

STATISTICAL INFERENCE FOR AUTOREGRESSIVE MODELS UNDER HETEROSCEDASTICITY OF UNKNOWN FORM

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This paper provides an entire inference procedure for the autoregressive model under (conditional) heteroscedasticity of unknown form with a finite variance. We first establish the asymptotic normality of the weighted least absolute deviations estimator (LADE) for the model. Second, we develop the random weighting (RW) method to estimate its asymptotic covariance matrix, leading to the implementation of the Wald test. Third, we construct a portmanteau test for model checking, and use the RW method to obtain its critical values. As a special weighted LADE, the feasible adaptive LADE (ALADE) is proposed and proved to have the same efficiency as its infeasible counterpart. The importance of our entire methodology based on the feasible ALADE is illustrated by simulation results and the real data analysis on three U.S. economic data sets.

1. Introduction. Consider a p th order heteroscedastic autoregressive (AR) model:

$$(1.1) \quad y_t = \mu_0 + \phi_{10}y_{t-1} + \cdots + \phi_{p0}y_{t-p} + \varepsilon_t,$$

where the error ε_t satisfies

$$(1.2) \quad \varepsilon_t = g_t u_t := g\left(\frac{t}{n}\right)u_t$$

for $t = 1, \dots, n$. Here, $g(\cdot)$ is a positive bounded scalar function with unknown form, u_t is a rescaled error with unknown form and a finite variance for all t , and n is the sample size. In model (1.1), y_t is allowed to be nonstationary with a finite variance due to the heteroscedasticity of ε_t , but it cannot be a unit root process, since the coefficients $\{\phi_{i0}\}_{i=1}^p$ are required to satisfy the stationarity condition of AR(p) model. In model (1.2), $\{u_t\}$ need not be independent and identically distributed (i.i.d.), nor even a martingale difference sequence (m.d.s.), allowing for model misspecification. This setting is important, in view of that the literature

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generally needs a non-i.i.d. sequence $\{u_t\}$ (see, e.g., Drost and Nijman (1993)). Under certain identification condition on u_t , g_t is identifiable. Particularly, when u_t is stationary, the (conditionally) heteroscedastic structure of ε_t is allowed to change abruptly, gradually or periodically according to the unspecified form of $g(\cdot)$. Similar specifications as for g_t have been widely adopted in the literature; see, for example, Robinson (1989, 2012), Fryzlewicz, Sapatinas and Subba Rao (2006), Zhou and Wu (2009) and Chen and Hong (2016) to name a few.

Model (1.1) is one of the often used models of empirical macroeconomics and statistics. Conventional statistical inference methods for this model are designed for homogenous error $\{\varepsilon_t\}$ (i.e., $E\varepsilon_t^2 = \text{a constant}$), which is either a sequence of i.i.d. random variables or an m.d.s.; see, for example, Gonçalves and Kilian (2004), Yao and Brockwell (2006), Zhu and Ling (2015) and references therein. However, homogenous error can be a crucial weakness in applications, since heteroscedasticity has been widely demonstrated by Tsay (1988) for social science data, Watson (1999) for interest rates data, Buseti and Taylor (2003) and Sensier and van Dijk (2004) for macroeconomic data, Amado and Teräsvirta (2013, 2014) for stock index data and many others. As shown in Diebold (1986), Mikosch and Stáricá (2004) and Stáricá and Granger (2005), the presence of heteroscedasticity could mislead the conventional time series analysis procedure resulting in erroneous conclusions. Hence, it is necessary to develop a valid statistical inference procedure for model (1.1) under heteroscedasticity.

So far, few works have centered around this topic, and most of them are based on the least squares estimator (LSE); see Nicholls and Pagan (1983) for earlier works and Phillips and Xu (2006) for more recent ones. The seminal work in Carroll (1982) and Robinson (1987) demonstrated that the LSE in regression models is less efficient than the adaptive LSE (ALSE), which takes the unknown heteroscedastic form of the error term into account; see also Cragg (1983) for the study of the instrumental variable LSE. Motivated by this, Xu and Phillips (2008) constructed a feasible ALSE for model (1.1), and showed that this feasible ALSE is more efficient than the LSE. However, their theory does not allow u_t to be conditionally heteroscedastic, and their methodology may be limited in practice since no valid statistical inference tools (e.g., t -test, Wald test and model diagnostic checking) is provided. Moreover, a heavy-tailed error is an often observed feature in fitted model (1.1), and the LSE-based inference methods in this case are undesirable from the viewpoint of robustness.

This paper provides an entire statistical inference procedure for model (1.1) based on the weighted least absolute deviations estimator (LADE). We first establish the asymptotic normality of the weighted LADE. Second, since the asymptotic covariance matrix of the weighted LADE cannot be estimated directly from the sample, we develop the random weighting (RW) method to estimate this covariance matrix, leading to the implementation of the Wald test. The RW method initiated by Jin, Ying and Wei (2001) has been widely applied to provide statistical

inference; see [Zhu \(2016\)](#). Third, we construct a portmanteau test for model diagnostic checking, and use the RW method to obtain its critical values. As a special weighted LADE, the infeasible adaptive LADE (ALADE) with weight equal to g_t is desired in terms of efficiency, and its advantage in efficiency is demonstrated by a formal comparison. To circumvent the unobservable g_t , we further propose a feasible ALADE, prove that it has the same efficiency as its infeasible counterpart, and show that our entire inference procedure is still valid based on this feasible estimator. Under the identification condition $E|u_t| = 1$ for g_t , this feasible ALADE is constructed with weight equal to a kernel estimator of g_t . Unlike [Xu and Phillips \(2008\)](#), our entire inference procedure based on the weighted LADE does not rule out the conditional heteroscedasticity of u_t . The importance of our methodology is illustrated by simulation results and the real data analysis on three U.S. economic data sets.

We emphasize that the idea of the weighted LADE has been adopted for the linear regression model with heteroscedasticity of either known form in [Gutenbrunner and Jurečková \(1992\)](#) or unknown form in [Zhao \(2001\)](#). But our technique is different from theirs, since they neither considered a general heteroscedastic time series model as models (1.1)–(1.2), nor provided an entire valid inference procedure. When ε_t is stationary and conditionally heteroscedastic with unknown form, [Zhu and Ling \(2015\)](#) provided the statistical inference method for model (1.1) based on either the self-weighted LADE or the usual LADE. Our weighted LADE and their self-weighted LADE are not compatible due to different weighting mechanisms. The weight $w_{sw,t}$ for the self-weighted LADE is a random variable in the filtration generated by $\{y_k\}_{k \leq t-1}$, and it is useful to deal with the infinite variance AR model. For the finite variance AR model (as in our setting), $w_{sw,t}$ is not needed any more, since the self-weighted LADE is always less efficient than the usual LADE in this case. On the contrary, the weight w_t for our weighted LADE is a deterministic function (or its kernel estimate based on the whole sample $\{y_t\}_{t=1}^n$), and it is used to take the unknown heteroscedastic form of ε_t into account. As a result, the weighted LADE can be more efficient than the usual LADE by choosing an appropriate weight (e.g., the weight for the feasible ALADE) in many circumstances. Compared to the usual LADE in [Zhu and Ling \(2015\)](#), our weighted LADE has two more incremental contributions besides the advantage in efficiency. First, the results based on the weighted LADE allow ε_t to be heteroscedastic with unknown form, and hence the scope of applications is much wider in dealing with the finite variance AR model. Second, although the RW method is motivated by [Zhu and Ling \(2015\)](#), new proof techniques are used to handle the heteroscedasticity of ε_t , especially for the feasible ALADE.

The remainder of the paper is organized as follows. Section 2 obtains the asymptotic normality of the weighted LADE. Section 3 presents the RW method to estimate the asymptotic covariance matrix. Section 4 constructs a portmanteau test for model diagnostic checking. Section 5 considers the choice of the weight function, and discusses the asymptotic efficiency. Section 6 proposes the feasible ALADE.

Simulation results are reported in Section 7. Concluding remarks are offered in Section 8. Proofs of all theorems are relegated to Appendices. Additional simulation results, applications, some technical lemmas and the remaining proofs are provided in the Supplementary Material (Zhu (2019)).

Throughout the paper, $|\cdot|$ and $\|\cdot\|$ denote the absolute value and the vector l_2 -norm, respectively, A' is the transpose of matrix A , $\|\xi\|_\kappa = (E|\xi|^\kappa)^{1/\kappa}$ is the L^κ -norm of random variable ξ , plim denotes the convergence in probability, \rightarrow_d denotes the convergence in distribution, $I(\cdot)$ is the indicator function, and $\text{sgn}(a) = I(a > 0) - I(a < 0)$ is the sign of any $a \in \mathcal{R}$.

2. The weighted LADE. Let $\theta = (\mu, \phi_1, \dots, \phi_p)' \in \Theta$ be the unknown parameter of model (1.1), and $\theta_0 = (\mu_0, \phi_{10}, \dots, \phi_{p0})' \in \Theta$ be its true value, where Θ is the parameter space, and it is a compact subset of \mathcal{R}^{p+1} with $\mathcal{R} = (-\infty, \infty)$. Given the observations $\{y_t\}_{t=1}^n$ and the initial values $\{y_t\}_{t=-p}^0 \equiv 0$, we consider the weighted least absolute deviations estimator (LADE) for θ_0 :

$$\hat{\theta}_{wn} = \arg \min_{\theta \in \Theta} \sum_{t=1}^n \frac{|y_t - Y'_{t-1}\theta|}{w_t},$$

where $Y_{t-1} = (1, y_{t-1}, \dots, y_{t-p})'$ and $w_t := w(t/n)$ are weights with $w(\cdot)$ being a positive scalar function. Particularly, when $w(\cdot) \equiv 1$, $\hat{\theta}_{wn}$ becomes the usual LADE, denoted by $\hat{\theta}_n$. To study the asymptotic theory of $\hat{\theta}_{wn}$, we need the notation of near-epoch dependent (NED) random variables.

DEFINITION 1. $\{Z_{nt}\}$ is near-epoch dependent in L^κ -norm (L^κ -NED) on $\{V_{nt}\}$ if $\|Z_{nt} - E(Z_{nt}|\mathfrak{F}_{t-m}^{t+m})\|_\kappa \leq d_{nt}\psi_m$, where $\mathfrak{F}_{t-m}^{t+m} := \sigma(V_{ns}; t-m \leq s \leq t+m)$ is a filtration with $\sigma(\cdot)$ being the σ -field operator, $\{d_{nt}\}$ are positive constants, and $\psi_m \rightarrow 0$ as $m \rightarrow \infty$.

The definition of near-epoch dependence was first introduced in Billingsley (1968) and has been widely used in the literature. NED processes allow for considerable heterogeneity and also for dependence and include the mixing processes as a special case. Many nonlinear models are shown to be NED (see, e.g., Davidson (2002, 2004) for overviews), and hence the NED concept makes possible a convenient theory of inference for these models. Besides near-epoch dependence, the physical dependence in Wu (2005) is a parallel tool to depict the dependence of a dynamic process. It might be interesting to apply physical dependence to our setting, and we leave this for future study.

Let $f_t(x)$ be the conditional density of u_t given \mathcal{F}_{t-1} , where $\mathcal{F}_t := \sigma(u_s; s \leq t)$ is a filtration. We make the following five assumptions throughout the paper, where the definition of $f_t(x)$ is essential only for the last one.

ASSUMPTION 2.1. θ_0 is an interior point in Θ and for each $\theta \in \Theta$, $\phi(z) \neq 0$ when $|z| \leq 1$, where $\phi(z) := 1 - \sum_{i=1}^p \phi_i z^i$.

ASSUMPTION 2.2. $g(\cdot)$ and $w(\cdot)$ are continuous on the interval $[0, 1]$ except at a finite number of points, and

$$0 < \underline{C} \leq \inf_{x \in [0,1]} g(x) \leq \sup_{x \in [0,1]} g(x) \leq \overline{C} < \infty,$$

$$0 < \underline{C} \leq \inf_{x \in [0,1]} w(x) \leq \sup_{x \in [0,1]} w(x) \leq \overline{C} < \infty,$$

for some positive numbers \underline{C} and \overline{C} .

ASSUMPTION 2.3. $E[\text{sgn}(\varepsilon_t) | \mathcal{F}_{t-1}] = 0$.

ASSUMPTION 2.4. u_t is an α -mixing but not necessary a stationary process, and:

- (i) $\sup_t E|u_t|^{4+\delta_0} < \infty$ for some $\delta_0 > 0$;
- (ii) $m_r^{(1)} := E(u_{t-r})$ is uniformly bounded for all $t, r \geq 1$;
- (iii) $m_{r,s}^{(2)} := E(u_{t-r}u_{t-s})$ is uniformly bounded for all $t, r, s \geq 1$.

ASSUMPTION 2.5. $f_t(x)$ satisfies that:

- (i) $f_t(x)$ is continuous with $f_t(0) > 0$ and $\sup_{x \in \mathcal{R}} f_t(x) < \infty$ (a.s.) for all $t \geq 1$;
- (ii) $\{f_t(0)\}$ is $L^{2+\delta_1}$ -NED on $\{u_t\}$ for some $\delta_1 > 0$;
- (iii) $\tau_0 := E[f_t(0)]$ is uniformly bounded for all $t \geq 1$;
- (iv) $\tau_r^{(1)} := E[f_t(0)u_{t-r}]$ is uniformly bounded for all $t, r \geq 1$;
- (v) $\tau_{r,s}^{(2)} := E[f_t(0)u_{t-r}u_{t-s}]$ is uniformly bounded for all $t, r, s \geq 1$.

We offer some remarks on the aforementioned assumptions. Assumption 2.1 is the usual stationarity condition for $AR(p)$ model, and it implies that

$$(2.1) \quad y_t = \rho + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i},$$

where $\rho = \mu_0 / (1 - \phi_{10} - \dots - \phi_{p0})$ and $\sum_{i=0}^{\infty} |\alpha_i| < \infty$. Assumption 2.2 is sufficient to guarantee that both $g(\cdot)$ and $w(\cdot)$ are integrable on the interval $[0, 1]$ up to any finite order, and similar conditions have been used in [Cavaliere \(2004\)](#), [Cavaliere and Taylor \(2007\)](#), [Phillips and Xu \(2006\)](#), [Xu and Phillips \(2008\)](#) and many others. Assumption 2.3 is the identification condition for θ_0 based on the weighted LADE. Assumption 2.4 is adopted from [Kuersteiner \(2002\)](#) and [Gonçalves and Kilian \(2004\)](#) in a similar way. Assumption 2.4(i) is stronger than the condition that $E u_t^2 < \infty$, which is sufficient for the asymptotic normality of the LADE if ε_t is stationary; see, for example, [Zhu and Ling \(2015\)](#) and references therein. We resort to a stronger moment condition of u_t in Assumption 2.4(i)

to handle the heteroscedasticity of ε_t . Assumption 2.4(ii)–(iii) are less restrictive than the condition that u_t is covariance stationary with mean zero. Assumption 2.5 is required only for the LADE but not for the LSE, and it is stronger than the one in Bassett and Koenker (1978), Davis and Dunsmuir (1997), Pan, Wang and Yao (2007) and Zhu and Ling (2012, 2015) due to the heteroscedasticity of ε_t . When $f_t(0) \equiv f(0)$ is independent of t , Assumption 2.5(ii)–(iii) hold automatically, and Assumption 2.5(iv)–(v) become Assumption 2.4(ii)–(iii), respectively. When u_t follows a stationary autoregressive conditional heteroskedasticity (ARCH) type model, the NED condition in Assumption 2.5(ii) can be checked or even be not needed and the moment conditions in Assumption 2.5(iii)–(v) can be satisfied (see the discussions for Corollaries 2.1–2.3 below).

We shall point out that Assumption 2.4(ii)–(iii) and Assumption 2.5(iii)–(v) need $m_r^{(1)}$, $m_{r,s}^{(2)}$, τ_0 , $\tau_r^{(1)}$ and $\tau_{r,s}^{(2)}$ to be independent of t . In the general framework, these terms may depend on t . A detailed study for this general framework might be done in a separate paper.

Let $d_1 = \int_0^1 \frac{1}{w^2(x)} dx$, $d_2 = \tau_0 \int_0^1 \frac{1}{g(x)w(x)} dx$, and

$$(2.2) \quad \zeta_r^{(1)} = \rho d_1 + \left(\sum_{i=0}^{\infty} \alpha_i m_{i+r}^{(1)} \right) \int_0^1 \frac{g(x)}{w^2(x)} dx,$$

$$(2.3) \quad \zeta_r^{(2)} = \rho d_2 + \left(\sum_{i=0}^{\infty} \alpha_i \tau_{i+r}^{(1)} \right) \int_0^1 \frac{1}{w(x)} dx,$$

$$(2.4) \quad \xi_{r,s}^{(1)} = \rho[\zeta_r^{(1)} + \zeta_s^{(1)}] - \rho^2 d_1 + \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_i \alpha_j m_{i+r,j+s}^{(2)} \right) \int_0^1 \frac{g^2(x)}{w^2(x)} dx,$$

$$(2.5) \quad \xi_{r,s}^{(2)} = \rho[\zeta_r^{(2)} + \zeta_s^{(2)}] - \rho^2 d_2 + \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_i \alpha_j \tau_{i+r,j+s}^{(2)} \right) \int_0^1 \frac{g(x)}{w(x)} dx.$$

With this notation, define

$$(2.6) \quad \Sigma_l = \begin{pmatrix} d_l & K_l' \\ K_l & \Omega_l \end{pmatrix} \in \mathcal{R}^{(p+1) \times (p+1)} \quad \text{for } l = 1, 2,$$

where K_l is the $p \times 1$ column vector with r th element $\zeta_r^{(l)}$, and Ω_l is the $p \times p$ matrix with (r, s) th element $\xi_{r,s}^{(l)}$. Our first main result is given as follows.

THEOREM 2.1. *Suppose Assumptions 2.1–2.5 hold. Then*

$$\sqrt{n}(\hat{\theta}_{wn} - \theta_0) \rightarrow_d N\left(0, \frac{1}{4} \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1}\right)$$

as $n \rightarrow \infty$, where Σ_l ($l = 1, 2$) is defined as in (2.6).

REMARK 1. Theorem 2.1 implies that $\hat{\theta}_{wn}$ is asymptotically normal for general specifications of $g(\cdot)$ and u_t , which determine the asymptotic variance of $\hat{\theta}_{wn}$, and hence need be considered in all inferential methods. Since there is no guarantee that the structures of $g(\cdot)$ and u_t specified by the researcher provide a correct description of reality, we will propose valid inferential methods without specifying the forms of $g(\cdot)$ and u_t in the sequel.

When $g(\cdot) \equiv 1$, $w(\cdot) \equiv 1$ and u_t is stationary, Theorem 2.1 covers the existing results on the usual LADE in Bassett and Koenker (1978) for the i.i.d. u_t and Zhu and Ling (2015) for the conditionally heteroscedastic u_t with unknown form.

When $\mu_0 \equiv 0$ (i.e., $\rho \equiv 0$) and $f_t(0) \equiv f(0)$, the asymptotic variance of $\hat{\theta}_{wn}$ has the following simplified version:

$$(2.7) \quad \frac{1}{4} \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1} = \frac{c_{gw}}{4 f^2(0)} \Lambda^{-1},$$

where $c_{gw} = (\int_0^1 \frac{g^2(x)}{w^2(x)} dx) (\int_0^1 \frac{g(x)}{w(x)} dx)^{-2}$ and Λ is the $p \times p$ matrix with (r, s) th element $\sum_{i=0}^\infty \sum_{j=0}^\infty \alpha_i \alpha_j m_{i+r, j+s}^{(2)}$. Furthermore, if u_t is a sequence of i.i.d. random variables, the (r, s) th element of Λ becomes $\lambda_{|r-s|}$, where $\lambda_k = E u_t^2 (\sum_{i=0}^\infty \alpha_i \alpha_{i+k})$.

REMARK 2. Our weighted LADE $\hat{\theta}_{wn}$ requires $E[\text{sgn}(\varepsilon_t) | \mathcal{F}_{t-1}] = 0$, based on which model (1.1) gives the predicted median of future y_t under L_1 loss function. If one is interested to predict the mean of future y_t , the prediction can be conducted under L_2 loss function, assuming $E[\varepsilon_t | \mathcal{F}_{t-1}] = 0$. From the viewpoint of robustness, the predicted median of future y_t under L_1 loss function may be desired, and if $E[\text{sgn}(\varepsilon_t) | \mathcal{F}_{t-1}] = E[\varepsilon_t | \mathcal{F}_{t-1}] = 0$, it also forecasts the mean of future y_t .

We also note that though $\hat{\theta}_{wn}$ has the parametric convergence rate $n^{-1/2}$ in Theorem 2.1, it is not needed for the purpose of prediction. A less restricted model (1.1) with time-varying AR coefficients might be studied in the future to deliver a better prediction of y_t .

In the finite variance AR model, the technique in Zhu and Ling (2015) for the usual LADE requires the stationarity of y_t , which is against our setting for a either time-varying g_t or nonstationary u_t . Our proof techniques for the weighted LADE in Theorem 2.1 or its related inferential methods below rely on Theorem 1 of Andrews (1988), and they require a key technical condition that $\{f_t(0)\}$ is $L^{2+\delta_1}$ -NED on $\{u_t\}$ (see Assumption 2.5(ii)), which seems new to the study of the LADE. Moreover, in the present of g_t , we further propose a special weighted LADE with w_t equal to a kernel estimator of g_t ; see Section 6 below. In many situations, this special weighted LADE is “optimal” in terms of efficiency, but its related proof techniques (particularly for Propositions A.3–A.4 in Appendix A) are new and not available in Zhu and Ling (2015).

In the infinite variance AR model, [Zhu and Ling \(2015\)](#) constructed a self-weighted LADE, which is \sqrt{n} -consistent and asymptotically normal when y_t is stationary. Meanwhile, they pointed out that the convergence rate of the usual LADE is slower than $n^{-1/2}$ when ε_t follows the first-order (G)ARCH model, while [Davis \(1996\)](#) proved that the convergence rate of the usual LADE is faster than $n^{-1/2}$ when ε_t is i.i.d. This implies that the self-weighted LADE may not always outperform the usual LADE (or the weighted LADE) when y_t is stationary with an infinite variance. In general, when y_t is allowed to be nonstationary (as in our setting) with an infinite variance, the asymptotic properties of the weighted LADE and the self-weighted LADE are not clear at this stage, and we leave this topic for future study.

Next, we relax the technical conditions in [Theorem 2.1](#) for some special cases of model (1.1). We first consider the case that u_t has the ARCH-type structure ([Engle \(1982\)](#), [Bollerslev \(1986\)](#), [Giraitis, Kokoszka and Leipus \(2000\)](#), [Francq and Zakoïan \(2010\)](#)):

$$(2.8) \quad u_t = \sigma_t \eta_t,$$

where u_t is stationary, $\{\eta_t\}$ is a sequence of i.i.d. innovations with median zero, and $\sigma_t \in \mathcal{F}_{t-1}$ satisfies that $\sigma_t \geq \underline{c}$ for some $\underline{c} > 0$. This condition on σ_t directly holds for most of ARCH-type models as long as their intercept term has a positive lower bound. Since u_t is stationary, [Assumption 2.4\(ii\)–\(iii\)](#) are not needed any more if [Assumption 2.4\(i\)](#) holds.

Let $f_\eta(\cdot)$ be the density of η_t . Under model (2.8), $f_t(0) = f_\eta(0)/\sigma_t$. Since $\sigma_t \geq \underline{c}$, [Assumption 2.5\(ii\)](#) holds if $\{\sigma_t\}$ is $L^{2+\delta_1}$ -NED on $\{u_t\}$, and [Assumption 2.5\(iii\)–\(v\)](#) hold by [Assumption 2.4\(i\)](#). Hence, [Assumption 2.5](#) can be relaxed to [Assumption 2.6](#) in this case.

ASSUMPTION 2.6. u_t satisfies model (2.8), and:

- (i) $f_\eta(x)$ is continuous with $f_\eta(0) > 0$ and $\sup_{x \in \mathcal{R}} f_\eta(x) < \infty$;
- (ii) $\{\sigma_t\}$ is $L^{2+\delta_1}$ -NED on $\{u_t\}$ for some $\delta_1 > 0$.

COROLLARY 2.1. *Suppose [Assumptions 2.1–2.3](#), [2.4\(i\)](#) and [2.6](#) hold. Then the result in [Theorem 2.1](#) holds.*

Since $\sigma_t \geq \underline{c}$, by a minor extension of [Lemma 2.1](#) of [Davidson \(2002\)](#), we can show that [Assumption 2.6\(ii\)](#) holds if $\{\sigma_t^{\kappa_1}\}$ is $L^{2+\delta_1}$ -NED on $\{u_t\}$ for some $\kappa_1 \geq 1$. When σ_t in (2.8) satisfies that

$$(2.9) \quad \sigma_t^{\kappa_1} = \psi_0 + \sum_{i=1}^{\infty} \psi_i |u_{t-i}|^{\kappa_1}$$

with $\psi_0 > 0$ and $\psi_i \geq 0$ for all $i \geq 1$, it is straightforward to see that $\{\sigma_t^{\kappa_1}\}$ is L^{κ_2} -NED (for some $\kappa_2 \geq 1$) on $\{u_t\}$ if

$$(2.10) \quad \sum_{i=1}^{\infty} \psi_i < \infty \quad \text{and} \quad E|u_t|^{\kappa_1 \kappa_2} < \infty.$$

Hence, Assumption 2.6(ii) holds under (2.9)–(2.10) with $\kappa_2 = 2 + \delta_1$, which allow a general class of ARCH-type models, including (G)ARCH, ARCH(∞) (Robinson (1991)), power GARCH (Ding, Granger and Engle (1993)), hyperbolic GARCH (Davidson (2004)), and many others. Note that our NED condition in (2.10) is different from the one in Davidson (2004), since our filtration is based on $\{u_t\}$ instead of $\{\eta_t\}$.

Second, we consider the case that $f_t(0)$ depends on finite lags of u_t . In this case, Assumption 2.4(i) and Assumption 2.5(ii) can be further relieved as shown in Assumption 2.7, which does not need $\{f_t(0)\}$ to be NED.

ASSUMPTION 2.7. (i) $\sup_t E|u_t|^{2+\delta_0} < \infty$ for some $\delta_0 > 0$; (ii) $f_t(0) \in \sigma(u_s; t - c_0 \leq s \leq t)$ for some $c_0 > 0$ is uniformly bounded for all $t \geq 1$.

COROLLARY 2.2. *Suppose Assumptions 2.1–2.3, 2.4(ii)–(iii), 2.5(i) and (iii)–(v) and 2.7 hold. Then the result in Theorem 2.1 holds.*

Clearly, Assumption 2.7(ii) holds under model (2.9) with $\psi_i \equiv 0$ for all $i \geq i_0$ and some integer $i_0 \geq 1$. This implies that when u_t follows a stationary finite-lag ARCH model with $E|u_t|^{2+\delta_0} < \infty$ and $Eu_t^4 = \infty$, $\hat{\theta}_{wn}$ is asymptotically normal. However, the LSE in this case is asymptotically nonnormal with a slower convergence rate than $n^{-1/2}$ as shown in Zhang and Ling (2015). Hence, the weighted LADE seems to be more convenient and efficient than the LSE to tackle the heavy-tailedness of ε_t .

Third, we consider the case that $\phi_{i0} \equiv 0$, under which model (1.1) becomes a location model. For this location model, Assumptions 2.1 is redundant, and Assumptions 2.4–2.5 can be replaced by Assumption 2.8 below.

ASSUMPTION 2.8. $\phi_{i0} \equiv 0$ for all i in model (1.1), and either:

- (i) Assumption 2.5(i) and (iii) hold and $\{f_t(0)\}$ is L^1 -NED on $\{u_t\}$; or
- (ii) Assumption 2.6(i) holds and $\{\sigma_t\}$ is L^1 -NED on $\{u_t\}$ under (2.8).

COROLLARY 2.3. *Suppose Assumptions 2.2–2.3 and 2.8 hold. Then the result in Theorem 2.1 holds.*

As argued before, Assumption 2.8(ii) holds under (2.9)–(2.10) with $\kappa_2 = 1$. It is worth noting that the weaker technical assumptions in Corollaries 2.1–2.3 not only suffice for Theorem 2.1 but also for other asymptotic results subsequently.

3. The bootstrapped variance estimator. Since the asymptotic variance of $\widehat{\theta}_{wn}$ in Theorem 2.1 depends on the unknown forms of $g(\cdot)$ and u_t , it cannot be estimated directly by its sample counterpart. To solve this problem, we use the random weighting (RW) method to approximate the limiting distribution of $\widehat{\theta}_{wn}$ in Theorem 2.1. Let $\{w_t^*\}_{t=1}^n$ be a sequence of i.i.d. nonnegative random variables, with mean and variance both equal to 1. Define

$$(3.1) \quad \widehat{\theta}_{wn}^* = \arg \min_{\theta \in \Theta} \sum_{t=1}^n w_t^* \frac{|y_t - Y'_{t-1}\theta|}{w_t}.$$

Based on Assumption 3.1, we can show that the distribution of $\sqrt{n}(\widehat{\theta}_{wn} - \theta_0)$ can be approximated by the resampling distribution of $\sqrt{n}(\widehat{\theta}_{wn}^* - \widehat{\theta}_{wn})$.

ASSUMPTION 3.1. (i) $E(w_t^*)^{2+\delta_2} < \infty$ for some $\delta_2 > 0$; (ii) $\{w_t^*\}$ and $\{y_t\}$ are independent.

THEOREM 3.1. *Suppose Assumption 3.1 and the conditions in Theorem 2.1 hold. Then, conditional on $\{y_t\}_{t=1}^n$,*

$$\sqrt{n}(\widehat{\theta}_{wn}^* - \widehat{\theta}_{wn}) \rightarrow_d N\left(0, \frac{1}{4}\Sigma_2^{-1}\Sigma_1\Sigma_2^{-1}\right) \quad \text{in probability}$$

as $n \rightarrow \infty$, where Σ_l ($l = 1, 2$) is defined as in (2.6).

REMARK 3. The RW method can be viewed as a variant of the wild bootstrap in Wu (1986) and Liu (1988). It provides us with a valid tool to implement the statistical inference based on the weighted LADE, and some other bootstrap methods (see, e.g., Gonçalves and Kilian (2004)) may also be valid or even better under our setting. An interesting work is to find the optimal bootstrap method among all valid ones in theory. This will necessitate higher order asymptotic analysis and is beyond the scope of this paper.

According to Theorems 2.1 and 3.1, we can approximate the asymptotic covariance matrix of $\widehat{\theta}_{wn}$ by the following procedure:

1. Generate J replications of the i.i.d. random weights $\{w_t^*\}_{t=1}^n$ from the standard exponential distribution, which has mean and variance both equal to one.
2. Compute $\widehat{\theta}_{wn}^*$ for i th replication, and denote it as b_i .
3. Calculate the sample variance-covariance matrix of $\{b_i - \widehat{\theta}_{wn}\}_{i=1}^J$, denoted by \widehat{V}_{wn} , which provides a good approximation for the asymptotic covariance matrix of $\widehat{\theta}_{wn}$ for large J .

Based on \widehat{V}_{wn} , we can construct a Wald test statistic

$$(3.2) \quad W_{wn} = (\Gamma\widehat{\theta}_{wn} - r)'(\Gamma\widehat{V}_{wn}\Gamma')^{-1}(\Gamma\widehat{\theta}_{wn} - r)$$

to test the following linear null hypothesis:

$$(3.3) \quad H_0 : \Gamma\theta_0 = r,$$

where Γ is an $s \times (p + 1)$ constant matrix with rank s and r is an $s \times 1$ constant vector. If W_{wn} is larger than the upper-tailed critical value of χ_s^2 , the null hypothesis H_0 is rejected. Otherwise, H_0 is not rejected.

4. Model diagnostic checking. Model diagnostic checking is an important step in applications. Most of the existing methods, including the time domain correlation-based tests and frequency domain periodogram-based tests, require that the observed data are stationary with a homogenous innovation; see, for example, Hong (1996), Hong and Lee (2005), Escanciano (2006) and Zhu and Li (2015) for an overview. Hence, they are invalid for model (1.1), which calls for a new method for its diagnostic checking. In this section, we construct a sign-based Ljung–Box portmanteau test as in Zhu and Ling (2015) for this purpose.

Let $\varepsilon_t(\theta) = y_t - \theta'Y_{t-1}$. The idea of our sign-based portmanteau test is based on the fact that $\{\text{sgn}(\varepsilon_t(\theta_0))\}$ is a sequence of uncorrelated random variables under Assumption 2.3. Hence, if model (1.1) is correctly specified, it is expected that the sample autocorrelation function of $\{\text{sgn}(\varepsilon_t(\hat{\theta}_{wn}))\}$ at lag k , denoted by $\hat{r}_{wn,k}$, is close to zero, where

$$\hat{r}_{wn,k} = \frac{\sum_{t=k+1}^n [\text{sgn}(\varepsilon_t(\hat{\theta}_{wn})) - \bar{\varepsilon}(\hat{\theta}_{wn})][\text{sgn}(\varepsilon_{t-k}(\hat{\theta}_{wn})) - \bar{\varepsilon}(\hat{\theta}_{wn})]}{\sum_{t=1}^n [\text{sgn}(\varepsilon_t(\hat{\theta}_{wn})) - \bar{\varepsilon}(\hat{\theta}_{wn})]^2}$$

with $\bar{\varepsilon}(\theta) = \frac{1}{n} \sum_{t=1}^n \text{sgn}(\varepsilon_t(\theta))$. Let $\hat{r}_{wn} = (\hat{r}_{wn,1}, \hat{r}_{wn,2}, \dots, \hat{r}_{wn,M})'$ for some integer $M \geq 1$. To study the joint distribution of \hat{r}_{wn} , we need one more assumption below.

ASSUMPTION 4.1. u_t and $f_t(0)$ satisfy that:

- (i) $m_{r,s}^{(3)} := E[\text{sgn}(u_{t-r})u_{t-s}]$ is uniformly bounded for all $t, s, r \geq 1$;
- (ii) $\mu_r^{(1)} := E[f_t(0) \text{sgn}(u_{t-r})]$ is uniformly bounded for all $t, r \geq 1$;
- (iii) $\mu_{r,s}^{(2)} := E[f_t(0) \text{sgn}(u_{t-r})u_{t-s}]$ is uniformly bounded for all $t, s, r \geq 1$.

Let $d_3 = \int_0^1 \frac{1}{g(x)} dx$ and

$$K_{3k} = \begin{pmatrix} 0 \\ \sum_{i=0}^{\infty} \alpha_i m_{k,1+i}^{(3)} \\ \vdots \\ \sum_{i=0}^{\infty} \alpha_i m_{k,p+i}^{(3)} \end{pmatrix}, \quad \Omega_{3k} = \begin{pmatrix} d_3 \mu_k^{(1)} \\ \rho d_3 \mu_k^{(1)} + \sum_{i=0}^{\infty} \alpha_i \mu_{k,1+i}^{(2)} \\ \vdots \\ \rho d_3 \mu_k^{(1)} + \sum_{i=0}^{\infty} \alpha_i \mu_{k,p+i}^{(2)} \end{pmatrix}$$

for $k = 1, \dots, M$. With this notation, define

$$(4.1) \quad \Sigma_3 = \begin{pmatrix} I_M & K_3' \\ K_3 & \Sigma_1 \end{pmatrix} \in \mathcal{R}^{\bar{p} \times \bar{p}} \quad \text{and} \quad \Sigma_4 = (I_M, -\Omega_3 \Sigma_2^{-1}) \in \mathcal{R}^{M \times \bar{p}},$$

where $K_3 = (\int_0^1 \frac{g(x)}{w(x)} dx) \dot{K}_3$ with $\dot{K}_3 = (K_{31}, K_{32}, \dots, K_{3M}) \in \mathcal{R}^{(p+1) \times M}$, $\Omega_3 = (\Omega_{31}, \Omega_{32}, \dots, \Omega_{3M})' \in \mathcal{R}^{M \times (p+1)}$, and $\bar{p} = p + M + 1$. The following theorem gives the joint distribution of \hat{r}_{wn} .

THEOREM 4.1. *Suppose Assumption 4.1 and the conditions of Theorem 2.1 hold. Then, if model (1.1) is correctly specified,*

$$\sqrt{n} \hat{r}_{wn} \rightarrow_d N(0, \Sigma_4 \Sigma_3 \Sigma_4')$$

as $n \rightarrow \infty$, where Σ_l ($l = 3, 4$) is defined as in (4.1).

REMARK 4. When $g(\cdot) \equiv 1$, $w(\cdot) \equiv 1$, and u_t is stationary, Theorem 4.1 covers the existing result on the usual LADE in [Zhu and Ling \(2015\)](#) for the conditionally heteroscedastic u_t with unknown form.

When $\mu_0 \equiv 0$ and $f_t(0) \equiv f(0)$, the asymptotic variance of \hat{r}_{wn} has the following simplified version:

$$(4.2) \quad \Sigma_4 \Sigma_3 \Sigma_4' = I_M - (2 - c_{gw}) \ddot{K}_3' \Lambda^{-1} \ddot{K}_3,$$

where \ddot{K}_3 is the submatrix of \dot{K}_3 by removing its first row, and c_{gw} and Λ are defined as in (2.7).

Since the forms of $g(\cdot)$ and u_t are not specified, the asymptotic covariance matrix of \hat{r}_{wn} cannot be directly estimated from its sample counterpart. We use the similar RW method as in Section 3 to avoid this difficulty. Define

$$\hat{r}_{wn,k}^* = \frac{\sum_{t=k+1}^n w_t^* [\text{sgn}(\varepsilon_t(\hat{\theta}_{wn}^*)) - \bar{\varepsilon}(\hat{\theta}_{wn}^*)] [\text{sgn}(\varepsilon_{t-k}(\hat{\theta}_{wn}^*)) - \bar{\varepsilon}(\hat{\theta}_{wn}^*)]}{\sum_{t=1}^n [\text{sgn}(\varepsilon_t(\hat{\theta}_{wn}^*)) - \bar{\varepsilon}(\hat{\theta}_{wn}^*)]^2}.$$

Let $\hat{r}_{wn}^* = (\hat{r}_{wn,1}^*, \hat{r}_{wn,2}^*, \dots, \hat{r}_{wn,M}^*)'$. The following theorem indicates that the distribution of $\sqrt{n} \hat{r}_{wn}$ can be approximated by the resampling distribution of $\sqrt{n}(\hat{r}_{wn}^* - \hat{r}_{wn})$.

THEOREM 4.2. *Suppose Assumption 3.1 and the conditions in Theorem 4.1 hold. Then, if model (1.1) is correctly specified, conditional on $\{y_t\}_{t=1}^n$,*

$$\sqrt{n}(\hat{r}_{wn}^* - \hat{r}_{wn}) \rightarrow_d N(0, \Sigma_4 \Sigma_3 \Sigma_4') \quad \text{in probability}$$

as $n \rightarrow \infty$, where Σ_l ($l = 3, 4$) is defined as in (4.1).

According to Theorems 4.1 and 4.2, we can approximate the asymptotic covariance matrix of \hat{r}_{wn} by the following procedure:

1. Generate J replications of the i.i.d. random weights $\{w_t^*\}_{t=1}^n$ from the standard exponential distribution.
2. Compute \hat{r}_{wn}^* for i th replication, and denote it as c_i .
3. Calculate the sample variance-covariance matrix of $\{c_i - \hat{r}_{wn}\}_{i=1}^J$, denoted by \hat{U}_{wn} , which provides a good approximation for the asymptotic covariance matrix of \hat{r}_{wn} for large J .

Based on \hat{U}_{wn} , we can construct a portmanteau test statistic

$$(4.3) \quad S_{wn}(M) = \hat{r}'_{wn} \hat{U}_{wn}^{-1} \hat{r}_{wn}$$

to detect the adequacy of model (1.1). If $S_{wn}(M)$ is larger than the upper-tailed critical value of χ^2_M , the fitted model (1.1) is not adequate. Otherwise, it is adequate.

5. The choice of weight function. By choosing the weight function $w_t \equiv 1$, a systematical statistical inference procedure for model (1.1) based on the usual LADE $\hat{\theta}_n$ is already available in Sections 2–4. However, this choice of the weight function may lead to a deficient weighted LADE in terms of efficiency. In this section, we are interested in finding the “optimal” weight w_t to minimize the asymptotic covariance matrix in Theorem 2.1. Given $w_t = g_t$, we consider a weighted LADE defined by

$$\hat{\theta}_{an} = \arg \min_{\theta \in \Theta} \sum_{t=1}^n \frac{|y_t - Y'_{t-1}\theta|}{g_t}.$$

Clearly, $\hat{\theta}_{an}$ is infeasible in practice as g_t is not observable. However, Corollary 5.1 below shows that $\hat{\theta}_{an}$ is the desired one under three different scenarios.

COROLLARY 5.1. $\frac{1}{4} \Sigma_2^{-1} \Sigma_1 \Sigma_2^{-1}$ attains its minimum A when $w_t = g_t$ (up to a constant multiplier) under one of the following scenarios:

- (S1) $\mu_0 \equiv 0$;
- (S2) $f_t(0) \equiv f(0)$;
- (S3) $\phi_{i0} \equiv 0$ for all i .

Particularly, under (S1), A is the $p \times p$ matrix with (r, s) th element $a_{r,s}$, and $a_{r,s} = \frac{1}{4} [a_{r,s}^{(\tau)}]^{-1} a_{r,s}^{(m)} [a_{r,s}^{(\tau)}]^{-1}$ with $a_{r,s}^{(\tau)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_i \alpha_j \tau_{i+r, j+s}^{(2)}$ and $a_{r,s}^{(m)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_i \alpha_j m_{i+r, j+s}^{(2)}$.

Under (S1), the asymptotic covariance matrix of $\hat{\theta}_{an}$ is adaptive to the unknown form of $g(\cdot)$. From this point of view, we follow Xu and Phillips (2008) to call $\hat{\theta}_{an}$ the infeasible adaptive LADE (ALADE). Obviously, Corollary 5.1 indicates that $\hat{\theta}_{an}$ is more efficient than $\hat{\theta}_n$ under (S1), (S2) or (S3). Our finding that g_t is the

optimal weight is similar to those in Gutenbrunner and Jurečková (1992) and Zhao (2001) for the linear heteroscedastic regression model.

Next, we make a formal comparison of efficiency among the LADE $\widehat{\theta}_n$, the infeasible ALADE $\widehat{\theta}_{an}$, the LSE $\check{\theta}_n$ in Phillips and Xu (2006), and the infeasible adaptive LSE (ALSE) $\check{\theta}_{an}$ in Xu and Phillips (2008), where

$$\check{\theta}_n = \left(\sum_{t=1}^n Y_{t-1} Y'_{t-1} \right)^{-1} \left(\sum_{t=1}^n y_t Y_{t-1} \right) \quad \text{and}$$

$$\check{\theta}_{an} = \left(\sum_{t=1}^n g_t^{-2} Y_{t-1} Y'_{t-1} \right)^{-1} \left(\sum_{t=1}^n g_t^{-2} y_t Y_{t-1} \right).$$

We assume that $\mu_0 \equiv 0$ as in Xu and Phillips (2008), and that $\{u_t\}$ is an i.i.d. sequence with $\text{median}(u_t) = 0$, $E u_t = 0$ and $E u_t^2 = 1$ to make the comparison feasible. As $\{u_t\}$ is an i.i.d. sequence, the moment condition that $E|u_t|^{2+\delta_0} < \infty$ for some $\delta_0 > 0$ is sufficient for the asymptotic normality of $\widehat{\theta}_n$ and $\widehat{\theta}_{an}$ by Corollary 2.2, and this is also the case for $\check{\theta}_n$ and $\check{\theta}_{an}$ by a slight modification of the proof in Xu and Phillips (2008).

Define

$$b_1 = \frac{1}{4f^2(0)} \frac{\int_0^1 g^2(x) dx}{[\int_0^1 g(x) dx]^2}, \quad b_2 = \frac{1}{4f^2(0)},$$

$$b_3 = \frac{\int_0^1 g^4(x) dx}{[\int_0^1 g^2(x) dx]^2} \quad \text{and} \quad b_4 = 1,$$

where $f(\cdot)$ is the density function of u_t . Under the aforementioned conditions, result (2.7) implies that as $n \rightarrow \infty$,

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \rightarrow_d N(0, b_1 \Lambda^{-1}) \quad \text{and} \quad \sqrt{n}(\widehat{\theta}_{an} - \theta_0) \rightarrow_d N(0, b_2 \Lambda^{-1}),$$

respectively; and Phillips and Xu (2006) and Xu and Phillips (2008) showed that as $n \rightarrow \infty$,

$$\sqrt{n}(\check{\theta}_n - \theta_0) \rightarrow_d N(0, b_3 \Lambda^{-1}) \quad \text{and} \quad \sqrt{n}(\check{\theta}_{an} - \theta_0) \rightarrow_d N(0, b_4 \Lambda^{-1}),$$

respectively. The asymptotic efficiencies of these four estimators can be formally compared by just looking at the values of b_i under different scenarios of u_t and $g(\cdot)$. Below, Example 1 considers the case that $g(\cdot)$ has an abrupt change in the variance, when u_t is chosen to be an i.i.d. standardized t_3 (ST₃) or standardized Laplace(0, 1) (SL(0, 1)) sequence with $E(u_t^2) = 1$. For the cases that $g(\cdot)$ has gradual and periodical change in the variance, one can refer to Examples 2 and 3 in Zhu (2019).

EXAMPLE 1 (An abrupt change in the variance). Let $\tau \in [0, 1]$ and $g(\cdot)$ be the step function

$$(5.1) \quad g(x) = e_0 + (e_1 - e_0)I(x \geq \tau),$$

where $x \in [0, 1]$, $e_0 > 0$, and $e_1 > 0$. Under (5.1), the variance of ε_t is e_0^2 before the break point $[n\tau]$, and e_1^2 afterwards. Let $\delta = e_1/e_0$. Then some algebra shows that

$$b_1 = \frac{1}{4f^2(0)} \frac{\tau + (1 - \tau)\delta^2}{(\tau + (1 - \tau)\delta)^2} \quad \text{and} \quad b_3 = \frac{\tau + (1 - \tau)\delta^4}{(\tau + (1 - \tau)\delta^2)^2}.$$

Figure 1(a)–(f) plot the values of all b_i in terms of δ for $\tau = 0.1, 0.5$ and 0.9 , respectively. From this figure, our findings are as follows:

- (i) $\hat{\theta}_{an}$ is more efficient than $\check{\theta}_{an}$ under all considered cases. Meanwhile, $\hat{\theta}_n$ can be more efficient than $\check{\theta}_{an}$ when the break happens earlier with a large δ , the break happens later with a small δ or the break happens in the middle with δ not largely deviating from 1. All of these findings imply the efficiency advantage of the ALADE (or LADE) over the ALSA when u_t is heavy-tailed.
- (ii) $\hat{\theta}_{an}$ (or $\check{\theta}_{an}$) is always more efficient than $\hat{\theta}_n$ (or $\check{\theta}_n$) as expected, but the efficiency advantage is less significant when the break happens earlier with a large δ , the break happens later with a small δ , or the break happens in the middle with

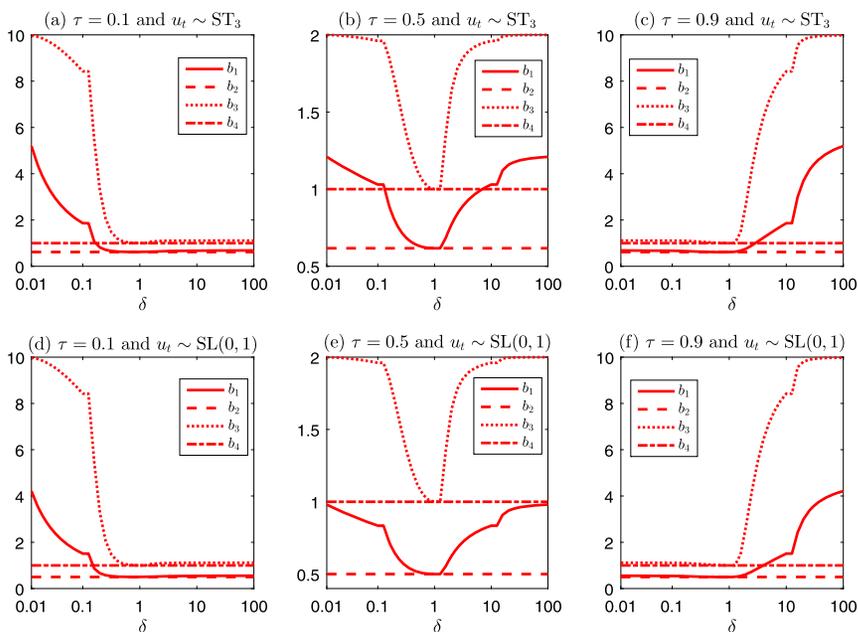


FIG. 1. The values of b_1 (solid line), b_2 (dashed line), b_3 (dash-dot line) and b_4 (dotted line) across δ .

$\delta \approx 1$ (see Xu and Phillips (2008), page 270) for a similar phenomenon and explanation). This indicates that the ALADE with an appropriate choice of the weight function can have a pronouncing efficiency advantage over the usual LADE.

Overall, Examples 1–3 demonstrate that (i) $\hat{\theta}_{an}$ and $\hat{\theta}_n$ can be more efficient than $\tilde{\theta}_{an}$ when u_t is heavy-tailed; (ii) $\hat{\theta}_{an}$ is more efficient than $\hat{\theta}_n$ in all considered cases. However, $\hat{\theta}_{an}$ is infeasible in practice, and this problem will be solved in the next section.

6. The feasible adaptive LADE. The objective of this section is to give a feasible ALADE, prove that this feasible ALADE calculated with estimated weights is equivalent to the infeasible ALADE $\hat{\theta}_{an}$ asymptotically, and show that our entire statistical inference procedure in Sections 2–4 is still valid based on this feasible ALADE.

We use a similar nonparametric method as in Xu and Phillips (2008) to achieve this goal. Let $\hat{\varepsilon}_t = y_t - Y'_{t-1}\hat{\theta}_n$ be the residual from the LADE. Denote

$$(6.1) \quad \hat{g}_t = \sum_{i=1}^n k_{ti} |\hat{\varepsilon}_i|,$$

where $k_{ti} = (\sum_{i=1}^n K_{ti})^{-1} K_{ti}$ with

$$K_{ti} = \begin{cases} K\left(\frac{t-i}{nb}\right) & \text{if } t \neq i, \\ 0 & \text{if } t = i, \end{cases}$$

where $K(\cdot)$ is the kernel function defined on \mathcal{R} , and $b > 0$ is the bandwidth. Of course, the role of \hat{g}_t is to deputize for g_t , which is the mean of $|\varepsilon_t|$ based on the following assumption.

ASSUMPTION 6.1. $E|u_t| = 1$.

Assumption 6.1 is used for the identification of g_t , and it does not rule out the conditional heteroscedasticity of u_t . For the feasible ALSE, Xu and Phillips (2008) resorted to the identification condition $E[u_t^2 | \mathcal{F}_{t-1}] = 1$; however, their identification condition is restrictive, since it does not allow u_t to be conditionally heteroscedastic. Moreover, we shall mention that besides (6.1), other methods could also be used to estimate g_t (see, e.g., Fan and Yao (1998), Yu and Jones (2004)), and this is a promising direction for the future study.

By using \hat{g}_t in (6.1), our feasible ALADE is defined as follows:

$$\tilde{\theta}_{an} := \arg \min_{\theta \in \Theta} \sum_{t=1}^n \frac{|y_t - Y'_{t-1}\theta|}{\hat{g}_t}.$$

To obtain the asymptotic theory of $\tilde{\theta}_{an}$, we need two more assumptions.

ASSUMPTION 6.2. u_t is an α -mixing process with $\alpha_u(k) \leq Ck^{-\delta_3}$ for some $C > 0$ and $\delta_3 \geq 2(4 + \delta_0)/\delta_0$, where $\alpha_u(k)$ is the sequence of strong mixing coefficients of u_t , and δ_0 is defined as in Assumption 2.4(i).

ASSUMPTION 6.3. (i) $K(\cdot) \geq 0$ is a bounded continuous function with $\int_{-\infty}^{\infty} K(x) dx = 1$; (ii) $b + 1/(nb^{5+\delta_4}) \rightarrow 0$ as $n \rightarrow \infty$ for some $\delta_4 > 0$.

Assumption 6.2 from Shao and Yu (1996) is a technical condition for some moment inequalities of the partial sums of strong mixing sequences. An exponentially fast decaying $\alpha_u(k)$, which holds for large classes of processes (see Carrasco and Chen (2002)), is sufficient for the validity of this assumption. Assumption 6.3(i) holds for commonly used kernels such as uniform, Epanechnikov, biweight and Gaussian. Assumption 6.3(ii) requires that b converges to zero at a slower rate than $n^{-1/(5+\delta_4)}$, and this rate can be improved under some stronger conditions; see Remark 5 below.

The following theorem shows that $\tilde{\theta}_{an}$ has the same asymptotic variance as $\hat{\theta}_{an}$, and hence it is the desired weighted LADE we are looking for.

THEOREM 6.1. *Suppose Assumptions 2.1–2.5 and 6.1–6.3 hold. Then*

$$\sqrt{n}(\tilde{\theta}_{an} - \theta_0) = \sqrt{n}(\hat{\theta}_{an} - \theta_0) + o_p(1) \rightarrow_d N\left(0, \frac{1}{4} \Sigma_{a2}^{-1} \Sigma_{a1} \Sigma_{a2}^{-1}\right)$$

as $n \rightarrow \infty$, where Σ_{al} ($l = 1, 2$) is defined in the same way as Σ_l in (2.6) with $w_t = g_t$.

REMARK 5. If $E[|u_t| | \mathcal{F}_{t-1}] = 1$ (a stronger condition than Assumption 6.1), we can similarly show that the result in Theorem 6.1 holds by Propositions A.3(ii') and A.4(ii') in Appendix A, without Assumption 6.2 and with the replacement of Assumption 6.3(ii) by the weaker condition that $b + 1/(nb^2) \rightarrow 0$ as $n \rightarrow \infty$. In this case, b can converge to zero at a slower rate than $n^{-1/2}$.

REMARK 6. When $\mu_0 \equiv 0$ and $f_t(0) \equiv f(0)$, the asymptotic variance of $\tilde{\theta}_{an}$ becomes $\frac{1}{4f^2(0)} \Lambda^{-1}$ by (2.7).

Since the implementation of $\tilde{\theta}_{an}$ depends on the bandwidth b , we use a similar cross-validation (CV) method as in Xu and Phillips (2008) to select b . That is, b is chosen to be the value of b^* which minimizes

$$(6.2) \quad \widehat{CV}(b) = \frac{1}{n} \sum_{t=1}^n (|\hat{\varepsilon}_t| - \hat{g}_t)^2.$$

In theory, it is unclear whether the bandwidth b^* chosen by this CV method satisfies Assumption 6.3(ii). In practice, we can set $b = Cn^{-1/(5+2\delta_4)}$ for a small δ_4 ,

and find C^* by a grid search such that $b^* = C^*n^{-1/(5+2\delta_4)}$ minimizes $\widehat{CV}(b)$. With b^* computed by this data-driven method, simulation studies in Section 7 show that our $\tilde{\theta}_{an}$ has a good finite sample performance.

Next, we consider the estimation of $g(\cdot)$ as an interest in its own.

COROLLARY 6.1. *Suppose (i) Assumptions 2.1–2.5, 6.1–6.2 and 6.3(i) hold; and (ii) $b + 1/(nb^2) \rightarrow 0$ as $n \rightarrow \infty$. Then*

$$\text{plim}_{n \rightarrow \infty} \widehat{g}_{[n\tau]} = \begin{cases} g(\tau-) \int_{-\infty}^0 K(x) dx + g(\tau+) \int_0^{\infty} K(x) dx & \text{for } \tau \in (0, 1), \\ g(1-) \int_{-\infty}^0 K(x) dx & \text{for } \tau = 1, \end{cases}$$

where $g(\tau-) = \lim_{\bar{\tau} \uparrow \tau} g(\bar{\tau})$ and $g(\tau+) = \lim_{\bar{\tau} \downarrow \tau} g(\bar{\tau})$ for $\tau \in (0, 1]$.

Corollary 6.1 shows that $\widehat{g}_{[n\tau]}$ converges in probability to $g(\tau)$ for interior points τ when $g(\cdot)$ is continuous, but in general to a point between $g(\tau-)$ and $g(\tau+)$ depending on the shape of the kernel function $K(\cdot)$. It is worth noting that the inconsistency of $\widehat{g}_{[n\tau]}$ at points of discontinuities has no impact on the asymptotic theory of $\tilde{\theta}_{an}$ as shown in Theorem 6.1.

Third, since the forms of $g(\cdot)$ and u_t are not specified, we use the RW method as before to estimate the asymptotic covariance matrix of $\tilde{\theta}_{an}$. Define

$$\tilde{\theta}_{an}^* = \arg \min_{\theta \in \Theta} \sum_{t=1}^n w_t^* \frac{|y_t - Y'_{t-1}\theta|}{\widehat{g}_t}.$$

The following theorem indicates that the distribution of $\sqrt{n}(\tilde{\theta}_{an} - \theta_0)$ can be approximated by the resampling distribution of $\sqrt{n}(\tilde{\theta}_{an}^* - \tilde{\theta}_{an})$.

THEOREM 6.2. *Suppose Assumption 3.1 and the conditions in Theorem 6.1 hold. Then, conditional on $\{y_t\}_{t=1}^n$,*

$$\sqrt{n}(\tilde{\theta}_{an}^* - \tilde{\theta}_{an}) \rightarrow_d N\left(0, \frac{1}{4} \Sigma_{a2}^{-1} \Sigma_{a1} \Sigma_{a2}^{-1}\right) \quad \text{in probability}$$

as $n \rightarrow \infty$, where Σ_{al} ($l = 1, 2$) is defined as in Theorem 6.1.

According to Theorems 6.1 and 6.2, we can approximate the asymptotic covariance matrix of $\tilde{\theta}_{an}$ by \tilde{V}_{an} , where \tilde{V}_{an} is calculated in the same way as \widehat{V}_{wn} with $\widehat{\theta}_{wn}$ and $\widehat{\theta}_{wn}^*$ replaced by $\tilde{\theta}_{an}$ and $\tilde{\theta}_{an}^*$, respectively. Then we can construct another Wald test statistic W_{an} based on \tilde{V}_{an} , to test the linear null hypothesis H_0 in (3.3), where

$$(6.3) \quad W_{an} = (\Gamma \tilde{\theta}_{an} - r)' (\Gamma \tilde{V}_{an} \Gamma')^{-1} (\Gamma \tilde{\theta}_{an} - r).$$

If W_{an} is larger than the upper-tailed critical value of χ^2_s , the null hypothesis H_0 is rejected. Otherwise, H_0 is not rejected.

In the end, we consider the sign-based portmanteau test based on $\tilde{\theta}_{an}$. Define $\tilde{r}_{an,k}$ and $\tilde{r}_{an,k}^*$ in the same way as $\hat{r}_{wn,k}$ and $\hat{r}_{wn,k}^*$, with $\hat{\theta}_{wn}$ and $\hat{\theta}_{wn}^*$ replaced by $\tilde{\theta}_{an}$ and $\tilde{\theta}_{an}^*$, respectively. Let $\tilde{r}_{an} = (\tilde{r}_{an,1}, \tilde{r}_{an,2}, \dots, \tilde{r}_{an,M})'$ and $\tilde{r}_{an}^* = (\tilde{r}_{an,1}^*, \tilde{r}_{an,2}^*, \dots, \tilde{r}_{an,M}^*)'$. The following theorem indicates that the distribution of $\sqrt{n}\tilde{r}_{an}$ can be approximated by the resampling distribution of $\sqrt{n}(\tilde{r}_{an}^* - \tilde{r}_{an})$.

THEOREM 6.3. *Suppose the conditions in Theorem 6.2 hold. Then, if model (1.1) is correctly specified:*

- (i) $\sqrt{n}\tilde{r}_{an} \rightarrow_d N(0, \Sigma_{a4}\Sigma_{a3}\Sigma'_{a4})$ as $n \rightarrow \infty$;
- (ii) conditional on $\{y_t\}_{t=1}^n$, $\sqrt{n}(\tilde{r}_{an}^* - \tilde{r}_{an}) \rightarrow_d N(0, \Sigma_{a4}\Sigma_{a3}\Sigma'_{a4})$ in probability as $n \rightarrow \infty$, where Σ_{al} ($l = 3, 4$) is defined in the same way as Σ_l in (4.1) with $w_t = g_t$.

REMARK 7. When $\mu_0 \equiv 0$ and $f_t(0) \equiv f(0)$, the asymptotic variance of \tilde{r}_{an} becomes $I_M - \check{K}'_3 \Lambda^{-1} \check{K}_3$ by (4.2).

According to Theorem 6.3, we can approximate the asymptotic covariance matrix of \tilde{r}_{an} by \tilde{U}_{an} , where \tilde{U}_{an} is calculated in the same way as \hat{U}_{wn} with $\hat{\theta}_{wn}$ and $\hat{\theta}_{wn}^*$ replaced by $\tilde{\theta}_{an}$ and $\tilde{\theta}_{an}^*$, respectively. Then we can construct another sign-based portmanteau test statistic $S_{an}(M)$, based on \tilde{U}_{an} , to detect the adequacy of model (1.1), where

$$(6.4) \quad S_{an}(M) = \tilde{r}'_{an} \tilde{U}_{an}^{-1} \tilde{r}_{an}.$$

If $S_{an}(M)$ is larger than the upper-tailed critical value of χ^2_M , the fitted model (1.1) is not adequate. Otherwise, it is adequate.

7. Simulation study. In this section, we first assess the performance of the LADE $\hat{\theta}_n$, the feasible ALADE $\tilde{\theta}_{an}$, and the corresponding RW approach in finite samples. We generate 1000 replications of sample size $n = 100$ and 200 from the following model:

$$(7.1) \quad y_t = \theta_0 y_{t-1} + \varepsilon_t \quad \text{and} \quad \varepsilon_t = g_t u_t,$$

where $\theta_0 = 0.5$, the error generating process for ε_t is designed as follows:

$$(7.2) \quad u_t = \eta_t \sigma_t \quad \text{with} \quad \sigma_t^2 = 0.1 + \alpha_{\dagger} u_{t-1}^2 + \beta_{\dagger} \sigma_{t-1}^2;$$

and $g_t = g(t/n)$ with $g(\cdot)$ satisfying one of the following structures:

$$(7.3) \quad [\text{Abrupt change}] \quad g(x) = 1 + (\delta - 1)I(x \geq 0.5),$$

$$(7.4) \quad [\text{Gradual change}] \quad g(x) = 1 + (\delta - 1)x^2,$$

$$(7.5) \quad [\text{Periodic change}] \quad g(x) = \sin(\delta x) + 2.$$

TABLE 1
The values of SE and AE of $\hat{\theta}_n$ and $\tilde{\theta}_{an}$ for model (7.1) with $g_t \sim$ model (7.3)

α_{\dagger}	β_{\dagger}	δ	n		$\eta_t \sim \text{SL}(0, 1)$		$\eta_t \sim \text{ST}_3$		$\eta_t \sim \text{N}(0, 1)$		
					$\hat{\theta}_n$	$\tilde{\theta}_{an}$	$\hat{\theta}_n$	$\tilde{\theta}_{an}$	$\hat{\theta}_n$	$\tilde{\theta}_{an}$	
0.0	0.0	0.2	100	SE	0.0813	0.0680	0.0893	0.0770	0.1322	0.1094	
				AE	0.0898	0.0758	0.0924	0.0800	0.1288	0.1118	
		200	SE	0.0588	0.0494	0.0632	0.0537	0.0923	0.0763		
			AE	0.0616	0.0520	0.0643	0.0551	0.0913	0.0783		
		5	100	SE	0.0897	0.0759	0.0942	0.0833	0.1352	0.1159	
				AE	0.0975	0.0825	0.1005	0.0869	0.1381	0.1189	
	200		SE	0.0564	0.0473	0.0615	0.0540	0.0930	0.0768		
			AE	0.0634	0.0527	0.0676	0.0574	0.0952	0.0813		
	0.1	0.8	0.2	100	SE	0.0962	0.0786	0.1020	0.0867	0.1304	0.1109
					AE	0.1012	0.0852	0.1088	0.0919	0.1372	0.1162
			200	SE	0.0631	0.0539	0.0767	0.0623	0.0951	0.0822	
				AE	0.0714	0.0579	0.0787	0.0652	0.0985	0.0826	
5			100	SE	0.1013	0.0836	0.1103	0.0954	0.1443	0.1200	
				AE	0.1103	0.0911	0.1174	0.0984	0.1489	0.1256	
		200	SE	0.0707	0.0568	0.0803	0.0647	0.0985	0.0854		
			AE	0.0731	0.0599	0.0806	0.0667	0.1056	0.0882		

Here, we use $(\alpha_{\dagger}, \beta_{\dagger}) = (0, 0)$ or $(0.1, 0.8)$, choose $\eta_t \sim$ i.i.d. $\text{SL}(0, 1)$, $\text{N}(0, 1)$ or ST_3 in model (7.2), and set $\delta \in \{0.2, 5\}$ in models (7.3)–(7.4), $\delta \in \{2\pi, 4\pi\}$ in model (7.5). Although u_t does not satisfy $E|u_t| = 1$ in the aforementioned set-up, the estimation property of θ_0 is unaffected by noting that $\varepsilon_t = g_t^{\dagger} u_t^{\dagger}$ with $g_t^{\dagger} = g_t E|u_t|$ and $u_t^{\dagger} = u_t/E|u_t|$. To save space, we only report the results for model (7.3) in what follows, and the results for models (7.4)–(7.5) are similar and can be found in Zhu (2019).

Table 1 reports the sample root mean squared error (SE) and the average bootstrapped sample root mean squared error (AE) of $\hat{\theta}_n$ and $\tilde{\theta}_{an}$, based on 1000 replications. In all calculations, $\tilde{\theta}_{an}$ is obtained by using Gaussian kernel and the CV method in (6.2) to choose b ; and the bootstrapped sample root mean squared errors for $\hat{\theta}_n$ and $\tilde{\theta}_{an}$ are computed by using the RW method with the bootstrap sample size $J = 500$. From Table 1, we can find that (i) the disparity between SE and AE in each case is small, indicating that our RW approach is reliable; (ii) the SE of $\tilde{\theta}_{an}$ is always smaller than the one of $\hat{\theta}_n$ as expected; (iii) the SE of each estimator becomes small as the sample size n increases; (iv) the SE of each estimator in the case of heavy-tailed η_t (i.e., $\eta_t \sim \text{SL}(0, 1)$ or ST_3) is smaller than the corresponding one in the case of light-tailed η_t (i.e., $\eta_t \sim \text{N}(0, 1)$). Besides SE, we also compute the sample median and sample median of absolute deviation of $\hat{\theta}_n$ and $\tilde{\theta}_{an}$ to see their variability, and the details are relegated to Zhu (2019).

Furthermore, we examine the performance of our CV method by calculating the value of the following ratio:

$$R_1 = \frac{\text{SD of } \tilde{\theta}_{an}}{\text{SD of } \hat{\theta}_{an}},$$

where SD stands for the sample standard deviation based on 1000 replications. If our CV method works well, the value of R_1 should be close to one. Meanwhile, we compare the efficiency of $\hat{\theta}_n$, $\tilde{\theta}_{an}$ and the LSE $\check{\theta}_n$ with the infeasible ALSE $\check{\theta}_{an}$ by looking at the values of the following three ratios:

$$R_2 = \frac{\text{SD of } \hat{\theta}_n}{\text{SD of } \check{\theta}_{an}}, \quad R_3 = \frac{\text{SD of } \tilde{\theta}_{an}}{\text{SD of } \check{\theta}_{an}} \quad \text{and} \quad R_4 = \frac{\text{SD of } \check{\theta}_n}{\text{SD of } \check{\theta}_{an}}.$$

Clearly, $\hat{\theta}_n$, $\tilde{\theta}_{an}$ and $\check{\theta}_n$ are more efficient than $\check{\theta}_{an}$ when the values of R_2 , R_3 and R_4 are smaller than one, respectively. Also, the most efficient estimator among $\hat{\theta}_n$, $\tilde{\theta}_{an}$ and $\check{\theta}_n$ is related to the smallest value of R_2 , R_3 and R_4 .

Table 2 reports the values of these four ratios. From this table, we first find that the value of R_1 is close to one, and hence it implies that our CV method works satisfactorily. Second, we can see that $\tilde{\theta}_{an}$ is more efficient than $\check{\theta}_{an}$ (with values of R_3 less than one) in general when η_t is heavy-tailed, and this efficiency advantage in the case of $(\alpha_{\dagger}, \beta_{\dagger}) = (0, 0)$ is less substantial than the one in the case of $(\alpha_{\dagger}, \beta_{\dagger}) = (0.1, 0.8)$; on the other hand, as expected, $\check{\theta}_{an}$ is generally more efficient than $\tilde{\theta}_{an}$ when η_t is light-tailed. For $\hat{\theta}_n$, it can still be more efficient than $\check{\theta}_{an}$ (with values of R_2 less than one) in most of examined cases especially when $\eta_t \sim \text{SL}(0, 1)$, and this is not the case when $\eta_t \sim \text{N}(0, 1)$. Third, we note that $\check{\theta}_n$ is always less efficient than $\tilde{\theta}_{an}$ and $\check{\theta}_{an}$, and it is more efficient than $\hat{\theta}_n$ only when $\eta_t \sim \text{N}(0, 1)$.

In summary, our numerical results in Tables 1–2 show that the RW method can provide reliable estimators of the standard deviations for both $\hat{\theta}_n$ and $\tilde{\theta}_{an}$, and $\tilde{\theta}_{an}$ calculated by the CV method shall be recommended for the heavy-tailed η_t .

Next, we examine the performance of the Wald tests W_{wn} and W_{an} , the sign-based portmanteau tests $S_{wn}(M)$ and $S_{an}(M)$, and the corresponding RW approach in finite samples. We generate 1000 replications of sample size $n = 100$ and 200 from the following model:

$$(7.6) \quad y_t = \phi_{10}y_{t-1} + \phi_{20}y_{t-2} + \varepsilon_t \quad \text{and} \quad \varepsilon_t = g_t u_t,$$

where $(\phi_{10}, \phi_{20}) = (0.5, \kappa)$ with $\kappa = 0, 0.2$ or 0.4 , and the error generating process for ε_t is given by model (7.1). We fit each replication by an AR(2) model with the LADE (or the feasible ALADE) of (ϕ_{10}, ϕ_{20}) , and then apply W_{wn} in (3.2) (or W_{an} in (6.3)) to test the hypothesis of $\phi_{20} = 0$ (i.e., $\Gamma = (0, 1)$, $\theta_0 = (\phi_{10}, \phi_{20})'$ and $r = 0$ in (3.3)). Furthermore, we fit each replication by an AR(1) model with the LADE (or the feasible ALADE) of ϕ_{10} , and then apply $S_{wn}(M)$ in (4.3) (or $S_{an}(M)$ in (6.4)) to check whether this fitted AR(1) model is adequate. In all cases, we set the

TABLE 2
The values of R_i ($i = 1, 2, 3, 4$) for model (7.1) with $g_t \sim$ model (7.3)

α_{\dagger}	β_{\dagger}	δ	n	$\eta_t \sim \text{SL}(0, 1)$				$\eta_t \sim \text{ST}_3$				$\eta_t \sim \text{N}(0, 1)$			
				R_1	R_2	R_3	R_4	R_1	R_2	R_3	R_4	R_1	R_2	R_3	R_4
0.0	0.0	0.2	100	1.0341	1.0375	0.8653	1.4035	1.0914	1.1754	1.0140	1.4778	1.0705	1.6754	1.3871	1.5480
			200	1.0296	0.9774	0.8220	1.3821	1.0583	1.0652	0.9050	1.4160	1.0734	1.6254	1.3449	1.4804
		5	100	1.0302	0.9944	0.8415	1.2922	1.0516	1.1202	0.9909	1.3496	1.0420	1.5078	1.2974	1.3422
			200	1.0284	0.9634	0.8073	1.3432	1.0334	1.0061	0.8879	1.3140	1.0186	1.5142	1.2521	1.3577
0.1	0.8	0.2	100	1.0457	1.0456	0.8581	1.3895	1.0062	1.0844	0.9240	1.3134	1.0428	1.5114	1.2841	1.4655
			200	1.0040	0.9013	0.7696	1.3530	0.9998	1.0026	0.8142	1.3148	1.0281	1.5076	1.3065	1.4204
		5	100	1.0049	0.9727	0.8048	1.2610	0.9939	1.0219	0.8870	1.2561	0.9917	1.4868	1.2356	1.3240
			200	1.0050	0.9728	0.7821	1.3056	0.9922	1.0144	0.8189	1.2813	1.0006	1.4119	1.2257	1.3133

TABLE 3
The power ($\times 100$) of all tests for model (7.6) with $g_t \sim$ model (7.3) and $\eta_t \sim \text{SL}(0, 1)$

α_{\dagger}	β_{\dagger}	δ	n	$\kappa = 0$				$\kappa = 0.2$				$\kappa = 0.4$				
				W_{wn}	W_{an}	$S_w(6)$	$S_a(6)$	W_{wn}	W_{an}	$S_w(6)$	$S_a(6)$	W_{wn}	W_{an}	$S_w(6)$	$S_a(6)$	
0.0	0.0	0.2	100	4.2	4.4	1.8	2.2	45.8	60.6	8.3	9.3	94.5	97.9	42.9	48.8	
			200	4.6	4.7	2.9	3.1	76.8	87.2	26.5	27.8	99.8	99.9	87.2	89.6	
			5	100	3.5	3.8	1.3	2.0	39.0	52.3	7.1	7.6	91.2	96.4	37.6	39.3
				200	4.7	4.9	2.8	2.8	71.9	85.7	25.2	25.0	99.4	99.9	85.0	86.1
0.1	0.8	0.2	100	3.3	3.2	1.7	1.8	42.3	53.5	6.8	8.7	92.7	98.0	43.2	48.6	
			200	5.2	4.4	3.0	2.9	66.6	81.2	23.3	24.6	99.4	99.9	84.5	87.0	
			5	100	4.2	3.2	1.6	1.7	32.0	43.9	6.7	6.9	86.2	94.4	35.4	37.8
				200	4.7	4.4	2.5	3.2	59.9	78.0	22.0	22.1	99.0	99.8	83.2	83.9

significance level $\alpha = 0.05$ and $M = 6$. Based on 1000 replications, the empirical power of all the tests are reported in Table 3 for the case that $\eta_t \sim \text{SL}(0, 1)$, and their sizes correspond to the results for the case that $\kappa = 0$. Since the results for other two distributions of η_t are similar, they are not provided here for saving the space. From Table 3, it is clear that the sizes of the two Wald tests are close to their nominal ones, and the sizes of the two portmanteau tests are conservative especially when n is small. For the power of all tests, we find that all the power becomes large as the value of n or κ increases; the power of W_{an} and S_{an} based on the feasible ALADE is larger than that of W_{wn} and S_{wn} based on the LADE, respectively, and this power advantage is more distinct for the Wald test. Overall, all tests based on the RW approach have a good performance especially when the sample size is large, and we recommend W_{an} and S_{an} based on the feasible ALADE in practice.

8. Concluding remarks. This paper provides an entire statistical inference procedure for the $\text{AR}(p)$ model under (conditional) heteroscedasticity of unknown form by establishing the asymptotic normality of the weighted LADE, developing the RW method to estimate its asymptotic covariance matrix, and constructing a portmanteau test for the model diagnostic checking. This entire procedure can either be based on the usual LADE or the feasible ALADE, and as demonstrated by the simulation results, the ALADE is the better choice in terms of the estimation efficiency and testing power. Applications to three U.S. economic data sets are considered in the Supplementary Material (Zhu (2019)), and the results indicate that after the financial crisis in 2007–2008, the monetary policies may become less prudent, and their control on the PPI and CPI seems to be weaker. The methodology developed in this paper provides a new way to handle the heteroscedastic time series data and shall have a large applicable scope in practice. Extensions to other time series models (e.g., autoregressive moving average models) and estimation

methods (e.g., M- and quantile estimations) could be potentially interesting future works.

APPENDIX A: PROOFS OF THEOREMS

This appendix gives the proofs of all the theorems, and the proofs of all the corollaries are offered in [Zhu \(2019\)](#). Define

$$(A.1) \quad z_n(\xi) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{Y_{t-1}[\text{sgn}(\varepsilon_t)]}{\xi_t} \quad \text{and}$$

$$(A.2) \quad Z_n(\xi) = \sum_{t=1}^n \frac{1}{\xi_t} \int_0^{v'Y_{t-1}/\sqrt{n}} \{I(\varepsilon_t \leq s) - I(\varepsilon_t \leq 0)\} ds,$$

where $v \in \mathcal{R}^{p+1}$. To facilitate the proof of [Theorem 2.1](#), the following two propositions are needed, and their proofs are given in [Zhu \(2019\)](#).

PROPOSITION A.1. *Suppose Assumptions 2.1–2.2 and 2.4–2.5 hold. Then:*

- (i) $\text{plim}_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n w_t^{-2} Y_{t-1} Y'_{t-1} = \Sigma_1$;
- (ii) $\text{plim}_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n (g_t w_t)^{-1} f_t(0) Y_{t-1} Y'_{t-1} = \Sigma_2$,

where Σ_1 and Σ_2 are defined as in [\(2.6\)](#).

PROPOSITION A.2. *Suppose Assumptions 2.1–2.4 hold. Then, $z_n(w) \rightarrow_d N(0, \Sigma_1)$ as $n \rightarrow \infty$, where $z_n(w)$ and Σ_1 are defined as in [\(A.1\)](#) and [\(2.6\)](#), respectively.*

PROOF OF THEOREM 2.1. Denote $H_n(v) = \sum_{t=1}^n \frac{1}{w_t} (|\varepsilon_t - \frac{v'}{\sqrt{n}} Y_{t-1}| - |\varepsilon_t|)$, where $v \in \mathcal{R}^{p+1}$. Then $\hat{v}_{wn} := \sqrt{n}(\hat{\theta}_{wn} - \theta_0)$ is the the minimizer of $H_n(v)$ over \mathcal{R}^{p+1} .

Using the identity,

$$(A.3) \quad |x - y| - |x| = -y \{ \text{sgn}(x) \} + 2 \int_0^y \{ I(x \leq s) - I(x \leq 0) \} ds,$$

which holds when $x \neq 0$ (see [Knight \(1998\)](#)), it follows that

$$(A.4) \quad H_n(v) = -v' z_n(w) + 2Z_n(w),$$

where $z_n(w)$ and $Z_n(w)$ are defined as in [\(A.1\)](#) and [\(A.2\)](#), respectively. Write $Z_n(w) := \sum_{t=1}^n K_t = \sum_{t=1}^n E(K_t | \mathcal{F}_{t-1}) + \sum_{t=1}^n [K_t - E(K_t | \mathcal{F}_{t-1})]$. Subsequently, we can show that for each $v \in \mathcal{R}^{p+1}$,

$$(A.5) \quad \sum_{t=1}^n E(K_t | \mathcal{F}_{t-1}) = v' \frac{\Sigma_2}{2} v + o_p(1),$$

$$(A.6) \quad \sum_{t=1}^n [K_t - E(K_t | \mathcal{F}_{t-1})] = o_p(1);$$

see [Zhu \(2019\)](#) for details. By (A.4)–(A.6), $H_n(v) = -v'z_n(w) + v'\Sigma_2v + o_p(1)$ for each $v \in \mathcal{R}^{p+1}$. As $H_n(v)$ is convex for each v , Theorem 2 in [Kato \(2009\)](#) implies that

$$(A.7) \quad \widehat{v}_{wn} = (2\Sigma_2)^{-1}z_n(w) + o_p(1).$$

Now, the conclusion follows from (A.7) and Proposition A.2. \square

PROOF OF THEOREM 3.1. Denote $H_n^*(v) = \sum_{t=1}^n \frac{w_t^*}{w_t} (|\varepsilon_t - \frac{v'}{\sqrt{n}}Y_{t-1}| - |\varepsilon_t|)$, where $v \in \mathcal{R}^{p+1}$. Then $\widehat{v}_{wn}^* = \sqrt{n}(\widehat{\theta}_{wn}^* - \theta_0)$ is the minimizer of $H_n^*(v)$ over \mathcal{R}^{p+1} . As for (A.4), we can obtain that $H_n^*(v) = -v'z_n^*(w) + 2Z_n^*(w)$, where

$$z_n^*(w) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{w_t^* Y_{t-1} [\text{sgn}(\varepsilon_t)]}{w_t} \quad \text{and}$$

$$Z_n^*(w) = \sum_{t=1}^n \frac{w_t^*}{w_t} \int_0^{v'Y_{t-1}/\sqrt{n}} \{I(\varepsilon_t \leq s) - I(\varepsilon_t \leq 0)\} ds.$$

Since $\{w_t^*\}$ is independent of $\{y_t\}$ with $E(w_t^*) = 1$ by Assumption 3.1, a similar argument as for (A.5)–(A.6) entails that $Z_n^*(w) = v'\frac{\Sigma_2}{2}v + o_p(1)$. Then, for each $v \in \mathcal{R}^{p+1}$, $H_n^*(v) = -v'z_n^*(w) + v'\Sigma_2v + o_p(1)$, which implies that

$$(A.8) \quad \widehat{v}_{wn}^* = (2\Sigma_2)^{-1}z_n^*(w) + o_p(1)$$

by a similar argument as for (A.7). Hence, by (A.7) and (A.8), we have

$$(A.9) \quad \begin{aligned} \sqrt{n}(\widehat{\theta}_{wn}^* - \widehat{\theta}_{wn}) &= \frac{[2\Sigma_2]^{-1}}{\sqrt{n}} \sum_{t=1}^n \frac{(w_t^* - 1)Y_{t-1} \text{sgn}(\varepsilon_t)}{w_t} + o_p(1) \\ &=: [2\Sigma_2]^{-1} \sum_{t=1}^n J_{tn} + o_p(1). \end{aligned}$$

Let E^* be the conditional expectation on $\{y_t\}_{t=1}^n$ and $\lambda \in \mathcal{R}^{p+1}$ be a nonzero constant vector. We now study the conditional distribution of $\sum_{t=1}^n \lambda' J_{tn}$. First, since $\{w_t^*\}$ is independent of $\{y_t\}$ with $E w_t^* = 1$, we have

$$(A.10) \quad E^*[\lambda' J_{tn}] = 0.$$

Next, since $\text{var}(w_t^*) = 1$, by the independence of $\{w_t^*\}$ and $\{y_t\}$, we have

$$(A.11) \quad \begin{aligned} \sum_{t=1}^n E^*[\lambda' J_{tn} J_{tn}' \lambda] &= \lambda' \left\{ \frac{1}{n} \sum_{t=1}^n \frac{Y_{t-1} Y_{t-1}' [\text{sgn}(\varepsilon_t)]^2}{w_t^2} \right\} \lambda \\ &= \lambda' \Sigma_1 \lambda + o_p(1) \end{aligned}$$

by Proposition A.1(i). Finally, we claim that the Lindeberg condition holds by showing that

$$(A.12) \quad \sum_{t=1}^n E^*[\lambda' J_{tn} J'_{tn} \lambda I(|\lambda' J_{tn}| > \eta)] = o_p(1);$$

see Zhu (2019) for details. By (A.10)–(A.12), the Cramér–Wold device and the central limit theorem in Pollard (1984, Theorem VIII.1) yield that conditional on $\{y_t\}_{t=1}^n$, $\sum_{t=1}^n J_{tn} \rightarrow_d N(0, \Sigma_1)$ in probability as $n \rightarrow \infty$. Now, the conclusion follows directly from (A.9). \square

PROOF OF THEOREM 4.1. By Lemma B.4(i) with $s_n = \widehat{v}_{wn}$, we have

$$(A.13) \quad \sqrt{n} \widehat{r}_{wn,k} = \frac{1}{\sqrt{n}} \sum_{t=k+1}^n \text{sgn}(\varepsilon_t(\widehat{\theta}_{wn})) \text{sgn}(\varepsilon_{t-k}(\widehat{\theta}_{wn})) + o_p(1).$$

Rewrite $\text{sgn}(\varepsilon_t(\widehat{\theta}_{wn})) \text{sgn}(\varepsilon_{t-k}(\widehat{\theta}_{wn})) - \text{sgn}(\varepsilon_t) \text{sgn}(\varepsilon_{t-k}) = P_{nt}(\widehat{v}_{wn}) + Q_{nt}(\widehat{v}_{wn})$, where $P_{nt}(v)$ and $Q_{nt}(v)$ are defined as in Lemma B.4. By (A.13) and Lemma B.4(ii)–(iv), it follows that

$$(A.14) \quad \sqrt{n} \widehat{r}_{wn,k} = \left[\frac{1}{\sqrt{n}} \sum_{t=k+1}^n \text{sgn}(\varepsilon_t) \text{sgn}(\varepsilon_{t-k}) \right] - 2\Omega'_{3k} \widehat{v}_{wn} + o_p(1).$$

Hence, by (A.7) and (A.14), we have

$$(A.15) \quad \sqrt{n} \widehat{r}_{wn} = \Sigma_4 S_n + o_p(1),$$

where

$$S_n = \frac{1}{\sqrt{n}} \left(\sum_{t=2}^n \text{sgn}(\varepsilon_t) \text{sgn}(\varepsilon_{t-1}), \dots, \sum_{t=M+1}^n \text{sgn}(\varepsilon_t) \text{sgn}(\varepsilon_{t-M}), \right. \\ \left. \sum_{t=1}^n \frac{Y'_{t-1} \text{sgn}(\varepsilon_t)}{w_t} \right)'.$$

Finally, the conclusion holds by the central limit theorem for m.d.s. and a similar argument as for Proposition A.1. \square

PROOF OF THEOREM 4.2. By Assumption 3.1 and Lemma B.4(i) with $s_n = \widehat{v}_{wn}^*$, we have

$$\sqrt{n} \widehat{r}_{wn,k}^* = \frac{1}{\sqrt{n}} \sum_{t=k+1}^n w_t^* \text{sgn}(\varepsilon_t(\widehat{\theta}_{wn}^*)) \text{sgn}(\varepsilon_{t-k}(\widehat{\theta}_{wn}^*)) + o_p(1).$$

By Assumption 3.1 and a similar argument as for (A.14), we can obtain that

$$(A.16) \quad \sqrt{n} \widehat{r}_{wn,k}^* = \left[\frac{1}{\sqrt{n}} \sum_{t=k+1}^n w_t^* \text{sgn}(\varepsilon_t) \text{sgn}(\varepsilon_{t-k}) \right] - 2\Omega'_{3k} \widehat{v}_{wn}^* + o_p(1).$$

Hence, by (A.8) and (A.16), we have

$$(A.17) \quad \sqrt{n}\widehat{r}_{wn}^* = \Sigma_4 S_n^* + o_p(1),$$

where

$$S_n^* = \frac{1}{\sqrt{n}} \left(\sum_{t=2}^n w_t^* \operatorname{sgn}(\varepsilon_t) \operatorname{sgn}(\varepsilon_{t-1}), \dots, \sum_{t=M+1}^n w_t^* \operatorname{sgn}(\varepsilon_t) \operatorname{sgn}(\varepsilon_{t-M}), \right. \\ \left. \sum_{t=1}^n \frac{w_t^* Y'_{t-1} \operatorname{sgn}(\varepsilon_t)}{w_t} \right)'.$$

Finally, the conclusion holds by (A.15), (A.17) and a similar argument as for Theorem 3.1. \square

Following a similar terminology as in Robinson (1987), we let

$$\widetilde{g}_t = \sum_{i=1}^n k_{ti} |\varepsilon_i| \quad \text{and} \quad \bar{g}_t = \sum_{i=1}^n k_{ti} g_i.$$

To prove Theorems 6.1–6.3, two crucial propositions are needed, and their proofs are given in Zhu (2019).

PROPOSITION A.3. *Suppose Assumptions 2.1–2.5, 6.1–6.2 and 6.3(i) hold. Then:*

- (i) $z_n(\widehat{g}) - z_n(\widetilde{g}) = O_p(\frac{1}{\sqrt{nb}})$;
- (ii) $z_n(\widetilde{g}) - z_n(\bar{g}) = O_p(\frac{1}{\sqrt{nb}^{\frac{3+\delta_4}{2}}} + \frac{1}{nb^{3+\delta_4}} + b^{\delta_4})$ for some $\delta_4 > 0$;
- (ii') $z_n(\widetilde{g}) - z_n(\bar{g}) = O_p(\frac{1}{\sqrt{nb}})$ if $E[|u_t| | \mathcal{F}_{t-1}] = 1$;
- (iii) $z_n(\bar{g}) - z_n(g) = o_p(1)$,

where $z_n(\cdot)$ is defined as in (A.1).

PROPOSITION A.4. *Suppose the conditions in Proposition A.3 hold. Then, for each $v \in \mathcal{R}^{p+1}$:*

- (i) $Z_n(\widehat{g}) - Z_n(\widetilde{g}) = O_p(\frac{1}{n^{3/4}b})$;
- (ii) $Z_n(\widetilde{g}) - Z_n(\bar{g}) = O_p(\frac{1}{n^{1/4}b})$;
- (ii') $Z_n(\widetilde{g}) - Z_n(\bar{g}) = O_p(\frac{1}{n^{1/4}b^{1/2}})$ if $E[|u_t| | \mathcal{F}_{t-1}] = 1$;
- (iii) $Z_n(\bar{g}) - Z_n(g) = o_p(1)$,

where $Z_n(\cdot)$ is defined as in (A.2).

PROOF OF THEOREM 6.1. Denote $\widehat{G}_n(v) = \sum_{t=1}^n \frac{1}{g_t} (|\varepsilon_t - \frac{v'}{\sqrt{n}} Y_{t-1}| - |\varepsilon_t|)$, where $v \in \mathcal{R}^{p+1}$. Then $\widetilde{v}_{an} := \sqrt{n}(\widetilde{\theta}_{an} - \theta_0)$ is the minimizer of $\widehat{G}_n(v)$ over \mathcal{R}^{p+1} .

As for (A.4), $\widehat{G}_n(v) = -v'z_n(\widehat{g}) + 2Z_n(\widehat{g})$. By Assumption 6.3(ii) and Propositions A.3–A.4, it follows that for each $v \in \mathcal{R}^{p+1}$, $\widehat{G}_n(v) = -v'z_n(g) + 2Z_n(g) + o_p(1)$. Hence, as for (A.7), we can deduce that

$$(A.18) \quad \widetilde{v}_{an} = (2\Sigma_{a2})^{-1}z_n(g) + o_p(1),$$

and the conclusion follows by Proposition A.2. \square

PROOF OF THEOREM 6.2. Denote $\widehat{G}_n^*(v) = \sum_{t=1}^n \frac{w_t}{\widehat{g}_t} (|\varepsilon_t - \frac{v'}{\sqrt{n}}Y_{t-1}| - |\varepsilon_t|)$, where $v \in \mathcal{R}^{p+1}$. Then $\widetilde{v}_{an}^* := \sqrt{n}(\widehat{\theta}_{an}^* - \theta_0)$ is the minimizer of $\widehat{G}_n^*(v)$ over \mathcal{R}^{p+1} . As for (A.4), $\widehat{G}_n^*(v) = -v'z_n^*(\widehat{g}) + 2Z_n^*(\widehat{g})$. By Assumptions 3.1 and 6.3(ii) and similar arguments as for Propositions A.3–A.4, it follows that for each $v \in \mathcal{R}^{p+1}$, $\widehat{G}_n^*(v) = -v'z_n^*(g) + 2Z_n^*(g) + o_p(1)$. Hence, as for (A.8), we can deduce that

$$(A.19) \quad \widetilde{v}_{an}^* = (2\Sigma_{a2})^{-1}z_n^*(g) + o_p(1).$$

Now, the conclusion holds by (A.18)–(A.19) and the similar arguments as for Theorem 3.1. \square

PROOF OF THEOREM 6.3. By noting that $\widetilde{v}_{an} = O_p(1)$ and $\widetilde{v}_{an}^* = O_p(1)$, the proofs of (i) and (ii) follow the same arguments as for Theorems 4.1 and 4.2, respectively. \square

APPENDIX B: SOME TECHNICAL LEMMAS

This appendix gives Lemmas B.1–B.6, which can be found in the Supplementary Material [Zhu \(2019\)](#).

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SUPPLEMENTARY MATERIAL

Supplement to “Statistical inference for autoregressive models under heteroscedasticity of unknown form” (DOI: [10.1214/18-AOS1775SUPP](https://doi.org/10.1214/18-AOS1775SUPP); .pdf). The supplement includes additional simulation results, applications, some technical lemmas and the remaining proofs.

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