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ON MAXIMA OF PERIODOGRAMS OF STATIONARY PROCESSES[1]

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We consider the limit distribution of maxima of periodograms for stationary processes. Our method is based on m-dependent approximation for stationary processes and a moderate deviation result.

1. Introduction. Let $\{\varepsilon_n; n \in Z\}$ be independent and identically distributed (i.i.d.) random variables and g be a measurable function such that

$$X_n = g(\dots, \varepsilon_{n-1}, \varepsilon_n) \tag{1.1}$$

is a well-defined random variable. Then, $\{X_n; n \in Z\}$ presents a huge class of processes. In particular, it contains the linear process and nonlinear processes including the threshold AR (TAR) models, ARCH models, random coefficient AR (RCA) models, exponential AR (EAR) models and so on. Wu and Shao [21] argued that many nonlinear time series are stationary causal with one-sided representation (1.1). Let

$$I_{n,X}(\omega) = n^{-1}\left|\sum_{k=1}^{n} X_k \exp(\mathrm{i}\omega k)\right|^2, \qquad \omega \in [0, \pi],$$

be the periodogram of random variables $X_1, \dots, X_n$ and denote

$$M_n(X) = \max_{1 \le j \le q} I_{n,X}(\omega_j), \qquad \omega_j = 2\pi j/n,$$

where $q = q_n = \max\{j : 0 < \omega_j < \pi\}$ so that $q \sim n/2$.

If $X_1, X_2, \dots$ are i.i.d. random variables with $N(0, 1)$ distribution, then $\{I_{n,X}(\omega_j); 1 \le j \le q\}$ is a sequence of i.i.d. standard exponential random variables. It is well known that (cf. Brockwell and Davis [2])

$$M_n(X) - \log q \Rightarrow G, \tag{1.2}$$

where $\Rightarrow$ means convergence in distribution, and G has the standard Gumbel distribution $\Lambda(x) = \exp(-\exp(-x))$, $x \in R$. However, in the non-Gaussian case, the

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independence of $I_{n,X}(\omega_j)$ is not guaranteed in general, and, therefore, (1.2) is not trivial. When $X_1, X_2, \dots$ are i.i.d. random variables, Davis and Mikosch [4] established (1.2) with the assumptions that $\mathsf{E}X_1 = 0$, $\mathsf{E}X_1^2 = 1$ and $\mathsf{E}|X_1|^s < \infty$ for some $s > 2$. They also conjectured that the condition $\mathsf{E}X_1^2 \log^+ |X_1| < \infty$ is sufficient for (1.2). Moreover, a similar result was established in their paper for the two-sided linear process $X_n = \sum_{j\in Z} a_j \varepsilon_{n-j}$ under the conditions that $\mathsf{E}|\varepsilon_0|^s < \infty$ for some $s > 2$ and

$$\sum_{j\in Z} |j|^{1/2} |a_j| < \infty. \tag{1.3}$$

The key step in Davis and Mikosch [4] is the approximation that (cf. Walker [17])

$$\max_{\omega\in[0,\pi]} \left| \frac{I_{n,X}(\omega)}{2\pi f(\omega)} - I_{n,\varepsilon}(\omega) \right| \to_{\mathsf{P}} 0. \tag{1.4}$$

Generally, it is very difficult to check (1.4) for the stationary process defined in (1.1). In this paper, we shall establish (1.2), or an analogous result, for (1.1) under some regularity conditions. Let us take a look at the linear process first. In this case, $X_n = \sum_{j=-m}^{m} a_j \varepsilon_{n-j} + \sum_{|j|>m} a_j \varepsilon_{n-j}$, $m > 0$. Under the assumptions of $\sum_{j\in Z} |a_j| < \infty$ and $\mathsf{E}|\varepsilon_0| < \infty$, $\sum_{|j|>m} a_j \varepsilon_{n-j} \to 0$, in probability, as $m \to \infty$. This implies that the linear process behaves like a process that is block-wise independent. In fact, many time series, such as the GARCH model, have such a property. Such an analysis suggests that we approximate X_n by $\mathsf{E}[X_n|\varepsilon_{n-m}, \dots, \varepsilon_n]$. This method has been employed in Hsing and Wu [11] to establish the asymptotic normality of a weighted U-statistic.

By the m-dependent approximation developed in Section 3, we show that, for proving (1.2), the condition (1.3) can be weakened to $\sum_{|j|\ge n} |a_j| = o(1/\log n)$. Meanwhile, the moment condition on ε_0 can also be weakened to $\mathsf{E}\varepsilon_0^2 I\{|\varepsilon_0| \ge n\} = o(1/\log n)$. This in turn proves that the conjecture by Davis and Mikosch [4] is true. Furthermore, it is shown that (1.2) still holds for the general process defined in (1.1).

Below, we explain how (1.2) (or the analogous result) can be used for detecting periodic components in a time series (see also Priestley [14]). Let us consider the model

$$Z_t = \mu + S(t) + X_t, \qquad t = 1, 2, \dots, n,$$

where X_t is a stationary time series with mean zero, and the deterministic part

$$S(t) = A_1 \cos(\gamma_1 t + \phi_1)$$

is a sinusoidal wave at frequency $\gamma_1 \neq 0$ with the amplitude $A_1 \neq 0$ and the phase ϕ_1. Without loss of generality, we assume that $\mu = 0$. A test statistic for the null hypothesis $H_0 : S(t) \equiv 0$ against the alternative $H_1 : S(t) = A_1 \cos(\gamma_1 t + \phi_1)$ is

$$g_n(Z) = \frac{\max_{1\le i\le q} I_{n,Z}(\omega_i)/\hat{f}(\omega_i)}{q^{-1} \sum_{i=1}^{q} I_{n,Z}(\omega_i)/\hat{f}(\omega_i)}, \tag{1.5}$$

where $\hat{f}(\omega)$ is an estimator of $f(\omega)$, which is the spectral density of Z_t. This statistic was proposed by Fisher [6], who assumed that X_t is a white Gaussian series and thus chose $\hat{f}(\omega) \equiv 1$. Often, however, it is not reasonable, as a null hypothesis, to assert that the observations are independent. Hence, Hannan [9] assumed that $X_t = \sum_{j \in Z} a_j \varepsilon_{t-j}$ with ε_t being i.i.d. normal and $\{a_j\}$ satisfying some conditions. The results in Section 2 make it possible to obtain the asymptotic distribution of $g_n(Z)$ under H_0 for a class of general processes rather than the linear process and without the requirement of the normality for ε_t (see Remark 2.4 for more details).

Sometimes, we might suspect that the series might contain several periodic components. In this case, we should test $H_0 : S(t) \equiv 0$ against the alternative $H_1 : S(t) = \sum_{k=1}^{r} A_k \cos(\gamma_k t + \phi_k)$, where $r (> 1)$ is the possible number of peaks. Assuming that X_t is a white Gaussian series, Shimshoni [16] and Lewis and Fieller [8] proposed the statistic

$$U_Z(r) = \frac{I_{n,q-r+1}(Z)}{\sum_{i=1}^{q} I_{n,Z}(\omega_i)}$$

for detecting r peaks. Here, $I_{n,1}(Z) \le I_{n,2}(Z) \le \cdots \le I_{n,q}(Z)$ are the order statistics of the periodogram ordinates $I_{n,Z}(\omega_i)$, $1 \le i \le q$. The exact (and asymptotic) null distribution of $U_Z(r)$ can be found in Hannan [10] and Chiu [3]. In the latter paper, the test statistic $R_Z(\beta) = I_{n,q}(Z) / \sum_{j=1}^{[q\beta]} I_{n,j}(Z)$, $0 < \beta < 1$, was given. Our results may be useful for obtaining the asymptotic distribution of $R_Z(\beta)$, when X_n is defined in (1.1).

The paper is organized as follows. Our main results, Theorems 2.1 and 2.2, will be presented in Section 2. In Section 3, we develop the m-dependent approximation for the Fourier transforms of stationary processes. The proofs of main results will be given in Sections 4 and 5. Throughout the paper, we let C, $C_{(\cdot)}$ denote positive constants, and their values may be different in different contexts. When δ appears, it usually means every $\delta > 0$ and may be different in every place. For two real sequences $\{a_n\}$ and $\{b_n\}$, write $a_n = O(b_n)$ if there exists a constant C such that $|a_n| \le C|b_n|$ holds for large n, $a_n = o(b_n)$ if $\lim_{n \to \infty} a_n / b_n = 0$ and $a_n \asymp b_n$ if $C_1 b_n \le a_n \le C_2 b_n$. With no confusion, we let $|\cdot|$ denote the d-dimensional Euclidean norm ($d \ge 1$) or the norm of a $d \times d$ matrix A, which is defined by $|\mathrm{A}| = \max_{|x| \le 1, x \in R^d} |\mathrm{A}x|$.

2. Main results.

We first consider the two-sided linear process. Let

$$Y_n = \sum_{j \in Z} a_j \varepsilon_{n-j} \quad \text{and} \quad X_n = h(Y_n) - \mathsf{E} h(Y_n), \tag{2.1}$$

where $\sum_{j \in Z} |a_j| < \infty$ and h is a Lipschitz continuous function. Let us redefine

$$I_{n,1}(X) \le I_{n,2}(X) \le \cdots \le I_{n,q}(X)$$

as the order statistics of the periodogram ordinates $I_{n,X}(\omega_j)/(2\pi f(\omega_j))$, $1 \le j \le q$, where $f(\omega)$ is the spectral density function of $\{X_n\}$, which is defined by

$$f(\omega) = \frac{1}{2\pi} \sum_{k \in Z} \mathsf{E} X_0 X_k \exp(\mathrm{i}k\omega)$$

and satisfies

$$f^* := \min_{\omega \in R} f(\omega) > 0. \tag{2.2}$$

Note that $f(\omega) \equiv \mathsf{E}X_1^2/(2\pi)$, if $X_1, X_2, \dots$ are i.i.d. centered random variables.

THEOREM 2.1. *Let X_n be defined in* (2.1). *Suppose that* (2.2) *holds and*

$$\mathsf{E}\varepsilon_0 = 0, \qquad \mathsf{E}\varepsilon_0^2 = 1 \quad \textit{and} \quad \sum_{|j| \ge n} |a_j| = o(1/\log n). \tag{2.3}$$

(i) *Suppose that $h(x) = x$ and*

$$\mathsf{E}\varepsilon_0^2 I\{|\varepsilon_0| \ge n\} = o(1/\log n). \tag{2.4}$$

Then,

$$I_{n,q}(X) - \log q \Rightarrow G, \tag{2.5}$$

where G has the standard Gumbel distribution $\Lambda(x) = \exp(-\exp(-x))$, $x \in R$.

(ii) *Suppose that h is a Lipschitz continuous function on R. If* (2.4) *is strengthened to* $\mathsf{E}\varepsilon_0^2 I\{|\varepsilon_0| \ge n\} = o(1/(\log n)^2)$, *then* (2.5) *holds.*

REMARK 2.1. From Theorem 2.1, we derive the asymptotic distribution of the maximum of the periodogram. Note that (2.4) is implied by $\mathsf{E}\varepsilon_0^2 \log^+ |\varepsilon_0| < \infty$. Hence, the conjecture in Davis and Mikosch [4] is true. In order to show $\max_{1 \le j \le q} I_{n,X}(\omega_j)/(2\pi f(\omega_j)) - \log q \Rightarrow G$ when $X_n = \sum_{j \in Z} a_j \varepsilon_{n-j}$, Davis and Mikosch [4] used the approximation

$$\max_{\omega \in [0,\pi]} \left| \frac{I_{n,X}(\omega)}{2\pi f(\omega)} - I_{n,\varepsilon}(\omega) \right| \to_{\mathsf{P}} 0, \tag{2.6}$$

which requires the condition (1.3). Obviously, our condition in (2.3) is weaker than (1.3). They also required $\mathsf{E}|\varepsilon_0|^s < \infty$ for some $s > 2$, which is stronger than (2.4). Moreover, it is difficult to prove (2.6) for the nonlinear transforms of linear processes considered in (ii).

REMARK 2.2. The (weak) law of logarithm for the maximum of the periodogram is a simple consequence of Theorem 2.1. Under conditions on the smoothness of the characteristic function of ε_n, An, Chen and Hannan [1] proved the (a.s.) law of logarithm for the maximum of the periodogram.

In the following, we will give a theorem when X_n satisfies the general form in (1.1). Of course, we should impose some dependency conditions on X_n. For the reader's convenience, we list the following notation:

- $\mathcal{F}_{i,j} := (\varepsilon_i, \dots, \varepsilon_j)$, $-\infty \le i \le j \le \infty$;
- $Z \in L^p$ if $\|Z\|_p := (\mathsf{E}|Z|^p)^{1/p} < \infty$;
- $\{\varepsilon_i^*, i \in Z\}$ is an independent copy of $\{\varepsilon_i, i \in Z\}$;
- $\theta_{n,p} := \|X_n - X_n^*\|_p$, where $X_n^* = g(\dots, \varepsilon_{-1}, \varepsilon_0^*, \mathcal{F}_{1,n})$;
- $\Theta_{n,p} := \sum_{i \ge n} \theta_{i,p}$.

REMARK 2.3. $\theta_{n,p}$ is called the physical dependence measure by Wu [19]. An advantage of such a dependence measure is that it is easily verifiable.

THEOREM 2.2. *Let X_n be defined as in* (1.1), *and let* (2.2) *hold. Suppose that* $\mathsf{E}X_0 = 0$, $\mathsf{E}|X_0|^s < \infty$ *for some* $s > 2$ *and* $\Theta_{n,s} = o(1/\log n)$. *Then*, (2.5) *holds*.

REMARK 2.4. To derive the asymptotic distribution (under H_0) of $g_n(Z)$ defined in (1.5) from Theorem 2.2, we should prove that

$$\left| q^{-1} \sum_{i=1}^{q} I_{n,Z}(\omega_i)/(2\pi f(\omega_i)) - 1 \right| = o_{\mathsf{P}}(1/\log n) \tag{2.7}$$

and choose $\hat{f}(\omega)$, an estimator of $f(\omega)$, to satisfy

$$\max_{1 \le j \le q} |\hat{f}(\omega_j) - f(\omega_j)| = o_{\mathsf{P}}(1/\log n). \tag{2.8}$$

Note that, under H_0, we have $Z_n = X_n$. For briefness, we assume that X_n satisfies $\mathsf{E}|X_n|^{4+\gamma} < \infty$ for some $\gamma > 0$, and the geometric-moment contraction (GMC) condition $\theta_{n,4+\gamma} = O(\rho^n)$ for some $0 < \rho < 1$ holds. Many nonlinear time series models (e.g., GARCH models, generalized random coefficient autogressive models, nonlinear AR models and bilinear models) satisfy GMC (see Section 5 in Shao and Wu [15] for more details). By Lemma A.4 in Shao and Wu [15], we have

$$\max_{j,k \le q} |\mathsf{Cov}(I_{n,X}(\omega_k), I_{n,X}(\omega_j)) - f(\omega_j)\delta_{j,k}| = O(1/n), \tag{2.9}$$

where $\delta_{j,k} = I_{j=k}$, and it follows that

$$q^{-1} \sum_{i=1}^{q} \big(I_{n,X}(\omega_i) - \mathsf{E}I_{n,X}(\omega_i)\big)/f(\omega_i) = O_{\mathsf{P}}\big(1/\sqrt{n}\big).$$

Moreover, since $I_{n,X}(\omega) = n^{-1} \sum_{k=-n+1}^{n-1} \sum_{t=1}^{n-|k|} X_t X_{t+|k|} \exp(-\mathrm{i}k\omega)$, we see that $\max_{\omega \in R} |\frac{\mathsf{E}I_{n,X}(\omega)}{2\pi f(\omega)} - 1| = O(1/n)$. This implies (2.7).

Now, we choose the estimator

$$\hat{f}(\omega) = \frac{1}{2\pi} \sum_{k=-B_n}^{B_n} \hat{r}(k) a(k/B_n) \exp(-\mathrm{i}k\omega),$$

where $\hat{r}(k) = n^{-1} \sum_{j=1}^{n-|k|} X_j X_{j+|k|}$, $|k| < n$, $a(\cdot)$ is an even Lipschitz continuous function, with support $[-1, 1]$, $a(0) = 1$ and $a(x) - 1 = O(x^2)$ as $x \to 0$, and B_n

is a sequence of positive integers with $B_n \to \infty$ and $B_n/n \to 0$. Now, suppose $B_n = O(n^\eta)$, $0 < \eta < \gamma/(4+\gamma)$, $0 < \gamma < 4$. Then, Theorem 3.2 in Shao and Wu [15] gives

$$\max_{\omega\in[0,\pi]} |\hat{f}(\omega) - \mathsf{E}\hat{f}(\omega)| = O_{\mathsf{P}}\big(\sqrt{B_n(\log n)/n}\big).$$

Moreover, simple calculations, as in Woodroofe and Van Ness [18], imply $\max_{\omega\in[0,\pi]} |\mathsf{E}\hat{f}(\omega) - f(\omega)| = O(B_n^{-2})$. Hence, (2.8) holds by letting $B_n \asymp n^\eta$, $0 < \eta < \gamma/(4+\gamma)$. Finally, Theorem 2.2 together with (2.7) and (2.8) yields, under H_0, $g_n(Z) - \log q \Rightarrow G$, where G has the standard Gumbel distribution.

3. Inequalities for Fourier transforms of stationary process. In this section, we prove some inequalities for X_n defined in (1.1). Suppose that $\mathsf{E}X_0 = 0$ and $\mathsf{E}X_0^2 < \infty$. Note that

$$X_n = \sum_{j\in Z}(\mathsf{E}[X_n|\mathcal{F}_{-j,\infty}] - \mathsf{E}[X_n|\mathcal{F}_{-j+1,\infty}]) =: \sum_{j\in Z}\mathcal{P}_j(X_n).$$

By virtue of Hölder's inequality, we have, for $u \geq 0$,

$$|r(u)| = |\mathsf{E}X_0X_u| = \left|\sum_{j\in Z}\mathsf{E}\mathcal{P}_j(X_0)\mathcal{P}_j(X_u)\right| \leq \sum_{j=0}^{\infty}\theta_{j,2}\theta_{u+j,2}, \tag{3.1}$$

and, hence, $\sum_{u\geq n}|r(u)| \leq \Theta_{0,2}\Theta_{n,2}$.

Next, we approximate the Fourier transforms of X_n by the sum of m-dependent random variables. Set

$$X_k(m) = \mathsf{E}[X_k|\varepsilon_{k-m},\dots,\varepsilon_k], \qquad k\in Z,\ m\geq 0.$$

LEMMA 3.1. *Suppose that $\mathsf{E}|X_0|^p < \infty$ for some $p \geq 2$ and $\Theta_{0,p} < \infty$. We have*

$$\sup_{\omega\in R}\mathsf{E}\left|\sum_{k=1}^{n}\big(X_k - X_k(m)\big)\exp(\mathrm{i}\omega k)\right|^p \leq C_p n^{p/2}\Theta_{m,p}^p,$$

where C_p is a constant only depending on p.

REMARK 3.1. This lemma, together with Proposition 1 in Wu [20], would lead to the maximal inequality, for $p > 2$,

$$\sup_{\omega\in R}\mathsf{E}\max_{1\leq j\leq n}\left|\sum_{k=1}^{j}\big(X_k - X_k(m)\big)\exp(\mathrm{i}\omega k)\right|^p \leq C_p n^{p/2}\Theta_{m,p}^p.$$

PROOF OF LEMMA 3.1. We decompose $X_k - X_k(m)$ as

$$X_k - X_k(m) = \sum_{j=-k+m}^{\infty} (\mathsf{E}[X_k|\mathcal{F}_{-j-1,k}] - \mathsf{E}[X_k|\mathcal{F}_{-j,k}]) =: \sum_{j=-k+m}^{\infty} R_{k,j}.$$

Therefore,

$$\sum_{k=1}^{n} \{X_k - X_k(m)\} \exp(\mathrm{i}\omega k) = \sum_{j=-n+m}^{\infty} \sum_{k=1\vee(-j+m)}^{n} R_{k,j} \exp(\mathrm{i}\omega k).$$

For every fixed n and m, $\{\sum_{k=1\vee(-j+m)}^{n} R_{k,j} \exp(\mathrm{i}\omega k), j \geq -n+m\}$ is a sequence of martingale differences. Hence, by the Marcinkiewicz–Zygmund–Burkholder inequality,

$$\begin{aligned} &\mathsf{E}\left|\sum_{j=-n+m}^{\infty} \sum_{k=1\vee(-j+m)}^{n} R_{k,j} \exp(\mathrm{i}\omega k)\right|^p \\ &\quad \leq C_p\left(\sum_{j=-n+m}^{\infty}\left(\sum_{k=1\vee(-j+m)}^{n} \|R_{k,j}\|_p\right)^2\right)^{p/2} \\ &\quad \leq C_p\left(\sum_{j=-n+m}^{\infty}\left(\sum_{k=1\vee(-j+m)}^{n} \theta_{j+1+k,p}\right)^2\right)^{p/2} \leq C_p n^{p/2} \Theta_{m,p}^p. \end{aligned}$$

This proves the lemma. □

Letting $m = 0$ in Lemma 3.1 and noting that $X_1(0), X_2(0), \ldots$ are i.i.d. random variables, we obtain the following moment inequalities.

LEMMA 3.2. *Under the conditions of Lemma 3.1, we have, for $p \geq 2$,*

$$\mathsf{E}\left|\sum_{k=1}^{n} X_k \exp(\mathrm{i}k\omega)\right|^p \leq Cn^{p/2} \quad \textit{and} \quad \mathsf{E}\left|\sum_{k=1}^{n} X_k(m) \exp(\mathrm{i}k\omega)\right|^p \leq Cn^{p/2},$$

where C is a constant that does not depend on ω and m.

Define $S_{n,j,1} = \sum_{k=1}^{n} X_k \cos(k\omega_j)$, $S_{n,j,2} = \sum_{k=1}^{n} X_k \sin(k\omega_j)$, $1 \leq j \leq q$.

LEMMA 3.3. *Suppose that $\mathsf{E}X_0 = 0$, $\mathsf{E}X_0^2 < \infty$ and $\Theta_{0,2} < \infty$. Then:*

(i) $$\max_{1\leq j\leq q}\left|\frac{\mathsf{E}S_{n,j,1}^2}{\pi n f(\omega_j)} - 1\right| \leq Cn^{-1}\sum_{k=0}^{n} \Theta_{k,2};$$

(ii) $$\max_{1\leq j\leq q}\left|\frac{\mathsf{E}S_{n,j,2}^2}{\pi n f(\omega_j)} - 1\right| \leq Cn^{-1}\sum_{k=0}^{n} \Theta_{k,2};$$

(iii) $\max_{1\le i,j\le q} |\mathsf{E}S_{n,i,1}S_{n,j,2}| \le C\sum_{k=0}^{n}\Theta_{k,2}$ *and* $\max_{1\le i\ne j\le q}|\mathsf{E}S_{n,i,l}\times S_{n,j,l}| \le C\sum_{k=0}^{n}\Theta_{k,2}$ *for* $l=1,2$.

PROOF. We only prove (i), since the others can be obtained in an analogous way. We recall the following propositions on the trigonometric functions:

(1) $\sum_{k=1}^{n}\cos(\omega_j k)\cos(\omega_l k) = \delta_{j,l}n/2$;
(2) $\sum_{k=1}^{n}\sin(\omega_j k)\sin(\omega_l k) = \delta_{j,l}n/2$;
(3) $\sum_{k=1}^{n}\cos(\omega_j k)\sin(\omega_l k) = 0$.

By applying the above propositions, it is readily seen that

$$\begin{aligned}\frac{\mathsf{E}S_{n,j,1}^2}{n} &= \frac{1}{2}\mathsf{E}X_1^2 + 2n^{-1}\sum_{k=2}^{n}\sum_{i=1}^{k-1}\mathsf{E}X_kX_i\cos(k\omega_j)\cos(\mathrm{i}\omega_j)\\ &= \frac{1}{2}\mathsf{E}X_1^2 + 2n^{-1}\sum_{k=1}^{n-1}r(k)\sum_{i=1}^{n-k}\cos(\mathrm{i}\omega_j)\cos\bigl((i+k)\omega_j\bigr)\\ &= \frac{1}{2}\mathsf{E}X_1^2 + \sum_{k=1}^{n-1}r(k)\cos(k\omega_j)\\ &\quad - 2n^{-1}\sum_{k=1}^{n-1}r(k)\sum_{i=n-k+1}^{n}\cos(\mathrm{i}\omega_j)\cos\bigl((i+k)\omega_j\bigr),\end{aligned}$$

which, together with (3.1) and the Abel lemma, implies

$$\begin{aligned}\left|\frac{\mathsf{E}S_{n,j,1}^2}{\pi n f(\omega_j)} - 1\right| &\le C\sum_{k=n}^{\infty}|r(k)| + Cn^{-1}\sum_{k=1}^{n-1}k|r(k)|\\ &\le C\Theta_{n,2} + Cn^{-1}\sum_{j=0}^{\infty}\theta_{j,2}\sum_{k=1}^{n}k(\Theta_{k+j,2}-\Theta_{k+j+1,2})\\ &\le Cn^{-1}\sum_{k=0}^{n}\Theta_{k,2}.\end{aligned}$$

The proof of the lemma is complete. □

Let $m=[n^\beta]$ for some $0<\beta<1$ and $J_{n,X}(\omega) = |\sum_{k=1}^{n}\{X_k - X_k(m)\}\times \exp(\mathrm{i}\omega k)|$.

LEMMA 3.4. *Suppose that* $\mathsf{E}X_0^2<\infty$ *and* $\Theta_{n,2}=o(1/\log n)$. *We have*, *for any* $0<\beta<1$,

$$\max_{1\le i\le q} J_{n,X}(\omega_i) = o_{\mathsf{P}}\bigl(\sqrt{n/\log n}\bigr).$$

PROOF. Since $\Theta_{m,2} = o((\log n)^{-1})$, there exists a sequence $\{\gamma_n\}$ with $\gamma_n > 0$ and $\gamma_n \to 0$ such that $\Theta_{m,2} \le \gamma_n (\log n)^{-1}$. By the decomposition used in the proof of Lemma 3.1, $J_{n,X}(\omega) = |\sum_{j=-n+m}^{\infty} \sum_{k=1\vee(m-j)}^{n} R_{k,j} \exp(\mathrm{i}k\omega)|$. Set

$$R_j(\omega) = \sum_{k=1\vee(m-j)}^{n} R_{k,j} \exp(\mathrm{i}k\omega),$$

$$\widetilde{R}_j(\omega) = R_j(\omega) I\left\{|R_j(\omega)| \le \gamma_n \sqrt{\frac{n}{(\log n)^3}}\right\},$$

$$\overline{R}_j(\omega) = \widetilde{R}_j(\omega) - \mathsf{E}[\widetilde{R}_j(\omega)|\mathcal{F}_{-j,\infty}], \qquad \widehat{R}_j(\omega) = R_j(\omega) - \overline{R}_j(\omega).$$

Using the fact $\max_{\omega\in R} |R_j(\omega)| \le \sum_{k=1\vee(m-j)}^{n} |R_{k,j}|$, we see that, for any $\delta > 0$,

$$\begin{aligned}
&\mathsf{P}\left(\max_{\omega\in R}\left|\sum_{j=-n+m}^{\infty} \widehat{R}_j(\omega)\right| \ge \delta\sqrt{n/\log n}\right) \\
&\quad\le C_\delta n^{-1/2}(\log n)^{1/2} \sum_{j=-n+m}^{\infty} \mathsf{E}\max_{\omega\in R} |\widehat{R}_j(\omega)| \\
&\quad\le 2C_\delta \frac{(\log n)^2 \gamma_n^{-1}}{n} \sum_{j=-n+m}^{\infty} \left(\sum_{k=1\vee(m-j)}^{n} \theta_{k+j+1,2}\right)^2 \\
&\quad\le 2C_\delta (\log n)^2 \gamma_n^{-1} \Theta_{m,2}^2 = o(1).
\end{aligned}$$

Hence, in order to prove the lemma, it is sufficient to show that

$$\max_{1\le i\le q}\left|\sum_{j=-n+m}^{\infty} \overline{R}_j(\omega_i)\right| = o_{\mathsf{P}}\big(\sqrt{n/\log n}\big). \tag{3.2}$$

Setting the event $A = \{\max_{\omega\in R} \sum_{j=-n+m}^{\infty} \mathsf{E}[|\overline{R}_j(\omega)|^2|\mathcal{F}_{-j,\infty}] \ge \gamma_n n/(\log n)^2\}$, we have

$$\begin{aligned}
\mathsf{P}(A) &\le C_\delta \frac{(\log n)^2 \gamma_n^{-1}}{n} \sum_{j=-n+m}^{\infty} \mathsf{E}\left(\sum_{k=1\vee(m-j)}^{n} |R_{k,j}|\right)^2 \\
&\le C_\delta (\log n)^2 \gamma_n^{-1} \Theta_{m,2}^2 = o(1).
\end{aligned}$$

Note that $\overline{R}_j(\omega)$, $j \ge -n+m$, are martingale differences. By applying Freedman's inequality [7], one concludes that

$$\begin{aligned}
&\mathsf{P}\left(\max_{1\le i\le q}\left|\sum_{j=-n+m}^{\infty} \overline{R}_j(\omega_i)\right| \ge \delta\sqrt{n/\log n}\right) \\
&\quad\le 2n\exp\left(-\frac{\delta^2 \log n}{\gamma_n(8+8\delta)}\right) + \mathsf{P}(A) = o(1).
\end{aligned}$$

This proves (3.2). □

REMARK 3.2. Let $X_n = g((\varepsilon_{n-i})_{i\in Z})$ be a two-sided process. For $n \in Z$, denote X_n^* by replacing ε_0 with ε_0^* in X_n. Define the physical dependence measure $\theta_{n,p} = \|X_n - X_n^*\|_p$ and $\Theta_{n,p} = \sum_{|i|\ge n} \theta_{i,p}$. Also, let $X_k(m) = \mathsf{E}[X_k|\varepsilon_{k-m}, \dots, \varepsilon_{k+m}]$. Then, Lemmas 3.1–3.4 still hold for $X_n = g((\varepsilon_{n-i})_{i\in Z})$. This can be proved similarly by observing that

$$\begin{aligned} X_k - X_k(m) &= \sum_{j=-k+m}^{\infty} (\mathsf{E}[X_k|\mathcal{F}_{-j-1,\infty}] - \mathsf{E}[X_k|\mathcal{F}_{-j,\infty}]) \\ &\quad + \sum_{j=m+k}^{\infty} (\mathsf{E}[X_k|\mathcal{F}_{k-m,j+1}] - \mathsf{E}[X_k|\mathcal{F}_{k-m,j}]) \\ &=: \sum_{j=-k+m}^{\infty} R_{k,j}^{(1)} + \sum_{j=m+k}^{\infty} R_{k,j}^{(2)}, \end{aligned} \tag{3.3}$$

$\|R_{k,j}^{(1)}\|_p \le \theta_{k+j+1,p}$ and $\|R_{k,j}^{(2)}\|_p \le \theta_{k-j-1,p}$. The details can be found in [12].

4. Proof of Theorem 2.1. Let h be a Lipschitz continuous function on R. Set

$$\varepsilon_i' = \varepsilon_i I\{|\varepsilon_i| \le \gamma_n\sqrt{n/\log n}\} - \mathsf{E}\varepsilon_i I\{|\varepsilon_i| \le \gamma_n\sqrt{n/\log n}\}, \qquad i \in Z,$$

where $\gamma_n \to 0$. Put $Y_k' = \sum_{i\in Z} a_i \varepsilon_{k-i}'$, $X_k' = h(Y_k') - \mathsf{E}h(Y_k')$ for $1 \le k \le n$. Since $\mathsf{E}\varepsilon_0^2 I\{|\varepsilon_0| \ge n\} = o(1/\log n)$, we can choose $\gamma_n \to 0$ sufficiently slowly such that

$$\sqrt{n\log n}\,\mathsf{E}|\varepsilon_0| I\{|\varepsilon_0| \ge \gamma_n\sqrt{n/\log n}\} \to 0.$$

This, together with the Lipschitz continuity of h, implies that

$$\begin{aligned} &\frac{\sqrt{\log n}\,\mathsf{E}\max_{1\le j\le q}|\sum_{k=1}^n (X_k - X_k')\exp(\mathrm{i}k\omega_j)|}{\sqrt{n}} \\ &\quad \le C\sqrt{n\log n}\sum_{j\in Z}|a_j|\mathsf{E}|\varepsilon_0| I\{|\varepsilon_0| \ge \gamma_n\sqrt{n/\log n}\} \to 0. \end{aligned}$$

In addition, note that, for $1 \le j \le q$,

$$\begin{aligned} |I_{n,X}(\omega_j) - I_{n,X'}(\omega_j)| &\le \sqrt{M_n(X')}\max_{1\le j\le q}\left|\sum_{k=1}^n (X_k - X_k')\exp(\mathrm{i}k\omega_j)\right| \Big/ \sqrt{n} \\ &\quad + \max_{1\le j\le q}\left|\sum_{k=1}^n (X_k - X_k')\exp(\mathrm{i}k\omega_j)\right|^2 \Big/ n. \end{aligned}$$

Then, in order to prove Theorem 2.1, we only need to show that

$$I_{n,q}(X') - \log q \Rightarrow G.$$

Recall that $m = [n^\beta]$ for some $0 < \beta < 1$. Let

$$X'_k(m) = \mathsf{E}[X'_k | \varepsilon_{k-m}, \dots, \varepsilon_{k+m}], \qquad 1 \le k \le n,$$

and

$$\widetilde{J}_{n,X}(\omega) = \left| \sum_{k=1}^{n} \big(X'_k - X'_k(m)\big) \exp(\mathrm{i}\omega k) \right|.$$

By Lemma 3.4 and Remark 3.2, it is readily seen that

$$\max_{1 \le i \le q} \widetilde{J}_{n,X}(\omega_i) = o_{\mathsf{P}}\big(\sqrt{n/\log n}\big). \tag{4.1}$$

We define the periodogram $I_{n,X'(m)}(\omega) = n^{-1} | \sum_{k=1}^{n} X'_k(m) \exp(\mathrm{i}k\omega)|^2$ and let $I_{n,1}(X'(m)) \le \cdots \le I_{n,q}(X'(m))$ be the order statistics of $I_{n,X'(m)}(\omega_j)/(2\pi f(\omega_j))$, $1 \le j \le q$. In view of (4.1), it is sufficient to prove that

$$I_{n,q}(X'(m)) - \log q \Rightarrow G. \tag{4.2}$$

For $0 < \beta < \alpha < 1/10$, let us split the interval $[1, n]$ into

$$\begin{aligned}
H_j &= [(j-1)(n^\alpha + 2n^\beta) + 1, (j-1)(n^\alpha + 2n^\beta) + n^\alpha], \\
I_j &= [(j-1)(n^\alpha + 2n^\beta) + n^\alpha + 1, j(n^\alpha + 2n^\beta)], \\
&\qquad\qquad 1 \le j \le m_n - 1,\ m_n - 1 = [n/(n^\alpha + 2n^\beta)] \sim n^{1-\alpha}, \\
H_{m_n} &= [(m_n - 1)(n^\alpha + 2n^\beta) + 1, n].
\end{aligned}$$

Here and below, the notation n^α is used to denote $[n^\alpha]$ for briefness. Put $v_j(\omega) = \sum_{k \in I_j} X'_k(m) \exp(\mathrm{i}k\omega)$, $1 \le j \le m_n - 1$. Then, $v_j(\omega)$, $1 \le j \le m_n - 1$, are independent and can be neglected by observing the following lemma.

LEMMA 4.1. *Under* (2.3), *we have* $\max_{1 \le l \le q} | \sum_{j=1}^{m_n - 1} v_j(\omega_l)| = o_{\mathsf{P}}(\sqrt{n/\log n})$.

PROOF. First, Corollary 1.6 of Nagaev [13], which is a Fuk–Nagaev-type inequality, shows that, for any large Q,

$$\begin{aligned}
&\sum_{l=1}^{q} \mathsf{P}\left(\left| \sum_{j=1}^{m_n - 1} v_j(\omega_l) \right| \ge \delta \sqrt{n/\log n} \right) \\
&\qquad \le C_{Q,\delta} \sum_{l=1}^{q} \left(\frac{\sum_{j=1}^{m_n - 1} \mathsf{E} v_j^2(\omega_l)}{n/\log n} \right)^Q \\
&\qquad\quad + C_Q \sum_{l=1}^{q} \sum_{j=1}^{m_n - 1} \mathsf{P}\big(|v_j(\omega_l)| \ge C_Q \delta \sqrt{n/\log n}\big).
\end{aligned}$$

By Lemma 3.2 and Remark 3.2, $\sum_{j=1}^{m_n-1} \mathsf{E}v_j^2(\omega_l) \le Cn^{1-\alpha+\beta}$. So, the first term above tends to zero. To complete the proof of Lemma 4.1, we shall show that the second term also tends to zero. In fact, using the fact that $|h(x)| \le C(|x|+1)$, we can get

$$\begin{aligned}
|v_j(\omega_l)| &\le C\left|\sum_{k\in I_j}\sum_{i=-m}^{m}|a_i|(|\varepsilon'_{k-i}| - \mathsf{E}|\varepsilon'_{k-i}|)\right| + C|I_j| \\
&=_d C\left|\sum_{k\in I_1}\sum_{i=-m}^{m}|a_i|(|\varepsilon'_{k-i}| - \mathsf{E}|\varepsilon'_{k-i}|)\right| + C|I_1| \qquad (4.3)\\
&= C\left|\sum_{t=-m}^{3m}\sum_{k=1\vee(t-m)}^{(m+t)\wedge(2m)}|a_{k-t}|(|\varepsilon'_t| - \mathsf{E}|\varepsilon'_t|)\right| + C|I_1|,
\end{aligned}$$

where $X =_d Y$ means X and Y have the same distribution. Hence

$$\begin{aligned}
&\sum_{l=1}^{q}\sum_{j=1}^{m_n-1}\mathsf{P}\big(|v_j(\omega_l)| \ge C_Q\delta\sqrt{n/\log n}\big) \\
&\quad\le \sum_{l=1}^{q}\sum_{j=1}^{m_n-1}\mathsf{P}\left(\left|\sum_{t=-m}^{3m}\sum_{k=1\vee(t-m)}^{(m+t)\wedge(2m)}|a_{k-t}|(|\varepsilon'_t| - \mathsf{E}|\varepsilon'_t|)\right| \ge C_Q\delta\sqrt{n/\log n}\right) \qquad (4.4)\\
&\quad\le C\sum_{l=1}^{q}\sum_{j=1}^{m_n-1}\left(\frac{m}{n/\log n}\right)^Q \to 0,
\end{aligned}$$

where the last inequality follows from the Fuk–Nagaev inequality, by noting that $|\varepsilon'_t| \le \gamma_n\sqrt{n/\log n}$. The desired conclusion is established, and the proof is now complete. $\square$

We now deal with the sum of large blocks. Let

$$u_j(\omega) = \sum_{k\in H_j} X'_k(m)\exp(\mathrm{i}k\omega), \qquad u'_j(\omega) = u_j(\omega)I\{|u_j(\omega)| \le \gamma_n^{1/2}\sqrt{n/\log n}\},$$

$$\overline{u}_j(\omega) = u'_j(\omega) - \mathsf{E}u'_j(\omega), \qquad 1 \le j \le m_n.$$

Noting that $|u_j(\omega)| \le \sum_{k\in H_j}|X'_k(m)| =: \xi_j$, $m_n \sim n^{1-\alpha}$ and using similar arguments to those employed in (4.3) and (4.4), it is readily seen that, for any large Q,

$$\begin{aligned}
&\frac{\sqrt{\log n}\sum_{j=1}^{m_n}\mathsf{E}\xi_j I\{\xi_j \ge \gamma_n^{1/2}\sqrt{n/\log n}\}}{\sqrt{n}} \\
&\quad\le C\sqrt{\log n}\,n^{1/2-\alpha}\sum_{k=n}^{\infty}\frac{1}{\sqrt{k\log k}}\mathsf{P}\big(\xi_1 \ge \gamma_n^{1/2}\sqrt{k/\log k}\big) \qquad (4.5)
\end{aligned}$$

$$
\begin{aligned}
&\quad + Cn^{1-\alpha}\mathsf{P}(\xi_1 \ge \gamma_n^{1/2}\sqrt{n/\log n}) \\
&\le C\sqrt{\log n}\, n^{1/2-\alpha}\sum_{k=n}^{\infty}\frac{1}{\sqrt{k\log k}}\left(\frac{n^\alpha}{\gamma_n k/\log k}\right)^Q \\
&\quad + Cn^{1-\alpha}(\gamma_n^{-1}n^{\alpha-1}\log n)^Q \\
&= o(1),
\end{aligned}
$$

which implies $\max_{1\le l\le q}|\sum_{j=1}^{m_n}(u_j(\omega_l)-\overline{u}_j(\omega_l))| = o_{\mathsf{P}}(\sqrt{n/\log n})$. Combining this and Lemma 4.1 yields that we only need to show

$$
I_{n,q}(\overline{X}) - \log q \Rightarrow G, \tag{4.6}
$$

where $I_{n,q}(\overline{X})$ denotes the maximum of

$$
\left|\sum_{k=1}^{m_n}\overline{u}_k(\omega_l)\right|^2 \Big/ (2\pi n f(\omega_l)), \qquad 1\le l\le q.
$$

In order to prove (4.6), we need the following moderate deviation result, whose proof is based on a Gaussian approximation technique due to Einmahl [5], Corollary 1(b), page 31 and remark on page 32. The detailed proof is given in [12].

LEMMA 4.2. *Let $\xi_{n,1},\dots,\xi_{n,k_n}$ be independent random vectors with mean zero and values in $\mathbb{R}^{2d}$, and let $S_n=\sum_{i=1}^{k_n}\xi_{n,i}$. Assume that $|\xi_{n,k}|\le c_n B_n^{1/2}$, $1\le k\le k_n$, for some $c_n\to 0$, $B_n\to\infty$ and*

$$
|B_n^{-1}\mathsf{Cov}(\xi_{n,1}+\cdots+\xi_{n,k_n}) - I_{2d}| = O(c_n^2),
$$

where I_{2d} is a $2d\times 2d$ identity matrix. Suppose that $\beta_n := B_n^{-3/2}\sum_{k=1}^{k_n}\mathsf{E}|\xi_{n,k}|^3\to 0$. Then,

$$
\begin{aligned}
&\big|\mathsf{P}(|S_n|_{2d}\ge x) - \mathsf{P}(|N|_{2d}\ge x/B_n^{1/2})\big| \\
&\quad \le o\big(\mathsf{P}(|N|_{2d}\ge x/B_n^{1/2})\big) \\
&\qquad + C\left(\exp\left(-\frac{\delta_n^2\min(c_n^{-2},\beta_n^{-2/3})}{16d}\right) + \exp\left(\frac{Cc_n^2}{\beta_n^2\log\beta_n}\right)\right),
\end{aligned}
$$

uniformly for $x\in[B_n^{1/2},\delta_n\min(c_n^{-1},\beta_n^{-1/3})B_n^{1/2}]$, with any $\delta_n\to 0$ and $\delta_n\times\min(c_n^{-1},\beta_n^{-1/3})\to\infty$. N is a centered normal random vector with covariance matrix I_{2d}. $|\cdot|_{2d}$ is defined by $|z|_{2d}=\min\{(x_i^2+y_i^2)^{1/2}: 1\le i\le d\}$, $z=(x_1,y_1,\dots,x_d,y_d)$.

We begin the proof of (4.6) by checking the conditions in Lemma 4.2. We define the following notation: $\overline{u}_k(\omega_l)/f^{1/2}(\omega_l) =: \overline{u}_{k,l}(1) + \mathrm{i}\overline{u}_{k,l}(2)$,

$$
\begin{aligned}
Z_k = (\overline{u}_{k,i_1}(1), \overline{u}_{k,i_1}(2), \dots, \overline{u}_{k,i_d}(1), \overline{u}_{k,i_d}(2)),& \\
1\le i_1<\cdots<i_d\le q,&
\end{aligned} \tag{4.7}
$$

and $U_n = \sum_{k=1}^{m_n} Z_k$. Then, it is easy to see that $Z_1, \dots, Z_{m_n}$ are independent.

LEMMA 4.3. *Under the conditions of Theorem 2.1, we have*

$$|\mathsf{Cov}(U_n)/(n\pi) - I_{2d}| = o(1/\log n)$$

uniformly for $1 \le i_1 < \cdots < i_d \le q$.

PROOF. Let $B_{n,i} = \sum_{k=1}^{m_n} \mathsf{E}(\overline{u}_{k,i}(1))^2$. Similar arguments to those in (4.5), together with some elementary calculations, give that $\max_{1\le l\le q} \mathsf{E}|u_j(\omega_l) - \overline{u}_j(\omega_l)|^2 = O(n^{-Q})$ for any large Q. This yields that, for any large Q,

$$\begin{aligned}
&\left|B_{n,i} - \sum_{j=1}^{m_n} \mathsf{E}\biggl(\sum_{k\in H_j} X'_k(m)\cos(k\omega_i)\biggr)^2\right| \\
&\qquad \le C\sum_{j=1}^{m_n} |H_j|^{1/2}\bigl(\mathsf{E}|u_j(\omega_i) - \overline{u}_j(\omega_i)|^2\bigr)^{1/2} \\
&\qquad\qquad + \sum_{j=1}^{m_n} \mathsf{E}|u_j(\omega_i) - \overline{u}_j(\omega_i)|^2 \\
&\qquad \le Cn^{-Q}.
\end{aligned} \tag{4.8}$$

Moreover, it follows from Lemmas 3.1 and 3.2 and Remark 3.2 that

$$\begin{aligned}
&\left|\mathsf{E}\biggl(\sum_{k=1}^{n} X'_k(m)\cos(k\omega_i)\biggr)^2 - \sum_{j=1}^{m_n} \mathsf{E}\biggl(\sum_{k\in H_j} X'_k(m)\cos(k\omega_i)\biggr)^2\right| \\
&\qquad \le Cn^{1-(\alpha-\beta)/2}, \\
&\left|\mathsf{E}\biggl(\sum_{k=1}^{n} X'_k(m)\cos(k\omega_i)\biggr)^2 - \mathsf{E}\biggl(\sum_{k=1}^{n} X'_k\cos(k\omega_i)\biggr)^2\right| \\
&\qquad = o(n/\log n).
\end{aligned} \tag{4.9}$$

In the case $h(x) \equiv x$, we have $\sum_{k=1}^{n} X'_k \cos(k\omega_i) = \sum_{t=-\infty}^{\infty} \sum_{k=1}^{n} a_{k+t}\cos(k\omega_i) \times \varepsilon'_{-t}$. Hence, condition (2.4) ensures that

$$\left|\mathsf{E}\biggl(\sum_{k=1}^{n} X'_k\cos(k\omega_i)\biggr)^2 - \mathsf{E}\biggl(\sum_{k=1}^{n} X_k\cos(k\omega_i)\biggr)^2\right| = o(n/\log n). \tag{4.10}$$

Suppose now that h is Lipschitz continuous. We write $\zeta_k = |\varepsilon_k| I\{|\varepsilon_k| \ge \gamma_n \times \sqrt{n/\log n}\}$. Then, since $|X_k - X'_k| \le C\sum_{j\in Z} |a_j|(\zeta_{k-j} + \mathsf{E}\zeta_{k-j})$, we have, from

$\mathsf{E}\varepsilon_0^2 I\{|\varepsilon_0| \geq n\} = o(1/(\log n)^2)$ and the fact $\gamma_n \to 0$ sufficiently slowly, that

$$\begin{aligned}
&\mathsf{E}\left(\sum_{k=1}^{n}(X_k - X_k')\cos(k\omega_i)\right)^2 \\
&\qquad \leq C\mathsf{E}\left(\sum_{k=1}^{n}\sum_{j\in Z}|a_j|(\zeta_{k-j} - \mathsf{E}\zeta_{k-j})\right)^2 + C\left(\sum_{k=1}^{n}\sum_{j\in Z}|a_j|\mathsf{E}\zeta_{k-j}\right)^2 \\
&\qquad \leq Cn\mathsf{E}\zeta_0^2 + Cn^2(\mathsf{E}\zeta_0)^2 = o\big(n/(\log n)^2\big),
\end{aligned}$$

which implies (4.10) by virtue of Lemma 3.2 and the inequality $|\mathsf{E}X^2 - \mathsf{E}Y^2| \leq \|X - Y\|_2\|X + Y\|_2$ for any random variables X and Y. From Lemma 3.3, Remark 3.2 and (4.8)–(4.10), we have $|B_{n,i}/(n\pi) - 1| = o(1/\log n)$ uniformly for $1 \leq i \leq q$.

In the following, we show that the off-diagonal elements in $\mathsf{Cov}(U_n)$ are $o(n/\log n)$. We only deal with $B_{n,i,j} := \mathsf{E}\{\sum_{k=1}^{m_n}\overline{u}_{k,i}(1)\sum_{k=1}^{m_n}\overline{u}_{k,j}(1)\}$, $i \neq j$, since the other elements can be estimated similarly. As in (4.8) and (4.9), we have

$$\begin{aligned}
&\left|B_{n,i,j} - (f(\omega_i)f(\omega_j))^{-1/2}\mathsf{E}\left(\sum_{k=1}^{n}X_k'(m)\cos(k\omega_i)\sum_{k=1}^{n}X_k'(m)\cos(k\omega_j)\right)\right| \\
&\quad \leq C\left|\mathsf{E}\left[\left\{\sum_{k=1}^{m_n}\overline{u}_{k,i}(1) - (f(\omega_i))^{-1/2}\sum_{k=1}^{n}X_k'(m)\cos(k\omega_i)\right\}\sum_{k=1}^{m_n}\overline{u}_{k,j}(1)\right]\right| \\
&\qquad + C|f(\omega_i)|^{-1/2}\left|\mathsf{E}\left[\sum_{k=1}^{n}X_k'(m)\cos(k\omega_i)\right.\right. \\
&\qquad\qquad \times\left\{\sum_{k=1}^{m_n}\overline{u}_{k,j}(1) - (f(\omega_j))^{-1/2}\right. \\
&\qquad\qquad\qquad \left.\left.\left.\times\sum_{k=1}^{n}X_k'(m)\cos(k\omega_j)\right\}\right]\right| \\
&\quad \leq Cn^{1-(\alpha-\beta)/2}.
\end{aligned}$$

Moreover, by virtue of Lemmas 3.1–3.3 and Remark 3.2, we have

$$\mathsf{E}\left(\sum_{k=1}^{n}X_k'(m)\cos(k\omega_i)\sum_{k=1}^{n}X_k'(m)\cos(k\omega_j)\right) = o(n/\log n).$$

Hence, $B_{n,i,j} = o(n/\log n)$, $i \neq j$. This proves the lemma. $\square$

LEMMA 4.4. *Under the conditions of Theorem* 2.1, *we have*, *uniformly for* $1 \le i_1 < \cdots < i_d \le q$, *that*

$$\overline{\beta}_n := n^{-3/2} \sum_{j=1}^{m_n} \mathsf{E}|Z_j|^3 = o\big(1/(\log n)^{3/2}\big).$$

PROOF. By the arguments in (4.3), the Fuk–Nagaev inequality and the fact that $\alpha < 1/10$ and $\gamma_n \to 0$ sufficiently slowly,

$$\begin{aligned}
&\sum_{j=1}^{m_n} \mathsf{E}|\overline{u}_j(\omega_i)|^3 \\
&\qquad \le \sum_{j=1}^{m_n}\sum_{k=1}^{n}\left(\frac{k}{\log k}\right)^{3/2} \mathsf{P}\left(\gamma_n^{1/2}\sqrt{\frac{k}{\log k}} < |u_j(\omega_i)| \le \gamma_n^{1/2}\sqrt{\frac{k+1}{\log(k+1)}}\right) \\
&\qquad \le Cn^{1+5\alpha} + C\sum_{j=1}^{m_n}\sum_{k=n^{4\alpha}}^{n} \frac{k^{1/2}}{(\log k)^{3/2}} \mathsf{P}\left(|u_j(\omega_i)| \ge \gamma_n^{1/2}\sqrt{\frac{k}{\log k}}\right) \\
&\qquad\quad + C\sum_{j=1}^{m_n} \frac{n^{6\alpha}}{(\log n)^{3/2}} \mathsf{P}\left(|u_j(\omega_i)| \ge \gamma_n^{1/2}\sqrt{\frac{n^{4\alpha}}{\log n^{4\alpha}}}\right) \\
&\qquad \le Cn^{1+5\alpha} + C\sum_{j=1}^{m_n}\sum_{k=n^{4\alpha}}^{n} \frac{k^{1/2}}{(\log k)^{3/2}} \left(\frac{n^{\alpha}}{\gamma_n k/\log k}\right)^{Q} \\
&\qquad\quad + C\sum_{j=1}^{m_n}\sum_{k=n^{4\alpha}}^{n} \frac{k^{1/2}n^{\alpha}}{(\log k)^{3/2}} \mathsf{P}\left(|\varepsilon_0| \ge C\gamma_n^{1/2}\sqrt{\frac{k}{\log k}}\right) \\
&\qquad\quad + C\sum_{j=1}^{m_n} \frac{n^{7\alpha}}{(\log n)^{3/2}} \mathsf{P}\left(|\varepsilon_0| \ge C\gamma_n^{1/2}\sqrt{\frac{n^{4\alpha}}{\log n^{4\alpha}}}\right) \\
&\qquad = o\big((n/\log n)^{3/2}\big), \qquad \text{uniformly for } 1 \le i \le q.
\end{aligned}$$

The desired result now follows. □

By Lemmas 4.3 and 4.4, we may write $\overline{\beta}_n = \nu_n^{3/2}(\log n)^{-3/2}$ and $|\mathsf{Cov}(U_n)/(n\pi) - I_{2d}| = \gamma_{n,1}(\log n)^{-1}$, where $\nu_n \to 0$, $\gamma_{n,1} \to 0$. Let us take $c_n = \{(4d\gamma_n \times (\pi f^*)^{-1})^{1/2} \vee \gamma_{n,1}^{1/2}\}(\log n)^{-1/2} =: \gamma_{n,2}^{1/2}(\log n)^{-1/2}$ and $\delta_n = \max\{\gamma_{n,2}^{1/4}, \nu_n^{1/4}\}$ in Lemma 4.2. Note that $\gamma_{n,2} \to 0$ sufficiently slowly. Then, simple calculations show that

$$\exp\left(-\frac{\delta_n^2 \min(c_n^{-2}, \overline{\beta}_n^{-2/3})}{16d}\right) \le Cn^{-4d}, \qquad \exp\left(\frac{Cc_n^2}{\overline{\beta}_n^2 \log \overline{\beta}_n}\right) \le Cn^{-4d}.$$

By virtue of Lemma 4.2, it holds that, for any fixed $x \in R$,

$$\begin{aligned} &\mathsf{P}\big((2n\pi)^{-1/2}|U_n|_{2d} \geq \sqrt{x+\log q}\big) \\ &\quad = \mathsf{P}\big(|N|_{2d} \geq \sqrt{2(x+\log q)}\big)\big(1+o(1)\big) \\ &\quad = q^{-d}\exp(-dx)\big(1+o(1)\big), \end{aligned} \tag{4.11}$$

uniformly for $1 \leq i_1 < \cdots < i_d \leq q$. We write $V_j := |\sum_{k=1}^{m_n} \overline{u}_k(\omega_j)|^2/(2\pi n \times f(\omega_j))$, $1 \leq j \leq q$, and

$$A := \{I_{n,q}(\overline{X}) \geq x + \log q\} = \bigcup_{j=1}^{q} \{V_i \geq x + \log q\} =: \bigcup_{j=1}^{q} A_j.$$

By the Bonferroni inequality, we have, for any fixed k satisfying $1 \leq k \leq q$,

$$\sum_{t=1}^{2k} (-1)^{t-1} E_t \leq \mathsf{P}(A) \leq \sum_{t=1}^{2k-1} (-1)^{t-1} E_t,$$

where $E_t = \sum_{1 \leq i_1 < \cdots < i_t \leq q} \mathsf{P}(A_{i_1} \cap \cdots \cap A_{i_t})$. In view of (4.11), it follows that $\lim_{n\to\infty} E_t = e^{-tx}/t!$. Since $\sum_{t=1}^{k} (-1)^{t-1} e^{-tx}/t! \to 1 - e^{-e^{-x}}$ as $k \to \infty$, the proof of Theorem 2.1 is complete.

5. Proof of Theorem 2.2. Recall that $m = [n^\beta]$, and β is sufficiently small. Let $S_{n,m}(\omega) = \sum_{k=1}^{n} X_k(m)\exp(\mathrm{i}\omega k)$ and $I_{n,1}(m) \leq \cdots \leq I_{n,q}(m)$ be the order statistics of $|S_{n,m}(\omega_j)|^2/(2\pi n f(\omega_j))$, $1 \leq j \leq q$. By Lemma 3.4, we only need to prove that

$$I_{n,q}(m) - \log q \Rightarrow G. \tag{5.1}$$

We use the same notation and blocking method as in the proof of Theorem 2.1 [replacing $X'_k(m)$ with $X_k(m)$]. For example, $v_j(\omega) = \sum_{k \in I_j} X_k(m)\exp(\mathrm{i}k\omega)$. As in Lemma 4.1, we claim that

$$\max_{1 \leq j \leq q} \left| \sum_{k=1}^{m_n - 1} v_k(\omega_j) \right| = o_{\mathsf{P}}\big(\sqrt{n/\log n}\big). \tag{5.2}$$

We come to prove it. Recall that $s > 2$ and $\beta < \alpha$. Then, we can choose α, β sufficiently small and τ sufficiently close to $1/2$ such that

$$(s-1)^{-1}(1 - \alpha + \alpha s - 1/2) < \tau < 1/2. \tag{5.3}$$

We define $\overline{v}_k(\omega_j) = v'_k(\omega_j) - \mathsf{E}v'_k(\omega_j)$, where $v'_k(\omega_j) = v_k(\omega_j) I\{|v_k(\omega_j)| \leq n^\tau\}$, $1 \leq j \leq q$, $1 \leq k \leq m_n - 1$. So,

$$\max_{1 \leq j \leq q} \left| \sum_{k=1}^{m_n-1} v_k(\omega_j) \right| \leq \max_{1 \leq j \leq q} \left| \sum_{k=1}^{m_n-1} \overline{v}_k(\omega_j) \right| + \max_{1 \leq j \leq q} \left| \sum_{k=1}^{m_n-1} \big(v_k(\omega_j) - \overline{v}_k(\omega_j)\big) \right|.$$

By the Fuk–Nagaev inequality and Lemma 3.2, we have, for any large Q,

$$\mathsf{P}\biggl(\max_{1\le j\le q}\biggl|\sum_{k=1}^{m_n-1}\bar{v}_k(\omega_j)\biggr|\ge\delta\sqrt{\frac{n}{\log n}}\biggr)\le Cn\biggl(\frac{n^{1-\alpha+\beta}}{n/\log n}\biggr)^Q\to 0. \tag{5.4}$$

Also, using (5.3), the condition $\mathsf{E}|X_0|^s<\infty$ and $|v_k(\omega)|\le\sum_{j\in I_k}|X_j(m)|$, we can get

$$\begin{aligned}
&\mathsf{E}\frac{\max_{1\le j\le q}|\sum_{k=1}^{m_n-1}(v_k(\omega_j)-\bar{v}_k(\omega_j))|}{\sqrt{n/\log n}}\\
&\quad\le\frac{2n^{1-\alpha}\mathsf{E}[\sum_{k=1}^{n^\beta}|X_k(m)|I\{\sum_{k=1}^{n^\beta}|X_k(m)|\ge n^\tau\}]}{\sqrt{n/\log n}}\\
&\quad\le Cn^{1-\alpha+\beta s-\tau(s-1)-1/2}(\log n)^{1/2}=o(1).
\end{aligned}\tag{5.5}$$

This, together with (5.4), implies (5.2).

Set

$$\begin{aligned}
u'_k(\omega_j)&=u_k(\omega_j)I\{|u_k(\omega_j)|\le n^\tau\},\\
\bar{u}_k(\omega_j)&=u'_k(\omega_j)-\mathsf{E}u'_k(\omega_j),\qquad 1\le j\le q,1\le k\le m_n.
\end{aligned}$$

By the similar arguments as (5.5), using (5.3), we can show that

$$\max_{1\le j\le q}\biggl|\sum_{k=1}^{m_n}(u_k(\omega_j)-\bar{u}_k(\omega_j))\biggr|=o_{\mathsf{P}}\bigl(\sqrt{n/\log n}\bigr).$$

So, in order to get (5.1), similarly to (4.6), it is sufficient to prove

$$I_{n,q}(\overline{X})-\log q\Rightarrow G. \tag{5.6}$$

In fact, (5.6) follows from Lemmas 5.1 and 5.2 and similar arguments to those employed in the proof of Theorem 2.1.

LEMMA 5.1. *Under the conditions of Theorem 2.2, we have*

$$|\mathsf{Cov}(U_n)/(n\pi)-I_{2d}|=o(1/\log n).$$

PROOF. The same arguments as those of Lemma 4.3 give that

$$\biggl|B_{n,i}-\mathsf{E}\biggl(\sum_{k=1}^n X_k\cos(k\omega_i)\biggr)^2\Big/(\pi f(\omega_i))\biggr|=o(n/\log n).$$

The lemma then follows from Lemma 3.3. □

LEMMA 5.2. *Under the conditions of Theorem 2.2, we have*

$$\overline{\beta}_n = n^{-3/2} \sum_{j=1}^{m_n} \mathsf{E}|Z_j|^3 = O(n^{t-1/2}),$$

where $t = \max\{(3-s)\tau + \alpha(s-2)/2, \alpha/2\} < \tau < 1/2$.

PROOF. Suppose that $2 < s < 3$. Then, by virtue of Lemma 3.2, we have

$$\overline{\beta}_n \le Cn^{-3/2+(3-s)\tau} \sum_{j=1}^{m_n} \mathsf{E}|Z_j|^s \le Cn^{-3/2+(3-s)\tau} \sum_{j=1}^{m_n} |H_j|^{s/2} \le Cn^{t-1/2}.$$

The case of $s \ge 3$ can be similarly proved. $\square$

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