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Independence properties of the Matsumoto–Yor type

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We define Letac–Wesolowski–Matsumoto–Yor (LWMY) functions as decreasing functions from $(0,\infty)$ onto $(0,\infty)$ with the following property: there exist independent, positive random variables X and Y such that the variables $f(X+Y)$ and $f(X)-f(X+Y)$ are independent. We prove that, under additional assumptions, there are essentially four such functions. The first one is $f(x)=1/x$. In this case, referred to in the literature as the *Matsumoto–Yor property*, the law of X is generalized inverse Gaussian while Y is gamma distributed. In the three other cases, the associated densities are provided. As a consequence, we obtain a new relation of convolution involving gamma distributions and Kummer distributions of type 2.

Keywords: gamma distribution; generalized inverse Gaussian distribution; Kummer distribution; Matsumoto–Yor property

1. Introduction

Many papers have been devoted to generalized inverse Gaussian (GIG) distributions since their definition by Good [5] (see, e.g., [1,8,15,16]).

The GIG distribution with parameters $\mu\in\mathbb{R}$, $a,b>0$ is the probability measure

$$\mathrm{GIG}(\mu,a,b)(\mathrm{d}x)=\left(\frac{b}{a}\right)^{\mu}\frac{x^{\mu-1}}{2K_\mu(ab)}\mathrm{e}^{-(a^2x^{-1}+b^2x)/2}\mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x, \tag{1.1}$$

where K_μ is the classical McDonald special function.

(1) We stress the close links between GIG, gamma distributions and the function $f_0(x)=1/x$ $(x>0)$.

(a) The family of GIG distributions is invariant under f_0: we can easily deduce from (1.1) that the image of $\mathrm{GIG}(\mu,a,b)$ by f_0 is $\mathrm{GIG}(-\mu,b,a)$.

(b) Barndorff-Nielsen and Halgreen [1] proved that

$$\mathrm{GIG}(-\mu,a,b)*\gamma\left(\mu,\frac{b^2}{2}\right)=\mathrm{GIG}(\mu,a,b),\qquad \mu,a,b>0, \tag{1.2}$$

where $\gamma(\mu,b^2/2)(\mathrm{d}x)=\frac{b^{2\mu}}{2^\mu\Gamma(\mu)}x^{\mu-1}\exp-\frac{b^2}{2}x\mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x$.

Therefore, if $X\sim\mathrm{GIG}(-\lambda,a,a)$ and $Y\sim\gamma(\lambda,a^2/2)$ are independent random variables, then

$$X\overset{(d)}{=}f_0(X+Y). \tag{1.3}$$

Letac and Seshadri [8] proved that (1.3) characterizes GIG distributions of the type GIG($-\lambda$, a, a).

(c) Almost sure realizations of (1.2) have been given by Bhattacharya and Waymire [3] in the case $\mu = \frac{1}{2}$, Vallois [16] for any $\mu > 0$ by means of a family of transient diffusions and Vallois [15], theorem on page 446, in terms of random walks.

(2) The so-called *Matsumoto–Yor property* is the following: let X and Y be two independent random variables such that

$$X \sim \mathrm{GIG}(-\mu, a, b), \qquad Y \sim \gamma(\mu, b^2/2), \qquad (\mu, a, b > 0). \tag{1.4}$$

Then,

$$U := \frac{1}{X+Y} = f_0(X+Y), \qquad V := \frac{1}{X} - \frac{1}{X+Y} = f_0(X) - f_0(X+Y) \tag{1.5}$$

are independent and

$$U \sim \mathrm{GIG}(-\mu, b, a), \qquad V \sim \gamma(\mu, a^2/2). \tag{1.6}$$

The case $a = b$ was proven by Matsumoto and Yor [11] and a nice interpretation of this property via Brownian motion was given by Matsumoto and Yor [12]. The case $\mu = -\frac{1}{2}$ of the Matsumoto–Yor property can be retrieved from an independence property established by Barndorff-Nielsen and Koudou [2] (see [7]).

Letac and Wesolowski [9] proved that the Matsumoto–Yor property holds for any $\mu, a, b > 0$ and characterizes the GIG distributions. More precisely, consider two independent and non-Dirac positive random variables X and Y such that U and V defined by (1.5) are independent. There then exist $\mu, a, b > 0$ such that (1.4) holds.

The starting point of this paper is to study the link between the function $f_0 : x \mapsto 1/x$ and the GIG distributions in the Matsumoto–Yor property.

Obviously, the Matsumoto–Yor property can be re-expressed as follows: the image of the probability measure (on $\mathbb{R}_+^2$) $\mathrm{GIG}(-\mu, a, b) \otimes \gamma(\mu, b^2/2)$ by the transformation $T_{f_0} : (x, y) \mapsto (f_0(x + y), f_0(x) - f_0(x + y))$ is the probability measure $\mathrm{GIG}(-\mu, b, a) \otimes \gamma(\mu, a^2/2)$. This formulation of the Matsumoto–Yor property, joined with the Letac and Wesolowski result, leads us to determine the triplets (μ_X, μ_Y, f) such that:

(a) μ_X, μ_Y are probability measures on $(0, \infty)$;
(b) $f : (0, \infty) \to (0, \infty)$ is bijective and decreasing;
(c) if X and Y are independent random variables such that $X \sim \mu_X$ and $Y \sim \mu_Y$, then the random variables $U = f(X + Y)$ and $V = f(X) - f(X + Y)$ are independent.

Unfortunately, we have not been able to solve this question without restriction. Our method can be applied provided that f is smooth and μ_X and μ_Y have smooth density functions (see Theorem 3.1 for details). After long and sometimes tedious calculations, we prove (see Theorem 2.2) that there are only four classes, $\mathcal{F}_1, \ldots, \mathcal{F}_4$, of functions f such that T_f keeps the independence property. Then, for any $f \in \mathcal{F}_i$, $1 \le i \le 4$, we have been able to give the corresponding distributions of X and Y and the related laws of U and V (for $\mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$, see Theorems 2.4, 2.14

and Remark 2.5). The first class, $\mathcal{F}_1 = \{\alpha/x; \alpha > 0\}$, corresponds to the known case $f = f_0$. This case, as mentioned in Remark 3.3, allows us recover, under stronger assumptions, the result of Letac and Wesolowski that the only possible distributions for X and Y are GIG and gamma, respectively. The proof of Letac and Wesolowski is completely different from ours since the authors make use of Laplace transforms and a characterization of the GIG laws as the distribution of a continued fraction with gamma entries. We have not been able to develop a proof as elegant as theirs because, with $f = f_0$, we have algebraic properties (e.g., continued fractions), while these properties are lost if we start with a general function f.

It is worth pointing out that one interesting feature of our analysis is an original characterization of the families of distributions $\{\beta_\alpha(a, b, c); a, b, \alpha > 0, c \in \mathbb{R}\}$ and the Kummer distributions $\{K^{(2)}(a, b, c); a, c > 0, b \in \mathbb{R}\}$ (see (2.14) and (2.29), respectively). The Kummer distributions appear as the laws of some random continued fractions (see [10], page 3393, mentioning a work by Dyson [4] in the setting of random matrices).

As by-products of our study, we obtain new relations for convolution. For simplicity, we only detail the case of Kummer distributions of type 2:

$$K^{(2)}(a, b, c) * \gamma(b, c) = K^{(2)}(a + b, -b, c). \tag{1.7}$$

Obviously, this relation is similar to (1.2).

Inspired by the result of Letac and Wesolowski [9] and Theorem 2.6, we can ask (for the purposes of future research) whether a characterization of Kummer distributions could be obtained via an “algebraic” method.

As recalled in the above item (c), there are various almost sure realizations of (1.2) and of the convolution coming from the Matsumoto–Yor property. One interesting open question derived from our study would be to determine a random variable Z with distribution $K^{(2)}(a + b, -b, c)$ which can be decomposed as the sum of two explicit independent random variables X and Y such that $X \sim K^{(2)}(a + b, -b, c)$ and $Y \sim \gamma(b, c)$.

The paper is organized as follows. We state our main results in Section 2. In Section 3 we give a key differential equation involving f and the log densities of the independent random variables X and Y such that $f(X + Y)$ and $f(X) - f(X + Y)$ are independent (see Theorem 3.1). Based on this equation, we prove (see Theorem 3.9) that there are only four classes of such functions f. The theorems stated in Section 2 are proved in Section 4; however, one technical proof has been postponed to the Appendix.

2. Main results

Definition 2.1. *Let $f : (0, \infty) \to (0, \infty)$ be a decreasing and bijective function.*

(1) *We consider the transformation associated with f*

$$\begin{aligned} T_f : (0, \infty)^2 &\to (0, \infty)^2, \\ (x, y) &\mapsto \big(f(x + y), f(x) - f(x + y)\big). \end{aligned} \tag{2.8}$$

The transformation T_f is one-to-one and if f^{-1} is the inverse of f, then

$$(T_f)^{-1} = T_{f^{-1}}. \tag{2.9}$$

(2) *Let X and Y be two independent and positive random variables. Let us define*

$$(U, V) = T_f(X, Y) = \big(f(X+Y), f(X) - f(X+Y)\big). \tag{2.10}$$

f is said to be an LWMY function with respect to (X, Y) *if the random variables U and V are independent. f is said to be an LWMY function if it is an LWMY function with respect to some random vector (X, Y).*

One aim of this paper is to characterize LWMY functions. Let us introduce

$$f_1(x) = \frac{1}{e^x - 1}, \qquad x > 0, \tag{2.11}$$

$$g_1(x) = f_1^{-1}(x) = \ln\left(\frac{1+x}{x}\right), \qquad x > 0 \tag{2.12}$$

and, for $\delta > 0$,

$$f_\delta^*(x) = \log\left(\frac{e^x + \delta - 1}{e^x - 1}\right), \qquad x > 0. \tag{2.13}$$

Theorem 2.2. *Let $f : (0, \infty) \to (0, \infty)$ be decreasing and bijective. Under some additional assumptions (see Theorem 3.1, (3.7) and (3.8)), f is an LWMY function if and only if either $f(x) = \frac{\alpha}{x}$, $f(x) = \frac{1}{\alpha} f_1(\beta x)$, $f(x) = \frac{1}{\alpha} g_1(\beta x)$ or $f(x) = \frac{1}{\alpha} f_\delta^*(\beta x)$ for some $\alpha, \beta, \delta > 0$.*

Remark 2.3. (1) The four classes of LWMY functions are $\mathcal{F}_1 = \{\alpha/x; \alpha > 0\}$, $\mathcal{F}_2 = \{\frac{1}{\alpha} f_1(\beta x); \alpha, \beta > 0\}$, $\mathcal{F}_3 = \{\frac{1}{\alpha} g_1(\beta x); \alpha, \beta > 0\}$ and $\mathcal{F}_4 = \{\frac{1}{\alpha} f_\delta^*(\beta x); \alpha, \beta > 0\}$.

(2) It is clear that if f is an LWMY function, then the functions f^{-1} and $x \mapsto \frac{1}{\alpha} f(\beta x)$, $\alpha, \beta > 0$, are LWMY functions.

(3) The image of $\mathcal{F}_2$ by the map $f \mapsto f^{-1}$ is $\mathcal{F}_3$. The functions $x \mapsto \alpha/x$ and f_δ are involutive.

In the sequel, we focus on the three new cases: either $f = f_1$, $f = g_1$ or $f = f_\delta^*$ and in each case, we determine the laws of the related random variables.

2.1. The cases $f = g_1$ and $f = f_1$

(a) Recall the definitions of the gamma distribution $\gamma(\lambda, c)(\mathrm{d}x) = \frac{c^\lambda}{\Gamma(\lambda)} x^{\lambda-1} \mathrm{e}^{-cx} \mathbf{1}_{(0,\infty)}(x)\, \mathrm{d}x$ $(\lambda, c > 0)$ and the beta distribution $\mathrm{Beta}(a, b)(\mathrm{d}x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \mathbf{1}_{\{0<x<1\}}\, \mathrm{d}x$ $(a, b > 0)$. Consider (see, e.g., [13], or [14] and the references therein) the *Kummer distribution* *of type 2*:

$$K^{(2)}(a,b,c) := \alpha(a,b,c)x^{a-1}(1+x)^{-a-b}\mathrm{e}^{-cx}\mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x, \qquad a,c>0, b\in\mathbb{R}, \tag{2.14}$$

where $\alpha(a,b,c)$ is a normalizing constant.

Associated with a couple (X,Y) of positive random variables, consider

$$(U,V) := T_{f_1}(X,Y) = \left(\frac{1}{\mathrm{e}^{X+Y}-1}, \frac{1}{\mathrm{e}^{X}-1} - \frac{1}{\mathrm{e}^{X+Y}-1}\right). \tag{2.15}$$

In Theorems 2.4 and 2.6 below, we suppose that all random variables have positive and twice differentiable densities.

First, we consider the case $f = f_1$. We determine the distributions of X and Y such that f_1 is an LWMY function associated with (X,Y).

Theorem 2.4. (1) *Consider two positive and independent random variables X and Y. The random variables U and V defined by* (2.15) *are independent if and only if the densities of Y and X are*, *respectively*,

$$p_Y(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(1-\mathrm{e}^{-y})^{b-1}\mathrm{e}^{-ay}\mathbf{1}_{\{y>0\}}, \tag{2.16}$$

$$\begin{aligned} p_X(x) &= \alpha(a+b,c,-a)\mathrm{e}^{-(a+b)x}(1-\mathrm{e}^{-x})^{-b-1} \\ &\qquad \times \exp\left(-c\frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}}\right)\mathbf{1}_{\{x>0\}}, \end{aligned} \tag{2.17}$$

where a, *b and c are constants such that $a,b,c>0$ and $\alpha(a+b,c,-a)$ is the constant from equation* (2.14). *Thus*, *the law of Y is the image of the* $\mathrm{Beta}(a,b)$ *distribution by the transformation $z\in(0,1)\mapsto -\log z\in(0,\infty)$*, *while the law of the variable $f_1(X)$ is $K^{(2)}(a+b,-b,c)$* (*see equation* (2.14)).

(2) *If* (1) *holds*, *then $U \sim K^{(2)}(a,b,c)$ and $V\sim\gamma(b,c)$.*

The proof of Theorem 2.4 will be given in Section 4.

Remark 2.5. Since $g_1 = f_1^{-1}$, Remark 2.3 and Theorem 2.4 imply that the random variables associated with the LWMY function g_1 are the random variables U and V distributed as in item 2 of Theorem 2.4.

(b) As suggested by identities (2.16) and (2.17), it is possible to simplify the statement of Theorem 2.4. Since $T_{g_1} = T_{f_1}^{-1}$, we have

$$(X,Y) = T_{g_1}(U,V) = \left(\log\left(\frac{1+U+V}{U+V}\right), \log\left(\frac{1+U}{U}\right) - \log\left(\frac{1+U+V}{U+V}\right)\right). \tag{2.18}$$

As (2.18) shows, it is useful to introduce

$$(U', V') = \left(\frac{1 + 1/(U+V)}{1+1/U}, U+V\right). \tag{2.19}$$

Obviously, the correspondence $(U, V) \mapsto (U', V')$ is one-to-one:

$$(U, V) = \left(\frac{U'V'}{V'+1-U'V'}, \frac{V'(V'+1)(1-U')}{V'+1-U'V'}\right). \tag{2.20}$$

Furthermore, (X, Y) can be easily expressed in terms of (U', V'):

$$X = \log(1 + 1/V') \quad \text{and} \quad Y = -\log U'. \tag{2.21}$$

Since it is easy to determine the density function of $\phi(\xi)$ knowing the density function of a random variable ξ, where ϕ is differentiable and bijective, Theorem 2.4 and its analog related to $f = g_1$ (see Remark 2.5) are equivalent to Theorem 2.6 below.

Theorem 2.6. **(a)** *Let U' and V' be two positive and independent random variables. The random variables U and V defined by* (2.20) *are independent if only if there exist some constants a, b, c such that*

$$U' \sim \text{Beta}(a, b) \quad \textit{and} \quad V' \sim K^{(2)}(a+b, -b, c). \tag{2.22}$$

If one of these equivalent conditions holds, then $U \sim K^{(2)}(a, b, p)$ and $V \sim \gamma(b, c)$.

(b) *Let U and V be two positive and independent random variables. The random variables U' and V' defined by* (2.19) *are independent if only if there exist some constants a, b, c such that*

$$U \sim K^{(2)}(a, b, c) \quad \textit{and} \quad V \sim \gamma(b, c). \tag{2.23}$$

Under (2.23), *$U' \sim \text{Beta}(a, b)$ and $V' \sim K^{(2)}(a+b, -b, c)$.*

We now formulate a simple consequence of Theorem 2.6.

Theorem 2.7. *For any $a, b, c > 0$, the transformation $(u, v) \mapsto (\frac{1+1/(u+v)}{1+1/u}, u+v)$ maps the probability measure $K^{(2)}(a, b, c) \otimes \gamma(b, c)$ to the probability measure $\text{Beta}(a, b) \otimes K^{(2)}(a+b, -b, c)$. In particular,*

$$K^{(2)}(a, b, c) * \gamma(b, c) = K^{(2)}(a+b, -b, c). \tag{2.24}$$

Remark 2.8. Note that (2.24) may be regarded as an analog of (1.2).

2.2. The case $f = f_\delta^*$

Recall that f_δ^* has been defined by (2.13). Due to the form of f_δ^*, a change of variables allows us to simplify the search for independent random variables X and Y such that the two components of $T_{f_\delta^*}(X, Y)$ are independent.

For any decreasing and bijective function $f:(0,\infty)\to(0,\infty)$, we define

$$\overline{f}(x)=\exp\{-f(-\log x)\},\qquad x\in(0,1), \tag{2.25}$$

$$T_f^m(x,y)=\left(f(xy),\frac{f(x)}{f(xy)}\right),\qquad x,y\in(0,1). \tag{2.26}$$

Observe that $\overline{f}$ is one-to-one and onto from $(0,1)$ to $(0,1)$, T_f^m is one-to-one and onto from $(0,1)^2$ to $(0,1)^2$ and

$$(T_f^m)^{-1}=T_{f^{-1}}^m. \tag{2.27}$$

Definition 2.9. *Let X and Y be two independent and $(0,1)$-valued random variables. We say that a decreasing and bijective function $f:(0,1)\to(0,1)$ is a multiplicative LWMY function with respect to (X,Y) if the random variables $U^m:=f(XY)$ and $V^m:=\frac{f(X)}{f(XY)}$ are independent.*

Remark 2.10. For any random vector (X,Y) in $(0,\infty)^2$, we consider $X'=\mathrm{e}^{-X}$ and $Y'=\mathrm{e}^{-Y}$. Then, f is an LWMY function with respect to (X,Y) if and only if $\overline{f}$ is a multiplicative LWMY function with respect to (X',Y').

The change of variable $x'=\mathrm{e}^{-x}$ is very convenient since the function

$$\phi_\delta(x):=\overline{f_\delta^*}(x)=\frac{1-x}{1+(\delta-1)x},\qquad x\in(0,1) \tag{2.28}$$

is homographic.

Note that $\overline{f_\delta^*}:(0,1)\to(0,1)$ is bijective, decreasing and equal to its inverse. First, let us determine the distribution of the couple (X',Y') of random variables such that ϕ_δ is a multiplicative LWMY function with respect to (X',Y').

For $a,b,\alpha>0$ and $c\in\mathbb{R}$, consider the probability measure

$$\beta_\alpha(a,b;c)(\mathrm{d}x)=k_\alpha(a,b;c)x^{a-1}(1-x)^{b-1}(\alpha x+1-x)^c\mathbf{1}_{(0,1)}(x)\,\mathrm{d}x. \tag{2.29}$$

Note that if $c=0$, then $\beta_\alpha(a,b;c)=\mathrm{Beta}(a,b)$.

Theorem 2.11. *Let X' and Y' be two independent random variables valued in $(0,1)$. Consider*

$$(U^m,V^m)=T_{\phi_\delta}^m(X',Y')=\left(\frac{1-X'Y'}{1+(\delta-1)X'Y'},\frac{1-X'}{1+(\delta-1)X'}\frac{1+(\delta-1)X'Y'}{1-X'Y'}\right)$$

for fixed $\delta>0$.

Then, U^m and V^m are independent if and only if there exist $a,b,\lambda>0$ such that

$$X'\sim\beta_\delta(a+b,\lambda;-\lambda-b),\qquad Y'\sim\mathrm{Beta}(a,b). \tag{2.30}$$

If this condition holds, then

$$U^m \sim \beta_\delta(\lambda + b, a; -a - b), \qquad V^m \sim \mathrm{Beta}(\lambda, b). \tag{2.31}$$

In the case $\delta = 1$, Theorem 2.11 takes a very simple form.

Proposition 2.12. *Let X' and Y' be two independent random variables valued in $(0, 1)$. Then,*

$$U^m = 1 - X'Y', \qquad V^m = \frac{1 - X'}{1 - X'Y'}$$

are independent if and only if there exist $a, b, \lambda > 0$ such that

$$X' \sim \mathrm{Beta}(a + b, \lambda) \quad \text{and} \quad Y' \sim \mathrm{Beta}(a, b).$$

If one of these conditions holds, then $U^m \sim \mathrm{Beta}(\lambda + b, a)$ and $V^m \sim \mathrm{Beta}(\lambda, b)$.

Remark 2.13. When $X' \sim \mathrm{Beta}(a + b, \lambda)$ and $Y' \sim \mathrm{Beta}(a, b)$, it can be proven that U^m and V^m are independent using the well-known property that if Z and Z' are independent with $Z \sim \gamma(a, 1)$ and $Z' \sim \gamma(b, 1)$, then $R := \frac{Z}{Z+Z'}$ and $Z + Z'$ are independent with $R \sim \mathrm{Beta}(a, b)$ and $Z + Z' \sim \gamma(a + b, 1)$ (see, e.g., [17]).

According to Remark 2.10, f_δ^* is an LWMY function with respect to (X, Y) if and only if ϕ_δ is a multiplicative LWMY function with respect to $(X', Y') = (\mathrm{e}^{-X}, \mathrm{e}^{-Y})$. Therefore, a classical change of variables allows us to deduce that Theorem 2.11 is equivalent to Theorem 2.14 below.

Theorem 2.14. (1) *Consider two positive and independent random variables X and Y. The random variables $U = f_\delta^*(X + Y)$, $V = f_\delta^*(X) - f_\delta^*(X + Y)$ are independent if and only if the densities of Y and X are, respectively,*

$$p_Y(y) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)}(1 - \mathrm{e}^{-y})^{b-1}\mathrm{e}^{-ay}\mathbf{1}_{\{y>0\}}, \tag{2.32}$$

$$\begin{aligned} p_X(x) = {} & k_\delta(a + b, \lambda, -\lambda - b)\mathrm{e}^{-(a+b)x}(\delta\mathrm{e}^{-x} + 1 - \mathrm{e}^{-x})^{-\lambda-b} \\ & \times (1 - \mathrm{e}^{-x})^{\lambda-1}\mathbf{1}_{x>0}, \end{aligned} \tag{2.33}$$

where $a, b > 0$, $\lambda \in \mathbb{R}$ and $k_\delta(a + b, \lambda, -\lambda - b)$ is the normalizing factor (see (2.29)). Thus, e^{-Y} is $\mathrm{Beta}(a, b)$-distributed and e^{-X} is $\beta_\delta(a + b, \lambda, -\lambda - b)$-distributed.

(2) *If* (1) *holds, then the densities of U and V are, respectively,*

$$\begin{aligned} p_U(u) = {} & k_\delta(\lambda + b, a; -a - b)\mathrm{e}^{-u(\lambda+b)}(1 - \mathrm{e}^{-u})^{a-1} \\ & \times \left(1 + (\delta - 1)\mathrm{e}^{-u}\right)^{-a-b}\mathbf{1}_{u>0}, \end{aligned} \tag{2.34}$$

$$p_V(v) = \mathrm{e}^{-\lambda v}(1 - \mathrm{e}^{-v})^{b-1}\mathbf{1}_{v>0}. \tag{2.35}$$

We omit the proof of Theorem 2.14 since it is similar to that of Theorem 2.4.

3. The set of all possible "smooth" LWMY functions

The following theorem gives a functional equation linking LWMY functions to the related densities.

Theorem 3.1. *Let X and Y be two independent and positive random variables whose densities p_X and p_Y are positive and twice differentiable. Define $\phi_X = \log p_X$ and $\phi_Y = \log p_Y$. Consider a decreasing function $f : (0,\infty) \mapsto (0,\infty)$, three times differentiable. Then, f is a LWMY function with respect to (X, Y) if and only if*

$$\phi_X''(x) - \phi_X'(x)\frac{f''(x)}{f'(x)} + \phi_Y''(y) f'(x)\left(\frac{1}{f'(x)} - \frac{1}{f'(x+y)}\right) + \phi_Y'(y)\frac{f''(x)}{f'(x)} + \frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = 0, \qquad x, y > 0. \tag{3.1}$$

Proof. Let $g = f^{-1}$ and $(U, V) = T_f(X, Y)$. By formula (2.9), $(X, Y) = T_g(U, V)$. X and Y being independent, the density of (U, V) is

$$p_{(U,V)}(u, v) = p_X\big(g(u+v)\big) p_Y\big(g(u) - g(u+v)\big) |J(u, v)| \mathbf{1}_{u,v>0}, \tag{3.2}$$

where J is the Jacobian of the transformation T_f. We get $|J(u, v)| = g'(u+v)g'(u)$, and then

$$p_{(U,V)}(u, v) = p_X\big(g(u+v)\big) p_Y\big(g(u) - g(u+v)\big) g'(u+v)g'(u). \tag{3.3}$$

The variables U and V are independent if and only if the function $H = \log p_{(U,V)}$ satisfies $\frac{\partial^2 H}{\partial u \partial v} = 0$. By equation (3.3) we obtain

$$\begin{aligned}\frac{\partial^2 H}{\partial u\, \partial v} &= \phi_X''(x)[g'(f(x))]^2 + \phi_X'(x) g''(f(x)) \\ &\quad - \phi_Y''(y) g'(f(x))\big[g'\big(f(x+y)\big) - g'(f(x))\big] \\ &\quad - \phi_Y'(y) g''(f(x)) + \frac{g'''g' - (g'')^2}{(g')^2}(f(x)),\end{aligned} \tag{3.4}$$

where $x = g(u+v)$ and $y = g(u) - g(u+v)$. Differentiating three times the relation $g(f(x)) = x$, we obtain $g''(f(x)) = -\frac{f''(x)}{f'(x)^3}$ and $g'''(f(x)) = -\frac{f'''(x)f'(x) - 3f''(x)^2}{f'(x)^5}$. As a result,

$$\frac{g'''g' - (g'')^2}{(g')^2}(f(x)) = \frac{2f''(x)^2 - f'''(x)f'(x)}{f'(x)^4}. \tag{3.5}$$

Therefore, $\frac{\partial^2 H}{\partial u\, \partial v} = 0$ leads to (3.1). □

We restrict ourselves to *smooth* LWMY functions f, that is, those satisfying

$$f:(0,\infty)\to(0,\infty)\text{ is bijective and decreasing,}\tag{3.6}$$

$$f\text{ is three times differentiable,}\tag{3.7}$$

$$F(x)=\sum_{n\geq 1}a_n x^n \qquad \forall x>0,\tag{3.8}$$

where $F := 1/f'$.

According to (3.6), $f'(0_+)=-\infty$. This implies that $F(0_+)=0$ and explains why the series in (3.8) starts with $n=1$.

The goal of this section is to prove half of Theorem 2.2: if f is a smooth LWMY function, then f belongs to one of the four classes $\mathcal{F}_1,\dots,\mathcal{F}_4$ introduced in Remark 2.3. First, in Theorem 3.2, we characterize all possible functions F. Second, we determine the associated functions f (see Theorem 3.9).

Theorem 3.2. *Suppose that f is a smooth LWMY function and the assumptions of Theorem 3.1 are satisfied.*

1. *If $F'(0_+)=0$, then $a_2<0$ and*

$$F(x)=\begin{cases}\dfrac{a_2^2}{6a_4}\left(\cosh\left(x\sqrt{\dfrac{12a_4}{a_2}}\right)-1\right), & \text{if } a_4<0,\\ a_2x^2, & \text{otherwise.}\end{cases}\tag{3.9}$$

2. *If $F'(0_+)\neq 0$, then*

$$F(x)=\begin{cases}\dfrac{a_1a_2}{3a_3}\left[\cosh\left(x\sqrt{\dfrac{6a_3}{a_1}}\right)-1\right] & \\ \qquad +a_1\sqrt{\dfrac{a_1}{6a_3}}\sinh\left(x\sqrt{\dfrac{6a_3}{a_1}}\right), & \text{if } a_1a_3>0,\\ a_1x+a_2x^2, & \text{otherwise.}\end{cases}\tag{3.10}$$

Remark 3.3. Unsurprisingly, the case $F(x)=a_2x^2$ corresponds to $f(x)=-\frac{1}{a_2}\frac{1}{x}$, that is, the case considered by Matsumoto and Yor, and Letac and Wesolowski. Thus, under stronger assumptions, we retrieve the result of Letac and Wesolowski. Indeed, writing the functional equation of Theorem 3.1 with $f:x\mapsto 1/x$ gives

$$\phi_X''(x)+\frac{2}{x}\phi_X'(x)+\phi_Y''(y)\frac{1}{x^2}\left(x^2-(x+y)^2\right)-\frac{2}{x}\phi_Y'(y)+\frac{2}{x^2}=0.$$

We then solve this differential equation and find that the laws of X and Y are necessarily GIG and gamma, respectively. We omit the details.

Throughout this subsection, we suppose that f satisfies (3.6)–(3.8) and that the assumptions of Theorem 3.1 are fulfilled. To simplify the statement of results below, we do not repeat these conditions.

Recall that ϕ_Y is the logarithm of the density of Y. Let us introduce

$$h := \phi_Y'. \tag{3.11}$$

Lemma 3.4.

1. *There exists a function* $\lambda : (0, \infty) \to \mathbb{R}$ *such that*

$$F(x+y) = \frac{\lambda(x) - h(y)F'(x)}{h'(y)} + F(x). \tag{3.12}$$

2. F *satisfies*

$$F(y) = \frac{\lambda(0_+) - h(y)F'(0_+)}{h'(y)}. \tag{3.13}$$

Remark 3.5. Suppose that we have been able to determine F. Then, $h = \phi_Y'$ solves the linear ordinary differential equation (3.13) and can therefore be determined. The remaining function ϕ_X is obtained by solving equation (3.1).

Proof of Lemma 3.4. Using (3.11) and $F = 1/f'$ in equation (3.1), we obtain

$$c(x) = h(y)\frac{F'(x)}{F(x)} + h'(y)\frac{1}{F(x)}\big(F(x+y) - F(x)\big),$$

where $c(x)$ depends only on x. Multiplying both sides by $F(x)$ and taking the y-derivative leads to

$$0 = F'(x)h'(y) + [F(x+y) - F(x)]h''(y) + h'(y)F'(x+y).$$

Fix $x > 0$. Then, $\theta(y) := F(x+y)$ is a solution of the differential equation in y

$$0 = F'(x)h'(y) + \big(\theta(y) - F(x)\big)h''(y) + h'(y)\theta'(y). \tag{3.14}$$

A solution of the related homogeneous equation in y is $\frac{\rho}{h'(y)}$, where ρ is a constant. It is easy to prove that $y \mapsto -F'(x)h(y) + F(x)h'(y)$ solves (3.14). Thus, the general solution of (3.14) is

$$\theta(y) = \frac{1}{h'(y)}[\lambda(x) - F'(x)h(y) + F(x)h'(y)].$$

Since $\theta(y) = F(x+y)$, (3.12) follows.

According to (3.8), $F(0_+)$ and $F'(0_+)$ exist. Therefore, taking the limit $x \to 0_+$ in (3.12) implies both the existence of $\lambda(0_+)$ and relation (3.13). $\square$

The following lemma shows that the function F (and thus f) solves a self-contained equation in which h, and thereby the densities of X and Y, are not involved.

Lemma 3.6. *F solves the delay equations*

$$F(x+y) = \frac{F(y)[\lambda(x) - h(y)F'(x)]}{\lambda(0_+) - h(y)F'(0_+)} + F(x) \qquad (x, y > 0), \tag{3.15}$$

$$F'(x+y) = \frac{F'(y) + F'(0_+)}{F(y)}[F(x+y) - F(x)] - F'(x) \qquad (x, y > 0). \tag{3.16}$$

Proof. By (3.13), we have

$$h'(y) = \frac{\lambda(0_+) - h(y)F'(0_+)}{F(y)}.$$

Equation (3.15) then follows by rewriting equation (3.12) and replacing $h'(y)$ with the expression above.

We differentiate (3.15) in y and use the fact that $\lambda(0_+) - h(y)F'(0_+) = h'(y)F(y)$ to obtain

$$F'(x+y) = [F'(y) + F'(0_+)]\frac{\lambda(x) - h(y)F'(x)}{F(y)h'(y)} - F'(x).$$

By (3.12), we have $\frac{\lambda(x)-h(y)F'(x)}{F(y)h'(y)} = \frac{F(x+y)-F(x)}{F(y)}$ and this gives (3.16). □

Remark 3.7. We can see (3.16) as a scalar neutral delay differential equation. Indeed, set $t = x + y$ and consider $y > 0$ as a fixed parameter. Then, (3.16) becomes

$$F'(t) = a\big(F(t) - F(t-y)\big) - F'(t-y), \qquad t \geq y, \tag{3.17}$$

where $a := \frac{F'(y)+F'(0_+)}{F(y)}$. Replacing $F(t)$ in (3.17) with $\mathrm{e}^{at}G(t)$ leads to

$$G'(t) + \mathrm{e}^{-ay}G'(t-y) + 2a\mathrm{e}^{-ay}G(t-y) = 0, \qquad t \geq y. \tag{3.18}$$

Equation (3.18) is called a *neutral delay differential equation* (see, e.g., Section 6.1, in [6]). These equations have been intensively studied, but the authors have only focused on the asymptotic behavior of the solution as $t \to \infty$. Unfortunately, these results do not help to solve explicitly either (3.16) or (3.18).

Lemma 3.8. *For all integers $k \geq 0$ and $l \geq 1$, we have*

$$\sum_{m=0}^{l-1}(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m = (l-2)(k+1)a_{k+1}a_l + a_1a_{l+k}C^k_{l+k}, \tag{3.19}$$

$$C^k_{k+3}a_{k+3}a_1 = (k+1)a_{k+1}a_3, \tag{3.20}$$

$$2C^k_{k+4}a_{k+4}a_1 + C^k_{k+3}a_{k+3}a_2 - C^k_{k+2}a_{k+2}a_3 - 2(k+1)a_{k+1}a_4 = 0, \tag{3.21}$$

where $C^p_n = \frac{n!}{(n-p)!p!}$.

Proof. Obviously, the equation (3.16) is equivalent to

$$\begin{aligned} F'(x+y)F(y) = F'(y)F(x+y) - F'(y)F(x) \\ - F(y)F'(x) + F'(0_+)F(x+y) - F'(0_+)F(x). \end{aligned} \tag{3.22}$$

Using the asymptotic expansion (3.8) of F, we can develop each term in (3.22) as a series with respect to x and y. Then, identifying the series on the right-hand side and the left-hand side, we get (3.19)–(3.21). The details are provided in the Appendix. □

Proof of Theorem 3.2. We will only prove item 1; the proof of item 2 is similar.

Since $a_1 = F'(0_+) = 0$, we necessarily have $a_2 \neq 0$. Indeed, if $a_2 = 0$, then, by (3.21) with $k = 1$, we would have $-3a_3^2 - 4a_2a_4 = 0$, that is, $a_3 = 0$. Again using (3.21) with $k = 3$ would imply that $a_4 = 0$ and finally that $a_k = 0$ for every $k \geq 0$, which is a contradiction because, by definition, $F = 1/f'$ does not vanish.

So, we have $a_1 = 0$ and $a_2 \neq 0$. Equation (3.20) with $k = 1$ reads $4a_4a_1 = 2a_2a_3$, which implies that $a_3 = 0$. Applying (3.20) to $k = 2n$ provides, by induction on n, $a_{2n+1} = 0$ for every $n \geq 0$.

Therefore, equation (3.21) reduces to $(k+3)(k+2)(k+1)a_{k+3}a_2 = 12(k+1)a_{k+1}a_4, k \geq 0$, that is, $a_{k+3} = \frac{12a_4}{a_2}\frac{1}{(k+3)(k+2)}a_{k+1}$. This leads to

$$a_{2k} = \left(\frac{12a_4}{a_2}\right)^{k-1} \frac{2}{(2k)!} a_2, \qquad k \geq 1. \tag{3.23}$$

Then, $F(x) = a_2x^2$ if $a_4 = 0$, and if $a_4 \neq 0$, we have

$$F(x) = \sum_{k \geq 1} \left(\frac{12a_4}{a_2}\right)^{k-1} \frac{2}{(2k)!} a_2 x^{2k}.$$

If $a_4a_2 < 0$, then $F(x) = \frac{a_2^2}{6a_4}[\cos(x\sqrt{\frac{-12a_4}{a_2}}) - 1]$. This implies $F(2\pi\sqrt{\frac{-12a_4}{a_2}}) = 0$, which is impossible since $F(x) = 1/f'(x) < 0$. Consequently,

$$F(x) = \frac{a_2^2}{6a_4}\left[\cosh\left(x\sqrt{\frac{12a_4}{a_2}}\right) - 1\right].$$

□

Now, in each case of Theorem 3.2, we compute the function f associated with F via the relation $F = 1/f'$. We do not detail the calculations since they reduce to getting a good primitive of $1/F$. Recall that we restrict ourselves to functions f satisfying (3.6)–(3.8) and work under the assumptions of Theorem 3.1.

Theorem 3.9.

1. *If* $F(x) = a_2 x^2$, *then* $f(x) = \frac{1}{a_2 x}$.
2. *If* $F(x) = \alpha(\cosh \beta x - 1), \alpha, \beta > 0$, *then* $f(x) = \frac{2}{\alpha\beta} f_1(\beta x)$.
3. *If* $F(x) = a_1 x + a_2 x^2$, *then* $f(x) = -\frac{1}{a_1} g_1(\frac{a_2}{a_1} x)$.
4. *If*

$$F(x) = \frac{a_1 a_2}{3a_3}\left[\cosh\left(x\sqrt{\frac{6a_3}{a_1}}\right) - 1\right] + a_1\sqrt{\frac{a_1}{6a_3}}\sinh\left(x\sqrt{\frac{6a_3}{a_1}}\right),$$

then

$$f(x) = -\frac{1}{\beta\gamma}\log\left(\frac{e^{\beta x} + \delta - 1}{e^{\beta x} - 1}\right),$$

where $\alpha = \frac{a_1 a_2}{3a_3}$, $\beta = \sqrt{\frac{6a_3}{a_1}}$ *and* $\gamma = a_1\sqrt{\frac{a_1}{6a_3}}$.

4. Proof of Theorem 2.4

Recall that $\phi_Y = \log p_Y$, $h = \phi_Y'$ and $F'(0_+) = 0$. It is easy to deduce from (3.13) that there exist constants λ and c_1 such that $h(y) = \lambda f(y) + c_1$, that is, $h(y) = \frac{\lambda \mathrm{e}^y}{\mathrm{e}^y - 1} + c_1 - \lambda$. This implies the existence of a constant d such that $\phi_Y(y) = \lambda\log(\mathrm{e}^y - 1) + (c_1 - \lambda)y + d$. Setting $M = \mathrm{e}^d$, we have, by integration, for all $y > 0$,

$$p_Y(y) = M(1 - \mathrm{e}^{-y})^\lambda \mathrm{e}^{c_1 y}. \tag{4.1}$$

To give more information on the normalizing constant M, we observe, for $a = -c_1$ and $b = \lambda + 1$, that

$$\int_0^\infty M(1 - \mathrm{e}^{-y})^{b-1}\mathrm{e}^{-ay}\,\mathrm{d}y = M\int_0^1 (1-u)^{b-1}u^{a-1}\,\mathrm{d}u,$$

which implies that $a > 0$, $b > 0$ and $M = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$. This proves (2.16).

To find the density of X, we return to equation (3.1) and compute each of its terms.

We have $f'(x) = \frac{-\mathrm{e}^x}{(\mathrm{e}^x-1)^2}$, $f''(x) = \frac{\mathrm{e}^{2x}+\mathrm{e}^x}{(\mathrm{e}^x-1)^3}$ and $f'''(x) = -\frac{\mathrm{e}^{3x}+4\mathrm{e}^{2x}+\mathrm{e}^x}{(\mathrm{e}^x-1)^4}$ so that $\frac{f'(x)}{f'(x+y)} = \frac{\mathrm{e}^{-y}(\mathrm{e}^{x+y}-1)^2}{(\mathrm{e}^x-1)^2}$ and $\frac{f''(x)}{f'(x)} = -\frac{\mathrm{e}^x+1}{\mathrm{e}^x-1}$. Calculations yield

$$\frac{2(f''(x))^2 - f'''(x)f'(x)}{f'(x)^2} = \frac{\mathrm{e}^{2x}+1}{(\mathrm{e}^x-1)^2}. \tag{4.2}$$

Moreover,

$$-\phi_Y'(y)\frac{f''(x)}{f'(x)} + \phi_Y''(y)\left(\frac{f'(x)}{f'(x+y)} - 1\right) = \frac{(c_1-\lambda)\mathrm{e}^{2x} - c_1}{(\mathrm{e}^x-1)^2}. \tag{4.3}$$

Equation (3.1) can then be written, using (4.2) and (4.3),

$$\phi_X''(x) + \frac{e^x+1}{e^x-1}\phi_X'(x) = \frac{(c_1-\lambda-1)e^{2x} - c_1 - 1}{(e^x-1)^2}.$$

Then, $h_0 := \phi_X'$ solves

$$h_0'(x) + \frac{e^x+1}{e^x-1}h_0(x) = \frac{(c_1-\lambda-1)e^{2x} - c_1 - 1}{(e^x-1)^2}. \tag{4.4}$$

Note that $x \mapsto \frac{K}{4\sinh^2(x/2)}$ solves (4.4) with the right-hand side equal to 0, and $x \mapsto \frac{(c_1-\lambda-1)e^x+(c_1+1)e^{-x}}{4\sinh^2(x/2)}$ is a particular solution of (4.4). Therefore, the solution of (4.4) is

$$h(x) = \frac{(c_1-\lambda-1)e^x + (c_1+1)e^{-x} + K}{4\sinh^2(x/2)}$$

for some constant K. This implies that

$$\phi_X'(x) = c_1 + 1 + \frac{(2c_1-\lambda+K)e^x}{(e^x-1)^2} - \frac{(\lambda+2)e^x}{e^x-1}.$$

As a consequence, there exists a constant δ such that

$$\phi_X(x) = (c_1+1)x - \frac{(2c_1-\lambda+K)e^x}{e^x-1} - (\lambda+2)\log(e^x-1) + \delta.$$

Thus, $p_X(x) = Ne^{(c_1+1)x}(e^x-1)^{-\lambda-2}\exp(-\frac{2c_1-\lambda+K}{e^x-1})\mathbf{1}_{\{x>0\}}$. Recall that $a = -c_1$ and $b = \lambda + 1$. With $c = 2c_1 - \lambda + K$, we get (2.17). More information on the constant N is obtained by observing that if we set $V' = f_1(X) = \frac{1}{e^X-1}$, then the density of V' is

$$f_{V'}(w) = N(w+1)^{-a}w^{a+b-1}\exp\{-cw\}\mathbf{1}_{\{w>0\}},$$

that is, the law of V' is $K^{(2)}(a+b, -b, c)$ (see equation (2.14)).

We have $g_1'(u) = -\frac{1}{u(u+1)}$. A computation of a Jacobian, together with (2.16) and (2.17), implies, for $u, v > 0$, that

$$p_{(U,V)}(u,v) = p_X\left(\log\left[\frac{u+v+1}{u+v}\right]\right)p_Y\left(\log\left[\frac{(u+1)(u+v)}{u(u+v+1)}\right]\right) \times \frac{1}{u(u+1)(u+v)(u+v+1)}.$$

We then get that $p_{(U,V)}(u,v)$ is the product of a function of u and a function of v, and this gives item 2 of Theorem 2.4.

Appendix

Proof of Lemma 3.8. We have

$$F'(x+y)F(y) = \sum_{k\geq 0} x^k \sum_{m\geq 0, n\geq 1+k} n a_n a_m C_{n-1}^k y^{n+m-1-k}.$$

Setting $l = m+n-1-k$ for fixed m gives

$$F'(x+y)F(y) = \sum_{k\geq 0, l\geq 0} x^k y^l \sum_{m=0}^{l} (l-m+1+k) C_{l-m+k}^k a_{l-m+1+k} a_m. \tag{5.5}$$

By the same method, we have

$$F'(y)F(x+y) = \sum_{k\geq 0, l\geq 0} x^k y^l \left(\sum_{m=0}^{l+1} m C_{l-m+k+1}^k a_{l-m+1+k} a_m \right). \tag{5.6}$$

As for the two other terms of (3.22), we get

$$F'(y)F(x) = \sum_{k\geq 0, l\geq 0} a_k a_{l+1} (l+1) x^k y^l, \tag{5.7}$$

$$F'(x)F(y) = \sum_{k\geq 0, l\geq 0} a_{k+1} a_l (k+1) x^k y^l. \tag{5.8}$$

Consequently,

$$F'(0_+)F(x+y) = a_1 \sum_{n\geq 0} a_n (x+y)^n = a_1 \sum_{k,l\geq 0} a_{l+k} C_{l+k}^k x^k y^l, \tag{5.9}$$

$$F'(0_+)F(x) = a_1 \sum_{k\geq 0} a_k x^k. \tag{5.10}$$

Identifying the coefficient of $x^k y^l$ in (3.22) and using (5.5)–(5.10), we have, for $k \geq 0$ and $l \geq 0$,

$$\begin{aligned} \sum_{m=0}^{l} (l-m+1+k) C_{l-m+k}^k a_{l-m+1+k} a_m &= -(l+1) a_k a_{l+1} - (k+1) a_{k+1} a_l \\ &\quad + \sum_{m=0}^{l+1} m C_{l-m+k+1}^k a_{l-m+1+k} a_m \\ &\quad + a_1 a_{l+k} C_{l+k}^k - a_1 a_k 1_{l=0}. \end{aligned} \tag{5.11}$$

Note that if $l = 0$, then both sides of (5.11) vanish. Therefore, we may suppose in the sequel that $l \geq 1$.

For $m = l + 1$, we have $mC^k_{l-m+k+1}a_{l-m+1+k}a_m = (l+1)a_k a_{l+1}$. Thus, equation (5.11) reads

$$\sum_{m=0}^{l}(l-m+1+k)C^k_{l-m+k}a_{l-m+1+k}a_m = -(k+1)a_{k+1}a_l + \sum_{m=0}^{l} mC^k_{l-m+k+1}a_{l-m+1+k}a_m + a_1 a_{l+k}C^k_{l+k}. \tag{5.12}$$

However, via a calculation involving the definition, we find that

$$(l-m+1+k)C^k_{l-m+k} - mC^k_{l-m+1+k} = (l-2m+1)C^k_{l-m+1+k},$$

so equation (5.12) is equivalent to

$$\sum_{m=0}^{l}(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m = -(k+1)a_{k+1}a_l + a_1 a_{l+k}C^k_{l+k}. \tag{5.13}$$

For $m = l$, we have $(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m = (1-l)(k+1)a_{k+1}a_l$. Consequently, equation (5.13) may be written as

$$\sum_{m=0}^{l-1}(l-2m+1)C^k_{l-m+1+k}a_{l-m+1+k}a_m - (l-1)(k+1)a_{k+1}a_l = -(k+1)a_{k+1}a_l + a_1 a_{l+k}C^l_{l+k},$$

which implies (3.19).

(3.20) and (3.21) follow by applying (3.19) to $l = 3$ and $l = 4$, respectively. □

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