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Self-normalized Cramér type moderate deviations for the maximum of sums

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Let $X_1, X_2, \ldots$ be independent random variables with zero means and finite variances, and let $S_n = \sum_{i=1}^n X_i$ and $V_n^2 = \sum_{i=1}^n X_i^2$. A Cramér type moderate deviation for the maximum of the self-normalized sums $\max_{1\le k\le n} S_k/V_n$ is obtained. In particular, for identically distributed $X_1, X_2, \ldots,$ it is proved that $\mathsf{P}(\max_{1\le k\le n} S_k \ge xV_n)/(1-\Phi(x)) \to 2$ uniformly for $0 < x \le \mathrm{o}(n^{1/6})$ under the optimal finite third moment of X_1.

Keywords: independent random variables; maximum of self-normalized sums

1. Introduction and main results

Let $X_1, X_2, \ldots$ be a sequence of independent non-degenerate random variables with zero means. Set

$$S_n = \sum_{j=1}^n X_j \quad \text{and} \quad V_n^2 = \sum_{j=1}^n X_j^2.$$

The past decade has brought significant developments in the limit theorems for the so-called "self-normalized" sum, S_n/V_n. It is now well understood that the limit theorems for S_n/V_n usually require fewer moment assumptions than those for their classical standardized counterpart, and thus have much wider applicability. For examples, for identically distributed $X_1, X_2, \ldots,$ a self-normalized large deviation holds without any moment assumption (Shao [11]), and a Cramér type moderate deviation (Shao [12]),

$$\lim_{n\to\infty} \frac{\mathsf{P}(S_n \ge xV_n)}{1-\Phi(x)} = 1, \tag{1.1}$$

holds uniformly for $x \in [0, \mathrm{o}(n^{1/6}))$ provided that $\mathsf{E}|X_1|^3 < \infty$, whereas a finite moment-generating condition of $\sqrt{|X_1|}$ is necessary for a similar result for the standard sum $S_n/\sqrt{\mathsf{Var}(S_n)}$ (see, e.g., Linnik [9]). For more related results, we refer to de la Peña, Lai and Shao [5] for a systematic treatment of the theory and applications of self-normalization and Wang [13] for some refined self-normalized moderate deviations.

As for the Cramér type moderate deviations for the maximum of self-normalized sums, namely for $\max_{1\le k\le n} S_k/V_n$, Hu, Shao and Wang [7] were the first to prove that if $X_1, X_2, \dots$ is a sequence of i.i.d. random variables with $\mathsf{E}X_1^4 < \infty$, then

$$\lim_{n\to\infty} \frac{\mathsf{P}(\max_{1\le k\le n} S_k \ge xV_n)}{1-\Phi(x)} = 2, \tag{1.2}$$

uniformly for $x \in [0, \mathrm{o}(n^{1/6}))$. This contrasts with the moderate deviation result for the maximum of partial sums of Aleshkyavichene [1,2], where a finite moment-generating condition is required. However, in view of the result given in (1.1), it is natural to ask whether a finite third moment suffices for (1.2). The main purpose of this paper is to provide an affirmative answer to this question. Indeed, we have the following more general result for independent random variables.

Theorem 1. *Assume that* $\max_{k\ge 1} \mathsf{E}|X_k|^{2+r} < \infty$ *and* $\min_{k\ge 1} \mathsf{E}X_k^2 > 0$, *where* $0 < r \le 1$. *Then* (1.2) *holds uniformly in* $0 \le x \le \mathrm{o}(n^{r/(4+2r)})$.

As in the moderate deviation result for self-normalized sum S_n/V_n, Theorem 1 is sharp in both the moment condition and the range in which the result (1.2) holds true. Examples can be constructed similarly as done by Chistyakov and Götze [4] and Shao [12]. In particular, for $r = 1$ and identically distributed $X_1, X_2, \dots,$ Theorem 1 establishes (1.2) under the optimal finite third moment of X_1.

Theorem 1 can be extended further; in fact, it is a direct consequence of Theorem 2 below. Set $B_n^2 = \sum_{i=1}^n \mathsf{E}X_i^2$, $L_{n,r} = \sum_{i=1}^n \mathsf{E}|X_i|^{2+r}$ and $d_{n,r} = B_n/L_{n,r}^{1/(2+r)}$, where $0 < r \le 1$.

Theorem 2. *For* $0 < r \le 1$, *suppose that* $d_{n,r} \to \infty$ *as* $n \to \infty$, *and that*

$$\max_{1\le k\le n} \frac{\sum_{j=k}^n \mathsf{E}|X_j|^{2+r}}{\sum_{j=k}^n \mathsf{E}|X_j|^2} \le \frac{\tau L_{n,r}^{r/(2+r)}}{d_{n,r}^{\delta}} \qquad \text{for some } \delta, \tau > 0. \tag{1.3}$$

Then (1.2) *holds uniformly in* $0 \le x \le \min\{B_n, \mathrm{o}(d_{n,r})\}$.

Remark 1. For i.i.d. random variables with $\mathsf{E}X_i = 0$ and $\mathsf{E}|X_i|^3 < \infty$, Jing, Shao and Wang [8] proved that (1.1) can be refined as

$$\frac{\mathsf{P}(S_n \ge xV_n)}{1-\Phi(x)} = 1 + \mathrm{O}(1)(1+x^3)\mathsf{E}|X_1|^3/(\mathsf{E}X_1^2)^{3/2}$$

uniformly in $x \in [0, n^{1/6}(\mathsf{E}X_1^2)^{1/3}/(\mathsf{E}X_1^2)^{1/2})$, where $\mathrm{O}(1)$ is bounded by an absolute constant. We conjecture that a similar result holds for $\max_{1\le k\le n} S_k/V_n$, that is,

$$\frac{\mathsf{P}(\max_{1\le k\le n} S_k \ge xV_n)}{1-\Phi(x)} = 2 + \mathrm{O}(1)(1+x^3)\mathsf{E}|X_1|^3/(\mathsf{E}X_1^2)^{3/2}$$

uniformly in $x \in [0, n^{1/6}(\mathsf{E}X_1^2)^{1/2}/(\mathsf{E}|X_1|^3)^{1/3})$.

This paper is organized as follows. The proof of the main theorems is given in the next section. The proofs of two technical propositions are deferred to Sections 3 and 4, respectively. Throughout the paper, $A, A_1, \ldots$ denotes absolute constants and $C_{\delta,\tau}$ denotes a constant depending only on δ and τ, which might be different at each appearance.

2. Proofs of theorems

Proof of Theorem 1. Simple calculations show that if

$$\max_{k\ge 1} \mathsf{E}|X_k|^{2+r} < \infty \quad \text{and} \quad \min_{k\ge 1} \mathsf{E}X_k^2 > 0,$$

then $B_n^2 \asymp n$, $L_{n,r} \asymp n$, $d_{n,r} \asymp n^{r/(4+2r)}$ and (1.3) holds for $\delta = 1$ and some $\tau > 0$, where the notation $a_n \asymp b_n$ denotes $0 < \underline{\lim}_{n\to\infty} a_n/b_n < \overline{\lim}_{n\to\infty} a_n/b_n < \infty$. Therefore, Theorem 1 follows immediately from Theorem 2. □

Proof of Theorem 2. First note that for $\forall\epsilon > 0$,

$$\frac{1}{B_n^2}\sum_{k=1}^n \mathsf{E}X_k^2 I(|X_k| \ge \epsilon B_n) \le \epsilon^{-r} d_{n,r}^{-1/(2+r)} \to 0$$

whenever $d_{n,r} \to 0$. That is, the Lindeberg condition is satisfied for the sequence $X_1, X_2, \ldots$. On the other hand, routine calculations show that, given $d_{n,r} \to 0$, $V_n^2/B_n^2 \to 1$ in probability. Given these facts, the invariance principle (see Theorem 2 of Brown [3]) and the continuous mapping theorem imply that $\max_{1\le k\le n} S_k/V_n \to_D |N(0,1)|$. This yields (1.2) uniformly for $0 \le x \le M$, where M is an arbitrary constant. Thus, Theorem 2 will follow if we can prove

$$\lim_{M\to\infty}\lim_{n\to\infty}\sup_{M\le x\le \min\{B_n, \mathrm{o}(d_{n,r})\}}\left|\frac{\mathsf{P}(\max_{1\le k\le n} S_k \ge xV_n)}{1-\Phi(x)} - 2\right| = 0. \tag{2.1}$$

Toward this end, let

$$\Delta_{n,x} = \frac{x^2}{B_n^2}\sum_{i=1}^n \mathsf{E}X_i^2\{|X_i| > B_n/x\} + \frac{x^3}{B_n^3}\sum_{i=1}^n \mathsf{E}|X_i|^3 I\{|X_i| \le B_n/x\},$$

and write

$$n_0 \equiv n_0(x) = \max\left\{k\colon \sum_{j=k}^n \mathsf{E}X_j^2 \ge 192 B_n^2 \log(x\vee \mathrm{e})/x^2, 1 \le k \le n\right\}. \tag{2.2}$$

It can be readily seen that the condition (1.3), together with $0 < x \le \min\{B_n, \mathrm{o}(d_{n,r})\}$ and $d_{n,r} \to \infty$, imply the existence of an absolute constant A such that

$$0 \le x \le B_n, \qquad \Delta_{n,x} \le \min(\delta^{9/2}, 1)/A, \tag{2.3}$$

$\Delta_{n,x} \to 0$, and

$$\frac{\sum_{j=n_0+1}^{n} \mathsf{E}|X_j|^3 I\{|X_j| \le B_n/x\}}{\sum_{j=n_0+1}^{n} \mathsf{E}|X_j|^2} \le \frac{B_n}{x^{1+\delta}} \tag{2.4}$$

for all sufficiently large n, where δ is defined as in (1.3). The result (2.1) follows immediately from the following proposition.

Proposition 1. *For all $x \ge 2$ satisfying* (2.3) *and* (2.4), *we have*

$$\frac{\mathsf{P}(\max_{1\le k\le n} S_k \ge x V_n)}{1-\Phi(x)} = 2 + \mathrm{O}(1)\bigl(x^{-\min\{1/4,\delta/20\}} + \Delta_{n,x}^{1/9}\bigr), \tag{2.5}$$

where $\mathrm{O}(1)$ *is bounded by a constant C_δ that depends only on δ.*

The main idea of the proof of Proposition 1 is to use truncation and the maximum probability inequality and then apply a moderate deviation theorem of Sakhanenko [10] to the truncated variables. A suitable truncation level is ensured by using an inequality from Jing, Shao and Wang [8], page 2181. This avoids the conjugate argument of Hu, Shao and Wang [7], and makes it possible to prove the main result under an optimal moment assumption.

It remains to prove Proposition 1. In addition to the notation in the previous section, let $\gamma = 72^{-1}\min(\delta, 1)$,

$$\varepsilon = \max(2\Delta_{n,x}^{2/9}, \gamma x^{-1/2}, \gamma x^{-\delta/10}), \qquad m = [x^2/2],$$

$N_0 = \varnothing$ and, for $1 \le l \le m$, $N_l = \{j_1, j_2, \dots, j_l\} \subseteq \{1, 2, \dots, n\}$. Furthermore, write $\bar{X}_i = X_i I\{|X_i| \le \varepsilon B_n/x\}$, and for $0 \le l \le m$ and $1 \le k \le n$,

$$\bar{S}_k^{N_l} = \sum_{i=1, i\notin N_l}^{k} \bar{X}_i, \qquad (\bar{V}_n^{N_l})^2 = \sum_{i=1, i\notin N_l}^{n} \bar{X}_i^2, \qquad (\bar{B}_n^{N_l})^2 = \sum_{i=1, i\notin N_l}^{n} \mathsf{E}\bar{X}_i^2,$$

$$S_k^{N_l} = \sum_{i=1, i\notin N_l}^{k} X_i, \qquad (V_n^{N_l})^2 = \sum_{i=1, i\notin N_l}^{n} X_i^2, \qquad (B_n^{N_l})^2 = \sum_{i=1, i\notin N_l}^{n} \mathsf{E}X_i^2.$$

Note that if $s, t \in R^1$, $x \ge 1$, $c \ge 0$ and $s + t \ge x\sqrt{c + t^2}$, then $s \ge (x^2 - 1)^{1/2}\sqrt{c}$. Similar to the arguments in reported by Jing, Shao and Wang [8], page 2181, we have

$$\begin{aligned} &\mathsf{P}\Bigl(\max_{1\le k\le n} S_k \ge x V_n\Bigr) \\ &\quad\le \mathsf{P}\Bigl(\max_{1\le k\le n} \bar{S}_k^{N_0} \ge x \bar{V}_n^{N_0}\Bigr) \\ &\qquad + \sum_{j_1=1}^{n} \mathsf{P}\Bigl(\max_{1\le k\le n} S_k \ge x V_n, |X_{j_1}| \ge \varepsilon B_n/x\Bigr) \end{aligned} \tag{2.6}$$

$$\le \mathsf{P}\Big(\max_{1\le k\le n}\bar{S}_k^{N_0}\ge x\bar{V}_n^{N_0}\Big) + \sum_{j_1=1}^{n}\mathsf{P}\Big(\max_{1\le k\le n}S_k^{N_1}\ge\sqrt{x^2-1}\,V_n^{N_1}\Big)\mathsf{P}(|X_{j_1}|\ge\varepsilon B_n/x)$$

and

$$\mathsf{P}\Big(\max_{1\le k\le n}S_k\ge xV_n\Big) \ge \mathsf{P}\Big(\max_{1\le k\le n}\bar{S}_k^{N_0}\ge x\bar{V}_n^{N_0}\Big) - \sum_{j_1=1}^{n}\mathsf{P}\Big(\max_{1\le k\le n}\bar{S}_k^{N_1}\ge\sqrt{x^2-1}\,\bar{V}_n^{N_1}\Big)\mathsf{P}(|X_{j_1}|\ge\varepsilon B_n/x). \tag{2.7}$$

Repeating (2.6) m-times gives

$$\mathsf{P}\Big(\max_{1\le k\le n}S_k\ge xV_n\Big) \le \mathsf{P}\Big(\max_{1\le k\le n}\bar{S}_k^{N_0}\ge x\bar{V}_n^{N_0}\Big) + \sum_{k=1}^{m}Z_k(x)+\Big\{\sum_{k=1}^{n}\mathsf{P}(|X_k|\ge\varepsilon B_n/x)\Big\}^{m+1}, \tag{2.8}$$

where

$$Z_k(x)=\sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n}\Big[\prod_{i=1}^{k}\mathsf{P}(|X_{j_i}|\ge\varepsilon B_n/x)\Big]\times\mathsf{P}\Big(\max_{1\le j\le n}\bar{S}_j^{N_k}\ge\sqrt{x^2-k}\,\bar{V}_n^{N_k}\Big).$$

Note that

$$\begin{aligned}\sum_{k=1}^{n}&\mathsf{P}(|X_k|\ge\varepsilon B_n/x)\\ &\le\frac{x^2}{\varepsilon^2B_n^2}\sum_{k=1}^{n}\mathsf{E}X_k^2I\{|X_k|\ge\varepsilon B_n/x\}\\ &\le\frac{x^2}{\varepsilon^2B_n^2}\sum_{k=1}^{n}\mathsf{E}X_k^2I\{|X_k|\ge B_n/x\}+\frac{x^3}{\varepsilon^3B_n^3}\sum_{k=1}^{n}\mathsf{E}|X_k|^3I\{|X_k|\le B_n/x\}\\ &\le\varepsilon^{-3}\Delta_{n,x}\le\varepsilon^{3/2}/16\le 1/16.\end{aligned} \tag{2.9}$$

It follows from $m = [x^2/2]$ that

$$\left[\sum_{k=1}^{n} \mathsf{P}(|X_k| \geq \varepsilon B_n/x)\right]^{m+1} \leq \mathrm{e}^{-x^2}. \tag{2.10}$$

This, together with (2.7) and (2.8), implies that Proposition 1 will follow if we prove the following two propositions.

Proposition 2. *For all* $0 \leq l \leq m$, *all* $x/2 \leq y \leq x$, *and all* $x \geq 2$ *satisfying* (2.3) *and* (2.4), *we have*

$$\frac{\mathsf{P}(\max_{1\leq k\leq n} \bar{S}_k^{N_l} \geq y\bar{V}_n^{N_l})}{1-\Phi(y)} \leq 2 + C_{\delta,\tau}(\varepsilon^{-2}\Delta_{n,x} + \varepsilon). \tag{2.11}$$

Proposition 3. *For all* $x \geq 2$ *satisfying* (2.3) *and* (2.4), *we have*

$$\frac{\mathsf{P}(\max_{1\leq k\leq n} \bar{S}_k^{N_0} \geq x\bar{V}_n^{N_0})}{1-\Phi(x)} = 2 + C_{\delta,\tau}(\varepsilon^{-2}\Delta_{n,x} + \varepsilon). \tag{2.12}$$

Indeed, noting that

$$\frac{x}{\sqrt{2\pi}(1+x^2)}\mathrm{e}^{-x^2/2} \leq 1-\Phi(x) \leq \frac{1}{\sqrt{2\pi}x}\mathrm{e}^{-x^2/2}$$

for $x \geq 1$, we have that for $1 \leq k \leq m = [x^2/2]$ and $x \geq 1$,

$$\frac{1-\Phi(\sqrt{x^2-k})}{1-\Phi(x)} \leq 2\mathrm{e}^{k/2}.$$

This, together with (2.8)–(2.11), implies that for all $x \geq 2$ satisfying (2.3) and (2.4),

$$\begin{aligned}
&\mathsf{P}\Big(\max_{1\leq k\leq n} S_k \geq xV_n\Big) \\
&\quad \leq \mathrm{e}^{-x^2} + 2\left\{1-\Phi(x) + \sum_{k=1}^{m}\big\{1-\Phi\big(\sqrt{x^2-k}\big)\big\}\left\{\sum_{j=1}^{n}\mathsf{P}(|X_j| \geq \epsilon B_n/x)\right\}^k\right\} \\
&\qquad\quad \times \{1 + C_{\delta,\tau}(\epsilon^{-2}\Delta_{n,x} + \epsilon)\} \qquad (2.13)\\
&\quad \leq 2\big(1-\Phi(x)\big)\{1 + C_{\delta,\tau}(\epsilon^{-3}\Delta_{n,x} + \epsilon + x^{-1})\} \\
&\quad \leq 2\big(1-\Phi(x)\big)\big\{1 + C_{\delta,\tau}\big(x^{-\min\{1/4,\delta/20\}} + \Delta_{n,x}^{1/9}\big)\big\}.
\end{aligned}$$

Similarly, by (2.7), (2.11) and (2.12), we obtain that for all $x \geq 2$ satisfying (2.3) and (2.4),

$$\mathsf{P}\Big(\max_{1\leq k\leq n} S_k \geq xV_n\Big) \geq 2\big(1-\Phi(x)\big)\big\{1 - C_{\delta,\tau}(x^{-\min\{1/4,\delta/20\}} + \Delta_{n,x}^{1/9})\big\}. \tag{2.14}$$

Combining (2.13) and (2.14), we obtain (2.5), and thus Proposition 1.

It remains to prove Propositions 2 and 3, which we give in Sections 3 and 4, respectively. The proof of Theorem 2 is now complete. □

3. Proof of Proposition 2

Let $b = y/B_n^{N_l}$. First, note that

$$\mathsf{P}\Big(\max_{1\le k\le n} \bar{S}_k^{N_l} \ge y\bar{V}_n^{N_l}\Big) \le \mathsf{P}\Big(2b\max_{1\le k\le n} \bar{S}_k^{N_l} \ge (b\bar{V}_n^{N_l})^2 + y^2 - \varepsilon^2\Big) + \mathsf{P}\Big(\max_{1\le k\le n} \bar{S}_k^{N_l} \ge y\bar{V}_n^{N_l}, |b\bar{V}_n^{N_l} - y| \ge \varepsilon\Big). \tag{3.1}$$

Furthermore, we have

$$\begin{aligned} &\mathsf{P}\Big(\max_{1\le k\le n} \bar{S}_k^{N_l} \ge y\bar{V}_n^{N_l}, |b\bar{V}_n^{N_l} - y| \ge \varepsilon\Big) \\ &\quad\le \mathsf{P}\Big(\max_{1\le k\le n} \bar{S}_k^{N_l} \ge y\bar{V}_n^{N_l}, b^2(\bar{V}_n^{N_l})^2 > y^2 + \varepsilon y\Big) \\ &\qquad + \mathsf{P}\Big(\max_{1\le k\le n} \bar{S}_k^{N_l} \ge y\bar{V}_n^{N_l}, b^2(\bar{V}_n^{N_l})^2 < y^2 - \varepsilon y\Big) \\ &\quad =: I_1 + I_2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\mathsf{P}\Big(2b\max_{1\le k\le n} \bar{S}_k^{N_l} \ge b^2(\bar{V}_n^{N_l})^2 + y^2 - \varepsilon^2\Big) \\ &\quad\le \mathsf{P}\Bigg(\bigcup_{k=1}^{n}\{2b\bar{S}_k^{N_l} \ge b^2(\bar{V}_n^{N_l})^2 + y^2 - \varepsilon^2, \\ &\qquad\qquad \mathsf{E}[(\bar{V}_n^{N_l})^2 - (\bar{V}_k^{N_l})^2] - [(\bar{V}_n^{N_l})^2 - (\bar{V}_k^{N_l})^2] \ge \varepsilon^2/b^2\}\Bigg) \\ &\qquad + \mathsf{P}\Bigg(\bigcup_{k=1}^{n}\{2b\bar{S}_k^{N_l} \ge b^2(\bar{V}_k^{N_l})^2 + b^2\mathsf{E}[(\bar{V}_n^{N_l})^2 - (\bar{V}_k^{N_l})^2] + y^2 - 2\varepsilon^2\}\Bigg) \\ &\quad =: I_3 + I_4. \end{aligned} \tag{3.3}$$

By (3.1)–(3.3), Proposition 2 follows from the following Lemma 1.

Lemma 1. *Under the conditions of Proposition 2, we have*

$$I_1 \le C_{\delta,\tau} y^{-2} \exp(-y^2/2), \tag{3.4}$$

$$I_2 \le C_{\delta,\tau} y^{-2} \exp(-y^2/2), \tag{3.5}$$

$$I_3 \le C_{\delta,\tau} y^{-2} \exp(-y^2/2), \tag{3.6}$$

$$I_4 \le 2[1-\Phi(y)][1+C_{\delta,\tau}(\varepsilon^{-2}\Delta_{n,x}+\varepsilon)]. \tag{3.7}$$

To prove Lemma 1, we start with some preliminaries. Note that

$$\gamma \max\{x^{-1/2}, x^{-\delta/10}\} \le \varepsilon \le \min\{1/24, \delta/72\}, \qquad \Delta_{n,x} \le (\varepsilon/2)^{9/2}. \tag{3.8}$$

This fact (3.8) is repeatedly used in the proof without further explanation. Define $k_0 = 0$, $k_T = n$ and k_i, $1 \le i < T$, by

$$k_i = \max\left\{k\colon \sum_{j=k_{i-1}+1}^{k} \mathsf{E}X_j^2 \le 2^{-1}\varepsilon^3 B_n^2/x^2\right\}.$$

By the definition of k_i,

$$\sum_{j=k_{i-1}+1}^{k_i} \mathsf{E}X_j^2 \le 2^{-1}\varepsilon^3 B_n^2/x^2 \quad \text{and} \quad \sum_{j=k_{i-1}+1}^{k_i+1} \mathsf{E}X_j^2 > 2^{-1}\varepsilon^3 B_n^2/x^2 \tag{3.9}$$

for any $1 \le i < T$. By (2.3) and (3.8),

$$\begin{aligned} x^2 \max_{1\le k\le n} \mathsf{E}X_k^2 &\le x^2 \max_{1\le k\le n} [\mathsf{E}X_k^2\{|X_k| > B_n/x\} + (\mathsf{E}|X_k|^3 I\{|X_k| \le B_n/x\})^{2/3}] \\ &\le B_n^2(\Delta_{n,x} + \Delta_{n,x}^{2/3}) \le \varepsilon^3 B_n^2/4, \end{aligned} \tag{3.10}$$

which, together with (3.9), implies that

$$\sum_{j=k_{i-1}+1}^{k_i} \mathsf{E}X_j^2 \ge 4^{-1}\varepsilon^3 B_n^2/x^2.$$

Therefore,

$$(T-1)4^{-1}\varepsilon^3 B_n^2/x^2 \le \sum_{i=1}^{T-1} \sum_{j=k_{i-1}+1}^{k_i} \mathsf{E}X_j^2 \le B_n^2,$$

which yields $T \le 4x^2/\varepsilon^3 + 1$. For $k_{i-1}+1 \le j \le k_i - 1$, define events

$$\mathbf{A}_j = \left\{\bar{S}_j^{N_l} \ge y\sqrt{(B_n^{N_l})^2(1+\varepsilon/y)}\right\}, \qquad \mathbf{C}_j = \left\{\sum_{k=j+1, k\notin N_l}^{k_i} (\bar{X}_k - \mathsf{E}\bar{X}_k) \ge -\varepsilon B_n^{N_l}/y\right\}.$$

Note that $\sum_{k\in N_l} \mathsf{E}X_k^2 \le \varepsilon^3 B_n^2/8$ for all $0 \le l \le m = [x^2/2]$ by (3.10), and thus

$$B_n^2 \ge (B_n^{N_l})^2 = \sum_{k=1}^{n} \mathsf{E}X_k^2 - \sum_{k\in N_l} \mathsf{E}X_k^2 \ge (1-\varepsilon^3/8)B_n^2 \ge \frac{7}{8}B_n^2. \tag{3.11}$$

Applying the Chebyshev inequality, we have, for any $k_{i-1} \le j \le k_i$ and $x/2 \le y \le x$,

$$\mathsf{P}(\mathbf{C}_j) \ge 1 - \frac{y^2 \sum_{k=j+1}^{k_i} \mathsf{E} X_k^2}{\varepsilon^2 (B_n^{N_l})^2} \ge 1 - 4\varepsilon/7 \ge 1/2. \tag{3.12}$$

We are now ready to prove Lemma 1.

Proof of (3.4). It follows from (3.12) and the independence between $\mathbf{C}_j$ and $\{\mathbf{A}_l, l \le j\}$ that

$$\begin{aligned} I_1 &\le \sum_{i=1}^{T} \mathsf{P}\Biggl(\bigcup_{j=k_{i-1}+1}^{k_i} \mathbf{A}_j\Biggr) \\ &\le \sum_{i=1}^{T} \Biggl[\mathsf{P}(\mathbf{A}_{k_{i-1}+1}) + \sum_{j=k_{i-1}+2}^{k_i} \mathsf{P}(\mathbf{A}_{k_{i-1}}^c, \dots, \mathbf{A}_{j-1}^c, \mathbf{A}_j)\Biggr] \\ &\le 2\sum_{i=1}^{T} \Biggl[\mathsf{P}(\mathbf{A}_{k_{i-1}+1}, \mathbf{C}_{k_{i-1}+1}) + \sum_{j=k_{i-1}+2}^{k_i} \mathsf{P}(\mathbf{A}_{k_{i-1}}^c, \dots, \mathbf{A}_{j-1}^c, \mathbf{A}_j, \mathbf{C}_j)\Biggr] \\ &\le 2\sum_{i=1}^{T} \mathsf{P}\bigl(\bar{S}_{k_i}^{N_l} - \mathsf{E}\bar{S}_{k_i}^{N_l} \ge y\sqrt{(B_n^{N_l})^2(1+\varepsilon/y)} - \varepsilon B_n^{N_l}/y - D_{k_i}\bigr), \end{aligned} \tag{3.13}$$

where $D_{k_i} = \sum_{j=1}^{k_i} \mathsf{E}|X_j| I\{|X_j| > \varepsilon B_n/x\}$. Taking $t = y\sqrt{1+\varepsilon/y}/B_n^{N_l}$ and noting

$$t(\varepsilon B_n/y + D_{k_i}) \le 2\varepsilon + 1,$$

we have

$$\begin{aligned} &\mathsf{P}\bigl(\bar{S}_{k_i}^{N_l} - \mathsf{E}\bar{S}_{k_i}^{N_l} \ge y\sqrt{(B_n^{N_l})^2(1+\varepsilon/y)} - \varepsilon B_n^{N_l}/y - D_{k_i}\bigr) \\ &\quad\le 9\exp(-y^2 - \varepsilon y) \prod_{j=1, j\notin N_l}^{k_i} \mathsf{E}\exp\bigl(t(\bar{X}_j - \mathsf{E}\bar{X}_j)\bigr) \\ &\quad\le 9\exp(-y^2 - \varepsilon y) \prod_{j=1, j\notin N_l}^{k_i} \biggl(1 + \frac{\mathsf{E}X_j^2}{2} t^2 + 8t^3 \mathsf{E}|\bar{X}_j|^3 \mathrm{e}^{2t\varepsilon B_n/x}\biggr) \\ &\quad\le 9\exp(-y^2/2 - \varepsilon y/2 + A\Delta_{n,x}). \end{aligned}$$

Submitting this estimate into (3.13) and recalling $T \le 4x^2/\varepsilon^3 + 1$, $x/2 \le y \le x$ and $\varepsilon \ge \gamma x^{-1/2}$, we obtain

$$\begin{aligned} I_1 &\le (4\varepsilon^{-3}x^2 + 1)\exp(-y^2/2 - \varepsilon y/2 + A\Delta_{n,x}) \\ &\le C_{\delta,\tau} y^{-2} \mathrm{e}^{-y^2/2}. \end{aligned} \tag{3.14}$$

This proves (3.4). □

Proof of (3.5). For this part, let $Y_{k_i} = \sum_{j=k_{i-1}+1, j\notin N_l}^{k_i} \bar{X}_j^2$, and define

$$\bar{\mathbf{A}}_j = \{\bar{S}_j^{N_l} \ge y\sqrt{(\bar{V}_n^{N_l})^2 - Y_{k_i}}, b^2[(\bar{V}_n^{N_l})^2 - Y_{k_i}] < y^2 - \varepsilon y\}, \qquad 1 \le j \le n.$$

From (3.12) and the independence between $\mathbf{C}_j$ and $\{\bar{\mathbf{A}}_l, l \le j\}$, it follows that

$$\begin{aligned}
I_2 &\le \sum_{i=1}^{T}\left[\mathsf{P}(\bar{\mathbf{A}}_{k_{i-1}+1}) + \sum_{j=k_{i-1}+2}^{k_i}\mathsf{P}(\bar{\mathbf{A}}_{k_{i-1}}^c, \dots, \bar{\mathbf{A}}_{j-1}^c, \bar{\mathbf{A}}_j)\right] \\
&\le 2\sum_{i=1}^{T}\left[\mathsf{P}(\bar{\mathbf{A}}_{k_{i-1}+1}, \mathbf{C}_{k_{i-1}+1}) + \sum_{j=k_{i-1}+2}^{k_i}\mathsf{P}(\bar{\mathbf{A}}_{k_{i-1}}^c, \dots, \bar{\mathbf{A}}_{j-1}^c, \bar{\mathbf{A}}_j, \mathbf{C}_j)\right] \\
&\le 2\sum_{i=1}^{T}\mathsf{P}\big(\bar{S}_{k_i}^{N_l} - \mathsf{E}\bar{S}_{k_i}^{N_l} \ge y\sqrt{(\bar{V}_n^{N_l})^2 - Y_{k_i}} - \varepsilon B_n^{N_l}/y - D_{k_i}, b^2[(\bar{V}_n^{N_l})^2 - Y_{k_i}] < y^2 - \varepsilon y\big) \\
&=: 2\sum_{i=1}^{T} I_{2,i},
\end{aligned} \tag{3.15}$$

where, as before, $D_{k_i} = \sum_{j=1}^{k_i}\mathsf{E}|X_j|I\{|X_j| > \varepsilon B_n/x\}$. Furthermore, for $i = 1, \dots, T$,

$$\begin{aligned}
I_{2,i} &\le \mathsf{P}\big((\bar{V}_n^{N_l})^2 - Y_{k_i} < (1-\varepsilon)(B_n^{N_l})^2\big) \\
&\quad + \sum_{k=1}^{[y]}\mathsf{P}\big(\bar{S}_{k_i}^{N_l} - \mathsf{E}\bar{S}_{k_i}^{N_l} \ge y\sqrt{(B_n^{N_l})^2[1-(k+1)\varepsilon/y]} - \varepsilon B_n^{N_l}/y - D_{k_i}, \\
&\qquad (B_n^{N_l})^2[1-(k+1)\varepsilon/y] < (\bar{V}_n^{N_l})^2 - Y_{k_i} < (B_n^{N_l})^2[1-k\varepsilon/y]\big) \\
&=: I_{2,i,0} + \sum_{k=1}^{[y]} I_{2,i,k}.
\end{aligned}$$

Note that, for any $t_1 \ge 0$ and $t_2 \ge 0$,

$$\begin{aligned}
&\mathsf{E}\exp\big(t_1(\bar{X}_k - \mathsf{E}\bar{X}_k) + t_2(\mathsf{E}\bar{X}_k^2 - \bar{X}_k^2)\big) \\
&\quad \le 1 + \tfrac{1}{2}\mathsf{E}\big(t_1(\bar{X}_k - \mathsf{E}\bar{X}_k) + t_2(\mathsf{E}\bar{X}_k^2 - \bar{X}_k^2)\big)^2 \\
&\qquad + (8t_1^3\mathsf{E}|\bar{X}_k|^3 + 8t_2^3\mathsf{E}|\bar{X}_k|^6)e^{2t_1\varepsilon B_n/x + t_2\mathsf{E}X_k^2} \\
&\quad \le \exp\big(\tfrac{1}{2}t_1^2\mathsf{E}\bar{X}_k^2 + \tfrac{1}{2}(4t_1t_2 + t_2^2\varepsilon B_n/x)\mathsf{E}|\bar{X}_k|^3 \\
&\qquad + (8t_1^3 + 8t_2^3\varepsilon^3 B_n^3/x^3)\mathsf{E}|\bar{X}_k|^3 e^{2t_1\varepsilon B_n/x + t_2\max_{1\le k\le n}\mathsf{E}X_k^2}\big).
\end{aligned} \tag{3.16}$$

Let $t_1 = y\sqrt{1-(k+1)\varepsilon/y}/B_n^{N_l}$ and $t_2 = \varepsilon^{-1}y^2/(B_n^{N_l})^2$ in (3.16). Noting that

$$t_1(\varepsilon B_n^{N_l}/y + D_{k_i}) \le \varepsilon + 1,$$

we have for $1 \le k \le [x]$,

$$\begin{aligned}
I_{2,i,k} &\le \mathsf{P}\big(t_1(\bar{S}_{k_i}^{N_l} - \mathsf{E}\bar{S}_{k_i}^{N_l}) + t_2\{\mathsf{E}[(\bar{V}_n^{N_l})^2 - Y_{k_i}] - [(\bar{V}_n^{N_l})^2 - Y_{k_i}]\} \\
&\qquad \ge y^2 - (k+1)\varepsilon/y + t_2 k\varepsilon(B_n^{N_l})^2/y - 2\big) \\
&\le \exp\big(-y^2 + (k+1)\varepsilon/y - t_2 k\varepsilon(B_n^{N_l})^2/y + 2\big) \\
&\quad \times \prod_{k=1, k\notin N_l}^{k_{i-1}} \mathsf{E}\exp\big(t_1(\bar{X}_k - \mathsf{E}\bar{X}_k) + t_2(\mathsf{E}\bar{X}_k^2 - \bar{X}_k^2)\big) \\
&\quad \times \prod_{k=k_{i-1}+1, k\notin N_l}^{k_i} \mathsf{E}\exp\big(t_1(\bar{X}_k - \mathsf{E}\bar{X}_k)\big) \times \prod_{k=k_i+1, k\notin N_l}^{n} \mathsf{E}\exp\big(t_2(\mathsf{E}\bar{X}_k^2 - \bar{X}_k^2)\big) \\
&\le \exp\Bigg(-y^2/2 + 2^{-1}(k+1)\varepsilon/y - t_2 k\varepsilon(B_n^{N_l})^2/y + 2 \\
&\qquad + A(t_1t_2 + t_2^2\varepsilon B_n/x + t_1^3 + t_2^3\varepsilon^3 B_n^3/x^3)\sum_{k=1}^{n}\mathsf{E}|\bar{X}_k|^3\Bigg) \\
&\le \exp\big(-y^2/2 + 2^{-1}(k+1)\varepsilon/y - ky + A\varepsilon^{-1}\Delta_{n,x} + 2\big) \\
&\le A\exp(-y^2/2 - y/2).
\end{aligned}$$

Similarly, by (3.16) with $t_1 = 0$, we have

$$\begin{aligned}
I_{2,i,0} &\le \mathsf{P}\big(t_2\{\mathsf{E}[(\bar{V}_n^{N_l})^2 - Y_{k_i}] - [(\bar{V}_n^{N_l})^2 - Y_{k_i}]\} \ge t_2\varepsilon(B_n^{N_l})^2 - \varepsilon^2\big) \\
&\le A\exp(-y^2 + A\varepsilon^{-1}\Delta_{n,x}) \le A_1\exp(-y^2/2 - y).
\end{aligned}$$

Combining above inequalities yields

$$I_2 \le A(4x^2/\varepsilon^3 + 1)\mathrm{e}^{-y^2/2-y} \le A_1 y^{-2}\mathrm{e}^{-y^2/2}. \tag{3.17}$$

The proof of (3.5) is now complete. □

Proof of (3.6). Following the arguments in the estimates of I_1 and I_2, we have

$$\begin{aligned}
I_3 \le \sum_{i=1}^{T}\mathsf{P}\Bigg(\bigcup_{j=k_{i-1}+1}^{k_i}&\{2b\bar{S}_j^{N_l} \ge b^2(\bar{V}_n^{N_l})^2 + y^2 - \varepsilon^2, \\
&\mathsf{E}[(\bar{V}_n^{N_l})^2 - (\bar{V}_j^{N_l})^2] - [(\bar{V}_n^{N_l})^2 - (\bar{V}_j^{N_l})^2] \ge \varepsilon^2(B_n^{N_l})^2/y^2\}\Bigg)
\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{T} \mathsf{P}\Bigg(\bigcup_{j=k_{i-1}+1}^{k_i} \{2b\bar{S}_j^{N_l} \ge b^2[(\bar{V}_n^{N_l})^2 - Y_{k_i}] + y^2 - \varepsilon^2\}, \\
&\qquad \sum_{k=k_i+1, k\notin N_l}^{n} (\mathsf{E}\bar{X}_k^2 - \bar{X}_k^2) \ge 2^{-1}\varepsilon^2 (B_n^{N_l})^2/y^2 \Bigg) \\
&\le 2\sum_{i=1}^{T} \mathsf{P}\Bigg(2b\bar{S}_{k_i}^{N_l} \ge b^2[(\bar{V}_n^{N_l})^2 - Y_{k_i}] + y^2 - 2\varepsilon, \\
&\qquad \sum_{k=k_i+1, k\notin N_l}^{n} (\mathsf{E}\bar{X}_k^2 - \bar{X}_k^2) \ge 2^{-1}\varepsilon^2 (B_n^{N_l})^2/y^2 \Bigg) \\
&=: 2\sum_{i=1}^{T} I_{3i}.
\end{aligned}
$$

As in the proof of (3.16), it can be easily shown that for $\alpha \ge 0$,

$$
\mathsf{E}e^{b\bar{X}_j - \alpha b^2 \bar{X}_j^2} \le \exp\{(1/2 - \alpha) b^2 \mathsf{E}\bar{X}_j^2 + A\Delta_{n,x}^{(j)}\}, \tag{3.19}
$$

where

$$
\begin{aligned}
\Delta_{n,x}^{(j)} &= \frac{x^2}{B_n^2}\mathsf{E}X_j^2 I\{|X_j| \ge \varepsilon B_n/x\} + \frac{x^3}{B_n^3}\mathsf{E}|X_j|^3 I\{|X_j| \le \varepsilon B_n/x\} \\
&\le \varepsilon^{-1}\left(\frac{x^2}{B_n^2}\mathsf{E}X_j^2 I\{|X_j| \ge B_n/x\} + \frac{x^3}{B_n^3}\mathsf{E}|X_j|^3 I\{|X_j| \le B_n/x\}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathsf{E}e^{\alpha(\mathsf{E}\bar{X}_j^2 - \bar{X}_j^2) - b^2\bar{X}_j^2/2} \\
&\quad \le \exp\{-\tfrac{1}{2}b^2\mathsf{E}\bar{X}_j^2 + (2\alpha^2 B_n/x + x^3/B_n^3)\varepsilon \mathsf{E}|\bar{X}_j|^3 e^{\alpha \max_{1\le k\le n}\mathsf{E}X_k^2}\}.
\end{aligned} \tag{3.20}
$$

Next, let t satisfy

$$
t e^{t\max_{1\le k\le n}\mathsf{E}X_k^2} = \frac{\varepsilon B_n}{24x\sum_{j=k_i+1}^{n}\mathsf{E}|\bar{X}_j|^3}.
$$

Clearly t exists. Furthermore, we have $t \ge x^2/B_n^2$. Indeed, if $t\max_{1\le k\le n}\mathsf{E}X_k^2 \ge \varepsilon$, then by (3.10) and recalling $\varepsilon \le 1/24$,

$$
t \ge \varepsilon / \max_{1\le k\le n} \mathsf{E}X_k^2 \ge 4\varepsilon^{-2}x^2/B_n^2 \ge x^2/B_n^2.
$$

If $t \max_{1\le k\le n} \mathsf{E}X_k^2 \le \varepsilon$, then

$$t \ge \frac{\varepsilon B_n}{24\mathrm{e}^{\varepsilon} x \sum_{j=k_i+1}^n \mathsf{E}|\bar{X}_j|^3} \ge \frac{\varepsilon x^2}{30 B_n^2 \Delta_{n,x}} \ge \frac{1}{15}\Delta_{n,x}^{-7/9} x^2/B_n^2 \ge x^2/B_n^2.$$

Now it follows from (3.19) and (3.20) with $\alpha = t$ that

$$\begin{aligned}
I_{3i} &\le \mathsf{P}\Bigg(b\bar{S}_{k_i}^{N_l} - 2^{-1}b^2[(\bar{V}_n^{N_l})^2 - Y_{k_i}] + t\sum_{k=k_i+1, k\notin N_l}^{n} (\mathsf{E}\bar{X}_k^2 - \bar{X}_k^2) \\
&\qquad \ge y^2/2 - 2\varepsilon + 2^{-1}t\varepsilon^2(B_n^{N_l})^2/y^2\Bigg) \\
&\le \exp\left[2\varepsilon - y^2/2 - \frac{\varepsilon^2(B_n^{N_l})^2 t}{2y^2}\right] \prod_{j=1, j\notin N_l}^{k_{i-1}} \mathsf{E}\mathrm{e}^{b\bar{X}_j - 2^{-1}b^2\bar{X}_j^2} \\
&\quad \times \prod_{j=k_{i-1}+1, j\notin N_l}^{k_i} \mathsf{E}\mathrm{e}^{b\bar{X}_j} \times \prod_{j=k_i+1, j\notin N_l}^{n} \mathsf{E}\mathrm{e}^{-2^{-1}b^2\bar{X}_j^2 + t(\mathsf{E}\bar{X}_j^2 - \bar{X}_j^2)} \\
&\le A\exp(-y^2/2)\exp\Bigg(\varepsilon^{-1}\Delta_{n,x} - \frac{\varepsilon^2 B_n^2 t}{3x^2} - \frac{y^2\sum_{j=k_i+1}^n \mathsf{E}X_j^2}{2(B_n^{N_l})^2} + \frac{y^2\sum_{j=k_{i-1}+1}^{k_i} \mathsf{E}X_j^2}{2(B_n^{N_l})^2} \\
&\qquad + (2t^2 B_n/x + x^3/B_n^3)\varepsilon \sum_{j=k_i+1}^{n} \mathsf{E}|\bar{X}_j|^3 \mathrm{e}^{t\max_{1\le k\le n}\mathsf{E}X_k^2}\Bigg) \\
&\le A_1\exp(-y^2/2)\exp\left(-\frac{\varepsilon^2 B_n^2 t}{4x^2} - \frac{y^2\sum_{j=k_i+1}^n \mathsf{E}X_j^2}{4B_n^2} + \frac{\varepsilon^2 x^2}{12 B_n^2 t}\right) \\
&\le A_1\exp(-y^2/2)\exp\left(-\frac{\varepsilon^2 B_n^2 t}{4x^2} - \frac{y^2\sum_{j=k_i+1}^n \mathsf{E}X_j^2}{4B_n^2}\right).
\end{aligned} \tag{3.21}$$

Note that when $t \le \frac{2\delta x^2 \log x}{B_n^2 \varepsilon^3}$, $t \max_{1\le k\le} \mathsf{E}X_k^2 < \frac{\delta}{2}\log x$ by (3.10). Thus, by the definition of t,

$$t \ge \frac{\varepsilon B_n}{24 x^{1+\delta/2} \sum_{j=k_i+1}^n \mathsf{E}|\bar{X}_j|^3}.$$

Now considering $t \le \frac{2\delta x^2 \log x}{B_n^2 \varepsilon^3}$ and $t \ge \frac{2\delta x^2 \log x}{B_n^2 \varepsilon^3}$, we have, by (3.21),

$$\begin{aligned}
I_{3i} &\le A y^{-\delta/(3\varepsilon)} \exp(-y^2/2) \\
&\quad + A\mathrm{e}^{-y^2/2}\exp\left(-\frac{y^2\sum_{j=k_i+1}^n \mathsf{E}X_j^2}{4B_n^2} - \frac{\varepsilon^3 B_n^3}{144 x^{3+\delta/2}\sum_{j=k_i+1}^n \mathsf{E}|\bar{X}_j|^3}\right).
\end{aligned} \tag{3.22}$$

From the definition of n_0, $\sum_{j=n_0+1}^{n} \mathsf{E}X_j^2 \le 192 B_n^2 x^{-2} \log x$ and thus by (2.4)

$$\sum_{j=n_0+1}^{n} \mathsf{E}|\bar{X}_j|^3 \le \frac{192\tau B_n^3 \log x}{x^{3+\delta}}.$$

For $i < i_0$, where $i_0 = \max\{i\colon k_i + 1 \le n_0\}$, we have

$$y^2 \sum_{j=k_i}^{n} \mathsf{E}X_j^2 \ge x^2 \sum_{j=n_0}^{n} \mathsf{E}X_j^2/4 \ge 24 B_n^2 \log x.$$

It now follows from (3.22), (3.8) and the fact $T \le 4x^2/\varepsilon^3 + 1$ that

$$\begin{aligned}
I_3 &\le 2\sum_{i=1}^{T} I_{3i} \\
&\le 2ATy^{-\delta/(3\varepsilon)} \mathrm{e}^{-y^2/2} + 2A\mathrm{e}^{-y^2/2} i_0 \mathrm{e}^{-6\log x} \\
&\quad + 2A\mathrm{e}^{-y^2/2} \sum_{i=i_0+1}^{T} \exp\biggl(-\frac{\varepsilon^3 B_n^3}{144 x^{3+\delta/2} \sum_{j=k_i+1}^{n} \mathsf{E}|\bar{X}_j|^3}\biggr) \qquad (3.23) \\
&\le A_1(4x^2/\varepsilon^3 + 1)\mathrm{e}^{-y^2/2}(y^{-6} + \mathrm{e}^{-Ax^{\delta/2}\varepsilon^3/\log x}) \\
&\le C_{\delta,\tau} y^{-2} \mathrm{e}^{-y^2/2}.
\end{aligned}$$

This completes the proof of (3.6). □

Proof of (3.7). For this result, we need the following moderate deviation theorem for the standardized sum due to Sakhanenko [10] (also see Heinrich [6]).

Lemma 2. *Suppose that $\eta_1, \ldots, \eta_n$ are independent random variables such that $\mathsf{E}\eta_j = 0$ and $|\eta_j| \le 1$ for $j \ge 1$. Write $\sigma_n^2 = \sum_{j=1}^{n} \mathsf{E}\eta_j^2$ and $\mathcal{L}_n = \sum_{j=1}^{n} \mathsf{E}|\eta_j|^3/\sigma_n^3$. Then there exists an absolute constant $A > 0$ such that for all $1 \le x \le \min\{\sigma_n, \mathcal{L}_n^{-1/3}\}/A$,*

$$\frac{\mathsf{P}(\sum_{j=1}^{n} \eta_j \ge x\sigma_n)}{1 - \Phi(x)} = 1 + \mathrm{O}(1) x^3 \mathcal{L}_n, \qquad (3.24)$$

where $|\mathrm{O}(1)|$ is bounded by an absolute constant.

To prove (3.7), write

$$\begin{aligned}
\xi_j &= 2b\bar{X}_j - b^2 \bar{X}_j^2 + b^2 \mathsf{E}\bar{X}_j^2, \\
\mathbf{E}_j &= \Biggl\{\sum_{k=1, k\notin N_l}^{j} \xi_k \ge z\Biggr\}, \qquad \text{where } z = 2(y^2 - \varepsilon^2).
\end{aligned}$$

Note that $|\xi_j - \mathsf{E}\xi_j| \le 4\varepsilon + 2\varepsilon^2 \le 5\varepsilon$, and by the non-uniform Berry–Esseen bound, there exists an absolute constant A_0 such that for any $1 \le k \le n$ and $c > 0$,

$$\begin{aligned}
&\mathsf{P}\Bigg(\sum_{j=k, j\notin N_l}^{n} (\xi_j - \mathsf{E}\xi_j) \le -c\varepsilon\Bigg)\\
&\quad\le 1 - \Phi(t) + \frac{A_0 \sum_{j=k, j\notin N_l}^{n} \mathsf{E}|\xi_j - \mathsf{E}\xi_j|^3}{(1+t)^3 s_{n,k}^3}\\
&\quad\le \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\int_0^t \mathrm{e}^{-s^2/2}\,\mathrm{d}s + \frac{5A_0}{c}(1+t)^{-3}t,
\end{aligned}$$

where $s_{n,k}^2 = \sum_{j=k, j\notin N_l}^{n} \mathsf{Var}(\xi_j)$ and $t = c\varepsilon/s_{n,k}$. Because $\int_0^t \mathrm{e}^{-s^2/2}\,\mathrm{d}s \ge t(1+t)^{-3}/2$ for any $t \ge 0$, we may choose $c_0 \ge 10A_0\sqrt{2\pi}$ such that for all $1 \le k \le n$,

$$\mathsf{P}\Bigg(\sum_{j=k, j\notin N_l}^{n} (\xi_j - \mathsf{E}\xi_j) \le -c_0\varepsilon\Bigg) \le 1/2. \tag{3.25}$$

By virtue of (3.25), we obtain that

$$\begin{aligned}
I_4 &= \mathsf{P}(\mathbf{E}_1) + \sum_{k=2}^{n} \mathsf{P}(\mathbf{E}_1^c, \ldots, \mathbf{E}_{k-1}^c, \mathbf{E}_k)\\
&\le 2\mathsf{P}\Bigg(\mathbf{E}_1, \sum_{j=2, j\notin N_l}^{n} (\xi_j - \mathsf{E}\xi_j) \ge -c_0\varepsilon\Bigg)\\
&\quad + 2\sum_{k=2}^{n} \mathsf{P}\Bigg(\mathbf{E}_1^c, \ldots, \mathbf{E}_{k-1}^c, \mathbf{E}_k, \sum_{j=k+1, j\notin N_l}^{n} (\xi_j - \mathsf{E}\xi_j) \ge -c_0\varepsilon\Bigg)\\
&\le 2\mathsf{P}\Bigg(\sum_{k=1, k\notin N_l}^{n} (\xi_k - \mathsf{E}\xi_k) \ge z - c_0\varepsilon - D_n\Bigg),
\end{aligned} \tag{3.26}$$

where $D_n = \sum_{j=1, j\notin N_l}^{n} |\mathsf{E}\xi_j|$. Write $z' = z - c_0\varepsilon - D_n$. It is not difficult to show that

$$\begin{aligned}
D_n &\le 2b \sum_{j=1, j\notin N_l}^{n} \mathsf{E}|X_j| I\{|X_j| \ge \varepsilon B_n/x\} \le 4\varepsilon^{-2}\Delta_{n,x},\\
s_{n,1}^2 &= \sum_{j=1, j\notin N_l}^{n} \mathsf{Var}(\xi_j) = 4b^2 \sum_{j=1, j\notin N_l}^{n} \mathsf{E}X_j^2 + \mathrm{O}(1)\varepsilon^{-1}\Delta_{n,x}\\
&= 4y^2 + \mathrm{O}(1)\varepsilon^{-2}\Delta_{n,x},
\end{aligned}$$

where $|\mathrm{O}(1)| \le 30$. This yields that

$$\frac{z'}{s_{n,1}} = y + \mathrm{O}(1)[(\varepsilon + \varepsilon^{-2}\Delta_{n,x})/y],$$

where $|\mathrm{O}(1)| \le 40$. Therefore, by Lemma 2 with $\eta_j = \xi_j - \mathsf{E}\xi_j$

$$\begin{aligned} I_4 &\le 2[1 - \Phi(z'/s_{n,1})]\left[1 + A(z'/s_{n,1})^3 s_{n,1}^{-3} \sum_{j=1, j\notin N_l}^{n} \mathsf{E}|\xi_j|^3\right] \\ &\le 2[1 - \Phi(y)][1 + A(\varepsilon + \varepsilon^{-2}\Delta_{n,x})], \end{aligned} \tag{3.27}$$

where we have used the fact that whenever $x\theta_n \to 0$,

$$\frac{1 - \Phi(x + \theta_n)}{1 - \Phi(x)} = 1 + \mathrm{O}(1)x\theta_n.$$

This proves (3.7), and also completes the proof of Proposition 2. □

4. Proof of Proposition 3

By Proposition 2, it suffices to show that

$$\mathsf{P}\Big(\max_{1\le k\le n} \bar{S}_k \ge x\bar{V}_n\Big) \ge 2\big(1 - \Phi(x)\big)\big(1 - C_{\delta,\tau}(\varepsilon^{-2}\Delta_{n,x} + \varepsilon)\big). \tag{4.1}$$

Toward this end, let $b = x/B_n^{N_0}$ throughout this section. Recall (3.8), which we use repeatedly in the proof without further explanation. Let n_0 be defined as in (2.2). It can be readily seen that

$$\begin{aligned} &\mathsf{P}\Big(\max_{1\le k\le n} \bar{S}_k \ge x\bar{V}_n\Big) \\ &\ge \mathsf{P}\Big(2b \max_{n_0\le k\le n} \bar{S}_k \ge b^2\bar{V}_n^2 + x^2\Big) \\ &\ge \mathsf{P}\left(\bigcup_{k=n_0}^{n} \{2b\bar{S}_k \ge b^2\bar{V}_k^2 + b^2\mathsf{E}(\bar{V}_n^2 - \bar{V}_k^2) + x^2 + \varepsilon\}\right) \\ &\quad - \mathsf{P}\left(\bigcup_{k=n_0}^{n} \{2b\bar{S}_k \ge b^2\bar{V}_k^2 + x^2 + \varepsilon, (\bar{V}_n^2 - \bar{V}_k^2) - \mathsf{E}(\bar{V}_n^2 - \bar{V}_k^2) \ge \varepsilon B_n^2/x^2\}\right) \\ &=: I_5 - I_6. \end{aligned} \tag{4.2}$$

To complete the proof of Proposition 3, we only need to show the following lemma.

Lemma 3. *Under the conditions of Proposition 3, we have*

$$I_5 \ge 2\bigl(1-\Phi(x)\bigr)\bigl(1-C_{\delta,\tau}(\varepsilon^{-2}\Delta_{n,x}+\varepsilon)\bigr), \tag{4.3}$$

$$I_6 \le C_{\tau,\delta}x^{-2}\mathrm{e}^{-x^2/2}. \tag{4.4}$$

Proof of (4.3). We have

$$\begin{aligned} I_5 \ \ge\ & \mathsf{P}\Biggl(\bigcup_{k=1}^{n}\{2b\bar S_k \ge b^2\bar V_k^2 + b^2\mathsf{E}(\bar V_n^2-\bar V_k^2)+x^2+\varepsilon\}\Biggr) \\ & -\mathsf{P}\Biggl(\bigcup_{k=1}^{n_0}\{2b\bar S_k \ge b^2\bar V_k^2 + b^2\mathsf{E}(\bar V_n^2-\bar V_k^2)+x^2+\varepsilon\}\Biggr) \\ =:\ & I_{5,1}-I_{5,2}. \end{aligned} \tag{4.5}$$

Write

$$\xi_j = 2b\bar X_j - b^2\bar X_j^2 + b^2\mathsf{E}\bar X_j^2,$$

$$\mathbf{F}_j = \Biggl\{\sum_{k=1}^{j}\xi_k \ge y\Biggr\}, \qquad \text{where } y = 2x^2+\varepsilon.$$

As in the proof of (3.25), there exists a constant c_0 such that for all $0\le k\le n-1$,

$$\mathsf{P}\Biggl(\sum_{j=k+1}^{n}(\xi_j-\mathsf{E}\xi_j)\ge c_0\varepsilon\Biggr)\le 1/2.$$

This, together with the independence of ξ_j, yields that

$$\begin{aligned} I_{5,1} &= \mathsf{P}(\mathbf{F}_1) + \sum_{k=2}^{n}\mathsf{P}(\mathbf{F}_1^c,\ldots,\mathbf{F}_{k-1}^c,\mathbf{F}_k) \\ &\ge \mathsf{P}(\mathbf{F}_1, y\le \xi_1\le y+4\varepsilon) + \sum_{k=2}^{n}\mathsf{P}\Biggl(\mathbf{F}_1^c,\ldots,\mathbf{F}_{k-1}^c,\mathbf{F}_k, y\le\sum_{j=1}^{k}\xi_j\le y+4\varepsilon\Biggr) \\ &\ge 2\mathsf{P}\Biggl(\mathbf{F}_1, y\le\xi_1\le y+4\varepsilon, \sum_{j=2}^{n}(\xi_j-\mathsf{E}\xi_j)\ge c_0\varepsilon\Biggr) \\ &\quad + 2\sum_{k=2}^{n}\mathsf{P}\Biggl(\mathbf{F}_1^c,\ldots,\mathbf{F}_{k-1}^c,\mathbf{F}_k, y\le\sum_{j=1}^{k}\xi_j\le y+4\varepsilon, \sum_{j=k+1}^{n}(\xi_j-\mathsf{E}\xi_j)\ge c_0\varepsilon\Biggr) \\ &\ge 2\mathsf{P}\Biggl(\sum_{k=1}^{n}(\xi_k-\mathsf{E}\xi_k)\ge y+(c_0+4)\varepsilon+D_n\Biggr), \end{aligned}$$

where $D_n = \sum_{j=1}^n |\mathsf{E}\xi_j|$. Similarly to the proofs of (3.26)–(3.27), it follows from Lemma 2 with $\eta_j = \xi_j - \mathsf{E}\xi_j$ that

$$\begin{aligned} I_{5,1} &\geq 2\mathsf{P}\left(\sum_{k=1}^n (\xi_k - \mathsf{E}\xi_k) \geq y + (c_0+4)\varepsilon + D_n\right) \\ &\geq 2\big(1-\Phi(x)\big)\big(1 - A(\varepsilon + \varepsilon^{-2}\Delta_{n,x})\big). \end{aligned} \tag{4.6}$$

On the other hand, similar to the proofs of (3.26) and (3.27), we have

$$\begin{aligned} I_{5,2} &\leq 2\mathsf{P}\left(\sum_{j=1}^{n_0} \xi_j \geq 2x^2 + (1-c_0)\varepsilon - D_n\right) \\ &\leq Cx^{-1}\exp\left(-\frac{x^2}{2} - \frac{x^2\sum_{j=n_0+1}^n \mathsf{E}X_j^2}{2B_n}\right) \\ &\leq Cx^{-2}\mathrm{e}^{-x^2/2}. \end{aligned} \tag{4.7}$$

This, together with (4.6), implies (4.3). □

Proof of (4.4). Define $k'_0 = 1$, and $k'_i = k'_{i-1} + 1$ if $\mathsf{E}X^2_{k'_{i-1}+1} > \varepsilon^2 B_n^2/x^6$, and otherwise

$$k'_i = \max\left\{k \leq n: \sum_{j=k'_{i-1}+1}^{k} \mathsf{E}X_j^2 \leq \frac{\varepsilon^2 B_n^2}{x^6}\right\} + 1.$$

Let m satisfy $k'_{m-1} < n \leq k'_m$ and define

$$k_i = k'_i \qquad \text{for } i < m, \quad \text{and} \quad k_m = n.$$

Because $\sum_{j=k_{i-1}+1}^{k_i} \mathsf{E}X_j^2 > \varepsilon^2 B_n^2/x^6$ for $i < m$, we have

$$B_n^2 \geq \sum_{i=1}^{m-1} \sum_{j=k_{i-1}+1}^{k_i} \mathsf{E}X_j^2 > (m-1)\varepsilon^2 B_n^2/x^6,$$

which implies that $m \leq \varepsilon^{-2}x^6 + 1$. Furthermore, suppose that i_0 satisfies $k_{i_0-1} < n_0 \leq k_{i_0}$, where n_0 is defined as in (2.2). Set

$$\begin{aligned} \check{X}_k &= X_k I\{|X_k| \leq 16^{-1}\varepsilon B_n/x^3\}, \\ \hat{X}_k &= X_k I\{16^{-1}\varepsilon B_n/x^3 < |X_k| \leq \varepsilon B_n/x\}, \qquad \hat{Z}_{k_i} = \sum_{k=k_{i-1}+1}^{k_i-1} |\hat{X}_k|. \end{aligned}$$

Note that $2b|\bar{X}_k| \le 2\varepsilon$. Simple calculations show that

$$\begin{aligned}
I_6 &\le \sum_{i=i_0}^{m} \mathsf{P}\Bigg(\bigcup_{k=k_{i-1}+1}^{k_i} \{2b\bar{S}_k \ge b^2\bar{V}_k^2 + x^2 + \varepsilon, \\
&\qquad\qquad (\bar{V}_n^2 - \bar{V}_k^2) - \mathsf{E}(\bar{V}_n^2 - \bar{V}_k^2) \ge \varepsilon B_n^2/x^2\}\Bigg) \\
&\le \sum_{i=i_0}^{m} \mathsf{P}\Bigg(\bigcup_{k=k_{i-1}+1}^{k_i-1} \{2b\bar{S}_k \ge b^2\bar{V}_k^2 + x^2 - \varepsilon, \\
&\qquad\qquad (\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) - \mathsf{E}(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) \ge 2^{-1}\varepsilon B_n^2/x^2\}\Bigg) \\
&\le \sum_{i=i_0}^{m} \mathsf{P}\big(2b(\bar{S}_{k_{i-1}} + \hat{Z}_{k_i} + \mathsf{E}\hat{Z}_{k_i}) \ge b^2\bar{V}_{k_i}^2 + x^2 - 2\varepsilon, \\
&\qquad (\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) - \mathsf{E}(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) \ge 2^{-1}\varepsilon B_n^2/x^2\big) \\
&\quad + \sum_{i=i_0}^{m} \mathsf{P}\Bigg(\max_{k_{i-1}+1\le j\le k_i-1} 2b \sum_{k=k_{i-1}+1}^{j} (\breve{X}_k - \mathsf{E}\breve{X}_k) \ge \varepsilon\Bigg) \\
&=: I_{6,1} + I_{6,2}.
\end{aligned} \tag{4.8}$$

Noting that

$$\sigma_{ni}^2 := \sum_{k=k_{i-1}+1}^{k_i-1} \mathsf{E}\breve{X}_k^2 \le \frac{\varepsilon^2 B_n^2}{x^6} \quad \text{and} \quad |\breve{X}_k| \le 16^{-1}\varepsilon B_n/x^3,$$

it follows from $m \le \varepsilon^{-2}x^6 + 1$ and Lévy's inequality that with $t = 2bx^2/\varepsilon$

$$\begin{aligned}
I_{6,2} &\le \sum_{i=i_0}^{m} \mathsf{P}\Bigg(\sum_{k=k_{i-1}+1}^{k_i-1} (\breve{X}_k - \mathsf{E}\breve{X}_k) \ge \varepsilon/(2b) - \sqrt{2}\sigma_{ni}\Bigg) \\
&\le \sum_{i=i_0}^{m} \mathrm{e}^{-t(\varepsilon/(2b)-\sqrt{2}\sigma_{ni})} \prod_{k=k_{i-1}+1}^{k_i-1} E\mathrm{e}^{t(\breve{X}_k - \mathsf{E}\breve{X}_k)} \\
&\le A\mathrm{e}^{-x^2} \sum_{i=i_0}^{m} \exp\{At^2\sigma_{ni}^2\} \\
&\le 2A_1(\varepsilon^{-2}x^6 + 1)\mathrm{e}^{-x^2} \le C_{\tau,\delta}x^{-2}\mathrm{e}^{-x^2/2},
\end{aligned} \tag{4.9}$$

where we used the fact that $\varepsilon \ge \gamma x^{-1/2}$.

To estimate $I_{6,1}$, let $t = 24\varepsilon^{-1}x^2 B_n^{-2}\log x$. Note that

$$2b\mathsf{E}\hat{Z}_{k_i} \le \frac{32x^4}{\varepsilon B_n^2}\sum_{k=k_{i-1}+1}^{k_i-1}\mathsf{E}X_k^2 \le \frac{32\varepsilon}{x^2} \le 8\varepsilon.$$

Similar to the estimate for I_3 in (3.4), we obtain

$$\begin{aligned}
I_{6,1} &\le \sum_{i=i_0}^{m}\mathsf{P}\big(2b(\bar{S}_{k_{i-1}} + \hat{Z}_{k_i} - \mathsf{E}\hat{Z}_{k_i}) \ge b^2\bar{V}_{k_{i-1}}^2 + x^2 - 18\varepsilon, \\
&\qquad (\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) - \mathsf{E}(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) \ge 2^{-1}\varepsilon B_n^2/x^2\big) \\
&\le \sum_{i=i_0}^{m}\mathsf{P}\big(b(\bar{S}_{k_{i-1}} - \mathsf{E}\bar{S}_{k_{i-1}} + \hat{Z}_{k_i} - \mathsf{E}\hat{Z}_{k_i}) - b^2\bar{V}_{k_{i-1}}^2/2 \\
&\qquad + t(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) - t\mathsf{E}(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) \ge x^2/2 + 12\log x - 9\varepsilon\big) \\
&\le Ax^{-12}\mathrm{e}^{-x^2/2}\sum_{i=i_0}^{m}\Bigg\{\prod_{j=1}^{k_{i-1}}\mathsf{E}\mathrm{e}^{b(\bar{X}_j - \mathsf{E}\bar{X}_j) - 2^{-1}b^2\bar{X}_j^2} \\
&\qquad \times \prod_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}\mathrm{e}^{b(|\hat{X}_j| - \mathsf{E}|\hat{X}_j|) + t(\bar{X}_j^2 - \mathsf{E}\bar{X}_j^2)} \times \prod_{j=k_i}^{n}\mathsf{E}\mathrm{e}^{t(\bar{X}_j^2 - \mathsf{E}\bar{X}_j^2)}\Bigg\} \\
&\le Ax^{-12}\mathrm{e}^{-x^2/2}\sum_{i=i_0}^{m}\exp\Bigg(A\Delta_{n,x} + A\frac{x^2\sum_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}X_j^2}{B_n^2}\mathrm{e}^{24\varepsilon\log x} \\
&\qquad + A\frac{x^3\sum_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}|\bar{X}_j|^3}{B_n^3}\mathrm{e}^{24\varepsilon\log x}\varepsilon^{-1}\log x \\
&\qquad + A\frac{x^4\sum_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}\bar{X}_j^4}{B_n^4}\mathrm{e}^{24\varepsilon\log x}(\varepsilon^{-1}\log x)^2 \\
&\qquad + A\frac{x^4\sum_{j=k_i}^{n}\mathsf{E}\bar{X}_j^4}{B_n^4}\mathrm{e}^{24\varepsilon\log x}(\varepsilon^{-1}\log x)^2\Bigg).
\end{aligned} \tag{4.10}$$

Recall, by the definition of k_i,

$$\frac{x^2}{B_n^2}\sum_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}X_j^2 \le \varepsilon^2 x^{-4},$$

$$\frac{x^4}{B_n^4}\sum_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}\bar{X}_j^4 \le \frac{\varepsilon x^3}{B_n^3}\sum_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}|\bar{X}_j|^3 \le \frac{\varepsilon^2 x^2}{B_n^2}\sum_{j=k_{i-1}+1}^{k_i-1}\mathsf{E}X_j^2 \le \varepsilon^4 x^{-4}.$$

On the other hand, we have

$$\varepsilon^{-1}x^{24\varepsilon} \le \gamma^{-1}\min\{x^{\delta/10+\delta/3}, x^{3/2}\} \le \gamma^{-1}x^{\min\{\delta/2,3/2\}},$$

and by (2.2)–(2.4) and the inequality $\sum_{j=n_0+1}^n \mathsf{E}X_j^2 \le 192x^{-2}B_n^2\log x$, for all $i \ge i_0$,

$$\begin{aligned}\frac{x^4}{B_n^4}\sum_{j=k_i}^n \mathsf{E}\bar{X}_j^4 &\le (\varepsilon B_n/x)\sum_{j=n_0+1}^n \mathsf{E}|\bar{X}_j|^3 \\ &\le (\varepsilon\tau B_n^2/x^{2+\delta})\sum_{j=n_0+1}^n \mathsf{E}X_j^2 \le C_{\tau,\delta}\varepsilon x^{-\delta}\log x.\end{aligned}$$

Substituting these estimates into (4.10) gives

$$\begin{aligned}I_{6,1} &\le C_{\delta,\tau}(\varepsilon^{-2}x^6+1)x^{-12}\mathrm{e}^{-x^2/2}\exp(C_{\tau,\delta}x^{-1/2}\log^2 x + C_{\tau,\delta}x^{-\delta/2}\log^3 x) \\ &\le C_{\delta,\tau}x^{-2}\mathrm{e}^{-x^2/2}.\end{aligned} \tag{4.11}$$

This proves (4.4), which also completes the proof of Lemma 3. □

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