, since $P = P_{ts}$ on $\mathcal{I}$:

$$E(1_C) = E_{ts}\big(1_C/(\alpha_0\overline{N})\big) = E_{ts}\left(\frac{1}{\overline{N}}E_{ts}\left(\frac{1}{\alpha_0}\Big|\mathcal{I}\right)1_C\right) = E\left(\frac{1}{\overline{N}}E_{ts}\left(\frac{1}{\alpha_0}\Big|\mathcal{I}\right)1_C\right).$$

We conclude: P is EAMS(P_{es}) and $\frac{1}{\overline{N}}E_{ts}(\frac{1}{\alpha_0}|\mathcal{I}) = 1$ P_{ts}-, P- and P_{es}-a.s. $\square$

Remark. Let P_{ts} be a TS point process distribution with $\lambda_{ts} := E_{ts}(N(0,1]) < \infty$ and P^0_{ts} the accompanying TS Palm distribution. Note that P_{ts} is TAMS(P_{ts}) and P^0_{ts} is EAMS(P^0_{ts}). By Theorem 4.2 it follows that P_{ts} is EAMS and P^0_{ts} is TAMS. By respectively using the results (4.4), (2.7a) and (4.5), (2.7b), (2.8a), (2.8c), (1.1) in 1 and 2 below, we obtain:

1. P_{ts} is EAMS$(\tilde{P}^0_{ts})$ with $\tilde{P}^0_{ts}(A) := \lambda_{ts}E^0_{ts}(\overline{\alpha}1_A)$ for $A \in \mathcal{M}^\infty$, where $\overline{\alpha} = E^0_{ts}(\alpha_0|\mathcal{I})$;
2. P^0_{ts} is TAMS$(\tilde{P}_{ts})$ with $\tilde{P}_{ts}(A) := E_{ts}(\overline{N}1_A)/\lambda_{ts}$ for $A \in \mathcal{M}^\infty$ with $\overline{N} = E_{ts}(N(0,1]|\mathcal{I})$;
3. $\tilde{P}^0_{ts} = P^0_{ts} \Leftrightarrow P^0_{ts}$ is pseudo-ergodic $\Leftrightarrow \tilde{P}_{ts} = P_{ts}$.

The validity of EAMS (respectively, TAMS) is equivalent to the validity of the ergodic result under the shift transformation η_1 (respectively under the flow $\{\theta_y\colon y \in \mathbb{R}\}$).

Theorem 4.3. *Let Φ be a point process with distribution P for which $P(M^\infty) = 1$.*

(a) *P is EAMS* $\Leftrightarrow$ $\forall_{A\in\mathcal{M}^\infty}$: $\displaystyle\frac{1}{n}\sum_{i=1}^{n} 1_A \circ \eta_i$ *converges P-a.s. (as $n \to \infty$);*

(b) *P is EAMS with limit distribution P_{es} such that $P_{es}\big[0 < E_{es}(\alpha_0|\mathcal{I}) < \infty\big] = 1$*

$\Leftrightarrow$ $\forall_{A\in\mathcal{M}^\infty}$: $\displaystyle\frac{1}{x}N_A(0,x]$ *converges P-a.s. (as $x \to \infty$) and*

the limit of $\displaystyle\frac{1}{x}N(0,x]$ *belongs P-a.s. to $(0,\infty)$;*

(c) *P is TAMS* $\Leftrightarrow$ $\forall_{A\in\mathcal{M}^\infty}$: $\displaystyle\frac{1}{x}\int_0^x 1_A \circ \theta_y\,\mathrm{d}y$ *converges P-a.s. (as $x \to \infty$).*

Proof. For (a), the implication '⇒' follows since by Birkhoff's ergodic theorem the right-hand convergence holds a.s. under the ES limit distribution of P, and hence under P itself by Theorem 4.1. The implication '⇐' follows from the theorem of Vitali–Hahn–Saks. For '⇒' of (c), apply Birkhoff's ergodic theorem to the flow $\{\theta_y\}$ and the TS limit distribution of P, and apply Theorem 4.1. For '⇐' of (c), apply again Vitali–Hahn–Saks. So, only (b) is left. If P is EAMS(P_{es}) and $P_{es}[0 < E_{es}(\alpha_0|\mathcal{I}) < \infty] = 1$, then application of (a) and Theorem 4.2 yields:

$$N_A(0,x]\frac{1}{N(0,x]} = \frac{1}{N(0,x]}\sum_{i=1}^{N(0,x]} 1_A \circ \eta_i \qquad \text{converges } P\text{-a.s. (as } x \to \infty), \tag{4.7a}$$

$$\frac{1}{x}N_A(0,x] = \frac{1}{x}N(0,x] \cdot N_A(0,x]\frac{1}{N(0,x]} \qquad \text{converges } P\text{-a.s. (as } x \to \infty). \tag{4.7b}$$

Implication '⇐' of (b) follows from (a), since the expression below converges P-a.s.:

$$\frac{1}{n}\sum_{i=1}^{n} 1_A \circ \eta_i = \frac{1}{n}N_A(0,T_n] = \frac{T_n}{N(0,T_n]} \cdot \frac{N_A(0,T_n]}{T_n}.$$

□

Since P is EAMS iff P_0 is EAMS, part (a) also follows from Theorem 1 of Gray and Kieffer [5]. By Theorem 4.3(a), (c) it follows immediately that, for point process distributions P and Q:

$$Q \ll P \quad \text{and} \quad P \text{ is EAMS} \quad \Rightarrow \quad Q \text{ is EAMS}, \tag{4.8a}$$

$$Q \ll P \quad \text{and} \quad P \text{ is TAMS} \quad \Rightarrow \quad Q \text{ is TAMS}. \tag{4.8b}$$

Theorems 4.1–4.3 yield several related limit results for AMS point processes; we mention a few. Suppose that P is EAMS(P_{es}) and $P_{es}[0 < E_{es}(\alpha_0|\mathcal{I}) < \infty] = 1$. Let P_{ts} be the TS limit-distribution of P with $\lambda_{ts} = E_{ts}(N(0,1]) < \infty$ and accompanying event-stationary PD P_{ts}^0. Then, it follows by (4.7a), (4.7b) that, for all $A \in \mathcal{M}^\infty$:

(a) $E(\frac{1}{x}N_A(0,x]) = \frac{1}{x}\int_{(0,x]} P^{0,y}(A)\,\mathrm{d}\nu(y) \to \lambda_{ts}P_{ts}^0(A)$ as $x \to \infty$;
(b) $E[N_A(0,x]/N(0,x]] \to P_{es}(A)$ while $E(N_A(0,x])/E(N(0,x]) \to P_{ts}^0(A)$ as $x \to \infty$; cf. (2.4).

Example 4.4. We will use Theorem 4.3(a) to construct a point process distribution P which is not EAMS. Set:

$$a(1) = 4 \quad \text{and} \quad a(k) = \begin{cases} a(k-1), & \text{if } k \text{ is even,} \\ \sum_{i=1}^{k-1} a(i), & \text{if } k \text{ is odd} \end{cases} \qquad \text{for } k = 2,3,\ldots,$$

$$b(0) = 0 \quad \text{and} \quad b(k) = \sum_{i=1}^{k} a(i) \qquad \text{for } k = 1,2,\ldots.$$

A sequence (x_i) of $\{0,1\}$-numbers is defined as follows:

$$x_i = \begin{cases} 1, & \text{if } i \in \{b(k)+1,\dots,b(k+1)\} \text{ for } k \text{ even},\\ 0, & \text{if } i \in \{b(k)+1,\dots,b(k+1)\} \text{ for } k \text{ odd} \end{cases} \qquad \text{for } k = 0,1,2,\dots.$$

Note that the sequence (m_n) with $m_n = \frac{1}{n}\sum_{i=1}^n x_i$ has no limit for $n \to \infty$ since:

$$m_{b(2n)} \to \tfrac{1}{2} \quad \text{and} \quad m_{b(2n+1)} \to \tfrac{3}{4}.$$

A point process distribution P which P-a.s. experiences a fixed eventuality A at the times T_i with $x_i = 1$ and A^c at the times T_i with $x_i = 0$, is *not* EAMS.

Unless stated otherwise, we will always assume that the conditions about $\overline{\alpha}$ and $\overline{N}$ in Theorem 4.2 are satisfied.

5. Absolute continuity properties equivalent to AMS

It is proven that AMS is equivalent to AC properties for $\{P_n\}$ and P^*, and also to a weak AC property for $\{P^{0,x}\}$. Thus, the class of AMS point processes is characterized in three ways. With (3.12a), (3.12b), we recognize the TS subclass within the AMS class.

If $P_m \ll P_{es}$ for a fixed $m \in \mathbb{Z}$, then $P_n \ll P_{es}$ for all $n \in \mathbb{Z}$ since $P_{es}(A) = 0$ implies $P_{es}[\eta_{n-m}(\varphi) \in A] = 0$ and hence $P_n(A) = P_m[\eta_{n-m}(\varphi) \in A] = 0$. We get, for each $m \in \mathbb{Z}$:

$$\{P_n\} \ll P_{es} \quad \Leftrightarrow_{\text{def}} \quad \forall_{n\in\mathbb{Z}}\colon\ P_n \ll P_{es} \quad \Leftrightarrow \quad P_m \ll P_{es}.$$

The theorem below shows that P is EAMS if and only if, for one (and hence all) $m \in \mathbb{Z}$, the intermediate distribution P_m is absolutely continuous w.r.t. an ES point process distribution.

Theorem 5.1. *Let P be a point process distribution. Then*:

(1) *P is* EAMS$(P_{es}) \Rightarrow \{P_n\} \ll P_{es}$,
(2) $\{P_n\} \ll P_{es}$ *with* $\delta_{-n} := \mathrm{d}P_n/\mathrm{d}P_{es} \Rightarrow P$ *is* EAMS$(\tilde{P}_{es})$.

Here, $\tilde{P}_{es}(A) = E_{es}(\overline{\delta}1_A)$ *with* $\overline{\delta} := E_{es}(\delta_0|\mathcal{I})$.

Proof. Suppose that P is EAMS(P_{es}), and that it holds for certain $A \in \mathcal{M}^\infty$ and $n \in \mathbb{Z}$ that $P_{es}(A) = 0$ but $P_n(A) = a > 0$. Set $\tilde{A} := A \cup (\bigcup_{k\in\mathbb{Z}} \eta_k^{-1}A)$. Note that $P_{es}(\tilde{A}) = 0$. However, for all $k \in \mathbb{Z}$ we have:

$$P_k(\tilde{A}) \geq P_k\bigl(\eta_{n-k}^{-1}A\bigr) = a.$$

Hence, $P_{es}(\tilde{A}) = \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^m P_k(\tilde{A}) \geq a > 0$, which leads to a contradiction. Hence, $\{P_n\} \ll P_{es}$. For the reversed implication, suppose that $\{P_n\} \ll P_{es}$ and $\delta_{-n} = \mathrm{d}P_n/\mathrm{d}P_{es}$. For all $n \in \mathbb{Z}$ and $C \in \mathcal{M}^\infty$ it holds that $P_n(C) = E_{es}(\delta_{-n}1_C)$, but also that:

$$P_n(C) = P_{-1}\bigl(\eta_{n+1}^{-1}C\bigr) = E_{es}(\delta_1 \cdot 1_C \circ \eta_{n+1}) = E_{es}(\delta_1 \circ \eta_{-n-1} \cdot 1_C).$$

So, $P_{es}[\delta_k = \delta_1 \circ \eta_{k-1}] = 1$ for all $k \in \mathbb{Z}$ and the discrete-time stochastic process $\{\delta_k\}$ is P_{es}-stationary. With $\overline{\delta} := E_{es}(\delta_0|\mathcal{I})$ we have by Birkhoff's ergodic theorem that, for all $A \in \mathcal{M}^\infty$:

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{-i} 1_A \to \overline{\delta} 1_A \qquad \text{as } n \to \infty \ P_{es}\text{-a.s.}$$

By taking P_{es}-expectation it follows that P is EAMS with ES limit distribution $\tilde{P}_{es}$. □

Remark. We conclude that P is AMS iff there exists an ES point process distribution P_{es} such that $P_0 \ll P_{es}$, which also follows by combining Theorems 2, 3 and 4 of Gray and Kieffer [5]. Comparison with (3.12a), (3.12b) learns that the class of TS point processes is (only) a relatively small part of the class of all AMS point processes.

Theorem 5.2. *Let P be a point process distribution, and P_{ts} a TS point process distribution with finite intensity λ_{ts} and associated PD P_{ts}^0. Let P^* be defined as in* (3.11). *Then*:

(a) *If $P \ll P_{ts}$ with $\sigma := \mathrm{d}P/\mathrm{d}P_{ts}$, then*:

$$P_0 \ll P_{ts}^0 \qquad \textit{with } \delta_0 := \mathrm{d}P_0/\mathrm{d}P_{ts}^0 = \lambda_{ts}\int_0^{\alpha_0} \sigma \circ \theta_y \,\mathrm{d}y;$$

(b) $P^* \ll P_{ts} \Leftrightarrow P_0 \ll P_{ts}^0$;
for $\sigma^* := \mathrm{d}P^*/\mathrm{d}P_{ts}$ and $\delta_0 := \mathrm{d}P_0/\mathrm{d}P_{ts}^0$ we have:

$$P_{ts}\big[\sigma^* = \delta_0 \circ \eta_0/(\lambda_{ts}\alpha_0)\big] = 1 \quad \textit{and} \quad P_{ts}^0\left[\delta_0 = \lambda_{ts}\int_0^{\alpha_0} \sigma^* \circ \theta_y \,\mathrm{d}y\right] = 1,$$

so σ^* satisfies $P_{ts}[\sigma^* \circ \eta_0 = \sigma^*] = 1$.

Proof. Part (a) follows immediately by applying (2.8a) and (2.7a) to P_{ts}, $P_{ts,0}$ and P_{ts}^0. Implication '⇒' of (b) is a consequence of (a) and Theorem 3.1(1). For '⇐' of (b), note that by (3.11), (2.7b) and (2.8a) we obtain for $A \in \mathcal{M}^\infty$ that $P^*(A)$ equals:

$$\begin{aligned} E_{ts}^0\left(\delta_0/\alpha_0 \cdot \int_0^{\alpha_0} 1_A \circ \theta_y \,\mathrm{d}y\right) &= \frac{1}{\lambda_{ts}} E_{ts}\left(\delta_0 \circ \eta_0/\alpha_0 \cdot \frac{1}{\alpha_0}\int_{T(0)}^{T(1)} 1_A \circ \theta_y \,\mathrm{d}y\right) \\ &= \frac{1}{\lambda_{ts}} E_{ts}\left(\delta_0 \circ \eta_0 \frac{1}{\alpha_0} 1_A\right) \end{aligned}$$

□

Hence, P is EAMS ⇔ there exists a TS point process distribution P_{ts} such that $P^* \ll P_{ts}$.

Theorem 5.3. *Suppose that $P(M) = 1$. For all $A \in \mathcal{M}$, the following holds*:

(1) $P(A) = 0 \quad \Leftrightarrow \quad P^x(A) = 0$ *for ν-a.e.* $x \in \mathbb{R}$;
(2) $P_n(A) = 0$ *for all $n \in \mathbb{Z}$ with* $P(F_n) > 0 \quad \Leftrightarrow \quad P^{0,x}(A) = 0$ *for ν-a.e.* $x \in \mathbb{R}$.

Proof. The left-hand sides of (1) and (2) are, respectively, equivalent to $\nu_A(B)$ being 0 for all $B \in \mathrm{Bor}(\mathbb{R})$ and to $\mu_A(B)$ being 0 for all $B \in \mathrm{Bor}(\mathbb{R})$. Next, use (3.1) and (3.3). □

Below, we will consider the following absolute continuity properties:

$$
\begin{aligned}
\{P^{0,x}\} \ll P_{es} \quad &\Leftrightarrow_{\text{def}} \quad \text{for } \nu\text{-a.e. } x \in \mathbb{R} \text{ and } \forall_{A\in\mathcal{M}}\text{:} \quad P_{es}(A) = 0 \quad \Rightarrow \quad P^{0,x}(A) = 0 \\
&\Leftrightarrow_{\text{def}} \quad \{P^{0,x}\} \text{ is } \textit{absolute continuous w.r.t. } P_{es}, \\
\{P^{0,x}\} \ll_w P_{es} \quad &\Leftrightarrow_{\text{def}} \quad \forall_{A\in\mathcal{M}} \text{ and for } \nu\text{-a.e. } x \in \mathbb{R}\text{:} \quad P_{es}(A) = 0 \quad \Rightarrow \quad P^{0,x}(A) = 0 \\
&\Leftrightarrow_{\text{def}} \quad \{P^{0,x}\} \text{ is } \textit{weakly absolute continuous w.r.t. } P_{es}, \\
\{P^{x}\} \ll P \quad &\Leftrightarrow_{\text{def}} \quad \text{for } \nu\text{-a.e. } x \in \mathbb{R} \text{ and } \forall_{A\in\mathcal{M}}\text{:} \quad P(A) = 0 \quad \Rightarrow \quad P^{x}(A) = 0.
\end{aligned}
$$

The next result is an immediate consequence of Theorem 5.3.

Corollary 5.4. *Let P be a point process distribution. It holds for event-stationary P_{es} that*: $\{P^{0,x}\} \ll P_{es} \Rightarrow \{P^{0,x}\} \ll_w P_{es} \Leftrightarrow \{P_n\} \ll P_{es}$.

The example below shows that Theorem 5.3 does not necessarily imply that $\{P^x\} \ll P$ and also not that $\{P^{0,x}\} \ll P_n$. It also shows that $\{P^{0,x}\} \ll_w P_{es}$ does not necessarily imply $\{P^{0,x}\} \ll P_{es}$.

Example 5.5. Let P be the distribution of an ES Poisson point process. Note that the eventualities $A_x := [\varphi\{x\} = 1]$ have P-probability 0 and P^x-probability 1 as long as $x \neq 0$. So, $\{P^x\} \ll P$ is *not* valid. Also $\{P^{0,x}\} \ll P_n$ is *not* valid since $P_n = P$ and the eventualities $C_x := [\varphi\{-x\} = 1]$ have $P^{0,x}$-probability 1 and P_n-probability 0 for $x \neq 0$. By Corollary 5.4, we have $\{P^{0,x}\} \ll_w P$. However, $\{P^{0,x}\} \ll P$ is not valid since for all $x \neq 0$ we have $P(C_x) = 0$ while $P^{0,x}(C_x) = 1$.

If P satisfies $P(M^0) = 1$, then $P = P_0$. So, the property $P \ll P_{es}$ is not interesting for further investigation about AMS. However, the property $P \ll P_{ts}$ is interesting since, by (2.8a), it only implies that $P^* \ll P_{ts}$ (i.e., no equivalence) and hence that P is AMS. By Corollary 5.4, the property $\{P^{0,x}\} \ll P_{es}$ also (only) implies that P is AMS. In Sections 6–8, we will characterize the properties $\{P^{0,x}\} \ll P_{es}$ and $P \ll P_{ts}$ and derive relationships between them.

6. Absolute continuity of $\{P^{0,x}\}$ w.r.t. P_{es}

The property $\{P^{0,x}\} \ll P_{es}$ implies that P can be expressed in P_{es}. The property is stronger than AMS; we characterize it. Below, we will use that:

$$P_{(\eta_n, T_n)}(A \times B) := P[\eta_n\varphi \in A; T_n\varphi \in B] \qquad \text{for } A \in \mathcal{M}^\infty \text{ and } B \in \mathrm{Bor}(\mathbb{R}).$$

Suppose that $\{P^{0,x}\} \ll P_{es}$ with RN derivatives $\{\rho_x\}$. Then we have, for ν-a.e. $x \in \mathbb{R}$:

$$P^{0,x}(A) = E_{es}(\rho_x 1_A) \quad \text{for all } A \in \mathcal{M}. \tag{6.1}$$

Since $P_{es}(M^\infty)=1$, it follows that $P^{0,x}(M^\infty)=1$ for ν-a.e. $x\in\mathbb{R}$. Hence, $P(M^\infty)=1$ by Theorem 5.3(1). By (3.6c) and (6.1) we can express P in P_{es}:

$$P(A)=E_{es}\left(\int_{(-T_{-k+1},-T_{-k}]}\rho_y\cdot 1_A\circ\theta_{-y}\,\mathrm{d}\nu(y)\right),\qquad A\in\mathcal{M}^\infty \text{ and } k\in\mathbb{Z}. \tag{6.2}$$

Theorem 6.1. *Let P_{es} be an ES distribution on $(M^0,\mathcal{M}^0)$. Then*:

$$\begin{aligned}&\{P^{0,x}\}\ll P_{es} \text{ on } (M^0,\mathcal{M}^0)\\ &\quad\Leftrightarrow\quad \forall_{n\in\mathbb{Z}}\colon\ \{P_{(\eta_n,T_n)}\}\ll P_{es}\times\nu \text{ on } \big(M^0\times\mathbb{R},\mathcal{M}^0\otimes\mathrm{Bor}(\mathbb{R})\big).\end{aligned}$$

The RN-derivatives

$$\rho_x(\varphi):=\frac{\mathrm{d}P^{0,x}}{\mathrm{d}P_{es}}(\varphi)\quad\text{and}\quad \tau_{-n}(\varphi,x):=\frac{\mathrm{d}P_{(\eta_n,T_n)}}{\mathrm{d}(P_{es}\times\nu)}(\varphi,x),$$

$\varphi\in M^0$, $x\in\mathbb{R}$ and $n\in\mathbb{Z}$, are related as follows:

(1) $\tau_{-n}(\varphi,x)=\rho_x(\varphi)\cdot 1_{[T(-n)\le -x<T(-n+1)]}(\varphi)$ $(P_{es}\times\nu)$*-a.e.*
(2) $\rho_x(\varphi)=\sum_{k\in\mathbb{Z}}\tau_{-k}(\varphi,x)$ P_{es}*-a.s. for ν-a.e. $x\in\mathbb{R}$.*

Proof. The implication '$\Rightarrow$' and (1) follow from (3.9) and (6.1). Next, suppose that for all $n\in\mathbb{Z}$ the P-distribution of (η_n,T_n) is dominated by $P_{es}\times\nu$, with RN-derivative denoted as $\tau_{-n}(\varphi,x)$. Set $Q^{0,x}(A):=\int_A(\sum_{n\in\mathbb{Z}}\tau_{-n}(\varphi,x))\,\mathrm{d}P_{es}(\varphi)$, for $A\in\mathcal{M}^0$. Note that

$$\int_B Q^{0,x}\big(M^0\big)\,\mathrm{d}\nu(x)=\sum_{n\in\mathbb{Z}}\int_B\int_{M^0}\tau_{-n}(\varphi,x)\,\mathrm{d}P_{es}(\varphi)\,\mathrm{d}\nu(x)=\nu(B)=\int_B 1\,\mathrm{d}\nu(x)$$

for all $B\in\mathrm{Bor}(\mathbb{R})$. So, for ν-a.e. $x\in\mathbb{R}$ it holds that $Q^{0,x}(M^0)=1$ and $Q^{0,x}$ is a probability measure. The right-hand side of (3.2), with $P^x(\cdot)$ replaced by $Q^{0,x}[\theta_{-x}\varphi\in\cdot]$, equals

$$\begin{aligned}&\int_{\mathbb{R}}\int_{M^0}f(x,\theta_{-x}\varphi)\,\mathrm{d}Q^{0,x}(\varphi)\,\mathrm{d}\nu(x)\\ &\quad=\int_{\mathbb{R}}\int_{M^0}f(x,\theta_{-x}\varphi)\sum_{n\in\mathbb{Z}}\tau_{-n}(\varphi,x)\,\mathrm{d}P_{es}(\varphi)\,\mathrm{d}\nu(x)\\ &\quad=\sum_{n\in\mathbb{Z}}\int_{M^0\times\mathbb{R}}f(x,\theta_{-x}\varphi)\,\mathrm{d}P_{(\eta_n,T_n)}(\varphi,x)\\ &\quad=\sum_{n\in\mathbb{Z}}\int_M f\big(T_n\varphi,\theta_{-T_n(\varphi)}(\eta_n\varphi)\big)\,\mathrm{d}P(\varphi)=\sum_{n\in\mathbb{Z}}\int_M f(T_n\varphi,\varphi)\,\mathrm{d}P(\varphi),\end{aligned}$$

which is just the left-hand side of (3.2). Since the family of PD's of P is unique in the ν-a.e. sense, we have $P^{0,x}=Q^{0,x}$ for ν-a.e. $x\in\mathbb{R}$. The if-part and (2) follow. $\square$

Corollary 6.2. *Suppose that* $\{P^{0,x}\} \ll P_{es}$ *with RN-derivatives* $\{\rho_x\}$. *Then*:

(1) *For all* $n \in \mathbb{Z}$: $P_n \ll P_{es}$ *with RN-derivative* $\delta_{-n} = \int_{(-T(-n+1),-T(-n)]} \rho_y \, \mathrm{d}\nu(y)$.
(2) *For all* $m \in \mathbb{Z}$ *it holds that* $P_{es}[\delta_{m+1} = \delta_m \circ \eta_1] = 1$, *so* $\{\delta_n\}$ *is* P_{es}*-stationary.*
(3) *If it holds additionally that* $P_{es}[\delta_0 > 0] = 1$, *then*:

$$\{P^{0,x}\} \ll P_0 \quad \text{and} \quad P_0 \ll P_{es}; \qquad \text{here, } \mathrm{d}P^{0,x}/\mathrm{d}P_0 = \rho_x/\delta_0 \ P_0\text{-a.s.}$$

Proof. (1) follows immediately from Theorem 6.1. For (2), note that, for all $A \in \mathcal{M}^\infty$ and $m \in \mathbb{Z}$:

$$E_{es}(1_A \cdot \delta_{m+1}) = P_{-m}[\eta_{-1}\varphi \in A] = E_{es}(1_A \circ \eta_{-1} \cdot \delta_m) = E_{es}(1_A \cdot \delta_m \circ \eta_1).$$

Part (3) follows from (1) and from the fact that it holds for ν-a.e. $x \in \mathbb{R}$ that:

$$P^{0,x}(A) = E_{es}(\rho_x 1_A) = E_{es}(\rho_x 1_A 1_{[\delta_0 > 0]}) = E_0\left(\rho_x \frac{1}{\delta_0} 1_A\right) \qquad \text{for } A \in \mathcal{M}^0. \qquad \square$$

Note that Corollary 6.2(1) generalizes (2.7a). By (6.2), the additional assumption $P_{es}[\delta_0 > 0] = 1$ yields that P can be expressed in terms of P_0, a property that according to (2.8a) and (3.11) also holds for TS distributions P and, more generally, for distributions P with $P^* = P$.

7. Absolute continuity of P w.r.t. P_{ts}

The point processes with $P \ll P_{ts}$ are characterized within the class of AMS point processes. We also compare the properties $P \ll P_{ts}$, $P \ll P^*$, $P^* \ll P_{ts}$, $\nu \ll \text{Leb}$, and $P^* = P$. The equivalence of 'P is also TS' and '$\mathcal{I}$-measurability of $\mathrm{d}P/\mathrm{d}P_{ts}$' is proved. For time-stationary P, the property $P \ll P_{ts}$ holds equivalently for the associated event-stationary PDs.

Assume that $P \ll P_{ts}$. Hence, $P[\varphi\{0\} = 0] = 1$ and $P(M^\infty) = 1$. Set $\sigma := \mathrm{d}P/\mathrm{d}P_{ts}$, let λ_{ts} be the (finite) intensity of P_{ts} and let P_{ts}^0 be the event-stationary PD of P_{ts}. It follows that:

$$P(A) = E_{ts}(\sigma \cdot 1_A), \qquad A \in \mathcal{M}^\infty, \tag{7.1}$$

$$P(A) = \lambda_{ts} E_{ts}^0\left(\int_{(-T_{-k+1},-T_{-k}]} \sigma \circ \theta_{-y} \cdot 1_A \circ \theta_{-y} \, \mathrm{d}y\right), \qquad A \in \mathcal{M}^\infty \text{ and } k \in \mathbb{Z}. \tag{7.2}$$

Theorem 7.1. *Let P and P_{ts} be point process distributions and let P^* be as in* (3.11); *suppose that P_{ts} is time-stationary. Below, versions of RN-derivatives for $P \ll P_{ts}$, $P^* \ll P_{ts}$ and $P \ll P^*$ are* (*if existing*) *respectively denoted as σ, σ^* and τ.*

(a) $P \ll P_{ts} \Leftrightarrow P \ll P^*$ *and* $P^* \ll P_{ts}$; $\sigma^* = \frac{1}{\alpha_0}\int_{T(0)}^{T(1)} \sigma \circ \theta_y \, \mathrm{d}y$ *and* $\tau = \sigma/\sigma^*$;
(b) *If* $P \ll P_{ts}$, *then*: $P = P^* \Leftrightarrow P_{ts}[\sigma = \sigma \circ \eta_0] = 1$;
(c) *If* $P \ll P_{ts}$, *then* $\nu \ll \text{Leb}$.

Proof. The implication '$\Leftarrow$' of (a) is trivial. For the implication '$\Rightarrow$', suppose that $P \ll P_{ts}$. By (3.11), (7.1), and (2.8a) under P_{ts}, it follows that $P^* \ll P_{ts}$ with σ^* as indicated. Since

$\sigma^* \circ \eta_0 = \sigma^*$, we obtain by (3.11) that not only $P^*[\sigma^* = 0] = 0$, but also $P[\sigma^* = 0] = 0$. It follows that $P \ll P^*$ since, because of $P^* \ll P_{ts}$, we have for all $A \in \mathcal{M}^\infty$:

$$E^*\left(\frac{\sigma}{\sigma^*} 1_A\right) = E_{ts}(\sigma 1_{[\sigma^*>0]} 1_A) = P\big(A \cap [\sigma^* > 0]\big) = P(A).$$

Part (b) follows from (a). For (c), suppose that $P \ll P_{ts}$. If $B \in \mathrm{Bor}(\mathbb{R})$ satisfies $\mathrm{Leb}(B) = 0$, then $\nu(B) = 0$ since:

$$E_{ts} N(B) = \lambda_{ts} \cdot \mathrm{Leb}(B) = 0 \quad \text{and} \quad P_{ts}\big[\varphi(B) = 0\big] = 1 = P\big[\varphi(B) = 0\big]. \qquad \square$$

By Theorems 5.2(b) and 7.1(a) it follows that the point processes with $P \ll P_{ts}$ are just the AMS point processes for which it additionally holds that $P \ll P^*$.

If P is time-stationary too …

We will consider the consequences of $P \ll P_{ts}$ if P is also time-stationary.

Theorem 7.2. *Suppose that $P \ll P_{ts}$ and that $\lambda_{ts} < \infty$. Then*:

(a) *P is time-stationary too $\Leftrightarrow$ there exists an $\mathcal{I}$-measurable version of* $\mathrm{d}P/\mathrm{d}P_{ts}$.
(b) *If P is also time-stationary and P_{ts} is ergodic*, *then $P = P_{ts}$.*
(c) *If P is also time-stationary with intensity λ and P_{ts} is pseudo-ergodic*, *then P is also pseudo-ergodic and $\lambda = \lambda_{ts}$.*

Proof. For (a), suppose that P is also TS and set $\sigma = \mathrm{d}P/\mathrm{d}P_{ts}$. By Birkhoff's ergodic theorem and taking E-expectations, we obtain for $A \in \mathcal{M}^\infty$:

$$\frac{1}{x}\int_0^x 1_A \circ \theta_y \,\mathrm{d}y \to E_{ts}(1_A|\mathcal{I}) \qquad \text{as } x \to \infty \ P\text{-a.s.}, \tag{7.3}$$
$$P(A) = E\big(E_{ts}(1_A|\mathcal{I})\big) = E_{ts}\big(\sigma E_{ts}(1_A|\mathcal{I})\big) = E_{ts}(\overline{\sigma} 1_A).$$

Hence, $\overline{\sigma} := E_{ts}(\sigma|\mathcal{I})$ is an $\mathcal{I}$-measurable version of $\mathrm{d}P/\mathrm{d}P_{ts}$. The if-part follows from (1.1). Statement (b) follows from (7.3) since now the limit is $P_{ts}(A)$. For (c), note that $E(N(0,1]|\mathcal{I})$ and λ_{ts} are both the P-a.s. limit of $N(0,x]/x$ as $x \to \infty$. Hence, $P[E(N(0,1]|\mathcal{I}) = \lambda_{ts}] = 1$, P is pseudo-ergodic too, and $\lambda = \lambda_{ts}$. $\square$

Theorem 7.3. *Suppose that P and P_{ts} are both TS with respective* (*finite*) *intensities λ and λ_{ts}*, *and accompanying event-stationary PDs P^0 and P^0_{ts}. Then*:

$$P \ll P_{ts} \quad \Leftrightarrow \quad P^0 \ll P^0_{ts}.$$

Respective $\mathcal{I}$-measurable versions σ and σ_0 of the RNs satisfy: $\lambda\sigma_0 = \lambda_{ts}\sigma$ *P^0_{ts}-a.s. and P_{ts}-a.s.*

Proof. If $P \ll P_{ts}$, then, by Theorem 7.2(a), we can take an $\mathcal{I}$-measurable version σ for $\mathrm{d}P/\mathrm{d}P_{ts}$. By (2.7b), (1.1) and (2.7a) we obtain for all $A \in \mathcal{M}^\infty$:

$$P^0(A) = \frac{1}{\lambda} E\left(\frac{1}{\alpha_0} 1_A \circ \eta_0\right) = \frac{1}{\lambda} E_{ts}\left(\sigma \frac{1}{\alpha_0} 1_A \circ \eta_0\right) = \frac{\lambda_{ts}}{\lambda} E^0_{ts}(\sigma \cdot 1_A).$$

Hence, $P^0 \ll P^0_{ts}$, and $\sigma_0 = \mathrm{d}P^0/\mathrm{d}P^0_{ts}$ satisfies $P^0_{ts}[\lambda\sigma_0 = \lambda_{ts}\sigma] = 1$. If $P^0 \ll P^0_{ts}$, it can be proved (as in the proof of Theorem 7.2(a)) that σ_0 can be taken as an $\mathcal{I}$-measurable function. By (2.6), (2.7b) under P_{ts}, and (2.8c) under P_{ts}, we have for $C \in \mathcal{M}^\infty$:

$$\begin{aligned} P(C) &= \lambda E^0\left(\int_0^{\alpha_0} 1_C \circ \theta_y \,\mathrm{d}y\right) = \lambda E^0_{ts}\left(\sigma_0 \int_0^{\alpha_0} 1_C \circ \theta_y \,\mathrm{d}y\right) \\ &= \frac{\lambda}{\lambda_{ts}} E_{ts}\left(\sigma_0 \cdot \frac{1}{\alpha_0} \int_{T(0)}^{T(1)} 1_C \circ \theta_y \,\mathrm{d}y\right) = \frac{\lambda}{\lambda_{ts}} E_{ts}(\sigma_0 \cdot 1_C). \end{aligned}$$

□

8. Relationships between absolute continuity properties

The properties $P \ll P_{ts}$ and $\{P^{0,x}\} \ll P^0_{ts}$ are compared and the relationships between the accompanying RN derivatives are investigated. If $P^* = P$, then P is AMS iff there exist a time-stationary P_{ts} which dominates P.

Theorem 8.1. *Let P and P_{ts} be point process distributions, where P_{ts} is time-stationary, $\lambda_{ts} < \infty$ and P^0_{ts} is the accompanying Palm distribution. Then*:

$$P \ll P_{ts} \quad \Leftrightarrow \quad \nu \ll \mathrm{Leb} \quad \textit{and} \quad \left\{P^{0,x}\right\} \ll P^0_{ts}.$$

The RN-derivatives $\sigma := \mathrm{d}P/\mathrm{d}P_{ts}$, $\lambda(\cdot) := \mathrm{d}\nu/\mathrm{dLeb}$ and $\rho_x := \mathrm{d}P^{0,x}/\mathrm{d}P^0_{ts}$ satisfy:

(a) $\lambda(y) = \lambda_{ts} \cdot E^0_{ts}(\sigma \circ \theta_{-y})$ *for Leb-a.e.* $y \in \mathbb{R}$;
(b) $P^0_{ts}[\lambda(y) \cdot \rho_y = \lambda_{ts} \cdot \sigma \circ \theta_{-y}] = 1$ *for Leb-a.e.* $y \in \mathbb{R}$;
(c) $P^0_{ts}[\lambda(y) \cdot \rho_y = \lambda_{ts} \cdot \sigma \circ \theta_{-y}$ *for Leb-a.e.* $y \in \mathbb{R}] = 1$;
(d) $P_{ts}[\lambda_{ts} \cdot \sigma = \lambda(T_k) \cdot (\rho_{T_k} \circ \eta_k)] = 1$ *for all* $k \in \mathbb{Z}$.

Proof. Suppose that $P \ll P_{ts}$ with RN-density σ. First note that $\nu \ll \mathrm{Leb}$ by Theorem 7.1(c). Write $\lambda(\cdot)$ for the RN-density and note that $\lambda(x) > 0$ for ν-a.e. $x \in \mathbb{R}$. For $B \in \mathrm{Bor}(\mathbb{R})$ we have:

$$\int_B \lambda(x)\,\mathrm{d}x = EN(B) = \sum_{k\in\mathbb{Z}} P[T_k \in B].$$

For all $\varphi \in M^0$, $k \in \mathbb{Z}$, and $y \in \mathbb{R}$ with $T_{-k}(\varphi) < -y \le T_{-k+1}(\varphi)$, we have $T_k(\theta_{-y}\varphi) = y$ and $\eta_k(\theta_{-y}\varphi) = \varphi$; call this observation (*). Taking $A = [T_k \in B]$ in (7.2) yields

$$\int_B \lambda(x)\,\mathrm{d}x = \int_B \lambda_{ts} E^0_{ts}(\sigma \circ \theta_{-x})\,\mathrm{d}x,$$

which proves (a). As a consequence of (a), we have:

$$\mathrm{Leb}\big\{x \in \mathbb{R}\colon\ \lambda(x) = 0 \text{ and } P_{ts}^0\big[\sigma(\theta_{-x}\varphi) \neq 0\big] > 0\big\} = 0. \tag{8.1}$$

To prove that $\{P^{0,x}\} \ll P_{ts}^0$, we use (3.2). For ν-a.e. $x \in \mathbb{R}$, we define probability measures $Q^{0,x}$ on $(M, \mathcal{M})$ as follows: $Q^{0,x}(C) := \lambda_{ts} E_{ts}^0(\sigma \circ \theta_{-x} \cdot 1_C)/\lambda(x)$ for $C \in \mathcal{M}^\infty$. By (7.1), (2.6), the above observation (*), and (8.1), the left-hand side of (3.2) equals:

$$\begin{aligned}
E_{ts}\bigg[\sigma(\varphi)\sum_{k\in\mathbb{Z}} f(T_k\varphi, \varphi)\bigg] &= \sum_{k\in\mathbb{Z}} \lambda_{ts} E_{ts}^0\bigg[\int_{-T_{-k+1}}^{-T_{-k}} \sigma(\theta_{-x}\varphi) f\big(T_k(\theta_{-x}\varphi), \theta_{-x}\varphi\big)\,\mathrm{d}x\bigg] \\
&= \int_{-\infty}^{\infty}\int_M f(x, \theta_{-x}\varphi)\lambda_{ts}\sigma(\theta_{-x}\varphi)\,\mathrm{d}P_{ts}^0(\varphi)\,\mathrm{d}x \\
&= \int_{\{x\in\mathbb{R}\colon\ \lambda(x)>0\}}\int_M f(x, \theta_{-x}\varphi)\lambda_{ts}\sigma(\theta_{-x}\varphi)\frac{1}{\lambda(x)}\,\mathrm{d}P_{ts}^0(\varphi)\,\mathrm{d}\nu(x) \\
&= \int_{-\infty}^{\infty}\int_M f(x, \theta_{-x}\varphi)\,\mathrm{d}Q^{0,x}(\varphi)\,\mathrm{d}\nu(x).
\end{aligned} \tag{8.2}$$

Note that (8.2) is just the right-hand side of (3.2) if we take $P^x(A) = Q^{0,x}[\theta_{-x}\varphi \in A]$, $A \in \mathcal{M}^\infty$. Because of the uniqueness of $\{P^x\}$ it follows for ν-a.e. $x \in \mathbb{R}$ that $Q^{0,x} = P^{0,x}$, that $P^{0,x}$ is dominated by P_{ts}^0 for ν-a.e. $x \in \mathbb{R}$ and that the RNs ρ_x satisfy (b) with ν instead of Leb. Hence, $\int_{\mathbb{R}} P_{ts}^0[\lambda(x)\rho_x \neq \lambda_{ts}\sigma \circ \theta_{-x}]\,\mathrm{d}\nu(x) = 0$ and

$$\mathrm{Leb}\big\{x \in \mathbb{R}\colon\ \lambda(x) P_{ts}^0\big[\lambda(x)\rho_x \neq \lambda_{ts}\sigma \circ \theta_{-x}\big] > 0\big\} = 0.$$

By (8.1), we obtain that $P_{ts}^0[\lambda(x)\rho_x \neq \lambda_{ts}\sigma \circ \theta_{-x}] = 0$ for Leb-a.e. $x \in \mathbb{R}$, which proves (b). Since (b) can equivalently be formulated as

$$\int_{\mathbb{R}} P_{ts}^0\big[\lambda(y)\rho_y \neq \lambda_{ts}\sigma \circ \theta_{-y}\big]\,\mathrm{d}y = 0,$$

result (c) is just a consequence of Fubini's theorem. Next, suppose that $\nu \ll \mathrm{Leb}$ and $\{P^{0,x}\} \ll P_{ts}^0$. By (6.2) we have, for $A \in \mathcal{M}^\infty$ and $k \in \mathbb{Z}$:

$$P(A) = E_{ts}^0\bigg(\int_{(-T_{-k+1}, -T_{-k}]} \rho_y \cdot 1_A \circ \theta_{-y}\lambda(y)\,\mathrm{d}y\bigg). \tag{8.3}$$

By observation (*), we can replace ρ_y and $\lambda(y)$ by, respectively, $\rho_{T_k\circ\theta_{-y}} \circ \eta_k \circ \theta_{-y}$ and $\lambda(T_k \circ \theta_{-y})$. We obtain by (2.6) under P_{ts} that the right-hand side of (8.3) equals:

$$E_{ts}\big(\lambda(T_k) \cdot (\rho_{T_k} \circ \eta_k) \cdot 1_A\big)/\lambda_{ts}.$$

Hence, $P \ll P_{ts}$ and (d) follows. $\square$

Remark. Theorem 8.1 generalizes Theorem 7.3. Also note that, by (a) and (c), it holds P^0_{ts}-a.s. that:

$$\rho_x = \frac{\sigma \circ \theta_{-x}}{E^0_{ts}(\sigma \circ \theta_{-x})} \qquad \text{for } \nu\text{-a.e. } x \in \mathbb{R}. \tag{8.4}$$

Starting with some preliminary TS model P_{ts}, each measurable function $\sigma : M^\infty \to [0, \infty)$ with $E_{ts}(\sigma) = 1$ can be used to transform P_{ts} into a (usually) not TS but AMS new model P via $P(A) = E_{ts}(\sigma \cdot 1_A)$, $A \in \mathcal{M}^\infty$. The accompanying family $\{P^{0,x}\}$ of shifted Palm distributions is then dominated by the (event-stationary) Palm distribution of P_{ts}. The family of RN-densities $\{\rho_x\}$ is given by (8.4).

By Theorems 8.1, 3.1(3), and Corollary 5.4 it follows that for point process distributions P with $P^* = P$, weak absolute domination of $\{P^{0,x}\}$ (and hence AMS) is equivalent to strong absolute domination:

Corollary 8.2. *If P is a point process distribution with $P^* = P$, then*:

$$P \ll P_{ts} \quad \Leftrightarrow \quad \left\{P^{0,x}\right\} \ll P^0_{ts} \quad \Leftrightarrow \quad \left\{P^{0,x}\right\} \ll_w P^0_{ts}.$$

The next results are immediate consequences of Corollary 6.2(1), Theorem 8.1 and (3.3).

Corollary 8.3. *Suppose that $P \ll P_{ts}$ with $\lambda_{ts} < \infty$ and $\sigma = \mathrm{d}P/\mathrm{d}P_{ts}$. Then*:

(a) *$\{P_n\} \ll P^0_{ts}$ with $\delta_{-n} := \mathrm{d}P_n/\mathrm{d}P^0_{ts} = \lambda_{ts} \int_{T_{-n}}^{T_{-n+1}} \sigma \circ \theta_y \, \mathrm{d}y$ for all $n \in \mathbb{Z}$.*

(b) *For ν-a.e. $x \in \mathbb{R}$ and all $A \in \mathcal{M}^\infty$ it holds that*:

$$P^{0,x}(A) = \frac{\lambda_{ts} E^0_{ts}(1_A \cdot \sigma \circ \theta_{-x})}{\lambda_{ts} E^0_{ts}(\sigma \circ \theta_{-x})}. \tag{8.5}$$

Here, numerator and denominator are just $\lambda_A(x)$ and $\lambda(x)$, RN-derivatives of μ_A and ν with respect to Leb.

(c) *If it holds additionally that $\sigma(\varphi) = \sigma(\eta_0(\varphi))$ for all $\varphi \in M^\infty$ (and hence $P^* = P$), then $\delta_{-n} = \sigma \circ \eta_{-n} \cdot \lambda_{ts} \alpha_{-n}$.*

In the example below, results of this paper are used to investigate consequences of a transformation via $P \ll P_{ts}$ if P_{ts} is TS and Poisson.

Example 8.4. Let P_{ts} be the distribution of a TS Poisson point process on $\mathbb{R}$ with intensity λ_{ts}. Suppose that $P \ll P_{ts}$ with $\sigma(\varphi) = \lambda_{ts}\alpha_0(\varphi)/2$ for $\varphi \in M^\infty$. It follows that, for $x \geq 0$:

$$P[\alpha_0 > x] = \mathrm{e}^{-\lambda_{ts} x}\left(\lambda^2_{ts} x^2/2 + \lambda_{ts} x + 1\right) \quad \text{and} \quad P_{ts}[\alpha_0 > x] = \mathrm{e}^{-\lambda_{ts} x}(\lambda_{ts} x + 1).$$

Hence, α_0 is under P stochastically larger than under P_{ts}. However, for $i \neq 0$ the distributions of α_i under P and P_{ts} are the same. By Theorems 7.1(b) and 3.1(2) it follows that, as under P_{ts}, it holds under P that T_1 is (conditionally) uniformly distributed on $(0, \alpha_0)$. Starting with Theorem 8.1(a), (8.4) and (8.5), we obtain after tough calculations that, for $x \in \mathbb{R}$:

$$\lambda(x) = \lambda_{ts}^2 E_{ts}^0(\alpha_0 \circ \theta_{-x})/2 = \lambda_{ts} - \lambda_{ts}\exp(-\lambda_{ts}|x|)/2,$$
$$\rho_x = \lambda_{ts}\alpha_0 \circ \theta_{-x}/\bigl(2 - \exp\bigl(-\lambda_{ts}|x|\bigr)\bigr),$$
$$P^{0,x}(A) = \lambda_{ts} E_{ts}^0(1_A \cdot \alpha_0 \circ \theta_{-x})/\bigl(2 - \exp\bigl(-\lambda_{ts}|x|\bigr)\bigr).$$

The independence of the interval lengths under P_{ts}^0 yields that:

- if $x \le 0$ and $A \in \sigma\{\alpha_{-1}, \alpha_{-2}, \dots\}$ then $P^{0,x}(A) = P_{ts}^0(A)$;
- if $x > 0$ and $A \in \sigma\{\alpha_0, \alpha_1, \dots\}$ then $P^{0,x}(A) = P_{ts}^0(A)$.

More generally, it can be proven that, for each $n \in \mathbb{Z}$, the RN-derivative $\sigma_n := \alpha_n/E_{ts}(\alpha_n)$ transforms the TS Poisson distribution P_{ts} into an AMS distribution P that conserves the independence of the interval lengths α_k and for $k \neq n$ also their distributions, but making α_n stochastically larger. However, an RN-derivative of the form $\sigma = \gamma_0\alpha_0 + \gamma_1\alpha_1$ with $\gamma_0 \ge 0$, $\gamma_1 \ge 0$ and $\gamma_1 = \lambda_{ts} - 2\gamma_0$ transforms P_{ts} into a distribution under which α_0 and α_1 are independent if and only if $(\gamma_0, \gamma_1) = (0, \lambda_{ts})$ or $(\gamma_0, \gamma_1) = (\lambda_{ts}/2, 0)$.

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