RANDOM GLUING OF METRIC SPACES[1]

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We construct random metric spaces by gluing together an infinite sequence of pointed metric spaces that we call *blocks*. At each step, we glue the next block to the structure constructed so far by randomly choosing a point on the structure and then identifying it with the distinguished point of the block. The random object that we study is the completion of the structure that we obtain after an infinite number of steps. In (*Ann. Inst. Fourier (Grenoble)* **67** (2017) 1963–2001), Curien and Haas study the case of segments, where the sequence of lengths is deterministic and typically behaves like $n^{-\alpha}$. They proved that for $\alpha > 0$, the resulting tree is compact and that the Hausdorff dimension of its set of leaves is α^{-1}. The aim of this paper is to handle a much more general case in which the blocks are i.i.d. copies of the same random metric space, scaled by deterministic factors that we call $(\lambda_n)_{n \geq 1}$. We work under some conditions on the distribution of the blocks ensuring that their Hausdorff dimension is almost surely d, for some $d \geq 0$. We also introduce a sequence $(w_n)_{n \geq 1}$ that we call the *weights* of the blocks. At each step, the probability that the next block is glued onto any of the preceding blocks is proportional to its weight. The main contribution of this paper is the computation of the Hausdorff dimension of the set $\mathcal{L}$ of points which appear during the completion procedure when the sequences $(\lambda_n)_{n \geq 1}$ and $(w_n)_{n \geq 1}$ typically behave like a power of n, say $n^{-\alpha}$ for the scaling factors and $n^{-\beta}$ for the weights, with $\alpha > 0$ and $\beta \in \mathbb{R}$. For a large domain of α and β, we have the same behaviour as the one observed in (*Ann. Inst. Fourier (Grenoble)* **67** (2017) 1963–2001), which is that $\dim_H(\mathcal{L}) = \alpha^{-1}$. However, for $\beta > 1$ and $\alpha < 1/d$, our results reveal an interesting phenomenon: the dimension has a nontrivial dependence in α, β and d, namely

$$\dim_H(\mathcal{L}) = \frac{2\beta - 1 - 2\sqrt{(\beta - 1)(\beta - \alpha d)}}{\alpha}.$$

The computation of the dimension in the latter case involves new tools, which are specific to our model.

CONTENTS

Introduction. Let us recall Aldous' famous line-breaking construction of the Brownian CRT (Continuum Random Tree) in [3]. On the half-line $[0, \infty)$, consider $C_1, C_2, \ldots, C_n$ the points of a Poisson process with intensity $t\,dt$. Cut the half-line in closed intervals $[C_i, C_{i+1}]$, which we call *branches* (of length

$C_{i+1} - C_i$). Starting from $[0, C_1]$, construct a tree by recursively gluing the branch $[C_i, C_{i+1}]$ to a random point chosen uniformly on the tree already constructed (i.e., under the normalised length measure). Aldous' Brownian CRT is the completion of the tree constructed after an infinite number of steps. This process can be generalised by using any arbitrary sequence (λ_n) for the length of the successive branches. This model was introduced and studied by Curien and Haas in [7], who proved that when $\lambda_n = n^{-\alpha+o(1)}$ for some $\alpha > 0$, the tree obtained is a.s. compact and has Hausdorff dimension $(1 \vee \alpha^{-1})$. In [4], Amini et al. obtained a necessary and sufficient condition on the sequence (λ_n) for the almost sure compactness of the resulting tree, under the assumption that this sequence is nonincreasing. In [13], Haas describes how the height of the tree explodes when $n \to \infty$ under the assumption that $\lambda_n \approx n^\alpha$, with $\alpha \geq 0$.

Our goal is to define a more general version of this model, in which the branches are replaced by arbitrary (and possibly random) measured metric spaces, and to investigate the compactness and the Hausdorff dimension of the resulting metric space. As we will see, in this broader context, a striking phenomenon (absent from [7]) pops up. In this paper, we will work with

$$(\lambda_n)_{n \geq 1} \quad \text{and} \quad (w_n)_{n \geq 1},$$

two sequences of nonnegative real numbers that will be the *scaling factors* and *weights* of the metric spaces that we glue. All the scaling factors $(\lambda_n)_{n \geq 1}$ are considered strictly positive, but the weights, except for the first one w_1, can possibly be null.

Definition of the model and main results.   Let us first present a simpler version of our construction, in which we construct a tree through an aggregation of segments. For now the branches, which we denote $(\mathbf{b}_n)_{n \geq 1}$, are segments of length $(\lambda_n)_{n \geq 1}$, rooted at one end and endowed with the Lebesgue measure normalised so that their respective total measure is $(w_n)_{n \geq 1}$ (or endowed with the null measure for branches with vanishing weight). We then define a sequence $(\mathcal{T}_n)_{n \geq 1}$ of increasing trees by gluing those branches as follows. First, $\mathcal{T}_1 = \mathbf{b}_1$. Then, if $\mathcal{T}_n$ is constructed, we build $\mathcal{T}_{n+1}$ by first sampling a point X_n chosen proportionally to the measure μ_n obtained by aggregating the measures concentrated on the branches $\mathbf{b}_1, \dots, \mathbf{b}_n$ and then gluing $\mathbf{b}_{n+1}$ onto $\mathcal{T}_n$ by identifying its root with X_n. Let $\mathcal{T}^*$ be the increasing union of the trees $\mathcal{T}_n$ for $n \geq 1$ and $\mathcal{T}$ be the completion of $\mathcal{T}^*$. Note that if $(w_n) = (\lambda_n)$, this model coincides with the one studied in [7].

We can compute the Hausdorff dimension of the resulting tree in the case where (λ_n) and (w_n) behave like powers of n, say $\lambda_n = n^{-\alpha}$ and $w_n = n^{-\beta}$. We define $\mathcal{L} := (\mathcal{T} \setminus \mathcal{T}^*)$ to which we refer as the set of *leaves* of $\mathcal{T}$. In this particular case it coincides, up to a countable set, with the set of points x such that $\mathcal{T} \setminus \{x\}$ remains

connected. In the above context, a trivial consequence of our main theorem is that $\mathcal{T}$ is a.s. compact and

$$\dim_H(\mathcal{L}) = \frac{2\beta - 1 - 2\sqrt{(\beta-1)(\beta-\alpha)}}{\alpha} \quad \text{if } \beta > 1 \text{ and } \alpha < 1,$$

$$= \frac{1}{\alpha} \quad \text{otherwise,}$$

where $\dim_H(X)$ stands for the Hausdorff dimension of the metric space X; see Section A.2.

Note that, since we can check that the dimension of the *skeleton* $\mathcal{T}^*$ is always 1, we can recover the dimension of $\mathcal{T}$ as $\dim_H(\mathcal{T}) = \max(1, \dim_H(\mathcal{L}))$. We see that $\dim_H(\mathcal{L}) = \frac{1}{\alpha}$ as in [7] for most values of β, however, a new phenomenon, absent from [7], happens in the case $\beta > 1$ (the sum of the weights is finite) and $\alpha < 1$ (the total length is infinite). In this case, the Hausdorff dimension of $\mathcal{T}$ depends in a nontrivial manner on α and β.

Now we want to generalise it to sequences $(\mathbf{b}_n)$ of more general metric spaces that we call *blocks*, which can be random and possibly more elaborate than just segments. Specifically, our blocks are based on the distribution of a random pointed measured compact metric space, $(\mathsf{B}, \mathsf{D}, \rho, \nu)$, with underlying set B, distance D, distinguished point ρ and endowed with a probability measure ν. We sometimes denote it B by abuse of notation when no confusion is possible and we refer to it as the *underlying random block*. We consider a sequence $((\mathsf{B}_n, \mathsf{D}_n, \rho_n, \nu_n))_{n \geq 1}$ of i.i.d. random variables with the distribution of $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ and define our blocks by setting

$$(1) \qquad \forall n \geq 1, \quad (\mathbf{b}_n, \mathbf{d}_n, \boldsymbol{\rho}_n, \boldsymbol{\nu}_n) := (\mathsf{B}_n, \lambda_n \cdot \mathsf{D}_n, \rho_n, w_n \cdot \nu_n),$$

meaning that we dilate all the distances in the space B_n by the factor λ_n and scale the measure by w_n. We suppose that, $\lambda_n \approx n^{-\alpha}$ for some $\alpha > 0$, and $w_n \approx n^{-\beta}$ for some $\beta \in \mathbb{R}$, in some loose sense which we make precise in the sequel. For technical reasons, we have to separate the case $\beta < 1$, the case $\beta > 1$ and $\beta = 1$. This gives rise to the three hypotheses Hypothesis $\bigcirc_{\alpha,\beta}$, Hypothesis $\diamond_{\alpha,\beta}$ and Hypothesis $\square_{\alpha,1}$. For any $d \in [0, \infty)$, we will introduce the Hypothesis H_d and suppose that the distribution of our underlying random block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ satisfies this hypothesis for some $d \geq 0$. This hypothesis ensures that our random block exhibits a d-dimensional behaviour. We set out all of these hypotheses just below the statement of our theorem.

Except in Section 1.1, we will always assume that the blocks are of the form (1). This is implicit in all our results.

In this extended setting, we can perform the same gluing algorithm and build a sequence $(\mathcal{T}_n)_{n \geq 1}$ of random compact metric spaces by iteratively gluing the root of $\mathbf{b}_{n+1}$ onto a point chosen in $\mathcal{T}_n$ according to the measure μ_n obtained as the sum of the measures of the blocks $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Again $\mathcal{T}^* = \bigcup_{n \geq 1} \mathcal{T}_n$ is called the
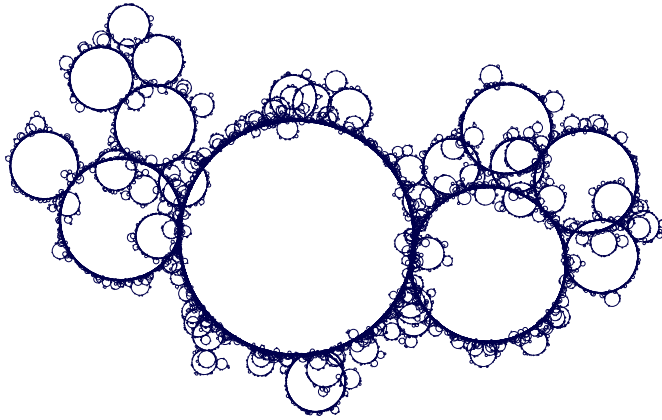
FIG. 1. *Gluing of circles of radii $\lambda_n = n^{-3/5}$, with weights $w_n = n^{-3/2}$. The Hausdorff dimension of the resulting metric space is $(\frac{10}{3} - \sqrt{5})$.*

skeleton of the construction and its completion is still denoted $\mathcal{T}$. See Figure 1 for nonisometric, nonproper representation in the plane of a simulation of this model, with B chosen to be almost surely a circle of unit length. As for the case of segments, we refer to $\mathcal{L} = (\mathcal{T} \setminus \mathcal{T}^*)$ as the set of leaves of the construction. We can now state our main theorem.

THEOREM 1. *Suppose that there exists $d \geq 0$, such that $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ satisfies Hypothesis H_d, and $\alpha > 0$ and $\beta \in \mathbb{R}$ such that the sequences (w_n) and (λ_n) satisfy either Hypothesis $\diamond_{\alpha,\beta}$ or Hypothesis $\bigcirc_{\alpha,\beta}$ or Hypothesis $\square_{\alpha,1}$. Then, almost surely, the structure $\mathcal{T}$ resulting from the construction is compact, and*

$$\dim_{\mathrm{H}}(\mathcal{L}) = \frac{2\beta - 1 - 2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha} \quad \text{if } \beta > 1 \text{ and } \alpha < \frac{1}{d},$$

$$= \frac{1}{\alpha} \quad \text{otherwise.}$$

The function $(\alpha, \beta) \mapsto \dim_{\mathrm{H}}(\mathcal{L})$ is plotted in Figure 2. Remark that for $\beta > 1$ the dimension of the set $\mathcal{L}$ depends on the geometry of the underlying random block through d, its dimension. For $\beta \leq 1$, it is not the case, and actually the theorem remains true under much weaker hypotheses for the distribution of $(\mathsf{B}, \mathsf{D}, \rho, \nu)$, namely that ν is not almost surely concentrated on $\{\rho\}$, and that $\forall k \geq 0$, $\mathbb{E}[(\mathrm{diam}(\mathsf{B}))^k] < \infty$, where $\mathrm{diam}(\cdot)$ denotes the diameter of a metric space. We could even replace the assumption that the blocks $((\mathsf{B}_n, \mathsf{D}_n, \rho_n, \nu_n))_{n \geq 1}$ are i.i.d. by some weaker assumption but we do not do it for the sake of clarity. The proofs when $\beta \leq 1$ are quite short and the interested reader can easily generalise them to a more general setting.
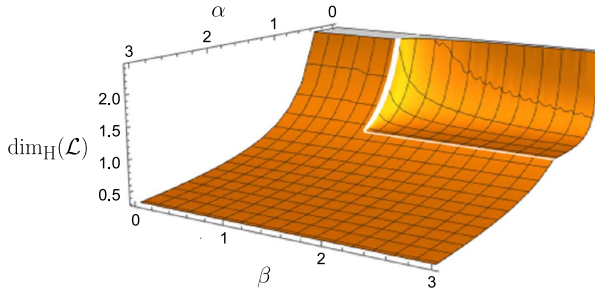
FIG. 2. *The plot represents the Hausdorff dimension of the leaves as a function of α and β, the dimension d being fixed to 1. The expression obtained for $\beta > 1$ and $\alpha < \frac{1}{d}$ can be rewritten as $d + \frac{(\sqrt{\beta - \alpha d} - \sqrt{\beta - 1})^2}{\alpha}$. This expression is always larger than d and smaller than $\frac{1}{\alpha}$, and it is decreasing in α and β on the domain on which we consider it. When $\beta \to 1$, it converges to the value $\frac{1}{\alpha}$ so that the function $(\alpha, \beta) \mapsto \dim_H(\mathcal{L})$ is continuous on the domain $\mathbb{R}_+^* \times \mathbb{R}$.*

Hypotheses of the theorem. Let us define and discuss the precise hypotheses of our theorem. First, let us describe the assumptions that we make on the sequences (λ_n) and (w_n). We define

$$W_n = \sum_{k=1}^n w_k,$$

and for all $\epsilon > 0$, we set

(2) $$G^\epsilon := \{k \geq 1 \mid w_k \geq k^{-\beta - \epsilon}, \lambda_k \geq k^{-\alpha - \epsilon}\},$$

and also

(3) $$G_n^\epsilon := \{k \in [\![n, 2n]\!] \mid w_k \geq n^{-\beta - \epsilon}, \lambda_k \geq n^{-\alpha - \epsilon}\}.$$

As said earlier, we separate the case $\beta < 1$, the case $\beta > 1$ and the case $\beta = 1$.

HYPOTHESIS $\bigcirc_{\alpha, \beta}$. *We have $\alpha > 0$ and $\beta < 1$ and for all $n \geq 1$, $\lambda_n \leq n^{-\alpha + o(1)}$ and $w_n \leq n^{-\beta + o(1)}$. Furthermore, $W_n = n^{1 - \beta + o(1)}$ and for all $\epsilon > 0$,*

$$\liminf_{n \to \infty} \frac{\sum_{k=1}^n w_k \mathbf{1}_{\{k \in G^\epsilon\}}}{\sum_{k=1}^n w_k} > 0.$$

The last display ensures that for all $\epsilon > 0$, the set G^ϵ contains asymptotically a positive proportion of the total weight.

HYPOTHESIS $\diamond_{\alpha, \beta}$. *We have $\alpha > 0$ and $\beta > 1$ and for all $n \geq 1$, $\lambda_n \leq n^{-\alpha + o(1)}$ and $w_n \leq n^{-\beta + o(1)}$. Furthermore, for all $\epsilon > 0$,*

$$\#G_n^\epsilon \underset{n \to \infty}{=} n^{1 + o(1)}.$$

Under the stronger assumption, $\lambda_n = n^{-\alpha+o(1)}$ and $w_n = n^{-\beta+o(1)}$, Hypothesis $\diamond_{\alpha,\beta}$ holds if $\beta > 1$ (resp., Hypothesis $\bigcirc_{\alpha,\beta}$, if $\beta < 1$). The case $\beta = 1$ is slightly different and in this case we set

HYPOTHESIS $\square_{\alpha,1}$.   *We have $\alpha > 0$ and $\beta = 1$ and for all $n \geq 1$, $\lambda_n \leq n^{-\alpha+o(1)}$ and $w_n \leq n^{-1+o(1)}$. Furthermore, for all $\epsilon > 0$,*

$$\frac{1}{\log\log\log N} \sum_{k=N}^{N^{1+\epsilon}} \frac{w_k}{W_k} \mathbf{1}_{\{k \in G^\epsilon\}} \underset{N\to\infty}{\longrightarrow} +\infty.$$

Note that this last hypothesis requires in particular that $W_n \to \infty$ as $n \to \infty$. Now let us define Hypothesis H_d, for any $d \geq 0$, which will ensure that our random underlying block has the appropriate d-dimensional behaviour.

HYPOTHESIS H_d.   *The law of the block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ satisfies the following conditions*:

(i)   • *If $d = 0$, the block B is a finite metric space which is not a.s. reduced to a single point and such that the measure ν satisfies $\nu(\{x\}) > 0$, for all points $x \in \mathsf{B}$.*
    • *If $d > 0$, there exists an increasing function $\varphi : [0, 1] \to [0, d/2]$, satisfying $\lim_{r\to 0} \varphi(r) = 0$, such that almost surely, there exists a (random) $r_0 \in (0, 1)$ such that*

$$(\star_{r_0}) \qquad \forall r \in [0, r_0), \forall x \in \mathsf{B}, \quad r^{d+\varphi(r)} \leq \nu(\mathsf{B}(x, r)) \leq r^{d-\varphi(r)}.$$

(ii) *Let $N_r(\mathsf{B})$ be the minimal number of balls of radius r needed to cover B. Then*

$$\mathbb{E}[N_r(\mathsf{B})] \leq r^{-d+o(1)} \quad \text{as } r \to 0.$$

(iii) *For all $k \geq 0$, we have $\mathbb{E}[\text{diam}(\mathsf{B})^k] < \infty$.*

Here, $\mathsf{B}(x, r)$ is the open ball centred at x with radius r and the notation $\text{diam}(\mathsf{B})$ denotes the diameter of B, defined as the maximal distance between two points of B. The conditions (i) and (ii) ensure that the blocks that we glue together have dimension d. The condition (iii) ensures that the blocks cannot be too big. In the paper, some results are stated under some weaker assumptions on the distribution of random block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ and they are hence all still valid under Hypothesis H_d.

Motivations.   The assumptions of Theorem 1 are rather general and various known models fall into our setting. First, let us cite two constructions that were already covered by the work presented in [7]. Of course, we have Aldous' line-breaking construction of the CRT but let us also cite the work of Ross and Wen in [19], in which the authors study a discrete model of growing trees and prove that its scaling limit can be described as a line-breaking procedure à la Aldous using

a Poisson process of intensity $t^l \mathrm{d}t$, with l an integer. The Hausdorff dimension of the resulting tree is then $(l + 1)/l$. Our extended setting now also includes the *Brownian looptree*, defined in [6], which appears as the scaling limit of the so-called *discrete looptree* associated with Barabási–Albert model. This random metric space also has a natural construction through an aggregation of circles, and our theorem proves that this object has almost surely Hausdorff dimension 2. These examples do not really use our theorem in its full generality since their underlying block is deterministic. In fact, Hypothesis H_d is very general and is satisfied (for the appropriate $d \geq 0$) by many distributions of blocks, including the Brownian CRT ($d = 2$); see [8], for the Brownian map ($d = 4$), see [15, 20], for the θ-stable trees ($d = \frac{\theta+1}{\theta}$), see [9]. Hence, our results can apply to a whole variety of such constructions, with a very general distributions for the blocks, and we are currently working on some examples in which this construction naturally arises as the limit of discrete models.

Indications on the proofs. The computations of the dimension in Theorem 1 differ, depending on the assumptions we make on α and β, and always consist of an upper bound, that we derive by providing explicit coverings, and a lower bound that arises from the construction of a probability measure satisfying the assumptions of Frostman's lemma; see Lemma 20 in the Appendix for a statement.

If we just assume that the scaling factors are smaller than $n^{-\alpha+o(1)}$, we can prove that the dimension is bounded above by $\frac{1}{\alpha}$ for rather general behaviours of the weights. To do so, we adapt arguments from [7] to our new setting. The essential idea behind the proof is that the substructure descending from a block $\mathbf{b}_n$ has size $n^{-\alpha+o(1)}$, and so that one only needs to cover every block $\mathbf{b}_n$ with a ball of radius $n^{-\alpha+o(1)}$ to cover the whole structure.

When $\alpha < \frac{1}{d}$ and $\beta > 1$, although the substructure descending from a block $\mathbf{b}_n$ may have diameter of order $n^{-\alpha+o(1)}$, we can also check that the index of the first block glued on block n has index roughly n^β, which is large compared to n. Hence, the diameter of the substructure descending from $\mathbf{b}_n$ is essentially due to $\mathbf{b}_n$ itself. This gives us a hint that we can cover the whole substructure descending from the block $\mathbf{b}_n$, using a covering of $\mathbf{b}_n$ with balls that are really small compared to the size of $\mathbf{b}_n$, and that it would lead to a more optimal covering. In fact we use these two observations to recursively construct a sequence of finer and finer coverings, which lead to the optimal upper bound. The idea of the proof is presented in more detail in Section 3.2.1.

Concerning the lower bounds, for all values of α and β, we can define a natural probability measure $\bar{\mu}$ on $\mathcal{T}$ as the limit of (a normalised version of) the measure μ_n defined on $\mathcal{T}_n$ for every $n \geq 1$, see Section 2. In the case $\beta \leq 1$, this probability measure only charges the leaves of $\mathcal{T}$, and an application of Lemma 20 gives the lower bound $\frac{1}{\alpha}$.

For $\beta > 1$, the measure $\bar{\mu}$ does not charge the leaves and so the preceding argument does not work. We construct another measure as the subsequential limit of a sequence of measures (π_k) which are concentrated on sets of the form $(\mathcal{T}_{2n_k} \setminus \mathcal{T}_{n_k})$ with (n_k) chosen appropriately; see Section 4.2.1 for a presentation of the idea of the proof. The limiting measure is then concentrated on a strict subset of leaves and again, using Lemma 20 yields the appropriate lower bound.

Related constructions. Let us also cite some other models that have been studied in the literature and which share some features with ours. First, the line-breaking construction of the scaling limit of critical random graphs in [2] by Addario-Berry, Broutin and Goldschmidt, that of the stable trees in [11] by Goldschmidt and Haas, and that of the stable graphs in [12] by Goldschmidt, Haas and Sénizergues (in preparation), use a gluing procedure that is identical to ours. Their constructions are not directly handled by Theorem 1 but they fall in a slightly more general setting, for which our proofs still hold. In [5], Borovkov and Vatutin study a discrete tree constructed recursively, which corresponds to the "genealogical tree" of the blocks in our model. Lastly, in [18], Rembart and Winkel study the distribution of random trees that satisfy a self-similarity condition (in law). They provide an iterative construction of those trees in which infinitely many branches are glued at each step.

Plan of the paper. In Section 1, we give a rigorous definition of our model, set up some useful notation, and discuss some general properties. In the second section, we study the (normalised) natural measure $\bar{\mu}_n$ on $\mathcal{T}_n$ and prove that it converges to a measure $\bar{\mu}$ on $\mathcal{T}$ under suitable assumptions. In Section 3.1, we prove the almost sure compactness of $\mathcal{T}$ and some upper bounds on its Hausdorff dimension under some relatively weak hypotheses. In Section 3.2, we develop a new (more involved) approach that allows us to obtain a better upper bound for some parameters for which the former fails to be optimal. In Section 4, we prove the lower bounds that match the upper bounds obtained in Section 3. It is again divided in two subsections, each providing a proof that is only valid for some choices of parameters α and β. The Appendix A.2 contains a short reminder of basic properties concerning Hausdorff dimension. The Appendices A.1, A.3 and A.4 contain some technical proofs that can be skipped at first reading.

1. General framework. In this section, we start by providing a precise definition of our model and then we investigate some of its general properties.

1.1. *Construction.* Consider $((\mathbf{b}_n, \mathbf{d}_n, \boldsymbol{\rho}_n, \boldsymbol{\nu}_n))_{n \geq 1}$ a sequence of compact pointed metric spaces endowed with a finite Borel measure. Recall from the Introduction the heuristics of our recursive construction. We define $\mathcal{T}_1$ as the first block $\mathbf{b}_1$ endowed with its measure $\boldsymbol{\nu}_1$. Then at each step, we construct $\mathcal{T}_{n+1}$ from $\mathcal{T}_n$ by gluing the root of the block $\mathbf{b}_{n+1}$ to a random point $X_n \in \mathcal{T}_n$, which has

distribution (a normalised version of) μ_n. The measure μ_{n+1} is defined as the sum of the measures μ_n and ν_{n+1}, the measure supported by $\mathbf{b}_{n+1}$. We define $\mathcal{T}^*$ as the increasing union of all the $\mathcal{T}_n$ for $n \geq 1$, and its completion is denoted $\mathcal{T}$. In the next paragraph, we describe formally how to construct such growing metric spaces as subsets of a larger ambient space. The definitions here are rather technical and the proofs in the paper do not use the details of the construction, so the reader can skip this part at first reading.

Embedded construction. We consider (U, δ) the Urysohn space, and fix a point $u_0 \in U$. The space U is defined as the only Polish metric space (up to isometry) which has the following extension property (see [14] for constructions and basic properties of U): given any finite metric space X, and any point $x \in X$, any isometry from $X \setminus \{x\}$ to U can be extended to an isometry from X to U. In the rest of the construction, we assume that the measured metric spaces $((\mathbf{b}_n, \mathbf{d}_n, \boldsymbol{\rho}_n, \boldsymbol{\nu}_n))_{n \geq 1}$ are all embedded in the space U and that their root is identified to u_0. From the properties of the Urysohn space, this is always possible (see Appendix A.1 for a construction in the case of random blocks).

We introduce

$$\ell^1(U, u_0) := \left\{ (x_n)_{n \geq 1} \in U^{\mathbb{N}^*} \;\middle|\; \sum_{n=1}^{\infty} \delta(x_n, u_0) < +\infty \right\}.$$

If we endow $\ell^1(U, u_0)$ with the distance $\mathrm{d}((x_n)_{n \geq 1}, (y_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \delta(x_n, y_n)$, it is an easy exercise to see that it makes this space Polish. We can now construct the $\mathcal{T}_n$ recursively, by $\mathcal{T}_1 = \{(x, u_0, u_0, \dots) \mid x \in \mathbf{b}_1\}$, and identifying $\mathcal{T}_1$ to the block $\mathbf{b}_1$, we set $\mu_1 = \nu_1$. For $n \geq 1$, the point X_n is sampled according to $\bar{\mu}_n$ a normalised version of μ_n with total mass 1. The point X_n is of the form $(x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}, u_0, \dots)$ and we set

$$\mathcal{T}_{n+1} := \mathcal{T}_n \cup \left\{ (x_1^{(n)}, x_2^{(n)}, \dots x_n^{(n)}, x, u_0, \dots) \mid x \in \mathbf{b}_{n+1} \right\}.$$

We set $\mu_{n+1} := \mu_n + \nu_n$, where as in the preceding section, we see $\mathbf{b}_{n+1}$ as the corresponding subset of $\mathcal{T}_{n+1}$. Then $\mathcal{T}^* = \bigcup_{n \geq 1} \mathcal{T}_n$ and $\mathcal{T} = \overline{(\mathcal{T}^*)}$ is its closure in the space $(\ell^1(U, u_0), \mathrm{d})$. At the end, $\mathcal{T}$ is a random closed subset of a Polish space.

In the rest of the paper, we will not refer to this formal construction of $\mathcal{T}$ and we will identify $\mathbf{b}_n$ with the corresponding subset in $\mathcal{T}$. We recall the notation $W_n = \sum_{k=1}^{n} w_n$ for the total mass of the measure μ_n.

1.2. *Some notation.* Let us introduce some notation that will be useful in the sequel, some of which is illustrated in Figure 3. Recall that, from now on, we always assume that the blocks are of the form (1).

- If (E, d, ρ) is a pointed metric space, and $x \in E$, we define $\mathrm{ht}(x)$, the height of x, as its distance to the root $\mathrm{d}(\rho, x)$. We also denote $\mathrm{ht}(E) = \sup_{x \in E} \mathrm{ht}(x)$, the
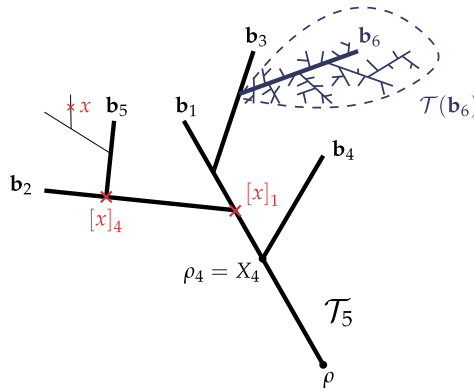
FIG. 3. *Substructure descending from a set, and projection on a substructure, illustrating some notation introduced in the paper in the case of the gluing of segments.*

height of E. Let us consider $(\mathsf{B}, \mathsf{D}, \rho, \nu)$, a random block of our model before scaling, and X a point of B which conditionally on $(\mathsf{B}, \mathsf{D}, \rho, \nu)$, has distribution ν. We denote

$$(4) \qquad\qquad \mathcal{H} := \mathrm{ht}(X) = \mathsf{D}(\rho, X),$$

the height of a uniform random point in the block. Remark that Hypothesis H_d implies that $\mathbb{E}[\mathcal{H}^2] < \infty$, and that $\mathbb{P}(\mathcal{H} > 0) > 0$. Some of our results are stated under these weaker assumptions.

- Whenever we sample the point X_n under $\bar{\mu}_n$, we do it in the following way: first, we sample K_n such that for all $1 \leq k \leq n$, $\mathbb{P}(K_n = k) = \frac{w_k}{W_n}$ and then, conditionally on $K_n = k$, the point X_n is chosen on the block $\mathbf{b}_k$ using the normalised version of the measure ν_k. Whenever $K_n = k$, we say that $\mathbf{b}_{n+1}$ is grafted onto $\mathbf{b}_k$ and write $\mathbf{b}_{n+1} \to \mathbf{b}_k$. Remark that this entails that $X_n \in \mathbf{b}_k$, but this condition is not sufficient in the case where X_n belongs to several blocks (which only happens if the measures carried by the blocks have atoms). We denote

$$(5) \qquad\qquad \bar{\mu}_n^* := \text{law of } (K_n, X_n),$$

seen as a measure on $\bigsqcup_{k=1}^n \{k\} \times \mathbf{b}_k$. In this way, the random variables $((K_n, X_n))_{n \geq 1}$ are independent with respective distributions $(\bar{\mu}_n^*)_{n \geq 1}$. We remain loose on the fact that we sometimes consider the blocks as abstract metric spaces and at other times we see them as subsets of $\mathcal{T}$. It is implicit in the preceding discussion that everything is expressed conditionally on the sequence of blocks $(\mathbf{b}_n)_{n \geq 1}$.

- We simultaneously construct a sequence of increasing discrete trees $(\mathsf{T}_n)_{n \geq 1}$ by saying that for $n \geq 1$, the tree T_n has n nodes labelled 1 to n and i is a child of j if and only if $\mathbf{b}_i \to \mathbf{b}_j$. Also define T their increasing union. We denote $\prec$ the genealogical order on $\mathbb{N}^*$ induced by this tree. We denote $\mathsf{d}_\mathsf{T}(i, j)$ for the graph

distance between the nodes with label i and j in this tree and $\text{ht}_\mathsf{T}(\cdot)$ for their height.

- For $x \in \mathcal{T}$, we define $[x]_n$, the projection of x on $\mathcal{T}_n$, as the unique point y of $\mathcal{T}_n$ that minimizes the distance $\text{d}(x, y)$.
- Similarly, for $k \geq 1$, we define $[k]_n$, the projection of k on T_n, as the unique node $i \leq n$ that minimizes the distance $\text{d}_\mathsf{T}(i, k)$.
- If S is a subset of a block $\mathbf{b}_n$ for some $n \geq 1$, then we define $\mathcal{T}(S)$, the substructure descending from S as

$$\mathcal{T}(S) := \overline{S \cup \bigcup_{\substack{i \succ n \\ [X_{i-1}]_n \in S}} \mathbf{b}_i}.$$

If $S = \mathbf{b}_n$, this reduces to

$$\mathcal{T}(\mathbf{b}_n) = \overline{\bigcup_{i \geq n} \mathbf{b}_i},$$

and we consider $(\mathcal{T}(\mathbf{b}_n), \text{d}, \boldsymbol{\rho}_n)$ as a rooted metric space.

- Remark that if $x \in \mathcal{T}(\mathbf{b}_k)$ for some $k \geq 1$ then we have $[x]_k \in \mathbf{b}_k$ and more generally, for any $n \leq k$, we have $[x]_n \in \mathbf{b}_{[k]_n}$.
- We often use the little-o notation and denote $o(1)$ a deterministic function that tends to 0 when some parameter tends to 0 or ∞, depending on the context. For such functions that are random, we write instead $o^\omega(1)$.

1.3. *Zero-one law for compactness, boundedness and Hausdorff dimension.* The main properties of $\mathcal{T}$ and $\mathcal{L}$ that we study are compactness and Hausdorff dimension. One can check that some of these properties are constants almost surely by an argument using Kolmogorov's zero-one law.

Indeed, take the whole construction $\mathcal{T}$ and contract the compact subspace $\mathcal{T}_n$ into a single point. We can easily check that the resulting space is compact (resp., bounded) iff the former is compact (resp., bounded). Also, the subset $\mathcal{L}$ and its image after the contraction of $\mathcal{T}_n$ have the same Hausdorff dimension. Now remark that the space that we just described only depends on the randomness of the blocks and the gluings after n steps. Indeed, if we start at time n with a unique point with weight W_n and then follow the procedure by gluing recursively $\mathbf{b}_{n+1}, \mathbf{b}_{n+2}, \ldots$, we get exactly the same space.

Hence, as this is true for all n, these properties only depend on the tail σ-algebra generated by the blocks and the gluings, and are therefore satisfied with probability 0 or 1.

REMARK 2. In the setting of [7], where the blocks are segments and the weights correspond to the lengths of those segments, the authors proved that the event of boundedness and compactness for $\mathcal{T}$ coincide almost surely. This is not

the case in our more general setting: consider the case of branches with weights and lengths defined as

$$w_n = 2^n, \qquad \lambda_n = 2^{-n} \quad \text{for } n \notin \{2^k \mid k \in \mathbb{N}\},$$
$$w_{2^k} = 1, \qquad \lambda_{2^k} = 1, \quad \text{for } k \in \mathbb{N}.$$

In this case, an application of the Borel–Cantelli lemma shows that a.s. for n large enough, no branch $\mathbf{b}_n$ is ever grafted onto a branch $\mathbf{b}_{2^k}$ for any k. It is then clear that the resulting tree is a.s. bounded since the sum of the lengths of the branches $\mathbf{b}_n$ for $n \notin \{2^k \mid k \in \mathbb{N}\}$ is finite, but it cannot be compact since there exists an infinite number of branches with length 1.

1.4. *Monotonicity of Hausdorff dimension.* Let us present an argument of monotonicity of the Hausdorff dimension of $\mathcal{L}$ with respect to the sequence (λ_n), on the event on which $\mathcal{T}$ is compact. Let (w_n) be a sequence of weights and (λ_n) and (λ'_n) be two sequences of scaling factors such that for all $n \geq 1$, we have $\lambda_n \geq \lambda'_n$. Suppose that $((\mathsf{B}_n, \mathsf{D}_n, \rho_n, \nu_n))_{n \geq 1}$ is a sequence of random compact metric spaces endowed with a probability measure. Then let $\mathcal{T}$ (resp., $\mathcal{T}'$) be the structure constructed using the blocks $(\mathbf{b}_n, \mathbf{d}_n, \boldsymbol{\rho}_n, \boldsymbol{\nu}_n) = (\mathsf{B}_n, \lambda_n \cdot \mathsf{D}_n, \rho_n, w_n \cdot \nu_n)$, for $n \geq 1$ (resp., $(\mathbf{b}'_n, \mathbf{d}'_n, \boldsymbol{\rho}'_n, \boldsymbol{\nu}'_n) = (\mathsf{B}_n, \lambda'_n \cdot \mathsf{D}_n, \rho_n, w_n \cdot \nu_n)$). Note that since we use the same sequence of weights we can couple the two corresponding gluing procedures.

Let f be the application that maps each of the block $\mathbf{b}_n$ to the corresponding $\mathbf{b}'_n$. Recall here that we see the blocks as subsets of the structure. We can verify that $f : \mathcal{T}^* \longrightarrow (\mathcal{T}')^*$, is 1-Lipschitz. We can then extend uniquely f to a function $\hat{f} : \mathcal{T} \longrightarrow \mathcal{T}'$, which is also 1-Lipschitz. Suppose $\mathcal{T}$ is compact. Then its image $\hat{f}(\mathcal{T})$ is compact, hence closed in $\mathcal{T}'$. Since $(\mathcal{T}')^* \subset \hat{f}(\mathcal{T})$ and $(\mathcal{T}')^*$ is dense in $\mathcal{T}'$, we have $\hat{f}(\mathcal{T}) = \mathcal{T}'$ and so $\hat{f}$ is surjective. Now since $(\mathcal{T}')^* = \hat{f}(\mathcal{T}^*)$, we also have $\mathcal{L}' = \hat{f}(\mathcal{L})$, and since $\hat{f}$ is Lipschitz,

$$(6) \qquad \qquad \dim_{\mathrm{H}}(\mathcal{L}') \leq \dim_{\mathrm{H}}(\mathcal{L}).$$

2. Study of a typical point. In this section, we study the height of a typical point of $\mathcal{T}_n$, that is, the distance from the root to a point sampled according to $\bar{\mu}_n$. The proofs in this section are really close to those of [7], Section 1, to which we refer for details.

2.1. *Coupling with a marked point.* We construct a sequence of points $(Y_n)_{n \geq 1}$ coupled with the sequence $(\mathcal{T}_n)_{n \geq 1}$ in such a way that for all $n \geq 1$, the point Y_n has distribution $\bar{\mu}_n$ conditionally on $\mathcal{T}_n$ and such that the distance from Y_n to the root is nondecreasing in n. For technical reasons, we in fact define a sequence $((J_n, Y_n))_{n \geq 1}$ such that for any $n \geq 1$, (J_n, Y_n) has distribution $\bar{\mu}_n^*$ conditionally on $(\mathsf{T}_n, \mathcal{T}_n)$, see (5). The properties of this construction are stated in the following lemma.

LEMMA 3. *We can couple the construction of $((\mathsf{T}_n, \mathcal{T}_n))_{n \geq 1}$ with a sequence $((J_n, Y_n))_{n \geq 1}$ such that for all $n \geq 1$:*

(i) *we have $J_n \in \{1, \ldots, n\}$ and $Y_n \in \mathbf{b}_{J_n}$,*
(ii) *conditionally on $(\mathsf{T}_n, \mathcal{T}_n)$, the couple (J_n, Y_n) has distribution $\bar{\mu}_n^*$,*
(iii) *for all $1 \leq k \leq n$, $([J_n]_k, [Y_n]_k) = (J_k, Y_k)$.*

Furthermore, under the assumption $H_d(\mathrm{iii})$, the sequence $(Y_n)_{n \geq 1}$ almost surely converges in $\mathcal{T}$ iff

$$(7) \qquad \sum_{n=1}^{\infty} \frac{\lambda_n w_n}{W_n} \mathbf{1}_{\{\lambda_n \leq 1\}} < \infty \quad and \quad \sum_{n=1}^{\infty} \frac{w_n}{W_n} \mathbf{1}_{\{\lambda_n > 1\}} < \infty.$$

Note that if either

$$W_\infty := \sum_{n=1}^{\infty} w_n < \infty \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{w_n \lambda_n}{W_n} < \infty,$$

then (7) is satisfied, and this is the case under the assumptions of Theorem 1. In this case, we let

$$Y := \lim_{n \to \infty} Y_n.$$

PROOF OF LEMMA 3. Let $n \geq 2$. Conditionally on $\mathcal{T}_n$ and T_n, sample a couple (J_n, Y_n) under the measure $\bar{\mu}_n^*$. Then two cases may happen:

- with probability $1 - w_n/W_n$: we have $J_n < n$, so the point Y_n belongs to $\mathcal{T}_{n-1}$, that is $[Y_n]_{n-1} = Y_n$, and conditionally on this event $([J_n]_{n-1}, [Y_n]_{n-1})$ has the same distribution as (J_{n-1}, Y_{n-1}),
- with probability w_n/W_n: we have $J_n = n$. In this case, the point Y_n is located on the last block $\mathbf{b}_n$ grafted on $\mathcal{T}_{n-1}$. Conditionally, on this event (if $w_n > 0$), Y_n is distributed on this block under the measure $\boldsymbol{\nu}_n$ and the couple $([J_n]_{n-1}, [Y_n]_{n-1})$ is independent of the location of Y_n on the nth block and has the same distribution as (J_{n-1}, Y_{n-1}).

From this observation, we deduce that

$$\big(\mathcal{T}_{n-1}, \mathsf{T}_{n-1}, [J_n]_{n-1}, [Y_n]_{n-1}\big) = (\mathcal{T}_{n-1}, \mathsf{T}_{n-1}, J_{n-1}, Y_{n-1})$$

in distribution and more generally, $(\mathcal{T}_k, \mathsf{T}_k, [J_n]_k, [Y_n]_k) = (\mathcal{T}_k, \mathsf{T}_k, J_k, Y_k)$ in distribution for all $1 \leq k \leq n$.

Reversing this observation, we can construct a sequence $(J_n, Y_n)_{n \geq 1}$ (coupled to the K_n and X_n involved in the construction of $\mathcal{T}^*$) such that conditionally on $\mathcal{T}_n$ and T_n, the couple (J_n, Y_n) has distribution $\bar{\mu}_n^*$ and that for all $1 \leq k \leq n$, we have $([J_n]_k, [Y_n]_k) = (J_k, Y_k)$. To do so, we consider:

- a sequence $(U_n)_{n \geq 1}$ of uniform random variables on $(0, 1)$,
- a sequence $(Z_n)_{n \geq 1}$ of points respectively sampled on $(\mathbf{b}_n)_{n \geq 1}$ with respective distribution (a normalised version of) the measure $(\boldsymbol{\nu}_n)_{n \geq 1}$ whenever it is nonzero, (set $Z_n = \boldsymbol{\rho}_n$ a.s. whenever $\boldsymbol{\nu}_n$ is trivial),
- a sequence $(I_n, P_n)_{n \geq 1}$, sampled with respective distributions $(\bar{\mu}_n^*)_{n \geq 1}$,

independently for all these random variables. Then we construct (K_n, X_n) and (J_n, Y_n) as follows. We set $(J_1, Y_1) = (1, Z_1)$. Then recursively for $n \geq 1$, we assume that X_{n-1} (if $n \neq 1$) and Y_n have been constructed:

- if $U_{n+1} \leq \frac{w_{n+1}}{W_{n+1}}$, then we set $(K_n, X_n) := (J_n, Y_n)$, $J_{n+1} := n + 1$ and $Y_{n+1} := Z_{n+1}$,
- if $U_{n+1} > \frac{w_{n+1}}{W_{n+1}}$, then we set $(K_n, X_n) := (I_n, P_n)$, $J_{n+1} := J_n$ and $Y_{n+1} := Y_n$.

We can check that with this construction, for all $1 \leq k \leq n$, we have $([J_n]_k, [Y_n]_k) = (J_k, Y_k)$, the $(K_n, X_n)_{n \geq 1}$ are independent with the appropriate distribution and for all $n \geq 1$ conditionally on $\mathcal{T}_n$ and T_n the couple (J_n, Y_n) has distribution $\bar{\mu}_n^*$. Notice that the distance from Y_n to the root ρ is nondecreasing. Denoting $\mathcal{T}_0 = \{\rho\}$ and $Y_0 = \rho$, for all $0 \leq m \leq n$, we have

$$(8) \qquad \mathrm{d}(Y_n, Y_m) = \mathrm{d}(Y_n, \mathcal{T}_m) = \sum_{k=m+1}^{n} \mathrm{d}(Z_k, \rho_k) \mathbf{1}_{\{U_k \leq \frac{w_k}{W_k}\}},$$

which is equal in distribution to

$$\sum_{k=m+1}^{n} \lambda_k \mathcal{H}_k \mathbf{1}_{\{U_k \leq \frac{w_k}{W_k}\}},$$

where the $(\mathcal{H}_k)_{k \geq 1}$ are i.i.d., independent of the $(U_k)_{k \geq 1}$ and have the law of $\mathcal{H}$, see (4). Under H_d(iii), the random variable $\mathcal{H}$ has a finite second moment, and an application of Kolmogorov's three series theorem tells us that the almost sure convergence of $\sum_{k \geq 1} \lambda_k \mathcal{H}_k \mathbf{1}_{\{U_k \leq \frac{w_k}{W_k}\}}$ is equivalent to (7). In this case, $(Y_n)_{n \geq 1}$ is a Cauchy sequence in the complete space $\mathcal{T}$, and hence it converges. $\qquad \square$

Also notice that with this construction, the discrete counterpart of (8) is

$$(9) \qquad \mathrm{d}_{\mathsf{T}}(J_n, J_m) = \sum_{k=m+1}^{n} \mathbf{1}_{\{U_k \leq \frac{w_k}{W_k}\}} \quad \text{and so} \quad \mathrm{ht}_{\mathsf{T}}(J_n) = \sum_{k=2}^{n} \mathbf{1}_{\{U_k \leq \frac{w_k}{W_k}\}}.$$

Remark that for any $\theta \in \mathbb{R}$,

$$
\begin{aligned}
\mathbb{E}\big[\exp(\theta \, \mathrm{ht}_\mathsf{T}(J_n))\big] &= \mathbb{E}\left[\exp\left(\theta \sum_{k=2}^n \mathbf{1}_{\{U_k \le \frac{w_k}{W_k}\}}\right)\right] \\
&= \prod_{k=2}^n \left(\frac{W_k - w_k}{W_k} \cdot 1 + \frac{w_k}{W_k} e^\theta\right) \\
&= \exp\left(\sum_{k=2}^n \log\left(1 + (e^\theta - 1)\frac{w_k}{W_k}\right)\right) \\
&\le \exp\left((e^\theta - 1)\sum_{k=2}^n \frac{w_k}{W_k}\right),
\end{aligned}
$$

(10)

where in the last line we use the inequality $\log(1 + x) \le x$, valid for all $x > -1$.

2.2. *Convergence of the measure $\bar{\mu}_n$.*

PROPOSITION 4. *Assume that $\mathbb{E}[\mathcal{H}^2] < \infty$ and $\mathbb{P}(\mathcal{H} > 0) > 0$ and that (7) holds. Then almost surely there exists a probability measure $\bar{\mu}$ on $\mathcal{T}$ such that*

$$
\bar{\mu}_n \underset{n \to \infty}{\longrightarrow} \bar{\mu} \quad \text{weakly.}
$$

Furthermore, conditionally on $(\mathcal{T}, \bar{\mu})$, the point Y is distributed according to $\bar{\mu}$ almost surely. If $W_\infty < \infty$, then $\bar{\mu} = \frac{1}{W_\infty}\mu_\infty$, and $\bar{\mu}$ is concentrated on $\mathcal{T}^*$. If $W_\infty = \infty$, then $\bar{\mu}$ is concentrated on $\mathcal{L}$.

The proof of the last proposition is very similar to the proof of [7], Theorem 4, and is left to the reader. We can easily check that the assumptions of Proposition 4 are satisfied under the hypotheses of Theorem 1. We now state an additional lemma that will be useful later in the paper.

LEMMA 5. *Suppose that the assumptions of Proposition 4 hold, that $\bar{\mu}$ is concentrated on the set $\mathcal{L}$ and that the sequence of weights satisfies $\frac{w_n}{W_n} \le n^{-1+o(1)}$. Then almost surely, we have*

$$
\bar{\mu}\big(\mathcal{T}(\mathbf{b}_n)\big) \le n^{-1+o^\omega(1)},
$$

where the random function $o^\omega(1)$ is considered as $n \to \infty$.

PROOF. Let us introduce some notation. If $i \ge n$, we set $M_i^{(n)} := \bar{\mu}_i(\mathcal{T}(\mathbf{b}_n))$ the relative mass of the tree descending from $\mathbf{b}_n$ in $\mathcal{T}_i$. As i varies, this sequence of random variables evolves like one of Pemantle's time-dependent Pólya urns (see [17]) and is therefore a martingale. The topological boundary of $\mathcal{T}(\mathbf{b}_n)$ in $\mathcal{T}$ is

either the empty set or the singleton $\{\boldsymbol{\rho}_n\}$, thus it has zero $\bar{\mu}$-measure.[2] It follows from Portmanteau theorem that the quantity of interest $\bar{\mu}(\mathcal{T}(\mathbf{b}_n))$ corresponds to $M_\infty^{(n)}$, the almost sure limit of this positive martingale. We can write

$$M_{i+1}^{(n)} = \left(\frac{W_i}{W_{i+1}}\right)M_i^{(n)} + \frac{w_{i+1}}{W_{i+1}}\mathbf{1}_{\{U_{i+1}\leq M_i^{(n)}\}},$$

with $(U_i)_{i\geq 1}$ a sequence of i.i.d. random variables, uniform on $(0,1)$. We are going to show by induction on $k \geq 1$ that there exists a function $o(1)$ as $n \to \infty$ such that for all $i \geq n$, we have

$$\mathbb{E}[(M_i^{(n)})^k] \leq n^{-k+o(1)}.$$

Note that we use the notation $o(1)$ for all such functions, but that in this proof, the corresponding functions can depend on k but not on i.

- For $k = 1$, the result follows from the fact that $(M_i^{(n)})_{i\geq n}$ is a martingale and that almost surely $M_n^{(n)} \leq \frac{w_n}{W_n} \leq n^{-1+o(1)}$.
- Let $k \geq 2$. Suppose that the result is true for all $1 \leq l \leq k-1$. Then

$$\mathbb{E}[(M_{i+1}^{(n)})^k \mid M_i^{(n)}]$$

$$= \mathbb{E}\left[\left(\left(\frac{W_i}{W_{i+1}}\right)M_i^{(n)} + \frac{w_{i+1}}{W_{i+1}}\mathbf{1}_{\{U_{i+1}\leq M_i^{(n)}\}}\right)^k \mid M_i^{(n)}\right]$$

$$= \left(\frac{W_i}{W_{i+1}}\right)^k (M_i^{(n)})^k$$

$$+ \mathbb{E}\left[\sum_{l=0}^{k-1}\binom{k}{l}\left(\frac{W_i}{W_{i+1}}\right)^l (M_i^{(n)})^l \left(\frac{w_{i+1}}{W_{i+1}}\mathbf{1}_{\{U_{i+1}\leq M_i^{(n)}\}}\right)^{k-l} \mid M_i^{(n)}\right]$$

$$= \left(\frac{W_i}{W_{i+1}}\right)^k (M_i^{(n)})^k + \sum_{l=0}^{k-1}\binom{k}{l}\left(\frac{W_i}{W_{i+1}}\right)^l (M_i^{(n)})^{l+1}\left(\frac{w_{i+1}}{W_{i+1}}\right)^{k-l}$$

$$\leq (M_i^{(n)})^k \left(\left(\frac{W_i}{W_{i+1}}\right)^k + k \cdot \frac{w_{i+1}}{W_{i+1}}\left(\frac{W_i}{W_{i+1}}\right)^{k-1}\right)$$

$$+ \sum_{l=0}^{k-2}\binom{k}{l}(M_i^{(n)})^{l+1}\left(\frac{w_{i+1}}{W_{i+1}}\right)^{k-l}.$$

---

[2]Indeed, under the assumptions of the lemma, $\bar{\mu}$ is carried on the leaves.

Now taking the expectation and using the fact that $\forall x \in [0,1]$, $(1-x)^k + k(1-x)^{k-1}x \leq 1$, we get, using the induction hypothesis,

$$\mathbb{E}[(M_{i+1}^{(n)})^k] \leq \mathbb{E}[(M_i^{(n)})^k] + \sum_{l=0}^{k-2} \binom{k}{l} \mathbb{E}[(M_i^{(n)})^{l+1}] \left(\frac{w_{i+1}}{W_{i+1}}\right)^{k-l}$$

$$\leq \mathbb{E}[(M_i^{(n)})^k] + \sum_{l=0}^{k-2} \binom{k}{l} n^{-(l+1)+o(1)} (i^{-1+o(1)})^{k-l}.$$

Summing over all i, we get that, for all $i \geq n$,

$$\mathbb{E}[(M_i^{(n)})^k] \leq \mathbb{E}[(M_n^{(n)})^k] + \sum_{j=n}^{\infty} \sum_{l=0}^{k-2} \binom{k}{l} n^{-l-1+o(1)} j^{-k+l+o(1)}$$

$$\leq \mathbb{E}[(M_n^{(n)})^k] + \sum_{l=0}^{k-2} \binom{k}{l} (n^{-l-1+o(1)}) \sum_{j=n}^{\infty} j^{-k+l+o(1)}$$

$$\leq n^{-k+o(1)} + \sum_{l=0}^{k-2} \binom{k}{l} n^{-l-1+o(1)} n^{-k+l+1+o(1)}$$

$$\leq n^{-k+o(1)}.$$

This completes the proof by induction. This property passes to the limit by dominated convergence so, for all $n \geq 1$, we have $\mathbb{E}[(M_\infty^{(n)})^k] \leq n^{-k+o(1)}$. For N an integer and $\epsilon > 0$,

$$\mathbb{P}(M_\infty^{(n)} \geq n^{-1+\epsilon}) \leq n^{N-N\epsilon} \mathbb{E}[(M_\infty^{(n)})^N]$$

$$\leq n^{-N\epsilon+o(1)}.$$

If we take N large enough, those quantities are summable and so, using the Borel–Cantelli lemma we get that with probability one, $M_\infty^{(n)} \leq n^{-1+\epsilon}$ for all n large enough. This completes the proof. $\square$

3. Upper bounds and compactness for the (α, β)-model. In this section, we compute upper bounds on the Hausdorff dimension of the set $\mathcal{L}$. We first prove Proposition 6, which tells us that, under the condition that $\lambda_n \leq n^{-\alpha+o(1)}$ for some $\alpha > 0$ and in a very general setting for the behaviour of the weights (w_n), the dimension is bounded above by $1/\alpha$. The techniques used in the proof are very robust, and do not depend on the geometry of the blocks nor on the sequence of weights. In a second step, in Proposition 7, we handle the more specific case where the underlying block satisfies Hypotheses H_d and that $\lambda_n \leq n^{-\alpha+o(1)}$ for some $0 < \alpha < 1/d$ and $w_n \leq n^{-\beta+o(1)}$ for some $\beta > 1$. In the proof of this proposition, a careful analysis allows us to refine some of the arguments of the previous proof and

prove upper bounds on the Hausdorff dimension of $\mathcal{L}$ that are below the "generic" value $1/\alpha$, given by Proposition 6. The techniques used for the proof are new and really take into account the behaviour of the weights and the geometry of the blocks.

3.1. *Upper bound independent of the weights and compactness.* Notice that under H_d(iii), the underlying block (B, D, ρ, ν) satisfies, for any $N > 0$,

$$\mathbb{P}(\mathrm{diam}(B) \geq n^\epsilon) \leq \frac{\mathbb{E}[\mathrm{diam}(B)^N]}{n^{-N\epsilon}},$$

which is summable if N is large enough. Hence, if (B_n) is an i.i.d. sequence with the same law as B, then using the Borel–Cantelli lemma we have almost surely,

$$\tag{11} \mathrm{diam}(B_n) \leq n^{o^\omega(1)}.$$

PROPOSITION 6. *Suppose $\lambda_n \leq n^{-\alpha+o(1)}$, with $\alpha > 0$, and that for all n, we have $W_n \leq n^\gamma$ for some $\gamma > 0$. Suppose also that* (11) *holds. Then the tree-like structure $\mathcal{T}$ is almost surely compact and we have*:

(i) $\mathrm{d_H}(\mathcal{T}_n, \mathcal{T}) \leq n^{-\alpha+o^\omega(1)}$,
(ii) $\dim_\mathrm{H}(\mathcal{L}) \leq \frac{1}{\alpha}$.

Since our model is invariant by multiplying all the weights by the same constant, we can always assume that $w_1 \leq 1$. Hence, the assumption in the lemma is always satisfied if W_n grows at most polynomially in n, which is the case if Hypothesis $\diamond_{\alpha,\beta}$, Hypothesis $\square_{\alpha,1}$ or Hypothesis $\bigcirc_{\alpha,\beta}$ is fulfilled, for any choice of $\alpha > 0$ and $\beta \in \mathbb{R}$.

PROOF OF PROPOSITION 6. We start with point (i). First,

$$\mathrm{d_H}(\mathcal{T}_{2^i}, \mathcal{T}_{2^{i+1}}) \leq \sup_{2^i+1 \leq k \leq 2^{i+1}} \lambda_k \,\mathrm{diam}(B_k) + \sup_{2^i+1 \leq k \leq 2^{i+1}} \mathrm{d}(\rho_k, \mathcal{T}_{2^i}).$$

For any $2^i \leq k \leq 2^{i+1} - 1$, the point ρ_{k+1} in the tree is identified with the point X_k, taken under the measure $\bar{\mu}_k$ on the tree $\mathcal{T}_k$. From our construction in Section 1.2, the point X_k belongs to some $\mathbf{b}_{K_k}$, and the couple (K_k, X_k) is sampled with measure $\bar{\mu}_k^*$. Bounding the contribution of every block along the ancestral line with their maximum, we get

$$\tag{12} \mathrm{d}(X_k, \mathcal{T}_{2^i}) \leq \Big(\sup_{2^i+1 \leq k \leq 2^i} \lambda_k \,\mathrm{diam}(B_k) \Big) \mathrm{ht}_\mathsf{T}(K_k).$$

Now, using Lemma 24 in Appendix A.4, we know that there exists a constant $C > 0$ such that $\sum_{i=1}^n \frac{w_i}{W_i} \leq C \log n$. Combining this with equation (10) (which

holds for K_k because it has the same distribution as J_k) and Markov inequality, we get for any $u > 0$,

$$\mathbb{P}\big(\mathrm{ht}_\mathsf{T}(K_n) \geq u \log n\big) \leq \exp\big((C(e-1) - u) \log n\big) = n^{C(e-1)-u}.$$

The last display is summable in n if we choose u large enough. Hence, using the Borel–Cantelli lemma, we almost surely have $\mathrm{ht}_\mathsf{T}(K_n) \leq u \log n$ for n large enough. Hence, in (12) we have $\mathrm{ht}_\mathsf{T}(K_k) = (2^i)^{o^\omega(1)}$. Combining this with (11) and the upper bound on λ_n, we get

$$\mathrm{d}_\mathrm{H}(\mathcal{T}_{2^i}, \mathcal{T}_{2^{i+1}}) \leq (2^i)^{-\alpha+o^\omega(1)}.$$

Replacing i by k and summing the last display over $k \geq i$,

$$\sum_{k=i}^{\infty} \mathrm{d}_\mathrm{H}(\mathcal{T}_{2^k}, \mathcal{T}_{2^{k+1}}) \leq (2^i)^{-\alpha+o^\omega(1)},$$

hence the sequence of compact sets $(\mathcal{T}_{2^i})$ is a.s. a Cauchy sequence for Hausdorff distance between compacts of the complete space $\mathcal{T}$. So the sequence $(\mathcal{T}_n)_{n \geq 1}$ is also Cauchy because of the increasing property of the construction, and $\mathcal{T}$ is then almost surely compact. Moreover, we have a.s.

$$\mathrm{d}_\mathrm{H}(\mathcal{T}_{2^i}, \mathcal{T}) \leq (2^i)^{-\alpha+o^\omega(1)},$$

and this entails (i). Remark that since $\mathrm{ht}(\mathcal{T}(\mathbf{b}_n)) \leq \mathrm{d}_\mathrm{H}(\mathcal{T}_{n-1}, \mathcal{T})$, this implies that a.s. we have

$$(13) \qquad\qquad \mathrm{ht}\big(\mathcal{T}(\mathbf{b}_n)\big) \leq n^{-\alpha+o^\omega(1)}.$$

We now prove point (ii). Let $\epsilon > 0$. From (13), the collection of balls $\mathrm{B}(\boldsymbol{\rho}_n, n^{-\alpha+\epsilon})$, for $n \geq N$, where N is an arbitrary number, is a covering of $\mathcal{L}$ whose maximal diameter tends to 0 as $N \to \infty$. Besides, if we fix δ, for N large enough, and $s > \frac{1}{\alpha-\epsilon}$, we have

$$\mathcal{H}_s^\delta(\mathcal{L}) \leq \sum_{n=N}^{\infty} \mathrm{diam}\big(\mathrm{B}(\boldsymbol{\rho}_n, n^{-\alpha+\epsilon})\big)^s \leq \sum_{n=N}^{\infty} 2^s n^{(-\alpha+\epsilon)s} \underset{N\to\infty}{\longrightarrow} 0.$$

Hence, for all such s, we have $\mathcal{H}_s(\mathcal{L}) = 0$ and so $\dim_\mathrm{H}(\mathcal{L}) \leq \frac{1}{\alpha-\epsilon}$. Letting $\epsilon \to 0$ completes the proof. $\square$

3.2. *Upper bound for $\alpha < 1/d$ and $\beta > 1$.* Now let us study the specific case where the blocks satisfy Hypothesis H_d and that $\lambda_n \leq n^{-\alpha+o(1)}$ for some $0 < \alpha < 1/d$ and $w_n \leq n^{-\beta+o(1)}$ for some $\beta > 1$. The preceding Proposition 6 still holds but it is not optimal in this specific case. As in the previous proof, we construct explicit coverings of the set $\mathcal{L}$ in order to bound its Hausdorff dimension. We construct them using an iterative procedure, which strongly depends on the dimension d and the exponent β. Starting from the covering given in the proof of Proposition 6, the procedure provides at each step a covering that is "better" in some sense than the preceding. In the limit, we prove the bound given in Proposition 7, which explicitly depends on β and d.

PROPOSITION 7. *Suppose $0 < \alpha < \frac{1}{d}$ and $\beta > 1$ and that for all $n \geq 1$, $\lambda_n \leq n^{-\alpha+o(1)}$ and $w_n \leq n^{-\beta+o(1)}$. Suppose also that H_d(iii) and H_d(ii) hold for some $d \geq 0$. Then the Hausdorff dimension of $\mathcal{L}$ almost surely satisfies*

$$\dim_{\mathrm{H}}(\mathcal{L}) \leq \frac{2\beta - 1 - 2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha}.$$

For our purposes, we will work with countable sets of balls of $\mathcal{T}$, that is, sets of the form

$$R = \{\mathrm{B}(x_i, r_i) \mid \forall i \geq 1, x_i \in \mathcal{T}, r_i > 0\},$$

where $\mathrm{B}(x, r)$ denotes the open ball centred at x with radius r. Let us introduce some notation. If R is such a set of balls of $\mathcal{T}$, we say that R is a *covering* of the subset $X \subset \mathcal{T}$ if $X \subset \bigcup_{B \in R} B$. We can also define the s-volume of R as

$$\mathrm{Vol}_s(R) := \sum_{B \in R} \mathrm{diam}(B)^s.$$

In this way, if the diameters of the balls that belong to R are bounded above by some $\delta > 0$, and R is a covering of X, then $\mathcal{H}_s^\delta(X) \leq \mathrm{Vol}_s(R)$; see Section A.2 in the Appendix for the definition of $\mathcal{H}_s^\delta(X)$. Also, if R and R' are collections of balls and R covers X and R' covers X', then obviously $R \cup R'$ is a countable set of balls that covers $X \cup X'$ and for any s, we have

(14) $\mathrm{Vol}_s(R \cup R') \leq \mathrm{Vol}_s(R) + \mathrm{Vol}_s(R').$

In what follows, we construct random sets of balls and we prove that they are coverings of our set $\mathcal{L}$, which allow us to prove upper bounds on the Hausdorff dimension of $\mathcal{L}$.

3.2.1. *An idea of the proof.* We briefly explain the idea of the proof before going into technicalities. The goal will be to provide a covering of each $\mathcal{T}(\mathbf{b}_n)$, for all n large enough. Since from the definition of $\mathcal{L}$, we have for any $N \geq 1$,

(15) $\mathcal{L} \subset \bigcup_{n \geq N} \mathcal{T}(\mathbf{b}_n),$

then the union over all n large enough of coverings of the $\mathcal{T}(\mathbf{b}_n)$ is indeed a covering of $\mathcal{L}$.

We recall how we derived the upper bound $\frac{1}{\alpha}$ for the Hausdorff dimension of $\mathcal{L}$ in the proof of Proposition 6(ii). The idea is to consider for every $n \geq 1$, a ball of radius $n^{-\alpha+\epsilon}$, say centred at $\boldsymbol{\rho}_n$. For n large enough, this ball covers $\mathcal{T}(\mathbf{b}_n)$ by (13). Thanks to (15), the set of balls $\{\mathrm{B}(\boldsymbol{\rho}_n, n^{-\alpha+\epsilon}) \mid n \geq N\}$, for any $N \geq 1$, is a covering of $\mathcal{L}$.

For $\beta \leq 1$, this covering is good because, as a block of index n has relative weight w_n/W_n which can be of order up to $n^{-1+o(1)}$ when it appears, the indices

of the first blocks that are glued on $\mathbf{b}_n$ can have also an index of the order of n, and so a height of order up to $n^{-\alpha}$. On the contrary, if $\beta > 1$, we will see that the first block to be grafted on $\mathbf{b}_n$ has index roughly of order n^β, and so a height at most of order $n^{-\alpha\beta}$, which is very small compared to $n^{-\alpha}$. This gives us a hint that we can provide a "better" covering using a big number of smaller balls to cover $\mathbf{b}_n$ instead of just a "big" one; see Figure 4. We will use this rough idea to provide an algorithm which will construct finer and finer (random) coverings. Let us fix $\beta > 1$ from now on and take $s > d$, and explain informally how the algorithm works.

Goal. At each step i of the algorithm, we want to construct for all $n \geq 1$ a set of balls $R_{n,i}^s$ such that, for n large enough, this set of balls is a covering of $\mathcal{T}(\mathbf{b}_n)$. Such a set of balls $R_{n,i}^s$ will have an s-volume of roughly $n^{f_i(s)}$, say. From step to step, we try to lower the s-volume of the set of balls constructed by the algorithm, which corresponds to lowering this exponent $f_i(s)$. Whenever we manage to get an exponent below -1, we stop the algorithm. We will see that it implies that the Hausdorff dimension of $\mathcal{L}$ is lower or equal to s.

Step 1. The first step of the algorithm is deterministic and corresponds to what we did in the proof of Proposition 6. For each n, we take a ball centred at $\boldsymbol{\rho}_n$ of radius roughly $n^{-\alpha}$ (in fact $n^{-\alpha+\epsilon}$ but let us not consider these technicalities for the moment). As seen before, for n large enough, it is a covering of $\mathcal{T}(\mathbf{b}_n)$. The s-volume of this covering is then of order $n^{-\alpha s}$. Denote $f_1(s) = -\alpha s$. If $f_1(s) < -1$, stop. Otherwise, proceed to step 2.

Step 2. As represented in Figure 4(a), decompose $\mathcal{T}(\mathbf{b}_n)$ as

$$\mathcal{T}(\mathbf{b}_n) = \mathbf{b}_n \cup \bigcup_{\mathbf{b}_k \to \mathbf{b}_n} \mathcal{T}(\mathbf{b}_k).$$

Since the first block grafted on the block $\mathbf{b}_n$ has typically an index, that is, very large compared to n, we design a covering using smaller balls. We fix $\gamma > 1$ and decide to cover $\mathbf{b}_n$ with balls of size $n^{-\alpha\gamma}$, so that the blocks (and their descending substructure) of index $> n^\gamma$ are included in these balls; see Figure 4(b). Since the blocks have dimension d, this covering uses roughly $(\frac{n^{-\alpha}}{n^{-\alpha\gamma}})^d$ balls, each with s-volume $n^{-\alpha\gamma s}$. So the total volume used is around $n^{-\alpha d + \alpha\gamma d - \alpha\gamma s}$.

But doing so, we forgot to cover the blocks $\mathbf{b}_k$ such that $\mathbf{b}_k \to \mathbf{b}_n$ and $k \leq n^\gamma$. To take care of them, we use the preceding step of the algorithm and cover each of them with a ball of radius $k^{-\alpha}$; see Figure 4(c). Recalling that $s \leq 1/\alpha$, we get that in expectation, these balls have a s-volume of order

$$\sum_{k=n+1}^{n^\gamma} \mathbb{P}(\mathbf{b}_k \to \mathbf{b}_n) k^{-\alpha s} \approx n^{-\beta} \sum_{k=n+1}^{n^\gamma} k^{-\alpha s} \approx n^{-\beta + \gamma(1 - \alpha s)}.$$
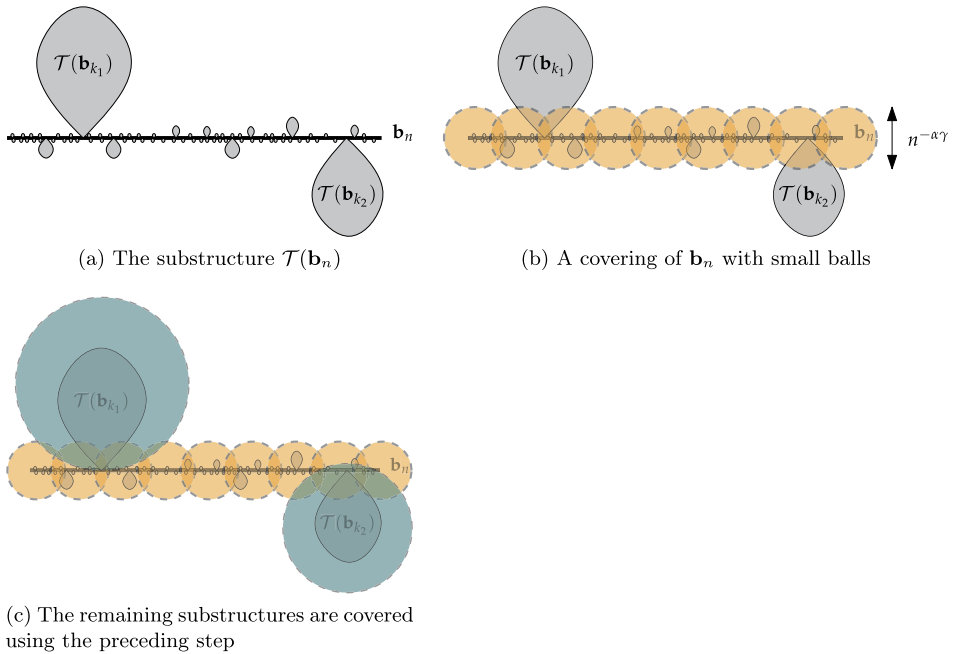
(a) The substructure $\mathcal{T}(\mathbf{b}_n)$

(b) A covering of $\mathbf{b}_n$ with small balls

(c) The remaining substructures are covered
using the preceding step

FIG. 4.   *Explanation of Step* 2 *of the algorithm.*

Consequently, the total s-volume of the balls used to cover $\mathcal{T}(\mathbf{b}_n)$ has order $n^{\max(-\beta+\gamma(1-\alpha s),-\alpha d+\alpha\gamma d-\alpha\gamma s)}$. Since we want to construct a covering having the smallest possible volume, we can optimise on γ the last exponent. Under our assumptions, one can check that it is minimal if we take $\gamma := \frac{\beta-\alpha d}{1-\alpha d} > 1$. We then get

$$\max(-\beta + \gamma(1-\alpha s), -\alpha d + \alpha\gamma d - \alpha\gamma s) = \frac{-\alpha d + \alpha\beta d - \alpha\beta s + \alpha^2 ds}{1-\alpha d}$$

$$:= f_2(s).$$

We can check that the new exponent $f_2(s)$ is smaller than $f_1(s) = -\alpha s$. Hence, we can cover $\mathcal{T}(\mathbf{b}_n)$ with balls using a total s-volume of a lower order than the preceding step. If $f_2(s) < -1$, stop. Otherwise, proceed to step 3.

Step i.   Now we recursively repeat the preceding step. Thanks to step $i-1$, we know that we can provide a covering of $\mathcal{T}(\mathbf{b}_n)$ for any n, using a s-volume of approximately $n^{f_{i-1}(s)}$. Now we fix a number $\gamma > 1$ and we cover the block $\mathbf{b}_n$ with balls of radius $n^{-\alpha\gamma}$. As in step 2, this covering has a s-volume of order $n^{-\alpha d+\alpha\gamma d-\alpha\gamma s}$. Then we take care of the $\mathbf{b}_k$ such that $\mathbf{b}_k \to \mathbf{b}_n$ and $k < n^\gamma$. To cover them, we use step $i-1$, which ensures that we can do that for each k with a s-volume roughly $k^{f_{i-1}(s)}$. Hence, the expectation on the s-volume for all these

balls is, if s is such that $f_{i-1}(s) \geq -1$,

$$\sum_{k=n+1}^{n^\gamma} \mathbb{P}(\mathbf{b}_k \to \mathbf{b}_n) k^{f_{i-1}(s)} \approx n^{-\beta} \sum_{k=n+1}^{n^\gamma} k^{f_{i-1}(s)} \approx n^{-\beta+\gamma(1+f_{i-1}(s))}.$$

We then choose the optimal $\gamma > 1$ that minimizes the maximum of the exponents

$$\max\left(-\alpha d + \alpha\gamma d - \alpha\gamma s, -\beta + \gamma(1 + f_{i-1}(s))\right).$$

We denote $\gamma_i(s)$ the value for which the minimum is obtained, which depends on s. The first exponent is linearly decreasing with γ, the other one is linearly increasing, and their value for γ tending to 1, satisfy $-\alpha s > -\beta + 1 + f_{i-1}(s)$. Hence, the value of $\gamma_i(s)$ is the value for which the two of them are equal, and this value is strictly greater than 1. We call this minimal exponent $f_i(s)$. If $f_i(s) < -1$, stop. Otherwise, proceed to step $i + 1$.

Upper bound on Hausdorff dimension. Now, suppose s is such that $f_i(s)$ is well defined and $f_i(s) < -1$, for some $i \geq 1$. If we cover every $\mathcal{T}(\mathbf{b}_n)$ using the covering provided by step i of the algorithm, then the union of all those coverings covers $\mathcal{L}$. Furthermore, we only need to cover all the $\mathcal{T}(\mathbf{b}_n)$ for n sufficiently large to cover $\mathcal{L}$, so we can have a covering of $\mathcal{L}$ using arbitrarily small balls. Hence, we get that for all $\delta > 0$, we have $\mathcal{H}_s^\delta(\mathcal{L}) < \sum_{n=1}^\infty n^{f_i(s)} < \infty$ and so $\mathcal{H}_s(\mathcal{L}) < \infty$, which proves that

$$\dim_{\mathrm{H}}(\mathcal{L}) \leq s.$$

This rough analysis is turned into a rigorous proof in what follows. We begin with elementary definitions and calculations that arise from what precedes.

3.2.2. *Study of a sequence of functions.* We begin by defining recursively the sequence of functions $(f_i)_{i \geq 1}$, together with a sequence $(s_i)_{i \geq 1}$ of real numbers.

DEFINITION-PROPOSITION 8. We set $s_0 := \infty$. We define a sequence $(f_i)_{i \geq 1}$ of functions as follows. We set

$$\forall s \in [d, \infty), \quad f_1(s) := -\alpha s,$$

and set $s_1 := \frac{1}{\alpha}$. Then for all $i \geq 1$, we recursively define

$$\forall s \in [d, s_i], \quad f_{i+1}(s) := \frac{\alpha(-d + \beta d - \beta s - f_i(s)d)}{1 + f_i(s) + \alpha s - \alpha d}.$$

Define s_{i+1} as the unique solution to the equation $f_{i+1}(s) = -1$.

Before proving the validity of this definition, let us state some properties of this sequence of functions.

PROPOSITION 9. *The following properties are satisfied*:

(i) *For all $i \geq 1$, the function f_i is continuous, strictly decreasing and $f_i(d) = -\alpha d$.*

(ii) *For all $i \geq 1$, for all $s \in (d, s_i]$, we have $f_{i+1}(s) < f_i(s)$.*

(iii) *Let $s_\infty := \frac{2\beta - 1 - 2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha}$. Then we have for all $s \in [d, s_\infty)$,*

$$f_i(s) \xrightarrow[i \to \infty]{} f_\infty(s),$$

where

$$f_\infty(s) = \frac{-(1 + \alpha s) + \sqrt{1 + 2\alpha s + \alpha^2 s^2 - 4\alpha d + 4\alpha\beta d - 4\alpha\beta s}}{2}.$$

(iv) *For all $s \in [d, s_\infty)$, we have $f_\infty(s) > -1$.*

(v) *The sequence $(s_i)_{i \geq 1}$ is strictly decreasing and*

$$s_i \xrightarrow[i \to \infty]{} s_\infty.$$

(vi) *For all $i \geq 1$, we have $f_{i+1}(s_i) < -1$.*

PROOF. We define the function F on the set $\{(s, x) \in \mathbb{R}^2 \mid d \leq s \leq \frac{1}{\alpha}, x > \alpha d - \alpha s - 1\}$ by the expression

$$F(s, x) = \frac{\alpha(-d + \beta d - \beta s - dx)}{1 + x + \alpha s - \alpha d}.$$

We have, for all $s > d$ and all $x > \alpha d - \alpha s - 1$,

$$\partial_x F(s, x) = \frac{\alpha(\beta - \alpha d)(s - d)}{(1 + x + \alpha s - \alpha d)^2} > 0.$$

This shows that for all $s > d$, the function $F(s, \cdot)$ is strictly increasing, and also strictly concave since the derivative is strictly decreasing.

From these facts, we can show by induction on i the points (i) and (ii) of Proposition 9, together with the validity of the definition of f_i and s_i, in Definition-Proposition 8.

- For $i = 1$, the function f_1 is well defined, s_1 is indeed the unique solution to $f_1(s) = -1$ and the point (i) is satisfied. Moreover, $f_2(s)$ is well defined for $s \in [d, s_1]$ by $f_2(s) = F(s, -\alpha s)$ and for all $s \in (d, s_1)$, we have

$$F(s, -\alpha s) + \alpha s = \frac{\alpha(\beta - 1)(d - s)}{1 - \alpha d} < 0,$$

which proves that (ii) holds for $i = 1$.

- By induction, if f_i and s_i are defined up to some $i \geq 1$ and satisfy (i), then one can verify that for all $s \in [d, s_i]$, the function f_{i+1} is well defined by the formula

$$f_{i+1}(s) = F(s, f_i(s)).$$

From the monotonicity of $F(s, \cdot)$ and f_i, this function is continuous and strictly decreasing. One can check that $F(d, x) = -\alpha d$ for any $x > -1$ so f_{i+1} satisfies (i). Then, if $i = 1$, the initialisation already gives us that (ii) holds. Otherwise, if $i \geq 2$, then using the induction hypothesis, for all $s \in (d, s_{i-1}]$ we have $f_i(s) < f_{i-1}(s)$. Using that $F(s, \cdot)$ is strictly increasing for $s > d$, we get that for all $s \in (d, s_i]$, $f_{i+1}(s) < f_i(s)$, and so (ii) holds. Since f_{i+1} is continuous and strictly decreasing and that $f_{i+1}(d) > -1$ and $f_{i+1}(s_i) < f_i(s_i) = -1$, then s_{i+1} is well defined. This completes our proof by induction.

Let us study at fixed $s > d$ the equation $F(s, x) = x$. We get the following second-order equation:

$$x^2 + x(1 + \alpha s) + (\alpha d - \alpha \beta d + \alpha \beta s) = 0,$$

for which the discriminant is $\Delta_s = 1 + 2\alpha s + \alpha^2 s^2 - 4\alpha d + 4\alpha \beta d - 4\alpha \beta s$. We can evaluate this quantity at d and at $\frac{1}{\alpha}$. We get

$$\Delta_d = (\alpha d - 1)^2 > 0 \quad \text{and} \quad \Delta_{1/\alpha} = 4(\beta - 1)(\alpha d - 1) < 0.$$

We can check that it vanishes exactly at $s = s_\infty$ so that in the end, Δ_s is strictly positive on $[d, s_\infty)$, null at s_∞ and strictly negative on $(s_\infty, \frac{1}{\alpha}]$. Hence, the function $F(s, \cdot)$ has 2 (resp., 1, resp., 0) fixed points on the corresponding intervals.

The convergence (iii) is a consequence of the fact that for $s \in [d, s_\infty)$, the function $F(s, \cdot)$ is strictly increasing and concave, has exactly two fixed points and that the initial value $f_1(s)$ is greater than the smallest fixed point. We then have a convergence of the sequence $f_i(s)$ towards the greatest fixed point of $F(s, \cdot)$, the value of which can be computed using the equation above. The property of the limit (iv) can be checked by proving that for all $s \in [d, s_\infty]$, we have $f_\infty(s) \geq f_\infty(s_\infty) = \sqrt{(\beta - 1)(\beta - \alpha d)} - \beta > -1$.

Let us prove the point (v). According to property (ii), we have $f_i(s_{i+1}) > f_{i+1}(s_{i+1}) = -1$, and since f_i is decreasing, we get $s_{i+1} < s_i$. Hence, the sequence $(s_i)_{i \geq 1}$ is strictly decreasing, bounded below by d, so it converges. Now let $s > s_\infty$. If the sequence $(f_i(s))_{i \geq 1}$ was well defined for all $i \geq 1$, then for all $i \geq 1$ we would have $f_i(s) > -1$, so it would be decreasing, bounded below; hence, it would have a limit, which would be a fixed point of $F(s, \cdot)$. It is impossible since $F(s, \cdot)$ has no fixed point, so the sequence is not well defined for all $i \geq 1$ and so for i large enough, $s > s_i$. We conclude that $\lim_{i \to \infty} s_i \leq s_\infty$. If we had $\lim_{i \to \infty} s_i < s_\infty$, then it would contradict the property (iv). In the end, $\lim_{i \to \infty} s_i = s_\infty$.

The last property (vi) follows from property (ii). Indeed, we have $f_{i+1}(s_i) < f_i(s_i) = -1$. $\quad \square$

3.2.3. *Construction of the coverings.*

Let us provide a rigorous proof of our upper bound, which follows the heuristics that we derived in the beginning of the section. Here, we distinguish two types of negligible functions, $o_n(1)$ and $o_\epsilon(1)$. A function denoted $o_n(1)$ (resp., $o_\epsilon(1)$) is negligible as $n \to \infty$ (resp., as $\epsilon \to 0$) and does not depend on ϵ (resp., on n).

PROPOSITION 10. *Fix $i \geq 1$ and $s \leq s_{i-1}$. For all $\epsilon > 0$, we can construct simultaneously for all $n \geq 1$ a set of balls $R_{n,i}^{s,\epsilon}$, such that the following holds*:

(i) *Almost surely, for n large enough, $R_{n,i}^{s,\epsilon}$ covers $\mathcal{T}(\mathbf{b}_n)$.*
(ii) *We have*

$$\mathbb{E}\big[\mathrm{Vol}_s\big(R_{n,i}^{s,\epsilon}\big)\big] \leq n^{f_i(s)+o_n(1)+o_\epsilon(1)}.$$

(iii) *The diameter of the balls used are such that $\max_{B \in R_{n,i}^{s,\epsilon}} \mathrm{diam}\, B \xrightarrow[n \to \infty]{} 0$.*

We will define the set of balls $R_{n,i}^{s,\epsilon}$ over the block $\mathbf{b}_n$ and its descendants in an algorithmic way, and each step of the algorithm only depends on the gluings that happen after time n. The proof of the upper bound will directly follow from this proposition. Let us first state an elementary result, the proof of which is left to the reader. Note that we allow the function $o_\epsilon(1)$ to be infinite for large values of ϵ.

LEMMA 11. *Let $\xi \geq -1$. Then for all $\gamma > 1$, we have*

$$\mathbb{E}\left[\sum_{k=n+1}^{n^\gamma} \mathbf{1}_{\{\mathbf{b}_k \to \mathbf{b}_n\}} k^{\xi+o_n(1)+o_\epsilon(1)}\right] \leq n^{-\beta+\gamma(\xi+1)+o_n(1)+o_\epsilon(1)}.$$

PROOF OF PROPOSITION 10. Let $s > 0$ and $\epsilon > 0$. We prove the proposition by induction on i. The first set of balls that we build is the following: for each block $\mathbf{b}_n$, we cover the block with a ball of radius $n^{-\alpha+\epsilon}$, centred on the point ρ_n. We write

$$R_{n,1}^{s,\epsilon} = \{\mathrm{B}(\rho_n, n^{-\alpha+\epsilon})\}.$$

According to (13), there exists a random N such that for all $n \geq N$, the set $R_{n,1}^{s,\epsilon}$ covers $\mathcal{T}(\mathbf{b}_n)$. The diameter of the ball of $R_{n,1}^{s,\epsilon}$ tend to 0 as $n \to \infty$. Besides we have

$$\mathbb{E}\big[\mathrm{Vol}_s\big(R_{n,1}^{s,\epsilon}\big)\big] \leq (2n)^{-\alpha s+\epsilon s} = n^{f_1(s)+o_n(1)+o_\epsilon(1)}.$$

The property is thus proved for $i = 1$.

Let $i \geq 1$ and $s < s_i$. Let us construct $(R_{n,i+1}^{s,\epsilon})_{n \geq 1}$, using the previous step i. We set $\gamma_{i+1}(s) > 1$ a positive real number that we will choose later, and $\epsilon > 0$. We define $R_{n,i+1}^{s,\epsilon}$ as follows: it is the union over all the blocks $\mathbf{b}_k$ for $k < n^{\gamma_i(s)}$ that are grafted on the block $\mathbf{b}_n$, of their covering $R_{k,i}^{s,\epsilon}$ of the preceding step, together with the union of a deterministic set of balls that we define hereafter.

We want to cover $\mathbf{b}_n$ with balls of radius $n^{-\alpha\gamma_{i+1}(s)}$, which is equivalent to covering B_n with balls of radius $\lambda_n^{-1} n^{-\alpha\gamma_{i+1}(s)}$. Under Hypothesis H_d, for any $d \geq 0$, using Lemma 21 and Lemma 23 in Appendix A.3, we can a.s. find a random collection $(x_m)_{1 \leq m \leq M_r(\mathrm{B}_n)}$ of points of B_n such that the balls centred on those

points with radius $r := \lambda_n^{-1} n^{-\alpha\gamma_{i+1}(s)}$ cover B_n, and such that $M_r(\mathsf{B}_n) \leq N_{r/4}(\mathsf{B}_n)$, where $N_r(\mathsf{B})$ is the minimal number of balls of radius r needed to cover B.

From the assumption on the sequence (λ_n), we have $r \geq n^{-\alpha\gamma_{i+1}(s)+\alpha+o_n(1)}$. Since $N_r(\mathsf{B}_n)$ is decreasing in r, using Hypothesis H_d(ii), we get that

$$\mathbb{E}[N_{r/4}(\mathsf{B}_n)] \leq n^{-\alpha d + \alpha\gamma_{i+1}(s)d + o_n(1)}.$$

In the end,

$$R_{n,i+1}^{s,\epsilon} := \left(\bigcup_{k \leq n^{\gamma_{i+1}(s)} : \mathbf{b}_k \to \mathbf{b}_n} R_{k,i}^{s,\epsilon} \right) \cup \{\mathsf{B}(x_m, n^{-\alpha\gamma_{i+1}(s)+\epsilon}) \mid 1 \leq m \leq M_r(\mathsf{B}_n)\}.$$

Now we compute the expectation of the s-volume of these sets of balls:

$$\mathbb{E}[\mathrm{Vol}_s(R_{n,i+1}^{s,\epsilon})]$$

$$= \mathbb{E}\left[\sum_{k=n+1}^{n^{\gamma_{i+1}(s)}} \mathbf{1}_{\{\mathbf{b}_k \to \mathbf{b}_n\}} \mathrm{Vol}_s(R_{k,i}^{s,\epsilon}) \right] + \mathbb{E}[M_r(\mathsf{B}_n)](2n^{(-\alpha\gamma_{i+1}(s)+\epsilon)})^s$$

$$\leq \mathbb{E}\left[\sum_{k=n+1}^{n^{\gamma_{i+1}(s)}} \mathbf{1}_{\{\mathbf{b}_k \to \mathbf{b}_n\}} \mathbb{E}[\mathrm{Vol}_s(R_{k,i}^{s,\epsilon})] \right] + \mathbb{E}[N_{r/4}(\mathsf{B}_n)](2n^{(-\alpha\gamma_{i+1}(s)+\epsilon)})^s$$

$$\leq \mathbb{E}\left[\sum_{k=n+1}^{n^{\gamma_{i+1}(s)}} \mathbf{1}_{\{\mathbf{b}_k \to \mathbf{b}_n\}} k^{f_i(s)+o_n(1)+o_\epsilon(1)} \right]$$

$$+ n^{-\alpha d + \alpha d \gamma_{i+1}(s) + (-\alpha\gamma_{i+1}(s)+\epsilon)s + o_n(1) + o_\epsilon(1)}$$

$$\leq n^{-\beta + \gamma_{i+1}(s)(f_i(s)+1) + o_n(1) + o_\epsilon(1)} + n^{-\alpha d + \alpha d \gamma_{i+1}(s) - \alpha\gamma_{i+1}(s)s + o_n(1) + o_\epsilon(1)},$$

where in the last line we used Lemma 11 which applies because $s \leq s_i$; hence, $f_i(s) \geq -1$. We then take $\gamma_{i+1}(s) := \frac{\beta - \alpha d}{f_i(s)+1 - \alpha d + \alpha s} > 1$, which yields

$$-\beta + \gamma_{i+1}(s)(f_i(s)+1) = -\alpha d + \alpha\gamma_{i+1}(s)d - \alpha\gamma_{i+1}(s)s = f_{i+1}(s).$$

We then have

$$\mathbb{E}[\mathrm{Vol}_s(R_{n,i+1}^{s,\epsilon})] \leq n^{f_{i+1}(s)+o_n(1)+o_\epsilon(1)}.$$

We can check that $\max_{B \in R_{n,i+1}^{s,\epsilon}} \mathrm{diam}\, B \xrightarrow[n\to\infty]{} 0$, and that almost surely, for n large enough, the collections of balls $R_{n,i+1}^{s,\epsilon}$ are indeed coverings of $\mathcal{T}(\mathbf{b}_n)$ thanks again to (13). This completes the proof. $\square$

We can now prove the main proposition of this section.

PROOF OF PROPOSITION 7. Let $i \geq 1$. For $\epsilon > 0$ small enough, we use Proposition 10 to get a set of balls $(R_{n,i+1}^{s_i,\epsilon})_{n\geq1}$, which satisfies

$$\mathbb{E}[\mathrm{Vol}_{s_i}(R_{n,i+1}^{s_i,\epsilon})] \leq n^{f_{i+1}(s_i)+o_n(1)+o_\epsilon(1)}.$$

From Proposition 9(vi), we have $f_{i+1}(s_i) < -1$, so we can choose ϵ small enough such that the exponent is eventually smaller than $\frac{f_{i+1}(s_i)-1}{2} < -1$ as $n \to \infty$. Then, for $N \geq 1$, we set $R_N = \bigcup_{n \geq N} R_{n,i+1}^{s_i,\epsilon}$. According to Proposition 10, the set of balls $R_{n,i+1}^{s_i,\epsilon}$ is a covering of $\mathcal{T}(\mathbf{b}_n)$ for n large enough and so R_N is a covering of $\mathcal{L}$, for all N. Since for any $\delta > 0$, we may choose N large enough so that $\max_{B \in R_N} \operatorname{diam} B < \delta$, and we get

$$\mathcal{H}_{s_i}(\mathcal{L}) = \lim_{\delta \to 0} \mathcal{H}_{s_i}^{\delta}(\mathcal{L}) \leq \limsup_{N \to \infty} \operatorname{Vol}_{s_i}(R_N) \leq \operatorname{Vol}_{s_i}(R_1) < +\infty, \quad \text{a.s.}$$

since

$$\mathbb{E}[\operatorname{Vol}_{s_i}(R_1)] \leq \sum_{n=1}^{\infty} n^{\frac{f_{i+1}(s_i)-1}{2}+o_n(1)} < +\infty.$$

This shows that the Hausdorff dimension of $\mathcal{L}$ satisfies $\dim_H(\mathcal{L}) \leq s_i$, almost surely. In the end, since the sequence $(s_i)_{i \geq 1}$ tends to s_∞, we conclude that almost surely,

$$\dim_H(\mathcal{L}) \leq s_\infty = \frac{2\beta - 1 - 2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha}. \qquad \square$$

4. Lower bounds for the (α, β)-model. In this section, we compute lower bounds on the Hausdorff dimension of the set $\mathcal{L}$. We do that by constructing Borel measures on $\mathcal{L}$ that satisfy the assumptions of Frostman's lemma (Lemma 20 in Appendix A.2). In the case where $\beta \leq 1$, we use the natural measure $\bar{\mu}$ on $\mathcal{T}$ which arises as the limit of the normalised weight measures on $\mathcal{T}_n$ (see Proposition 4). The case $\beta > 1$ is a bit more technical because the natural measure $\bar{\mu}$ is not concentrated on $\mathcal{L}$, so we have to construct another measure π, that we define as the subsequential limit of some well-chosen sequence of probability measures on $\mathcal{T}$.

4.1. *Case $\beta \leq 1$ and use of the measure $\bar{\mu}$.* In this subsection, we suppose that $\beta \leq 1$. Under the assumptions of Proposition 4, the sequences of measures $\bar{\mu}_n$ almost surely converges weakly to a measure $\bar{\mu}$, which is concentrated on the set of leaves $\mathcal{L}$. The existence of $\bar{\mu}$ will be useful for the proof of the next proposition. Recall from (4) the definition of the random variable $\mathcal{H}$ and the fact that the assumptions on $\mathcal{H}$ in the proposition are satisfied under Hypothesis H_d for any $d \geq 0$.

PROPOSITION 12. *Suppose that Hypothesis $\bigcirc_{\alpha,\beta}$ or Hypothesis $\square_{\alpha,1}$ is satisfied. Suppose also that $\mathbb{E}[\mathcal{H}^2] < \infty$ and that $\mathbb{P}(\mathcal{H} > 0) > 0$. Then the Hausdorff dimension of $\mathcal{L}$ almost surely satisfies*

$$\dim_H(\mathcal{L}) \geq \frac{1}{\alpha}.$$

As we said earlier, the idea is to prove this lower bound on the dimension using Frostman's lemma: we will thus prove that almost surely, for $\bar{\mu}$-almost all leaves $x \in \mathcal{L}$, we have an upper bound of the type

$$\bar{\mu}\big(\mathrm{B}(x,r)\big) \leq r^{1/\alpha - \epsilon},$$

for r sufficiently small, and for all ϵ. An application of Lemma 20 will then complete our proof.

In order to prove this control on the masses of the balls, we will use two lemmas. The first one allows us to compare $\bar{\mu}(\mathrm{B}(x,r))$ with a quantity of the form $\bar{\mu}(\mathcal{T}(\mathbf{b}_n))$ for an appropriate n. The second one, Lemma 5, provides a good control of the quantities $\bar{\mu}(\mathcal{T}(\mathbf{b}_n))$ for large n, such that the combination of the two will provide the upper bound that we want. Let $\epsilon > 0$. Recall from (2) the definition of G^ϵ.

LEMMA 13. *Set $n_0 = 2$ and $n_{k+1} = \lceil n_k^{1+\epsilon} \rceil$. Under the hypotheses of Proposition 12, almost surely for $\bar{\mu}$-almost every $x \in \mathcal{L}$, for all k large enough,[3] there exists $n \in [\![n_k, n_{k+1}]\!] \cap G^\epsilon$ such that*

$$x \in \mathcal{T}(\mathbf{b}_n) \quad and \quad \mathrm{d}(x, \boldsymbol{\rho}_n) \geq n^{-\alpha - 2\epsilon}.$$

PROOF. Note that in our setting, the hypothesis of Proposition 4 holds and so the random leaf Y constructed in Section 2 is defined a.s. Also, according to Proposition 4, conditionally on $(\mathcal{T}, \bar{\mu})$, the point Y has distribution $\bar{\mu}$. So it suffices to prove that the lemma holds for the the random leaf Y. We recall

$$\mathrm{d}(Y_n, Y_m) \overset{\mathrm{law}}{=} \sum_{k=m+1}^{n} \lambda_k \mathcal{H}_k \mathbf{1}_{\{U_k \leq \frac{w_k}{W_k}\}} \quad \text{and} \quad Y = \lim_{n \to \infty} Y_n.$$

Let us introduce a constant $c > 0$ and set

$$p := \mathbb{P}(\mathcal{H} > c).$$

For $\beta < 1$, thanks to our assumptions, we can fix c such that p is nonzero. We then have

$$\mathbb{P}\bigg(\forall i \in [\![n_k, n_{k+1}]\!] \cap G^\epsilon, U_i > \frac{w_i}{W_i} \text{ or } \mathcal{H}_i < c\bigg) = \prod_{\substack{i=n_k \\ i \in G^\epsilon}}^{n_k^{1+\epsilon}} \bigg(1 - p\frac{w_i}{W_i}\bigg)$$

$$= \exp\bigg(\sum_{\substack{i=n_k \\ i \in G^\epsilon}}^{n_k^{1+\epsilon}} \log\bigg(1 - p\frac{w_i}{W_i}\bigg)\bigg)$$

---

[3]The threshold depends on the realisation and on x.

$$\leq \quad \exp\left(-p \sum_{\substack{i=n_k \\ i \in G^\epsilon}}^{n_k^{1+\epsilon}} \frac{w_i}{W_i}\right)$$

$$\underset{\text{Lem.25}}{\leq} \exp(-pC_\epsilon \log(n_k)).$$

To write the last line, we use Lemma 25 in the Appendix and we can see that the last display is summable over k.

For the case $\beta = 1$, Hypothesis $\square_{\alpha,1}$ allows us to write

$$\mathbb{P}\left(\forall i \in [\![n_k, n_{k+1}]\!] \cap G^\epsilon, U_i > \frac{w_i}{W_i} \text{ or } \mathcal{H}_i < c\right) \leq \exp(-pf(k)\log\log\log(n_k)),$$

with a function $f(k)$ tending to infinity. Since $n_k \geq 2^{(1+\epsilon)^k}$, then $\log\log\log(n_k) \geq (1 + o(1))\log k$ and the last display is also summable in k. In both cases, an application of the Borel–Cantelli lemma shows that we have almost surely, for k large enough,

$$\exists n \in [\![n_k, n_{k+1}]\!] \cap G^\epsilon, \qquad U_n \leq \frac{w_n}{W_n} \quad \text{and} \quad \mathcal{H}_n \geq c.$$

Since $n \in G^\epsilon$, we have $\lambda_n \geq n^{-\alpha-\epsilon}$. Combined with the fact that $\mathcal{H}_n \geq c$, we get

$$d(\boldsymbol{\rho}_n, Y) \geq \lambda_n \mathcal{H}_n \geq cn^{-\alpha-\epsilon} \geq n^{-\alpha-2\epsilon},$$

for n (or equivalently k) large enough. $\square$

PROOF OF PROPOSITION 12.   Let $\epsilon > 0$. Let us fix a realisation of $\mathcal{T}$ and a leaf $x \in \mathcal{L}$ such that the conclusions of Lemma 13 and Lemma 5 hold. Note that thanks to Hypothesis $\bigcirc_{\alpha,\beta}$ or Hypothesis $\square_{\alpha,1}$, the condition of application of Lemma 5 are fulfilled. From the definition of n_{k+1}, we have $n_k^{1+\epsilon} < n_{k+1} \leq n_k^{1+\epsilon} + 1$ and so $n_{k+1} \underset{k\to\infty}{=} n_k^{1+\epsilon+o(1)}$. We know from Lemma 13 that for all k large enough, there exists $n \in [\![n_k, n_{k+1}]\!]$ such that $x \in \mathcal{T}(\mathbf{b}_n)$ and

$$d(\boldsymbol{\rho}_n, x) \geq n^{-\alpha-2\epsilon} \geq n_{k+1}^{-\alpha-2\epsilon} \geq n_k^{(1+\epsilon+o(1))(-\alpha-2\epsilon)}.$$

So if we take k large enough and $r \in [n_{k+2}^{-\alpha-2\epsilon}, n_{k+1}^{-\alpha-2\epsilon})$, then

$$\bar{\mu}(\mathrm{B}(x,r)) \leq \bar{\mu}(\mathcal{T}(\mathbf{b}_n)) \underset{\text{Lem. 5}}{\leq} n^{-1+\epsilon} \leq n_k^{-1+\epsilon} = n_{k+2}^{\frac{-1+\epsilon}{(1+\epsilon)^2}+o(1)}$$

$$\leq \left(r^{\frac{-1}{\alpha+2\epsilon}}\right)^{\frac{-1+\epsilon}{1+\epsilon}+o(1)}$$

$$\leq r^{\frac{1}{\alpha}+g(\epsilon)+o(1)},$$

with a function g tending to 0 as $\epsilon \to 0$.

Since the last display is true almost surely for all r sufficiently small, we use Lemma 20 (Frostman's lemma) to deduce that the Hausdorff dimension of $\mathcal{L}$ is a.s. larger than $\frac{1}{\alpha} + g(\epsilon)$. Taking $\epsilon \to 0$, we get that almost surely,

$$\dim_H(\mathcal{L}) \geq \frac{1}{\alpha}. \qquad \square$$

4.2. *Case $\beta > 1$ and construction of measures on the leaves.* The following section is devoted to prove the following proposition.

PROPOSITION 14. *Suppose that Hypothesis $\diamond_{\alpha,\beta}$ is satisfied and that the block* B *satisfies Hypothesis H_d for some $d \geq 0$. Then the Hausdorff dimension of $\mathcal{L}$ almost surely satisfies*

$$\dim_H(\mathcal{L}) \geq \frac{2\beta - 1 - 2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha} \quad \text{if } \alpha < \frac{1}{d},$$

$$\geq \frac{1}{\alpha} \quad \text{otherwise.}$$

In the case $\beta > 1$, we cannot use the natural measure $\bar{\mu}$ to get a good lower bound on the Hausdorff dimension of $\mathcal{L}$ since, as stated in Proposition 4, the measure $\bar{\mu}$ does not charge the leaves. So the goal of this subsection is to artificially construct a probability measure concentrated on the leaves that will give us, using Frostman's lemma, the appropriate lower bound on the Hausdorff dimension, that is, the one matching with the upper bound derived in Section 3. The measure will be obtained as a subsequential limit of a sequence of measures concentrated on the blocks, and will only charge a strict subset of $\mathcal{L}$.

First, let us fix some notation. Recall the definition of G_n^ϵ in (3). It follows from Hypothesis $\diamond_{\alpha,\beta}$ that there exists a function $h(n)$ tending to 0 such that $\#G_n^{h(n)} = n^{1+o(1)}$. We choose such a function h and let

$$(16) \qquad\qquad\qquad G_n := G_n^{h(n)}.$$

We will also use an increasing sequence of positive integers $(n_k)_{k \geq 0}$, such that for all $k \geq 0$, we have $n_{k+1} = \lceil n_k^\gamma \rceil$, with a fixed $\gamma > 1$, that we will optimise later. Also we suppose n_0 to be very large, with conditions that we will make explicit in what follows. For all $n \geq 1$, we set

$$\overline{\mathbf{b}}_n := \left\{ x \in \mathbf{b}_n \,\middle|\, \mathbf{d}_n(\rho_n, x) > \frac{\text{ht}(\mathbf{b}_n)}{2} \right\},$$

the "upper-half" of block $\mathbf{b}_n$. For technical reasons, we will only keep in our construction the blocks that behave reasonably well; see the forthcoming property (P_d), introduced in Section 4.2.2. We recursively define some random sets of integers. Let

$$(17) \qquad\qquad \widetilde{G}_{n_0} = \{n \in G_{n_0} \mid \mathsf{B}_n \text{ satisfies property } (P_d)\},$$

and for $k \geq 0$,

$$
\widetilde{G}_{n_{k+1}} = \{n \in G_{n_{k+1}} \mid \mathsf{B}_n \text{ satisfies property } (P_d);
$$

(18)

$$
\exists i \in \widetilde{G}_{n_k}, \mathbf{b}_n \to \mathbf{b}_i, X_{n-1} \in \overline{\mathbf{b}}_i\}.
$$

We then define for all $k \geq 0$,

$$
\mathcal{B}_k = \bigcup_{n \in \widetilde{G}_{n_k}} \overline{\mathbf{b}}_n.
$$

In other words, $\mathcal{B}_0$ is the union of all the upper-halves of the blocks $\mathbf{b}_n$, for n in G_{n_0} for which B_n behaves well, and $\mathcal{B}_{k+1}$ is defined to be the union of all the upper-halves of the blocks of index $n \in G_{n_{k+1}}$ that are grafted directly on $\mathcal{B}_k$, and such that B_n behaves well. Note that for the moment, $\mathcal{B}_k$ can be empty.

We define the measure $\sum_{n \in \widetilde{G}_{n_k}} \nu_n$ and refer to it as the *mass measure*[4] on the kth generation. To simplify notation, we denote it $|\cdot|$. We do not index it by k since the index for which we consider it is always clear from the context. We also define a sequence $(\pi_k)_{k \geq 0}$ of probability measures on $\mathcal{B}_k$ by

$$
\pi_k := \frac{|\cdot \cap \mathcal{B}_k|}{|\mathcal{B}_k|},
$$

the normalised mass measure on $\mathcal{B}_k$. Note that the sequence (π_k) is only well defined on the event where $\mathcal{B}_k$ has nonzero mass for all k. In what follows, we will ensure that it is the case for an event of strictly positive probability and only work conditionally on this event. Remark that, still conditionally on this event and on the event that $\mathcal{T}$ is compact, which has probability 1, the sequence $(\pi_k)_{k \geq 0}$ is a sequence of probability measures on a compact space, hence it admits at least one subsequential limit π for the Lévy–Prokhorov distance. We can check using [7], Lemma 17, which is essentially an application of the Portmanteau theorem, that π is concentrated on $\bigcap_{k \geq 0} \mathcal{T}(\mathcal{B}_k) \subset \mathcal{L}$.

4.2.1. *Idea of the proof.* Let us briefly explain how the measure π that we just constructed enables us to derive the appropriate lower bound for the Hausdorff dimension. We give the intuition for $\alpha < 1/d$; the idea for $\alpha > 1/d$ is very similar. We will be very rough for this sketch of proof and we keep the notation introduced above. Let us here forget that some blocks may not satisfy (P_d), and that we only deal with half-blocks.

---

[4]It may appear more natural to define the mass measure as simply $\sum_{n=1}^{\infty} \nu_n$ which gives $\mathcal{T}$ a finite mass. However, this definition would not have the property that for every set $S \subset \overline{\mathbf{b}}_n$, we have $|S| = \nu_n(S)$. Indeed, when blocks can have an atom at their root, which is possible in the case $d = 0$, the contribution of the mass added by the roots of future blocks should be counted in $|S|$.

Number of blocks in $\mathcal{B}_k$. Suppose that the number of blocks in $\mathcal{B}_k$ evolves like a power of n_k, say n_k^a. Then the total weight of $\mathcal{B}_k$ is $|\mathcal{B}_k| \approx n_k^a n_k^{-\beta}$, because all the blocks in $\mathcal{B}_k$ have weight $\approx n_k^{-\beta}$. Since the probability that any block with index in $[\![n_{k+1}, 2n_{k+1}]\!]$ is grafted on $\mathcal{B}_k$ is roughly $|\mathcal{B}_k|$, and since the number of blocks in $\mathcal{B}_{k+1}$ is roughly n_{k+1}^a, we have

$$n_k^{\gamma a} \approx n_{k+1}^a \approx |\mathcal{B}_k| \cdot n_{k+1} \approx n_k^a n_k^{-\beta} n_k^\gamma.$$

Hence, we have $a = \frac{\gamma - \beta}{\gamma - 1}$, and so $|\mathcal{B}_k| \approx n_k^{\frac{\gamma(1-\beta)}{\gamma-1}}$.

Estimation on π. For each $k \geq 0$, the set $\mathcal{B}_k$ is made of blocks of size $\approx n_k^{-\alpha}$. Let us suppose that the quantities of the form $\pi(\mathrm{B}(x,r))$ are well approximated by $\pi_k(\mathrm{B}(x,r))$, whenever $r \in [n_{k+1}^{-\alpha}, n_k^{-\alpha}]$. For x close to $\mathcal{B}_k$, such a ball typically intersects only one block of $\mathcal{B}_k$, with weight roughly $n_k^{-\beta}$, and since the block is d-dimensional, the ball covers a proportion $(r/n_k^{-\alpha})^d$ of this block. So $|\mathrm{B}(x,r) \cap \mathcal{B}_k| \approx n_k^{-\beta}(r/n_k^{-\alpha})^d$, and

$$\pi\big(\mathrm{B}(x,r)\big) \approx \pi_k\big(\mathrm{B}(x,r)\big)$$

$$= \frac{|\mathrm{B}(x,r) \cap \mathcal{B}_k|}{|\mathcal{B}_k|}$$

$$\approx r^d n_k^{-\beta + \alpha d - \gamma \frac{\beta-1}{\gamma-1}}$$

$$\approx r^d n_k^{\frac{1}{\gamma-1}(\gamma(\alpha d-1)+\beta-\alpha d)}.$$

Then, if $\alpha < 1/d$, the last exponent is negative for γ large enough. For such γ, using $r > n_{k+1}^{-\alpha} \approx n_k^{-\alpha\gamma}$ yields

$$(19) \qquad \pi\big(\mathrm{B}(x,r)\big) \leq r^{d - \frac{1}{\alpha\gamma(\gamma-1)}(\gamma(\alpha d-1)+\beta-\alpha d)}.$$

Optimisation. We then choose γ such that the exponent $d - \frac{\gamma(\alpha d-1)-\alpha d+\beta}{\alpha\gamma(\gamma-1)}$ is maximal. We get the value $\frac{2\beta-1-2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha}$ which matches our upper bound.

Plan of the proof. Our goal is now to make those heuristics rigorous. First, we will make some precise estimation on how the mass of the blocks with indices in $\widetilde{G}_{n_k}$ is spread on subsets of $\mathcal{T}_{n_k-1}$. Then we will decompose each block of each $\mathcal{B}_k$ into subsets that we call fragments for which our preceding estimation holds. After that, we use this decomposition to control the behaviour of the (π_k), and also how the measures π_k can approximate the limiting measure π. At the end, we conclude by optimising on the parameters.

We will distinguish the two cases $d = 0$ and $d > 0$ and mostly work on the latter. We then explain quickly how the proof can be adapted to $d = 0$, in which fewer technicalities are involved.

4.2.2. *Mass estimations.* Before proving our main proposition, we have to state some technical lemmas that will allow us to control how regularly the mass of $\mathcal{B}_{k+1}$ is spread on $\mathcal{B}_k$. Let us now define the property (P_d), in a different way whether $d = 0$ or $d > 0$. Let $C > 0$ be a positive number and remind the definition of $(\star_{r_0})$ in Hypothesis $H_d(\mathrm{i})$. For $d > 0$, we say that a pointed compact metric space endowed with a probability measure $(\mathbf{b}, \mathbf{d}, \boldsymbol{\rho}, \boldsymbol{\nu})$ satisfies (P_d) iff

$$(P_d) \qquad \begin{cases} C^{-1} \leq \mathrm{ht}(\mathbf{b}) \leq C, \\ \boldsymbol{\nu}\left(\left\{x \in \mathbf{b} \;\middle|\; \mathbf{d}(\boldsymbol{\rho}, x) \geq \dfrac{\mathrm{ht}(\mathbf{b})}{2}\right\}\right) \geq C^{-1}, \\ (\mathbf{b}, \mathbf{d}, \boldsymbol{\rho}, \boldsymbol{\nu}) \quad \text{satisfies } (\star_{r_0}) \text{ with } r_0 = C^{-1}. \end{cases}$$

For $d = 0$, we say that $(\mathbf{b}, \mathbf{d}, \boldsymbol{\rho}, \boldsymbol{\nu})$ satisfies (P_0) iff $\mathbf{b}$ is finite and

$$(P_0) \qquad \begin{cases} C^{-1} \leq \mathrm{ht}(\mathbf{b}) \leq C, \\ \#\mathbf{b} \leq C, \\ \forall x \in \mathbf{b}, \quad \nu(\{x\}) \geq C^{-1}. \end{cases}$$

In any case, under Hypothesis $H_d(\mathrm{i})$, for any $d \geq 0$, we can choose C such that the underlying block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ satisfies (P_d) with a positive probability $p > 0$.

$$p = \mathbb{P}\big(\mathsf{B} \text{ satisfies } (P_d)\big) > 0.$$

From now on, we fix such a constant C. We also set $\mathsf{M} := \nu(\{x \in \mathsf{B} \mid \mathsf{D}(\rho, x) \geq \frac{\mathrm{ht}(\mathsf{B})}{2}\})$, and $m := \mathbb{E}[\mathsf{M} \mid \mathsf{B} \text{ satisfies } (P_d)]$. We also denote by $\mathbb{M}$ a random variable with the law of M conditional on the event $\{\mathsf{B} \text{ satisfies } (P_d)\}$.

LEMMA 15. *Let S be a subset of some $\mathbf{b}_i$ with $i \leq n_k - 1$, measurable with respect to $\mathcal{F}_{n_k-1}$, the σ-field generated by the blocks and the gluings up to time $n_k - 1$. Let $\chi(S)$ be the total mass of the union of the sets $\{\overline{\mathbf{b}}_n \mid n \in G_{n_k}, \mathbf{b}_n \to \mathbf{b}_i, X_{n-1} \in S, \mathsf{B}_n \text{ satisfies } (P_d)\}$, namely, the total mass of the half-blocks that are grafted on S with index in G_{n_k}, and such that the corresponding blocks satisfy property (P_d). Then for all $x \in [0, 1]$,*

$$\mathbb{P}\big(\big|\chi(S) - a_k|S|\big| > x a_k|S| \mid \mathcal{F}_{n_k-1}\big) \leq 2\exp\big(-x^2 n_k^{1+o(1)}|S|\big),$$

where $a_k := pm \sum_{i \in G_{n_k}} \frac{w_i}{W_{i-1}}$, is such that $\mathbb{E}[\chi(S) \mid \mathcal{F}_{n_k-1}] = a_k|S|$ and the function $o(1)$ in the right-hand side does not depend on x.

This lemma roughly states that, for every subset $S \subset \mathbf{b}_i$ for $i \in G_{n_k}$, if the subset has enough mass to attract a substantial number of the blocks coming between time n_k and $2n_k$ then we have a good control on how the mass of $\mathcal{B}_k$ grafted on S can deviate from its expected value.

PROOF OF LEMMA 15. First, we write $\chi(S)$ as

$$\chi(S) = \sum_{i \in G_{n_k}} \mathbf{1}_{\{U_i \leq \frac{|S|}{W_{i-1}}\}} \mathbf{1}_{\{\mathsf{B}_i \text{ satisfies } (P_d)\}} \mathsf{M}_i w_i,$$

where the (U_i) are independent uniform variables on $[0, 1]$, independent of everything else. Then we can compute

$$\mathbb{E}\big[\chi(S) \mid \mathcal{F}_{n_k-1}\big] = \sum_{i \in G_{n_k}} \frac{|S|}{W_{i-1}} pw_i \cdot \mathbb{E}\big[\mathsf{M}_i \mid \mathsf{B}_i \text{ satisfies } (P_d)\big]$$

$$= |S| \cdot pm \bigg(\sum_{i \in G_{n_k}} \frac{w_i}{W_{i-1}} \bigg) = |S| \cdot a_k.$$

Let us bound the exponential moments of $\chi(S)$:

$$\mathbb{E}\big[\exp(\theta \chi(S)) \mid \mathcal{F}_{n_k-1}\big]$$

$$= \prod_{i \in G_{n_k}} \bigg(\frac{p|S|}{W_{i-1}} \cdot \mathbb{E}\big[e^{\theta w_i \mathsf{M}_i} \mid \mathsf{B}_i \text{ satisfies } (P_d)\big] + 1 - \frac{p|S|}{W_{i-1}} \bigg)$$

$$= \prod_{i \in G_{n_k}} \bigg(1 + \frac{p|S|}{W_{i-1}} (\mathbb{E}\big[e^{\theta w_i \mathsf{M}}\big] - 1) \bigg)$$

$$\leq \exp \bigg(p|S| \sum_{i \in G_{n_k}} \frac{1}{W_{i-1}} (\mathbb{E}\big[e^{\theta w_i \mathsf{M}}\big] - 1) \bigg),$$

where we used the inequality $e^z \geq 1 + z$, in the last line. Now,

$$\mathbb{E}\big[\exp(\theta(\chi(S) - a_k|S|)) \mid \mathcal{F}_{n_k-1}\big]$$

$$\leq \exp \bigg(p|S| \sum_{i \in G_{n_k}} \frac{1}{W_{i-1}} (\mathbb{E}\big[e^{\theta w_i \mathsf{M}}\big] - 1) - \theta|S| \sum_{i \in G_{n_k}} \frac{pm w_i}{W_{i-1}} \bigg)$$

$$\leq \exp \bigg(p|S| \bigg(\sum_{i \in G_{n_k}} \frac{1}{W_{i-1}} (\mathbb{E}\big[e^{\theta w_i \mathsf{M}}\big] - 1 - \theta m w_i) \bigg) \bigg)$$

$$\leq \exp \bigg(p|S| \bigg(\sum_{i \in G_{n_k}} \frac{1}{W_{i-1}} c(\theta w_i)^2 \bigg) \bigg).$$

Here, we used the fact that for $z \in [-1, 1]$, we have $e^z \leq 1 + z + 3z^2$ and so

$$\mathbb{E}\big[e^{z\mathsf{M}}\big] - 1 - z\mathbb{E}[\mathsf{M}] \leq \mathbb{E}\big[1 + z\mathsf{M} + 3(z\mathsf{M})^2\big] - 1 - z\mathbb{E}[\mathsf{M}] \leq 3\mathbb{E}[\mathsf{M}^2]z^2 \leq cz^2,$$

for c a constant. Since we ask that $z \in [-1, 1]$, the computation above is valid if we restrict ourselves to $|\theta| \leq (\sup_{i \in G_{n_k}} w_i)^{-1} = n_k^{\beta+o(1)}$. Note that we can use this inequality for negative values of θ. Hence, for $x \in [0, 1]$ we have

$$\mathbb{P}\big(|\chi(S) - a_k|S|| > xa_k|S| \mid \mathcal{F}_{n_k-1}\big)$$

$$\leq \mathbb{P}\big(\chi(S) - a_k|S| > xa_k|S| \mid \mathcal{F}_{n_k-1}\big)$$

$$+ \mathbb{P}\big(-\big(\chi(S) - a_k|S|\big) > x a_k|S| \mid \mathcal{F}_{n_k-1}\big)$$

$$\leq \mathbb{P}\big(\exp\big(\theta\big(\chi(S) - a_k|S|\big)\big) > e^{\theta x a_k|S|} \mid \mathcal{F}_{n_k-1}\big)$$

$$+ \mathbb{P}\big(\exp\big(-\theta\big(\chi(S) - a_k|S|\big)\big) > e^{\theta x a_k|S|} \mid \mathcal{F}_{n_k-1}\big)$$

$$\leq 2\exp\bigg(p|S|\bigg(\sum_{i \in G_{n_k}} \frac{1}{W_{i-1}} c(\theta w_i)^2\bigg) - \theta x a_k|S|\bigg)$$

$$= 2\exp\bigg(p|S|\bigg(\sum_{i \in G_{n_k}} \frac{1}{W_{i-1}}\big(c(\theta w_i)^2 - \theta x m w_i\big)\bigg)\bigg).$$

Taking $\theta = x n_k^{\beta-\epsilon}$ in the last inequality, which is possible for n_k large enough, this gives

$$\mathbb{P}\big(|\chi(S) - a_k|S|| > x a_k|S| \mid \mathcal{F}_{n_k-1}\big)$$

$$\leq 2\exp\bigg(p|S|x^2\bigg(\sum_{i \in G_{n_k}} \frac{1}{W_{i-1}}\big(n_k^{\beta-\epsilon} w_i\big)\big(c n_k^{\beta-\epsilon} w_i - m\big)\bigg)\bigg).$$

From our assumptions on the sequence (w_n), we have $n_k^{\beta-\epsilon} w_i \to 0$, and hence $(c n_k^{\beta-\epsilon} w_i - m)$ is eventually smaller than $-\frac{m}{2}$, uniformly for $i \in G_{n_k}$. Also $\frac{1}{W_n}$ is always greater than $\frac{1}{W_\infty}$. Combining this with the last display, we get for n_k large enough

$$\mathbb{P}\big(|\chi(S) - a_k|S|| > x a_k|S| \mid \mathcal{F}_{n_k-1}\big)$$

$$\leq 2\exp\bigg(-x^2|S| \cdot \frac{pm}{2W_\infty} \#G_{n_k}\Big(\inf_{i \in G_{n_k}} w_i\Big) n_k^{\beta-\epsilon}\bigg)$$

$$\leq 2\exp\big(-x^2|S|n_k^{1-\epsilon+o(1)}\big).$$

Now for every $\epsilon > 0$, this inequality is true for n_k large enough, so this proves the lemma. $\square$

Let us also state another technical lemma, the proof of which is in the Appendix A.4.

LEMMA 16. *Suppose that Hypothesis $\diamond_{\alpha,\beta}$ is satisfied. We have, for the sequence (a_i) defined in Lemma 15,*

$$\prod_{i=1}^{k} a_i = n_k^{\frac{\gamma(1-\beta)}{(\gamma-1)} + o(1)},$$

where the $o(1)$ is considered when $k \to \infty$.

4.2.3. *Construction of fragments*, *case $d > 0$.*

Fragments of a random block. Let us discuss how we can decompose a metric space into a partition of subsets that we call *r-fragments*, all of them having a diameter of order r. Suppose that the random block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ comes with a sequence of random points $(X_n)_{n \geq 1}$, which are i.i.d. with law ν, conditionally on $(\mathsf{B}, \mathsf{D}, \rho, \nu)$, and that this block satisfies Hypothesis $H_d(\mathrm{i})$, for some $d > 0$. The following lemma ensures that in this setting, we can construct a partition $F(\mathsf{B}, r) = (f_i^{(r)})_{1 \leq i \leq N}$ of r-fragments of $(\mathsf{B}, \mathsf{D}, \rho, \nu)$, which have approximately equal diameter and measure. Recall the function φ defined in Hypothesis $H_d(\mathrm{i})$, and the notation $N_r(\mathsf{B})$ for the minimal number of balls of radius r needed to cover B.

LEMMA 17. *Suppose that $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ satisfies Hypothesis $H_d(\mathrm{i})$ for some $d > 0$. For any $r \in [0, 1]$, we construct a finite partition of Borel subsets $(f_i^{(r)})_{1 \leq i \leq N}$ of the block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ in a deterministic way from $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ and the sequence of random points $(X_n)_{n \geq 1}$. There exists two functions ψ and ϕ defined on the interval $[0, 1]$, which tend to 0 at 0 such that the following holds almost surely on the event $\{(\mathsf{B}, \mathsf{D}, \rho, \nu)$ satisfies $(\star_{r_0})\}$, for any $r_0 > 3r$:*

(i) *For all $1 \leq i \leq N$,*

$$\operatorname{diam} f_i \leq 2r \quad and \quad \left(\frac{r}{4}\right)^{d + \varphi(r/4)} \leq \nu(f_i) \leq r^{d - \varphi(r)}.$$

(ii) *For all $r' < r_0/3$, we have*

$$\forall x \in \mathsf{B}, \quad \#\{1 \leq i \leq N \mid \mathsf{B}(x, r') \cap f_i \neq \varnothing\} \leq (r \vee r')^{d + \psi(r \vee r')} \cdot r^{-d + \phi(r)}.$$

(iii) *The (random) number N of fragments satisfies*

$$N \leq N_{r/4}(\mathsf{B}) \quad and \quad N \leq \left(\frac{r}{4}\right)^{-d - \varphi(r/4)}.$$

In the paper, we use this construction on $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, \nu_n)$, assuming that for all $n \geq 1$, a sequence $(X_{n,j})_{j \geq 1}$ is defined on the same probability space and that this sequence is i.i.d. with law ν_n, conditionally on $(\mathsf{B}_n, \mathsf{D}_n, \rho_n, \nu_n)$. For any $n \geq 1$ and $r > 0$, we denote $F(\mathsf{B}_n, r) = \{f_i^{(r)} \mid 1 \leq i \leq N\}$ the partition of B_n into (random) r-fragments which is given by the lemma. The proof of Lemma 17 can be found in the Appendix.

Decomposition of $\mathbf{b}_n$ into fragments. Fix a parameter $\eta \in (0, \frac{1}{d})$. We want to decompose every $\mathbf{b}_n$, for $n \in \widetilde{G}_{n_k}$, in fragments of size approximately $n_{k+1}^{-\eta}$. For that, it is sufficient to use $F(\mathsf{B}_n, r_n)$ the decomposition of B_n in r_n-fragments with $r_n = (\lambda_n^{-1} \cdot n_{k+1}^{-\eta})$, which is given by Lemma 17. Let us emphasise that these fragments are constructed as subsets of B_n, but we consider them as subsets of $\mathbf{b}_n$ in

what follows, without changing notation. We define the set F_k is as the collection of all these fragments coming from every $\mathbf{b}_n$ with $n \in \widetilde{G}_{n_k}$. We have, of course,

$$\bigcup_{f \in F_k} f = \bigcup_{n \in \widetilde{G}_{n_k}} \mathbf{b}_n. \tag{20}$$

In our construction, we decided to keep only the blocks that were sufficiently well behaved with respect to some properties that will be useful now. Recall the definition of the random set $\widetilde{G}_{n_k}$ in equations (17) and (18). Remark that, from the definition of G_{n_k}, we have

$$c_k := \min_{n \in G_{n_k}} (\lambda_n^{-1} \cdot n_{k+1}^{-\eta}) = n_k^{\alpha - \gamma \eta + o(1)} \quad \text{and}$$

$$C_k := \max_{n \in G_{n_k}} (\lambda_n^{-1} \cdot n_{k+1}^{-\eta}) = n_k^{\alpha - \gamma \eta + o(1)}.$$

If γ and η are such that $\gamma > \frac{\alpha}{\eta}$, then the last exponent is strictly negative, and so we can take n_0 sufficiently large so that $C_k < C^{-1}/3$, for all $k \geq 0$. For $n \in \widetilde{G}_{n_k}$, we know that B_n satisfies $(\star_{r_0})$ with $r_0 = C^{-1}$. Hence, for all $n \in \widetilde{G}_{n_k}$, we have $3r_n \leq r_0$, and so the conclusions of Lemma 17 hold simultaneously for all the decompositions $F(\mathsf{B}_n, r_n)$ for $n \in \widetilde{G}_{n_k}$.

Control on the mass and number of fragments. Recall the function h that we defined in (16), which tends to 0 at infinity, and the function φ specified in Hypothesis $H_d(\mathrm{i})$, which tends to 0 at 0. Thanks to Lemma 17, we get, for all $f \in F_k$ such that $f \subset \mathbf{b}_n$,

$$
\begin{aligned}
|f| \quad &= \quad w_n \cdot v_n(f) \\
&\underset{(16),\mathrm{Lem.17(i)}}{\geq} \quad n_k^{-\beta - h(n_k)} \cdot \left(\lambda_n^{-1} \cdot n_{k+1}^{-\eta}\right)^{d + \varphi(n_k^{-\alpha d + \alpha + o(1)})} \\
&\geq \quad n_k^{-\beta - h(n_k)} \cdot c_k^{d + \varphi(C_k)}.
\end{aligned}
$$

Note that the last quantity is deterministic and only depends on n_k, and so almost surely,

$$
\begin{aligned}
\min_{f \in F_k} |f| &\geq n_k^{-\beta - h(n_k)} \cdot c_k^{d + \varphi(C_k)} \\
&\geq n_k^{-\eta d \gamma + \alpha d - \beta + o(1)} = n_{k+1}^{-\eta d + \frac{1}{\gamma}(\alpha d - \beta) + o(1)}.
\end{aligned} \tag{21}
$$

Note that a similar computation using upper bounds instead of lower bounds also yields, almost surely,

$$\max_{f \in F_k} |f| \leq n_k^{-\beta + o(1)} \cdot C_k^{d - \varphi(c_k)} \leq n_k^{-\eta d \gamma + \alpha d - \beta + o(1)}, \tag{22}$$

where the right-hand side is deterministic. Also, from Lemma 17(iii), we get that the number of fragments obtained from the block $\mathbf{b}_n$ by that construction is

bounded above by $(r_n/4)^{-d-\varphi(r_n/4)}$, with $r_n = \lambda_n^{-1} \cdot n_{k+1}^{-\eta} = n_k^{\alpha-\gamma\eta+o(1)}$, and so at the end, the total number of fragments in F_k is bounded above by a deterministic quantity which grows at most polynomially in n_k.

4.2.4. *Construction of fragments, case $d = 0$.* In this case, we will consider the finite number of points of each block as a decomposition into fragments; hence, we set $F_k = \{\{x\} \mid x \in \mathbf{b}_n, n \in \widetilde{G}_{n_k}\}$. Note that

$$\forall f \in F_k, f \subset \mathbf{b}_n, \quad |f| = w_n \cdot v_n(f) \underset{(16),(P_0)}{\geq} n_k^{-\beta-h(n_k)} \cdot C^{-1},$$

and so the equations (21) and (22) are still valid when $d = 0$, and also the number of fragments in F_k grows linearly, hence polynomially in n_k.

4.2.5. *Using the mass estimations.* Recall that we fixed a parameter $\eta \in (0, \frac{1}{d})$. We let $0 < \epsilon < (1 - \eta d)$. If γ and η are such that $\gamma > \frac{\beta-\alpha d}{1-\eta d-\epsilon}$, then $-\eta d + \frac{1}{\gamma}(\alpha d - \beta) > -1 + \epsilon$. And so we get from (21) that $\min_{f \in F_k}|f| \geq n_{k+1}^{-1+\epsilon+o(1)}$. We can apply the result of Lemma 15 for every fragment $f \in F_k$, with $x = n_{k+1}^{-\epsilon/4}$,

$$\mathbb{P}\big(|\chi(f) - a_{k+1}|f|| > n_{k+1}^{-\epsilon/4}a_{k+1}|f| \mid \mathcal{F}_{n_{k+1}-1}\big)$$

$$\leq 2\exp\Big(-\big(n_{k+1}^{-\epsilon/4}\big)^2 n_{k+1}^{1+o(1)} \min_{f \in F_k}|f|\Big)$$

$$\leq 2\exp\big(-n_{k+1}^{-\epsilon/2} n_{k+1}^{1+o(1)} n_{k+1}^{-1+\epsilon+o(1)}\big)$$

$$\leq 2\exp\big(-n_{k+1}^{\epsilon/4}\big) \quad \text{for } n_{k+1} \text{ large enough.}$$

For that, again, we impose that n_0 is large enough such that the last display is true for all k and for all f. Now we can sum this over all fragments,

$$\mathbb{E}\Bigg[\sum_{k=0}^{\infty} \sum_{f \in F_k} \mathbb{P}\big(|\chi(f) - a_{k+1}|f|| > n_{k+1}^{-\epsilon/4}a_{k+1}|f| \mid \mathcal{F}_{n_{k+1}-1}\big)\Bigg]$$

$$\leq \sum_{k=0}^{\infty} \mathbb{E}[\#F_k] \cdot 2\exp\big(-n_{k+1}^{\epsilon/4}\big) \xrightarrow[n_0 \to \infty]{} 0,$$

since $(\#F_k)$ is almost surely bounded by a deterministic quantity which grows at most polynomially in n_k. The same is true for the $\mathcal{B}_k$,

$$\mathbb{E}\Bigg[\sum_{k=0}^{\infty} \mathbb{P}\big(|\chi(\mathcal{B}_k) - a_{k+1}|\mathcal{B}_k|| > n_{k+1}^{-\epsilon/4}a_{k+1}|\mathcal{B}_k| \mid \mathcal{F}_{n_{k+1}-1}\big)\Bigg] \xrightarrow[n_0 \to \infty]{} 0.$$

In the rest of Section 4.2, we will fix n_0 large enough and work on the event of large probability $\mathcal{E}$ on which we have, for all $k \geq 0$ and for all $f \in F_k$

$$(23) \qquad \begin{aligned} \left|\chi(\mathcal{B}_k) - a_{k+1}|\mathcal{B}_k|\right| &\leq n_{k+1}^{-\epsilon/4} a_{k+1}|\mathcal{B}_k| \quad \text{and} \\ \left|\chi(f) - a_{k+1}|f|\right| &\leq n_{k+1}^{-\epsilon/4} a_{k+1}|f|. \end{aligned}$$

Remark that thanks to Section 1.4, giving a lower bound of the Hausdorff dimension on a set of positive probability is enough to prove that the bound holds almost surely. Note that this construction depends on the parameters η and ϵ and γ. The parameters must satisfy

$$(24) \qquad \eta \in \left(0, \frac{1}{d}\right), \qquad \epsilon \in (0, 1 - \eta d), \qquad \gamma > \max\left(\frac{\alpha}{\eta}, \frac{\beta - \alpha d}{1 - \eta d - \epsilon}\right),$$

and we can choose them in this particular order.

4.2.6. *Control on the limiting measure.* In this section, the values of η and ϵ and n_0 are fixed in such a way that the construction of the previous section holds. Note that everything in the section implicitly depends on those values. On the event $\mathcal{E}$, if we consider a fragment $f \in F_k$, we have a very good control on the values of $\pi_i(\mathcal{T}(f))$ for $i \geq k$. Indeed set

$$c_1 = \prod_{k=0}^{\infty} \left(1 - n_{k+1}^{-\frac{\epsilon}{4}}\right) \quad \text{and} \quad c_2 = \prod_{k=0}^{\infty} \left(1 + n_{k+1}^{-\frac{\epsilon}{4}}\right).$$

Remark that both c_1 and c_2 are strictly positive real numbers. Using in cascade the estimations (23) which hold on the event $\mathcal{E}$, we get that for $f \in F_k$ and $i \geq k$,

$$(25) \qquad |\mathcal{T}(f) \cap \mathcal{B}_i| \leq c_2 |f| \left(\prod_{j=k+1}^{i} a_j\right).$$

In fact, we can use the same argument for $\mathcal{B}_k$, which is not empty on the event $\mathcal{E}$. For k large enough, we can write

$$(26) \qquad |\mathcal{B}_i| \in |\mathcal{B}_k| \left(\prod_{j=k+1}^{i} a_j\right) \cdot [c_1, c_2].$$

Remark that (26) combined with Lemma 16 yields that almost surely on $\mathcal{E}$,

$$(27) \qquad n_k^{\frac{\gamma(1-\beta)}{(\gamma-1)} + o(1)} \leq |\mathcal{B}_k| \leq n_k^{\frac{\gamma(1-\beta)}{(\gamma-1)} + o(1)},$$

where the upper and lower bound are both deterministic. For π_i the normalised mass measure on $\mathcal{B}_i$, we have

$$\pi_i(\mathcal{T}(f)) = \frac{|\mathcal{T}(f) \cap \mathcal{B}_i|}{|\mathcal{B}_i|} \underset{(25),(26)}{\leq} \frac{c_2}{c_1} \cdot \frac{|f|}{|\mathcal{B}_k|}.$$

If π is a subsequential limit of the (π_k), using Portmanteau theorem (remark that π is concentrated on the leaves and the leaves of $\mathcal{T}(f)$ belong to the interior of $\mathcal{T}(f)$), we get

$$\pi\big(\mathcal{T}(f)\big) \le \frac{c_2}{c_1} \cdot \frac{|f|}{|\mathcal{B}_k|}.$$

And then

$$\max_{f \in F_k} \pi\big(\mathcal{T}(f)\big) \le \frac{c_2}{c_1} \frac{1}{|\mathcal{B}_k|} \max_{f \in F_k} |f|.$$

We can now write, for all $r > 0$, for all $x \in \mathcal{T}$,

$$\pi\big(\mathrm{B}(x, r)\big) \le \sum_{f \in F_k, \, f \cap \mathrm{B}(x,r) \ne \varnothing} \pi\big(\mathcal{T}(f)\big)$$

$$\le \#\{f \in F_k, \, f \cap \mathrm{B}(x, r) \ne \varnothing\} \cdot \max_{f \in F_k} \pi\big(\mathcal{T}(f)\big).$$

Putting everything together, we get

$$(28) \qquad \pi\big(\mathrm{B}(x, r)\big) \le \#\{f \in F_k, \, f \cap \mathrm{B}(x, r) \ne \varnothing\} \cdot \frac{c_2}{c_1} \frac{1}{|\mathcal{B}_k|} \max_{f \in F_k} |f|.$$

4.2.7. *Control on the number of fragments intersecting a ball.* From (28), we see that the last thing that we have to estimate is $\#\{f \in F_k, \, f \cap \mathrm{B}(x, r) \ne \varnothing\}$, the number of fragments of F_k that have a nonempty intersection with a ball of radius r. Since the measure π only charges $\bigcap_{k \ge 1} \mathcal{T}(\mathcal{B}_k)$, we are only interested in balls centred around points belonging to this set. Let us fix some notation again. For all $k \ge 0$, we set

$$\Delta_k := \inf\{\mathrm{d}(x, y) \mid x \in \mathcal{B}_{k-1}, y \in \mathcal{B}_k\},$$

the set distance between $\mathcal{B}_{k-1}$ and level $\mathcal{B}_k$, for the integers k for which it is possible. On the event $\mathcal{E}$, this quantity is well defined for all $k \ge 1$ and the following upper and lower bounds are almost surely satisfied

$$(29) \qquad n_k^{-\alpha+o(1)} = \frac{C^{-1}}{2} \min_{n \in G_{n_k}} \lambda_n \le \Delta_k \le C \max_{n \in G_{n_k}} \lambda_n = n_k^{-\alpha+o(1)}.$$

Now let us state a lemma.

LEMMA 18. *Let $x \in \bigcap_{k \ge 1} \mathcal{T}(\mathcal{B}_k)$. For $k \ge 0$, we denote $x_k := [x]_{2n_k} \in \mathcal{B}_k$. If $\mathbf{b}_n$ is the block of $\mathcal{B}_k$ such that $x_k \in \mathbf{b}_n$, we have $\forall r \in [0, \Delta_k]$,*

$$(30) \qquad \{f \in F_k \mid f \cap \mathrm{B}(x, r) \ne \varnothing\} \subset \{f \in F_k \mid f \cap \mathrm{B}(x_k, r) \cap \mathbf{b}_n \ne \varnothing\}.$$

The proof of this lemma is simple and left to the reader. It tells us that, in fact, if r is small enough, then all the fragments $f \in F_k$ who intersect the ball of centre x and radius r belong to the same block. This will allow us in the sequel, combined with Lemma 17(ii), to bound the number of fragments involved, which is what we wanted.

4.2.8. *Obtaining the lower bound.* In order to get the lower bound on the Hausdorff dimension of $\mathcal{L}$ matching that of the theorem, we have to distinguish between the case $\alpha < \frac{1}{d}$ and the case $\alpha > \frac{1}{d}$. The case $\alpha = \frac{1}{d}$ can be recovered by a monotonicity argument, as seen in Section 1.4. Whenever $d = 0$, we have $\frac{1}{d} = +\infty$ and only the first case can happen.

Case $\beta > 1$ and $\alpha < 1/d$. We use the construction of Section 4.2.3 with $\eta = \alpha$. Recall (24) for the admissible parameters of the construction. In this case, if ϵ is fixed and small enough, the only condition on γ implied by (24) is $\gamma > \frac{\beta - \alpha d}{1 - \alpha d - \epsilon}$ since $\gamma > \frac{\alpha}{\eta}$ reduces to $\gamma > 1$, which is already contained in the previous inequality because $\frac{\beta - \alpha d}{1 - \alpha d - \epsilon} > 1$. We define

$$\Lambda_k := \Delta_k \wedge \left(\frac{\min_{n \in G_{n_k}} \lambda_n}{\log n_k} \right).$$

Using (29), we get $n_k^{-\alpha + o(1)} \leq \Lambda_k \leq n_k^{-\alpha + o(1)}$, almost surely, with deterministic bounds. Here, the choice of $\frac{1}{\log n_k}$ is rather arbitrary and we could change it to any quantity that tends to 0 and is $n_k^{o(1)}$ as $n_k \to \infty$. Now we claim the following.

LEMMA 19. *On the event $\mathcal{E}$, for all $k \geq 0$, for any $d \geq 0$, we almost surely have*

$$(31) \quad \forall r \in [\Lambda_{k+1}, \Lambda_k], \quad \#\{f \in F_k \mid f \cap \mathrm{B}(x, r) \neq \varnothing\} \leq r^{d + o(1)} n_k^{\alpha \gamma d + o(1)}.$$

Note that the the bounds Λ_k of the interval on which we consider r are random, but the upper bound given by (31) is deterministic.

PROOF OF LEMMA 19. For $r \in [\Lambda_{k+1}, \Lambda_k]$, we have $r \leq \Lambda_k \leq \Delta_k$ so using Lemma 18, with $x_k = [x]_{2n_k}$ and n such that $x_k \in \mathbf{b}_n$, we know that (30) holds. In the case $d > 0$, the fragments of F_k that come from $\mathbf{b}_n$ were constructed as fragments of B_n of size $\mathsf{r}_n := \lambda_n^{-1} n_{k+1}^{-\alpha}$. Recall that we denote $F(\mathrm{B}_n, \mathsf{r}_n)$ the set of these fragments, seen as subsets of B_n, and denote x_k the point of B_n corresponding to $x_k \in \mathbf{b}_n$. The analogue of the ball $\mathrm{B}(x_k, r)$ in B_n is then the ball of centre x_k and radius $\mathsf{r}'_n := \lambda_n^{-1} r$. From our definition of Λ_k, we have

$$\mathsf{r}'_n = \lambda_n^{-1} r \leq \lambda_n^{-1} \frac{\min_{n \in G_{n_k}} \lambda_n}{\log n_k} \leq \frac{1}{\log n_k} \xrightarrow[k \to \infty]{} 0,$$

as well as $\mathsf{r}_n := \lambda_n^{-1} n_{k+1}^{-\alpha} = n_k^{\alpha - \gamma \alpha + o(1)} \to 0$ when $k \to \infty$. Applying Lemma 17(ii) yields

$$\#\{f \in F(\mathrm{B}_n, \mathsf{r}_n) \mid f \cap \mathrm{B}(\mathsf{x}_k, \mathsf{r}_n) \neq \varnothing\} \underset{\text{Lem.17(ii)}}{\leq} (\mathsf{r}_n \vee \mathsf{r}'_n)^{d + \psi(\mathsf{r}_n \vee \mathsf{r}'_n)} \cdot \mathsf{r}_n^{-d + \phi(\mathsf{r}_n)}$$

$$\leq \left((\lambda_n^{-1} n_{k+1}^{-\alpha}) \vee (\lambda_n^{-1} r) \right)^{d + o(1)}$$

$$\cdot \left(\lambda_n^{-1} n_{k+1}^{-\alpha}\right)^{-d+o(1)}$$

$$\leq \quad r^{d+o(1)} n_k^{\alpha\gamma d+o(1)},$$

and the last quantity is deterministic. Since any fragment in $\{f \in F_k \mid f \cap B(x,r) \neq \varnothing\}$ corresponds to a fragment in $\{f \in F(B_n, r_n) \mid f \cap B(x_k, r_n) \neq \varnothing\}$, the cardinal of $\{f \in F_k \mid f \cap B(x,r) \neq \varnothing\}$ is almost surely bounded above by the last display, which proves that (31) holds whenever $d > 0$. In the case $d = 0$, from our definition of fragments and the property (P_0), we easily have

$$\#\{f \in F_k \mid f \cap B(x_k, r) \cap \mathbf{b}_n \neq \varnothing\} \leq C \leq r^{d+o(1)} n_k^{\alpha\gamma d+o(1)},$$

and so (31) also holds whenever $d = 0$. $\square$

We can compute

$$\pi\big(B(x,r)\big) \quad \underset{(28)}{\leq} \quad \#\{f \in F_k, f \cap B(x,r) \neq \varnothing\} \cdot \frac{c_2}{c_1} \cdot \frac{1}{|\mathcal{B}_k|} \cdot \max_{f \in F_k} |f|$$

$$\underset{(31),(27),(22)}{\leq} \left(r^{d+o(1)} n_k^{\alpha\gamma d+o(1)}\right) \cdot r^{o(1)} \cdot \left(n_k^{\frac{\gamma(\beta-1)}{(\gamma-1)}+o(1)}\right)$$

$$\cdot \left(n_k^{-\alpha\gamma d+\alpha d-\beta+o(1)}\right)$$

$$\leq \quad r^{d+o(1)} \cdot n_k^{\frac{1}{(\gamma-1)}(\gamma\alpha d-\alpha d+\beta-\gamma+o(1))}$$

$$\leq \quad r^{d-\frac{1}{\alpha\gamma(\gamma-1)}(\gamma\alpha d-\alpha d+\beta-\gamma)+o(1)}.$$

In the last line, we used that $r > \Lambda_{k+1} \geq n_{k+1}^{-\alpha+o(1)} = n_k^{-\gamma\alpha+o(1)}$ and so $n_k > r^{-\frac{1}{\alpha\gamma}+o(1)}$ and the fact that $\gamma\alpha d - \alpha d + \beta - \gamma < 0$ because $\gamma > \frac{\beta-\alpha d}{1-\alpha d}$. Let us now maximise the quantity $d - \frac{\gamma\alpha d-\alpha d+\beta-\gamma}{\alpha\gamma(\gamma-1)}$ for $\gamma \in (\frac{\beta-\alpha d}{1-\alpha d}, +\infty)$. It is an easy exercise to see that the maximum is attained at $\bar{\gamma} = \frac{\beta-\alpha d+\sqrt{(\beta-1)(\beta-\alpha d)}}{1-\alpha d}$, with value

$$\frac{2\beta - 1 - 2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha}.$$

If we fix ϵ small enough then, the value of γ that maximises the last display satisfies $\gamma > \frac{\beta-\alpha d}{1-\alpha d-\epsilon}$ and so, using this value to construct π, we get that on the event $\mathcal{E}$, for all $x \in \bigcap_{k \geq 1} \mathcal{T}(\mathcal{B}_k)$,

$$\pi\big(B(x,r)\big) \leq r^{\frac{2\beta-1-2\sqrt{(\beta-1)(\beta-\alpha d)}}{\alpha}+o(1)},$$

which allows us to conclude using Lemma 20.

Case $\beta > 1$ and $\alpha > 1/d$. Here, we suppose that $d > 0$. We fix $\eta < \frac{1}{d}$ which we suppose to be very close to $\frac{1}{d}$ and a small $\epsilon > 0$ and use the construction of Section 4.2.3 with these values, which satisfy (24) if we take $\gamma > \max(\frac{\beta - \alpha d}{1 - \eta d - \epsilon}, \frac{\alpha}{\eta})$.

- For $r \in [\Lambda_{k+1}, n_{k+1}^{-\eta}]$, we apply Lemma 19 to get

$$\#\{f \in F_k, f \cap B(x, r) \neq \varnothing\} \leq n_k^{o(1)}.$$

Now we can use the upper bound (28), replacing term by term,

$$\pi\big(B(x, r)\big) \underset{(28)}{\leq} \#\{f \in F_k, f \cap B(x, r) \neq \varnothing\} \cdot \frac{c_2}{c_1} \frac{1}{|\mathcal{B}_k|} \max_{f \in F_k} |f|.$$

$$\underset{(27),(22)}{\leq} n_k^{o(1)} \cdot \frac{c_2}{c_1} n_k^{\frac{\gamma(\beta-1)}{(\gamma-1)}+o(1)} n_k^{-\eta d \gamma + \alpha d - \beta + o(1)}$$

$$\leq n_k^{\frac{\gamma(\beta-1)}{(\gamma-1)} - \gamma \eta d + \alpha d - \beta + o(1)}.$$

Hence, using the fact that $r > \Lambda_{k+1} = n_k^{-\gamma\alpha+o(1)}$ and that the exponent in the last display is negative, we get

$$\pi\big(B(x, r)\big) \leq r^{-\frac{1}{\gamma\alpha} \cdot (\frac{\gamma(\beta-1)}{(\gamma-1)} - \gamma \eta d + \alpha d - \beta) + o(1)}$$

$$\leq r^{\frac{\eta d}{\alpha} - \frac{1}{\alpha\gamma}(\frac{\gamma(\beta-1)}{(\gamma-1)} + \alpha d - \beta) + o(1)}.$$

- For $r \in [n_{k+1}^{-\eta}, \Lambda_k]$, we have once again using Lemma 19,

$$\#\{f \in F_k, f \cap B(x, r) \neq \varnothing\} \leq r^{d+o(1)} n_{k+1}^{\eta d + o(1)}.$$

Replacing in (28) yields

$$\pi\big(B(x, r)\big) \leq r^{d+o(1)} n_{k+1}^{\eta d + o(1)} \cdot \frac{c_2}{c_1} \frac{1}{n_k^{\frac{\gamma(1-\beta)}{(\gamma-1)}+o(1)}} n_k^{-\eta d \gamma + \alpha d - \beta + o(1)}$$

$$\leq r^{d+o(1)} \cdot n_k^{\alpha d - 1 + \frac{\beta-1}{\gamma-1} + o(1)}.$$

Since $r \leq \Lambda_k \leq n_k^{-\alpha+o(1)}$, we have $n_k \leq r^{-\frac{1}{\alpha}+o(1)}$. Since the quantity $\alpha d - 1 + \frac{\beta-1}{\gamma-1}$ is positive, we can write

$$\pi\big(B(x, r)\big) \leq r^{d+o(1)} \cdot r^{-\frac{1}{\alpha}(\alpha d - 1 + \frac{\beta-1}{\gamma-1}) + o(1)}$$

$$\leq r^{d - \frac{\alpha d - 1}{\alpha} - \frac{\beta-1}{\alpha(\gamma-1)} + o(1)}$$

$$\leq r^{\frac{1}{\alpha} - \frac{\beta-1}{\alpha(\gamma-1)} + o(1)}.$$

Now the result is obtained by taking $\epsilon \to 0$ and $\eta \to \frac{1}{d}$ and $\gamma \to \infty$.

To conclude the proof of Proposition 14, we have to prove that in the case $\alpha = 1/d$, the dimension of $\mathcal{L}$ is bounded below by d. To that end, we use the monotonicity of the Hausdorff dimension of $\mathcal{L}$, with respect to the scaling factors (λ_n), proved in Section 1.4. Suppose the sequences (λ_n) and (w_n) satisfy Hypothesis $\diamond_{\alpha,\beta}$ for $\alpha = 1/d$. If for some $\epsilon > 0$, we set for all $n \geq 1$, $\lambda'_n = n^{-\epsilon}\lambda_n$, then the sequences (λ'_n) and (w_n) satisfy Hypothesis $\diamond_{\alpha+\epsilon,\beta}$. Now for $n \geq 1$, we have $\lambda_n \geq \lambda'_n$, and $\mathcal{T}$ is compact with probability 1 from Proposition 6. Hence, (6) holds and so we have a.s.

$$\dim(\mathcal{L}) \geq \frac{1}{\alpha + \epsilon}.$$

In the end, $\dim(\mathcal{L}) \geq d$.

PROOF OF THEOREM 1. Use Proposition 6, Proposition 7 for the upper bounds and Proposition 12 and Proposition 14 for the lower bounds. □

APPENDIX

A.1. Lifting to the Urysohn space. In this section, we prove that it is always possible to work with random measured metric spaces that are embedded in the Urysohn space. Let us first recall the definition of the Gromov–Hausdorff–Prokhorov distance. If (X, d) is a metric space, and $A \subset X$, then we denote $A^{(\epsilon)} := \{x \in X \mid d(x, A) < \epsilon\}$, the ϵ-fattening of A. Then d_H the Hausdorff distance on the set of nonempty compact subsets of X is defined as

$$d_H(K, K') := \inf\{\epsilon > 0 \mid K \subset (K')^{\epsilon}, K' \subset (K)^{\epsilon}\}.$$

Also we denote the so-called Lévy–Prokhorov distance on the Borel probability measures by

$$d_{LP}(\nu, \nu') := \inf\{\epsilon > 0 \mid \forall A \in \mathcal{B}(X), \nu(A) \leq \nu'((A)^{\epsilon}) + \epsilon \text{ and}$$
$$\nu'(A) \leq \nu((A)^{\epsilon}) + \epsilon\},$$

where $\mathcal{B}(X)$ is the set of Borel sets of X. Now let (X, d, ρ, ν) and (X', d', ρ', ν') be two compact, rooted, metric spaces endowed with a probability measure. Their Gromov–Hausdorff–Prokhorov distance is defined as

$$d_{GHP}((X, d, \rho, \nu), (X', d', \rho', \nu'))$$
$$:= \inf_{E,\phi,\phi'} \max(d(\rho, \rho'), d_H(\phi(X), \phi(X')), d_{LP}(\phi_*\nu, \phi'_*\nu')),$$

where the infimum is taken over all Polish spaces (E, δ) and all isometric embeddings $\phi : X \to E$ and $\phi' : X' \to E$, of respectively X and X' into E. The notation $\phi_*\nu$ denotes the push-forward of the measure μ through the map ϕ. As it is, this is only a pseudo-distance and becomes a distance on the set $\mathbb{K}$ of GHP-isometry

(root and measure preserving isometry) classes of compact, rooted, metric spaces endowed with a probability measure, which from [1], Theorem 2.5, is a Polish space. We consider all our blocks as (possibly random) elements of the set $\mathbb{K}$.

We would like to see all the blocks as compact subsets of the same space. To that end, we consider (U, δ) the Urysohn space, and fix a point $u_0 \in U$. The space U is defined as the only Polish metric space (up to isometry) which has the following extension property (see [14] for constructions and basic properties of U): given any finite metric space X, and any point $x \in X$, any isometry from $X \setminus \{x\}$ to U can be extended to an isometry from X to U. This property ensures in particular that any separable metric space can be isometrically embedded into U. In what follows, we will use the fact that if (K, d, ρ) is a rooted compact metric space, there exists an isometric embedding of K to U such that ρ is mapped to u_0. We set

$$\mathbb{K}(U) := \big\{(K, \nu) \mid K \subset U, K \text{ compact}, u_0 \in K,$$

$$\nu \text{ is a Borel measure and } \mathrm{supp}(\nu) \subset K\big\},$$

where $\mathrm{supp}(\nu)$ denotes the topological support of ν. We endow $\mathbb{K}(U)$ with the "Hausdorff–Prokhorov" distance

$$\mathrm{d}_{\mathrm{HP}}\big((K, \nu), (K', \nu')\big) = \max\big(\mathrm{d}_{\mathrm{H}}(K, K'), \mathrm{d}_{\mathrm{LP}}(\nu, \nu')\big).$$

It is easy to see that $(\mathbb{K}(U), \mathrm{d}_{\mathrm{HP}})$ is a Polish space. Now, we have a map $f : \mathbb{K}(U) \to \mathbb{K}$, which maps every (K, ν) to the isometry class of $(K, \delta_{|K}, u_0, \nu)$ in $\mathbb{K}$. This map is continuous, and hence measurable. The properties of U ensure that f is surjective. Using a theorem of measure theory from [16], every probability distribution τ on $\mathbb{K}$ can be lifted to a probability measure σ on $\mathbb{K}(U)$, such that $f_* \sigma = \tau$. Hence, for all $n \geq 1$, we can have a version of $(\mathbf{b}_n, \mathbf{d}_n, \boldsymbol{\rho}_n, \boldsymbol{\nu}_n) = (B_n, \lambda_n \cdot D_n, \rho_n, w_n \cdot \nu_n)$ that is embedded in the space U.

A.2. Hausdorff dimension.   We recall some notation and definitions that are in relation with Hausdorff dimension and that we use throughout the paper. Let (X, d) be a metric space and $\delta > 0$. We say that the family $(O_i)_{i \in I}$ of subsets of X is a δ-cover of X if it is a covering of X, and the set I is at most countable and for all $i \in I$, the set O_i is such that its diameter satisfies $\mathrm{diam}(O_i) < \delta$. We set

$$\mathcal{H}_s^\delta(X) := \inf\bigg\{\sum_{i \in I} \mathrm{diam}(O_i)^s \,\Big|\, (O_i)_{i \in I} \text{ is a } \delta\text{-cover of } X\bigg\}.$$

As this quantity increases when δ decreases to 0, we define its limit

$$\mathcal{H}_s(X) := \lim_{\delta \to 0} \mathcal{H}_s^\delta(X) \in [0, \infty],$$

the *s-dimensional Hausdorff measure* of X. Now the Hausdorff dimension of X is defined as

$$\dim_{\mathrm{H}}(X) := \inf\big\{s > 0 \mid \mathcal{H}_s(X) = 0\big\} = \sup\big\{s > 0 \mid \mathcal{H}_s(X) = \infty\big\}.$$

We refer to [10] for details. A useful tool for deriving lower bounds on the Hausdorff dimension of a metric space is the so called Frostman's lemma. In this paper, we use the following version.

LEMMA 20 (Frostman's lemma). *Let (X, d) be a metric space. If there exists a nonzero finite Borel measure μ on X and $s > 0$ such that for μ-almost every $x \in X$, we have*

$$\mu\big(B(x, r)\big) \underset{r \to 0}{\leq} r^{s + o(1)},$$

then

$$\dim_H(X) \geq s.$$

A.3. Decomposition into *fragments*. In this section, we prove Lemma 17. We first construct our fragments in a deterministic setting and then show how we can apply this to random blocks.

Decomposition of a deterministic block. Let $(\mathbf{b}, \mathbf{d}, \boldsymbol{\rho}, \boldsymbol{\nu})$ be a (deterministic) pointed compact metric space endowed with a Borel probability measure. We are interested in how we can decompose $\mathbf{b}$ into a partition of subsets that all have approximately the same diameter r. For $r > 0$, we set

$$\mathcal{P}_r(\mathbf{b}) := \left\{ \{x_1, x_2, \ldots, x_n\} \subset \mathbf{b} \,\middle|\, n \geq 1 \text{ and } \forall i \neq j, \mathbf{d}(x_i, x_j) \geq \frac{r}{2} \right\}.$$

It is easy to verify that we can find $p = \{x_1, x_2, \ldots, x_n\} \in \mathcal{P}_r(\mathbf{b})$ such that $\mathbf{b} \subset \bigcup_{i=1}^n B(x_i, r)$ and the balls $(B(x_i, \frac{r}{4}))_{1 \leq i \leq n}$ are disjoint. Indeed, any $\frac{r}{2}$-net of $\mathbf{b}$ belongs to $\mathcal{P}_r(\mathbf{b})$ (they are the maximal elements of $\mathcal{P}_r(\mathbf{b})$ for the order relation of inclusion). We denote $\mathcal{P}_r^*(\mathbf{b})$ the set

$$\mathcal{P}_r^*(\mathbf{b}) := \left\{ \{x_1, x_2, \ldots, x_n\} \in \mathcal{P}_r(\mathbf{b}) \,\middle|\, \mathbf{b} \subset \bigcup_{i=1}^n B(x_i, r) \right\},$$

which is nonempty from what precedes. Considering the collection of balls of radius r with centres in $p \in \mathcal{P}_r^*(\mathbf{b})$ gives rise to a covering of $\mathbf{b}$ which is close to optimal in a sense specified by the following lemma. We recall the notation $N_r(\mathbf{b})$ which denotes the minimal number of balls of radius r needed to cover $\mathbf{b}$.

LEMMA 21. *For any $p = \{x_1, x_2, \ldots, x_n\} \in \mathcal{P}_r^*(\mathbf{b})$, we have $n \leq N_{r/4}(\mathbf{b})$.*

PROOF. Let $p = \{x_1, x_2, \ldots, x_n\} \in \mathcal{P}_r^*(\mathbf{b})$. Remark that, for any set S such that the union of the balls $(B(s, \frac{r}{4}))_{s \in S}$ covers $\mathbf{b}$, each of the balls $B(x_i, r/4)$, for $1 \leq i \leq n$, contains at least a point of S. Since those balls are disjoint, the cardinality of S is at least n. $\square$

From any element $p = \{x_1, x_2, \ldots, x_n\} \in \mathcal{P}_r^*(\mathbf{b})$, we can then construct a partition of $\mathbf{b}$, into subsets $(f_i)_{1 \le i \le n}$ that we call fragments, and such that

$$\forall i \in [\![1, n]\!], \quad \mathrm{B}\left(x_i, \frac{r}{4}\right) \subset f_i \subset \mathrm{B}(x_i, r).$$

We define the (f_i) recursively as follows:

$$\begin{cases} f_1 := \left\{ x \in \mathbf{b} \mid \mathbf{d}(x, x_1) = \min_{1 \le i \le n} \mathbf{d}(x, x_i) \right\}, \\ f_{k+1} := \left\{ x \in \left(\mathbf{b} \setminus \bigcup_{i=1}^{k} f_i \right) \mid \mathbf{d}(x, x_{k+1}) = \min_{1 \le i \le n} \mathbf{d}(x, x_i) \right\}. \end{cases}$$

If we suppose that $\mathbf{b}$ satisfies the condition $(\star_{r_0})$, and that $r < r_0$ then, for $p = \{x_1, x_2, \ldots, x_n\} \in \mathcal{P}_r^*(\mathbf{b})$, we get

$$n\left(\frac{r}{4}\right)^{d+\varphi(r)} \le \sum_{i=1}^{n} \boldsymbol{\nu}\left(\mathrm{B}\left(x_i, \frac{r}{4}\right)\right) \le \boldsymbol{\nu}(\mathbf{b}) = 1,$$

so that we have

$$(32) \qquad\qquad n \le \left(\frac{r}{4}\right)^{-d-\varphi(r/4)},$$

and also, for all $i \in [\![1, n]\!]$,

$$(33) \qquad \operatorname{diam} f_i \le 2r \quad \text{and} \quad \left(\frac{r}{4}\right)^{d+\varphi(r/4)} \le \boldsymbol{\nu}(f_i) \le r^{d-\varphi(r)}.$$

Let us state another lemma.

LEMMA 22.   *Let $r_0 < 1$. Under the condition $(\star_{r_0})$, there exists two functions ψ and ϕ defined on $[0, r_0/3]$, which tend to 0 at 0 such that for all $r \in (0, r_0/3)$, for all $p = \{x_1, x_2, \ldots, x_n\} \in \mathcal{P}_r^*(\mathbf{b})$ and fragments (f_i) constructed as above, we have*

$$\forall x \in \mathbf{b}, \forall r' \in (0, r_1),$$

$$\#\{1 \le i \le n \mid \mathrm{B}(x, r') \cap f_i \ne \varnothing\} \le (r \vee r')^{d+\psi(r \vee r')} \cdot r^{-d+\phi(r)}.$$

PROOF.   Let $x \in \mathbf{b}$. If for an $i \in [\![1, n]\!]$, we have $y \in \mathrm{B}(x, r') \cap f_i \ne \varnothing$, then $\mathbf{d}(x, y) < r'$ and $\mathbf{d}(x_i, y) < r$ so $\mathbf{d}(x, x_i) < r + r'$, and so we get that $\mathrm{B}(x_i, r) \subset \mathrm{B}(x, r' + 2r)$. Then, using that $f_i \subset \mathrm{B}(x_i, r)$,

$$\left(\bigcup_{i: f_i \cap \mathrm{B}(x, r') \ne \varnothing} \mathrm{B}\left(x_i, \frac{r}{4}\right) \right) \subset \left(\bigcup_{i: f_i \cap \mathrm{B}(x, r') \ne \varnothing} f_i \right) \subset \mathrm{B}(x, r' + 2r).$$

We can use the condition $(\star_{r_0})$ to get, for all $r, r' \in [0, r_0/3]$,

$$\#\{1 \le i \le n \mid \mathrm{B}(x, r') \cap f_i \ne \varnothing\} \cdot \left(\frac{r}{4}\right)^{d+\varphi(r/4)} \le (r' + 2r)^{d-\varphi(r'+2r)}.$$

And so,

$$\#\{1 \le i \le n \mid B(x, r') \cap f_i \ne \varnothing\}$$

$$\le 4^{d+\varphi(r/4)} \frac{(r' + 2r)^{d-\varphi(r'+2r)}}{r^{d+\varphi(r/4)}}$$

$$\le 4^{d+\varphi(r/4)} \frac{(3(r \vee r'))^{d-\varphi(r'+2r)}}{r^{d+\varphi(r/4)}}$$

$$\le 4^{d+\varphi(r/4)} 3^{d-\varphi(r'+2r)} (r \vee r')^{d-\varphi(r'+2r)} r^{-d-\varphi(r/4)},$$

$$\le (r \vee r')^{d-\varphi(3(r \vee r'))} r^{-d-\varphi(r/4) + \frac{\log(12^{3d/2})}{\log r}},$$

which proves the lemma. $\square$

This lemma gives us an abstract result for the existence and the properties of these decompositions in fragments. The next paragraph explains a procedure to construct one using a sequence of i.i.d. random points, on a possibly random block.

Finding an element of $\mathcal{P}_r^*(\mathbf{b})$. Suppose that the measure ν charges all open sets. Remark that this is almost surely true for our random block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$ because they satisfy Hypothesis H_d(i). Let $(X_n)_{n \ge 1}$ be a sequence of i.i.d. random variables with law ν. Let us construct a random element of $\mathcal{P}_r^*(\mathbf{b})$ for some fixed $r > 0$. Define the set E_n recursively as follows:

$$E_1 := \{1\} \quad \text{and} \quad \begin{cases} E_{n+1} := E_n & \text{if } X_{n+1} \in \bigcup_{i \in E_n} B\left(X_i, \frac{r}{2}\right), \\ E_{n+1} := E_n \cup \{n + 1\} & \text{otherwise.} \end{cases}$$

We set $E_\infty = \bigcup_{n \ge 1} E_n$. Note that from the construction, $\{X_i, i \in E_\infty\} \in \mathcal{P}_r(\mathbf{b})$.

LEMMA 23. *Almost surely, we have $\{X_i, i \in E_\infty\} \in \mathcal{P}_r^*(\mathbf{b})$.*

PROOF. The fact that the balls $(B(X_i, \frac{r}{4}))_{i \in E_\infty}$ are disjoint follows directly from the construction. Now let $x \in \mathbf{b}$. Since $\nu(B(x, \frac{r}{4})) > 0$, by the Borel–Cantelli lemma there exists at least one n such that $X_n \in B(x, \frac{r}{4})$. If $n \in E_\infty$, then $x \in B(X_n, \frac{r}{4})$. Otherwise, $n \notin E_\infty$, in which case there exists $k \le n$ such that $X_n \in B(X_k, \frac{r}{2})$ and so $x \in B(X_k, \frac{3}{4}r)$. In both cases,

$$B\left(x, \frac{r}{4}\right) \subset \bigcup_{i \in E_\infty} B(X_i, r).$$

Since we can apply the same reasoning to every point of a countable dense sequence $(y_k)_{k \ge 1}$, the lemma is proved. $\square$

PROOF OF LEMMA 17.   This is just a consequence of Lemma 21, Lemma 22 and Lemma 23 and equations (32) and (33), which almost surely apply to the random block $(\mathsf{B}, \mathsf{D}, \rho, \nu)$.   □

A.4. Computations.

LEMMA 24.   *Suppose that there exists $\gamma \geq 0$ such that for all $n \in \mathbb{N}$, $W_n \leq n^\gamma$. Then there exists a constant C such that*

$$\sum_{k=1}^{n} \frac{w_k}{W_k} \leq C \log n.$$

PROOF.   If the series $\sum w_k$ converges, then the results is trivial so let us suppose that it diverges. For $k \geq 0$, we define $n_k := \inf\{i \geq 1 \mid W_i \geq 2^k\}$ and write

$$\sum_{k=n_0}^{n} \frac{w_k}{W_k} \leq \sum_{i=0}^{\lceil \frac{\log W_n}{\log 2} \rceil} \sum_{k=n_i}^{n_{i+1}-1} \frac{w_k}{W_k}$$

$$\leq \sum_{i=0}^{\lceil \frac{\log W_n}{\log 2} \rceil} \frac{1}{2^i} \sum_{k=n_i}^{n_{i+1}-1} w_k$$

$$\leq \sum_{i=0}^{\lceil \frac{\log W_n}{\log 2} \rceil} \frac{2^{i+1}}{2^i}$$

$$\leq 2 \left\lceil \frac{\log W_n}{\log 2} \right\rceil,$$

which grows at most logarithmically thanks to our assumption on the sequence (W_n).   □

LEMMA 25.   *Let $\beta < 1$ and assume that $w_n \leq n^{-\beta+o(1)}$ and $W_n = n^{1-\beta+o(1)}$, and that for some $\epsilon > 0$ we have*

$$\liminf_{n \to \infty} \frac{1}{W_n} \sum_{\substack{k=1 \\ k \in G^\epsilon}}^{n} w_k > 0.$$

Then there exists a constant C_ϵ such that for N large enough we have

$$\sum_{\substack{k=N \\ k \in G^\epsilon}}^{N^{1+\epsilon}} \frac{w_k}{W_k} \geq C_\epsilon \log N.$$

PROOF. Let c be such that, for n large enough, $\frac{1}{W_n}\sum_{k=1}^{n} w_k \mathbf{1}_{\{k\in G^\epsilon\}} > c$. Let $C := \frac{3}{c}$, note that $C > 1$ because $c \le 1$. For all $i \ge 1$, we set $n_i = \inf\{n \mid W_n \ge C^i\}$. For all $i \ge 1$, we have $W_{n_i-1} \le C^i \le W_{n_i} \le C^i + w_{n_i}$. We get

$$\sum_{k=n_i+1}^{n_{i+1}} w_k \mathbf{1}_{\{k\in G^\epsilon\}} \ge c W_{n_{i+1}} - W_{n_i}$$

$$\ge cC^{i+1} - C^i - w_{n_i}$$

$$\ge C^i\left(cC - 1 - \frac{w_{n_i}}{C^i}\right)$$

$$\ge C^i(1 + o(1))$$

for i tending to infinity. Now for N a large integer, we set

$$I_N := \inf\{i \mid n_i \ge N\} = \left\lceil \frac{\log W_N}{\log C} \right\rceil \quad \text{and}$$

$$J_N := \sup\{i \mid n_i \le N^{1+\epsilon}\} = \left\lfloor \frac{\log W_{\lfloor N^{1+\epsilon}\rfloor}}{\log C} \right\rfloor.$$

Then we compute

$$\sum_{k=N}^{N^{1+\epsilon}} \frac{w_k}{W_k} \mathbf{1}_{\{k\in G^\epsilon\}} \ge \sum_{i=I_N}^{J_N} \sum_{k=n_i+1}^{n_{i+1}} \frac{w_k}{W_k} \mathbf{1}_{\{k\in G^\epsilon\}}$$

$$\ge \sum_{i=I_N}^{J_N} \frac{1}{W_{n_{i+1}}} \sum_{k=n_i+1}^{n_{i+1}} w_k \mathbf{1}_{\{k\in G^\epsilon\}}$$

$$\ge \sum_{i=I_N}^{J_N} \frac{1}{C^{i+1}(1 + o(1))} C^i(1 + o(1))$$

$$\ge \frac{J_N - I_N}{C}(1 + o(1)).$$

We complete the proof by noting that, thanks to the hypothesis on the growth of W_n, the last display grows logarithmically in N. □

PROOF OF LEMMA 16. From our assumptions, it is easy to see that we have $a_k = n_k^{1-\beta+o(1)}$. For all k, we write

$$\log a_k = (1 - \beta + r_k)\log n_k,$$

with $r_k \to 0$ as $k \to \infty$. We write

$$(34) \quad \sum_{i=1}^{k} \log a_i = \sum_{i=1}^{k} (1 - \beta + r_i)\log n_i = \sum_{i=0}^{k-1} (1 - \beta + r_{k-i})\log n_{k-i}.$$

For any $k \geq 0$, from the recursive definition of the sequence (n_k), we have $n_{k+1} - 1 < n_k^\gamma \leq n_{k+1}$, which entails $\log n_k = \frac{1}{\gamma} \log n_{k+1} + s_k$, with $|s_k| \leq 1$. Using this recursively yields

$$\left| \log n_{k-i} - \frac{1}{\gamma^i} \log n_k \right| \leq \frac{\gamma}{1 - \gamma}.$$

Hence, using (34) and the fact that $\log n_k$ grows exponentially in k,

$$\sum_{i=1}^{k} \log a_i = \log n_k \left((1 - \beta) \sum_{i=0}^{k-1} \frac{1}{\gamma^i} + \underbrace{\sum_{i=0}^{k} \frac{r_{k-i}}{\gamma^i}}_{\to 0} \right)$$

$$+ \underbrace{\sum_{i=0}^{k-1} (1 - \beta + r_{k-i}) \left(\log n_{k-i} - \frac{1}{\gamma^i} \log n_k \right)}_{= O(k)}$$

$$= \log n_k \left(\frac{(1 - \beta)\gamma}{\gamma - 1} + o(1) \right),$$

which proves the lemma. $\quad \square$

REFERENCES

[1] ABRAHAM, R., DELMAS, J.-F. and HOSCHEIT, P. (2013). A note on the Gromov–Hausdorff–Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.* **18** 14. MR3035742

[2] ADDARIO-BERRY, L., BROUTIN, N. and GOLDSCHMIDT, C. (2010). Critical random graphs: Limiting constructions and distributional properties. *Electron. J. Probab.* **15** 741–775. MR2650781

[3] ALDOUS, D. (1991). The continuum random tree. I. *Ann. Probab.* **19** 1–28. MR1085326

[4] AMINI, O., DEVROYE, L., GRIFFITHS, S. and OLVER, N. (2017). Explosion and linear transit times in infinite trees. *Probab. Theory Related Fields* **167** 325–347. MR3602848

[5] BOROVKOV, K. A. and VATUTIN, V. A. (2006). On the asymptotic behaviour of random recursive trees in random environments. *Adv. in Appl. Probab.* **38** 1047–1070. MR2285693

[6] CURIEN, N., DUQUESNE, T., KORTCHEMSKI, I. and MANOLESCU, I. (2015). Scaling limits and influence of the seed graph in preferential attachment trees. *J. Éc. Polytech. Math.* **2** 1–34. MR3326003

[7] CURIEN, N. and HAAS, B. (2017). Random trees constructed by aggregation. *Ann. Inst. Fourier (Grenoble)* **67** 1963–2001. MR3732681

[8] DUQUESNE, T. and WANG, G. (2014). Exceptionally small balls in stable trees. *Bull. Soc. Math. France* **142** 223–254. MR3269345

[9] DUQUESNE, T. and WANG, M. (2017). Decomposition of Lévy trees along their diameter. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 539–593. MR3634265

[10] FALCONER, K. (2014). *Fractal Geometry*, 3rd ed. Wiley, Chichester. Mathematical foundations and applications. MR3236784

[11] GOLDSCHMIDT, C. and HAAS, B. (2015). A line-breaking construction of the stable trees. *Electron. J. Probab.* **20** 16. MR3317158

[12] GOLDSCHMIDT, C., HAAS, B. and SÉNIZERGUES, D. (2018). Stable graphs: Distributions and line-breaking construction. Avialable at arxiv.org/abs/1811.06940.

[13] HAAS, B. (2017). Asymptotics of heights in random trees constructed by aggregation. *Electron. J. Probab.* **22** 21. MR3622891

[14] HUŠEK, M. (2008). Urysohn universal space, its development and Hausdorff's approach. *Topology Appl.* **155** 1493–1501. MR2435145

[15] LE GALL, J.-F. (2010). Geodesics in large planar maps and in the Brownian map. *Acta Math.* **205** 287–360. MR2746349

[16] LUBIN, A. (1974). Extensions of measures and the von Neumann selection theorem. *Proc. Amer. Math. Soc.* **43** 118–122. MR0330393

[17] PEMANTLE, R. (1990). A time-dependent version of Pólya's urn. *J. Theoret. Probab.* **3** 627–637. MR1067672

[18] REMBART, F. and WINKEL, M. (2018). Recursive construction of continuum random trees. *Ann. Probab.* **46** 2715–2748. MR3846837

[19] ROSS, N. and WEN, Y. (2018). Scaling limits for some random trees constructed inhomogeneously. *Electron. J. Probab.* **23** 5. MR3761565

[20] SERLET, L. (1997). A large deviation principle for the Brownian snake. *Stochastic Process. Appl.* **67** 101–115. MR1445046

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