

CONE POINTS OF BROWNIAN MOTION IN ARBITRARY DIMENSION

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We show that the convex hull of the path of Brownian motion in n -dimensions, up to time 1, is a smooth set. As a consequence we conclude that a Brownian motion in any dimension almost surely has no cone points for any cone whose dual cone is nontrivial.

1. Introduction. Fix a dimension $n \geq 2$. Let $B(\cdot)$ be a standard Brownian motion in \mathbb{R}^n . Our main object of concern in this paper will be

$$K = \text{Conv}(\{B(t) | 0 \leq t \leq 1\}),$$

where $\text{Conv}(\cdot)$ denotes the convex hull.

For a convex set K and a point $x \in \partial K$, we say that x is *singular* if the supporting hyperplane to K at x is not unique. We say that K is smooth if none of its boundary points are singular. The main result of this paper is the following:

THEOREM 1. *K is smooth almost surely.*

In two dimensions the fact that the boundary of K is C_1 -smooth was first stated by Paul Lévy in 1948 [6] and was later rigorously established in [2] (see also [1]).

When $n = 2$ and $0 \leq \alpha \leq 2\pi$, we say that x is a α -cone point of the two-dimensional Brownian motion $B(t)$, $0 \leq t \leq 1$, if there exists $t_0 \in [0, 1]$ and $\delta > 0$ such that $B(t_0) = x$ and such that $B(t) \in W$, $\forall t \in [t_0 - \delta, t_0 + \delta]$, where W is a wedge whose tip lies at x and with opening angle α . In two dimensions the smoothness of K is related to the absence of cone points of angle smaller than π . Furthermore, it was proven in [4] that, almost surely, the Hausdorff dimension of the set of α -cone points is equal to $2 - \frac{2\pi}{\alpha}$.

The objective of the present paper is to initiate the investigation of cone points in higher dimension. By taking a countable intersection over rational δ , we can strengthen the aforementioned smoothness, proving that cone points in which the supporting cone is strictly contained in a half-space do not exist in any dimension:

Received May 2018; revised December 2018.

¹Incumbent of the Elaine Blond Career Development Chair. Supported by a European Research Council Starting Grant (ERC StG).

²Partially supported by the Israel Science Foundation (Grant no. 715/16)
MSC2010 subject classifications. 60J65, 52A22.

Key words and phrases. Brownian motion, cone points, convex hull.

COROLLARY 2. *Almost surely, there does not exist $t_0 \in [0, 1]$, $\delta > 0$ such that $\{B(t), |t - t_0| \leq \delta\}$ is contained in a convex cone \mathcal{C} whose tip lies at $B(t_0)$ and which is strictly contained in a half-space.*

PROOF. For any $t_1 < t_2$, define $K[t_1, t_2] := \text{Conv}(\{B(t), t_1 \leq t \leq t_2\})$. Because of Brownian scaling and the strong Markov property, Theorem 1 implies that $K[t_1, t_2] - B(t_1)$ (and hence also $K[t_1, t_2]$ itself) is almost surely smooth. Taking a countable intersection over all rational t_1, t_2 , we get that almost surely $K[t_1, t_2]$ is smooth for all $t_1, t_2 \in [0, 1] \cap \mathbb{Q}$. For every $t_0 \in [0, 1]$, $\delta > 0$, there exist a pair of rational numbers q_1, q_2 such that $t_0 \in [q_1, q_2]$ and $[q_1, q_2] \subseteq \{|t - t_0| \leq \delta\}$. The smoothness of $K[q_1, q_2]$ implies that $\{B(t), |t - t_0| \leq \delta\}$ cannot be contained in a cone whose tip lies at $B(t_0)$ which is strictly contained in a half-space. Indeed, such a cone would necessarily also be contained in the intersection of two distinct half-spaces. Thus, either $B(t_0)$ is in the interior of $K[q_1, q_2]$ or if it is on the boundary, that would contradict the uniqueness of the supporting hyperplanes to $K[q_1, q_2]$ at $B(t_0)$. \square

1.1. *Discussion and proof outline.* In the planar case ($n = 2$), the smoothness of K can be reduced to a countable intersection of events in the following way. For every pair of rational directions, consider the event that the supporting hyperplanes to K in these directions coincide at a point of the Brownian motion. If K is not smooth, there must exist two rational directions whose respective event occurs. This allows us to reduce the smoothness to bounds concerning the local behavior of Brownian motion which in turn relies on the decomposition of the Brownian motion to two independent coordinates and several classical bounds regarding the probability of a one dimensional Brownian motion to be contained in a small interval.

Dimensions higher than *two* seem to pose a significant additional difficulty. Unlike the planar case, it appears that one is not able to express the smoothness of the convex hull as the intersection of a countable family of “local” events. To put it differently, in order for a boundary point of the convex hull in three dimensions to be smooth, one needs to check that the convex hull is not contained in any wedge among a one-parameter family of wedges, and this event cannot be written as an intersection of a countable number of events that only depend on a two-dimensional behavior.

Alternatively, the smoothness of a convex body amounts to the smoothness of every two-dimensional projection; however, there is no hope of reducing the smoothness of K to the smoothness of a countable set of two-dimensional projection. To illustrate this, consider the set $\{(x_1, x_2, x_3); x_3 > x_1^2 + |x_2|\}$. This set is not smooth, while for all directions except a set of measure zero, the corresponding two-dimensional projection is smooth. Therefore there does not seem to be a way to take advantage of the fact that every fixed two-dimensional projection of K is smooth with probability one.

One of the key ideas that allows us to overcome this difficulty is to reduce non-smoothness of K to a quantitative estimate regarding the behavior of its discrete approximations by polytopes. Specifically, we will construct a family of discrete polytopes K_α that provide increasingly fine approximations to K as $\alpha \rightarrow \infty$. This will be done by sampling points from a family of Poisson point processes Λ_α on the interval $[0, 1]$, where Λ_α has intensity α and $\Lambda_\beta \subseteq \Lambda_\gamma$ for $\beta \leq \gamma$ and defining

$$K_\alpha = \text{Conv}(\{B(t) | t \in \Lambda_\alpha\}).$$

This construction will allow us to reduce the nonsmoothness of K to a more tractable property—the existence of pairs of “discordant” facets of K_α , that is, facets whose distance from each other is of order $1/\sqrt{\alpha}$, and such that the angle between their corresponding normal directions is bounded away from zero. This reduction is carried out in Section 2, whose main conclusion is that

$$\mathbb{P}(K \text{ is not smooth}) \leq \sup_{\kappa > 0} \limsup_{\alpha \rightarrow \infty} \alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{P}(\tilde{C}_{\alpha,\kappa}(r, s)) \, dr \, ds.$$

Let us now explain the right-hand side of the above formula. We define Δ_n to be the n -dimensional simplex, $\{(s_1, \dots, s_n), 0 \leq s_1 \leq \dots \leq s_n \leq 1\}$ which should be thought of as all possible ways to choose a set of times such that the points $B(s_1), \dots, B(s_n)$ potentially form a facet in the boundary of the approximating polytope. The event $\tilde{C}_{\alpha,\kappa}(r, s)$ is, roughly speaking, the event that the points corresponding to r and s form a pair of discordant facets of K_α with angle at least κ between the respective normal directions. Thus, the right-hand side should be understood of as the expectation of the number of pairs of discordant facets.

The following two sections are devoted to bounding from above the probability of the event $\tilde{C}_{\alpha,\kappa}(r, s)$. Specifically, our goal will be to obtain the estimate

$$\mathbb{P}(\tilde{C}_{\alpha,\kappa}(r, s)) \leq \alpha^{-2n-1} + \alpha^{-2n-c\kappa} \frac{1}{\sqrt{t_1(1-t_{2n})}} \prod_{i=2}^{2n} \frac{1}{t_i - t_{i-1}},$$

where for $(r, s) \in \Delta_n \times \Delta_n$, we denote by $\mathbf{t}(r, s) = (t_1, \dots, t_{2n})$ the points of $r \cup s$ in increasing order, and $c > 0$ is some constant that depends only on the dimension n . We will do this by decomposing $\tilde{C}_{\alpha,\kappa}(r, s)$ into an intersection of $2n + 1$ independent events, each requiring that the Brownian motion is contained in a wedge during the subinterval $[t_{i-1}, t_i]$ (this is necessary in order that the images of $r \cup s$ form facets in the boundary of K_α). Setting aside the dependence on t_i (which will prove to be immaterial to the result), the main point of Section 4 will then be to show that the edge subintervals $[0, t_1]$ and $[t_{2n}, 1]$ contribute factors of order $O(\alpha^{-1/2})$, each of the $2n - 1$ interior subintervals $[t_1, t_2], \dots, [t_{2n-1}, t_{2n}]$ contributes a factor of order $O(\alpha^{-1})$, and, crucially, that one of the $2n + 1$ subintervals will contribute a power of α strictly smaller than that (hence the $c\kappa$ term). The first two bounds are not hard to obtain and can be thought of as generalizations of classical results regarding random walks. Indeed, it is well known that the probability

that a $2n$ -step simple random walk stays nonnegative at all times is of order $n^{-1/2}$, and that the conditional probability that it stays nonnegative at all times given that it returns to the origin at time $2n$ is of order n^{-1} . Thus those bounds roughly follow by substituting the intensity α (or more precisely $\alpha(t_i - t_{i-1})$) for n .

The main difficulty lies in obtaining the extra power of α , which is where we use the fact that the Brownian motion is confined to be in a wedge as opposed to a half-space, and the facets forming the wedge are discordant on a carefully chosen subinterval. The estimates, which exist in the literature regarding exit probabilities of a Brownian motion from a wedge, do not quite suffice here, and we apply sharp bounds, derived in Section 4.1, for the probability that a Brownian bridge, *whose endpoints are close to the boundary of some wedge*, stays in that wedge. Once the estimate is established, Theorem 1 follows by a simple computation which is given in Section 5.

2. An approximating polytope. A key step in our proof is to consider an approximation of K by polytopes defined as the convex hull of an associated random walk partial to the range of $B(t)$. In Section 2.2 below we will see how to reduce the smoothness of K to quantitative behavior of the facets of those discrete approximations. In turn the behavior of those facets can be made tractable via a formula derived in Section 2.1. The construction as well as some of the formulas that make it accessible were used in [3] in order to compute several quantities related to K , such as its volume and its surface area.

We construct the random walk as follows. Let $P = ((x_1, y_1), (x_2, y_2), \dots)$ be a Poisson point process of intensity 1, independent of $B(\cdot)$, in the set $[0, 1] \times [0, \infty]$ and for all $\alpha \geq 0$, define

$$\Lambda_\alpha = \{x_i | y_i \leq \alpha, i \in \mathbb{N}\} \cup \{0, 1\}.$$

The process Λ can be thought of as a ‘‘Poisson rain’’ on the interval $[0, 1]$; note that for all $\alpha \geq 0$, Λ_α is a Poisson point process of intensity α on the unit interval and that the family Λ_α is increasing with α . For a fixed value of α , writing $\Lambda_\alpha = (t_1, \dots, t_N)$ where $0 = t_1 \leq \dots \leq t_N = 1$, we can think of $(B(t_1), B(t_2), \dots, B(t_N))$ as a random walk in \mathbb{R}^n . Finally, for all $\alpha > 0$, we define

$$K_\alpha = \text{Conv}(\{B(t) | t \in \Lambda_\alpha\}),$$

so K_α is a monotone family of discrete approximations of K .

2.1. A formula for the facets. In this section we recall some notions from [3] towards a formula which allows us to calculate the expectation of quantities related to the facets of K_α . We begin with some notation. Let Δ_n be the n -dimensional simplex, namely

$$\Delta_n = \{(r_1, \dots, r_n) \in [0, 1]^n; r_1 < \dots < r_n\}.$$

Next, for $r \in \Delta_n$ we define (by slight abuse of notation)

$$B(r) := \{B(r_1), \dots, B(r_n)\},$$

and $F_r := \text{Conv}(B(r))$ which is almost surely an $(n - 1)$ -dimensional simplex. Let n_r be a unit vector normal to F_r chosen such that $\langle n_r, B(r_1) \rangle \geq 0$.

Next, for a Borel subset $A \subset \Delta_n$ we define

$$(1) \quad q_\alpha(A) = \#\{r \in A; F_r \text{ is a facet in the boundary of } K_\alpha\}.$$

We also need the definition of the point process

$$w_\alpha(A) = \#\{r \in A; \{r_1, \dots, r_n\} \subset \Lambda_\alpha\},$$

which we can think of as points $r \in \Delta_n$ which are candidates to be facets of K_α in the sense that all their vertices are points of the random walk. Moreover, we define the (deterministic) measures

$$\mu_\alpha(\cdot) = \mathbb{E}[q_\alpha(\cdot)] \quad \text{and} \quad \nu_\alpha(\cdot) = \mathbb{E}[w_\alpha(\cdot)].$$

Let \mathcal{F}_B be the σ -algebra generated by the Brownian motion $B(\cdot)$ (so that a random variable is measurable with respect to \mathcal{F}_B if and only if it does not depend on the point process Λ). Let $f : \Delta_n \rightarrow \mathbb{R}$ be a function such that for all $r \in \Delta_n$, $f(r)$ is a random variable which is measurable with respect to \mathcal{F}_B . Then, by a calculation using the Campbell–Little–Mecke formula one obtains (see [3], equation (3.7))

$$\mathbb{E}\left[\int_{\Delta_n} f(r) dq_\alpha(r)\right] = \alpha^n \int_{\Delta_n} \mathbb{E}[f(r)\mathbf{1}_{E_\alpha(r)}] dr,$$

where

$$E_\alpha(r) := \{F_r \text{ is a facet in the boundary of } \text{Conv}(B(r) \cup K_\alpha)\}.$$

Following the exact same lines, this formula can be generalized in the following sense. Let $f : \Delta_n \times \Delta_n \rightarrow [0, \infty)$ be a function such that for all $(r, s) \in \Delta_n \times \Delta_n$, $f(r, s)$ is a random variable which is measurable with respect to \mathcal{F}_B . Define

$$E_\alpha(r, s) := \{F_r, F_s \text{ are facets in the boundary of } \text{Conv}(B(r) \cup B(s) \cup K_\alpha)\}.$$

Then, the generalized formula reads

$$\mathbb{E}\left[\int_{\Delta_n \times \Delta_n} f(r, s) dq_\alpha(r) dq_\alpha(s)\right] = \alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{E}[f(r, s)\mathbf{1}_{E_\alpha(r, s)}] dr ds.$$

Since $E_\alpha(r, s) \subseteq E_\alpha(r) \cap E_\alpha(s)$, we finally have

$$(2) \quad \begin{aligned} &\mathbb{E}\left[\int_{\Delta_n \times \Delta_n} f(r, s) dq_\alpha(r) dq_\alpha(s)\right] \\ &\leq \alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{E}[f(r, s)\mathbf{1}_{E_\alpha(r) \cap E_\alpha(s)}] dr ds. \end{aligned}$$

In the following subsection the function $f(r, s)$ will be taken to be, roughly speaking, the indicator of the event that angle between the facets corresponding to r and s is bounded away from zero and that one of these facets is close to the tip of the associated wedge.

2.2. *Reducing the smoothness of K to an approximate smoothness of K_α .* In this section lies the idea behind the definition of the approximating polytope, K_α . We will show that the nonsmoothness of K amounts, roughly, to the following asymptotic behavior of K_α , as $\alpha \rightarrow \infty$. Given that K is not smooth, for sufficiently large α the polytope K_α has two *discordant facets*, namely, facets with distance of order $1/\sqrt{\alpha}$ from each other, such that the angle between the corresponding normal directions is bounded away from zero. In the upcoming subsections our main goal will be to show that this is not possible.

Let us first introduce some notation. For $(r, s) \in \Delta_n \times \Delta_n$, define $L(r, s)$ to be the intersection of the $(n - 1)$ -dimensional affine subspaces spanned by F_r and F_s . Moreover, define

$$W(r, s) := \{x + y; x \in L(r, s), \langle y, n_r \rangle \leq 0 \text{ and } \langle y, n_s \rangle \leq 0\},$$

the wedge defined by the facets F_r, F_s and set $\theta(r, s) = \arccos(\langle n_r, n_s \rangle)$, the inner angle of the wedge $W(r, s)$. Define

$$\phi(\alpha) := e^{\sqrt{\log \alpha}}.$$

The function $\phi(\alpha)$ will be used throughout the proof as a function which is asymptotically subpolynomial and super-logarithmic. Its particular definition is of no importance as long as it has those asymptotics.

A key definition for us will be the event

$$(3) \quad C_{\alpha, \kappa}(r, s) := \{\theta(r, s) \geq \kappa/16\} \cap \left\{ \text{dist}(F_r \cup F_s, L(r, s)) \leq \frac{128\phi^2(\alpha)}{\kappa^2\sqrt{\alpha}} \right\},$$

for all $\alpha > 0, \kappa \in [0, \pi)$. When the event $C_{\alpha, \kappa}(r, s)$ holds, we will say that F_s and F_r are *discordant facets*.

For $0 \leq a \leq b \leq 1$, we consider the event

$$(4) \quad R_\alpha[a, b] := \{\forall t \in [a, b], \Lambda_\alpha \cap [a, b] \cap [t - \alpha^{-1}\phi(\alpha), t + \alpha^{-1}\phi(\alpha)] \neq \emptyset\} \\ \cap \left\{ \sup_{t_1, t_2 \in [a, b]} |B(t_1) - B(t_2)| \leq |t_2 - t_1|^{1/2}\phi(\alpha) + \alpha^{-2n-1} \right\}$$

and write $R_\alpha := R_\alpha[0, 1]$. Remark that for α larger than some constant depending only on n , we have the following implication:

$$(5) \quad R_\alpha \text{ holds} \Rightarrow \forall t \in [0, 1], \exists s \in \Lambda_\alpha \text{ such that } |B(s) - B(t)| \leq \frac{\phi(\alpha)^2}{\sqrt{\alpha}}.$$

The following lemma, whose proof is postponed to the [Appendix](#), is based on standard estimates.

LEMMA 3. *For every dimension $n \geq 2$, there exists $C > 0$ such that for all $\alpha > C$ we have*

$$(6) \quad \mathbb{P}(R_\alpha^C) \leq \alpha^{-2n-1}.$$

Next, we formulate a geometric lemma which will later allow us to reduce the nonsmoothness of K to the existence of two facets F_r, F_s for which the event $C_{\alpha,\kappa}(r, s)$ holds true. The lemma roughly states that if a polytope is contained in a wedge with opening angle bounded from below and such that the polytope contains a point not too far from the tip of the wedge, then the polytope must have two discordant facets.

LEMMA 4. *Let W be a wedge whose tip contains the origin and whose opening angle is $\pi - \kappa$ for some $\kappa \in (0, \pi)$. Let P be a convex polytope contained in W whose distance from the origin is at most s . Then there exist two facets F_1, F_2 of P with normal directions n_1, n_2 such that the angle between n_1 and n_2 is at least $\frac{\kappa}{16}$ and such that the following holds. Let L be the span of n_1, n_2 and let $W' \subset L$ be the wedge corresponding to F_1, F_2 with tip w' , then the distance between w' and the projection of F_1 on L is at most $\frac{128s}{\kappa^2}$.*

The proof of this lemma is postponed to the end of the section. As a direct corollary of this lemma, we have the following proposition:

PROPOSITION 5. *For every $\kappa > 0$ there exist a constant $C_\kappa > 0$ such that the following holds almost surely: For any $\alpha > C_\kappa$, suppose that there exists $x \in \partial K$ such that K is contained in a wedge of angle κ whose tip is at x and that R_α holds. Then, there exist $(r, s) \in \Delta_n \times \Delta_n$ such that F_r, F_s are facets on the boundary of K_α and such that the event $C_{\alpha,\kappa}(r, s)$ holds.*

PROOF. Using the implication (5), we can invoke Lemma 4 with $s = \frac{\phi^2(\alpha)}{\sqrt{\alpha}}$ for the wedge spanned by the supporting hyperplanes to K at x and with the polytope being K_α . The existence of the facets F_1, F_2 amounts to the required events for some r, s . \square

A consequence of the above proposition is that whenever K is not smooth, then necessarily, for all α larger than some constant, either the event R_α^C holds, or one has that for some $\kappa > 0$,

$$\int_{\Delta_n \times \Delta_n} \mathbf{1}_{C_{\alpha,\kappa}(r,s)} dq_\alpha(r) dq_\alpha(s) \geq 1,$$

with $q_\alpha(\cdot)$ defined as in equation (1). Using Markov's inequality, our main theorem will thus follow if we prove that for every $\kappa > 0$, one has

$$\mathbb{P}(R_\alpha^C) + \mathbb{E}\left[\int_{\Delta_n \times \Delta_n} \mathbf{1}_{C_{\alpha,\kappa}(r,s)} dq_\alpha(r) dq_\alpha(s)\right] \xrightarrow{\alpha \rightarrow \infty} 0.$$

We are now in position to apply equation (2), according to which it is enough to show that

$$(7) \quad \mathbb{P}(R_\alpha^C) + \alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{E}[\mathbf{1}_{C_{\alpha,\kappa}(r,s)} \mathbf{1}_{E_\alpha(r) \cap E_\alpha(s)}] dr ds \xrightarrow{\alpha \rightarrow \infty} 0.$$

At this point it will be more convenient to work with the following events:

$$(8) \quad \tilde{E}_\alpha(r) := \left\{ \forall t \in [0, 1], \langle B(t), n_r \rangle \leq \langle B(r_1), n_r \rangle + \frac{\phi(\alpha)^2}{\sqrt{\alpha}} \right\}.$$

This event is almost similar to $E_\alpha(r)$ in the following sense. Equation (5) implies that

$$(9) \quad R_\alpha \cap E_\alpha(r) \subset \tilde{E}_\alpha(r).$$

Finally, we consider the event

$$\tilde{C}_{\alpha,\kappa}(r, s) := \tilde{E}_\alpha(r) \cap \tilde{E}_\alpha(s) \cap C_{\alpha,\kappa}(r, s).$$

Using (9), we now have that equation (7) follows from

$$(10) \quad \alpha^{2n} \mathbb{P}(R_\alpha^C) + \alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{P}(\tilde{C}_{\alpha,\kappa}(r, s)) dr ds \xrightarrow{\alpha \rightarrow \infty} 0.$$

The proof of the main theorem now boils down to proving the last equation. Lemma 3 implies that $\alpha^{2n} \mathbb{P}(R_\alpha^C) \xrightarrow{\alpha \rightarrow \infty} 0$, and the rest of the paper is devoted to estimating the integral. The content of this section is summarized by the following statement:

PROPOSITION 6. *For every dimension n , one has*

$$\mathbb{P}(K \text{ is not smooth}) \leq \sup_{\kappa > 0} \limsup_{\alpha \rightarrow \infty} \alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{P}(\tilde{C}_{\alpha,\kappa}(r, s)) dr ds.$$

The rest of the proof is devoted to finding an upper bound for the quantities on the right-hand side.

PROOF OF LEMMA 4. Without loss of generality, we may assume that $\kappa \leq \frac{\pi}{2}$ (indeed, otherwise we may consider a wedge with a bigger opening angle that contains the wedge W). Let u_1, u_2 be the inner normal directions to the facets of W . Define $h = \frac{u_1 + u_2}{|u_1 + u_2|}$. Since, by assumption, P contains a point whose distance to the origin is at most s , then P must contain a vertex v such that $\langle v, h \rangle \leq s$.

Let F be an $(n - 1)$ -dimensional facet containing the vertex v . Denote by u the inner normal to P at F . We claim that there exists $i \in \{1, 2\}$ such that the angle between u and u_i is within the range $[\frac{\kappa}{2}, \pi - \frac{\kappa}{2}]$. Indeed, since the angle between u_1 and u_2 is κ , which is at most $\pi/2$, by the triangle inequality it cannot be the case that both vectors have angle less than $\kappa/2$ with either the vector u or with its antipodal. Assume without loss of generality that the vector u_1 satisfies the above, hence

$$(11) \quad \arccos(\langle u, u_1 \rangle) \in \left[\frac{\kappa}{2}, \pi - \frac{\kappa}{2} \right]$$

(otherwise we may switch between u_1 and u_2).

Consider the plane H spanned by u and u_1 and passing at v . Set $P' = P \cap H$. The set P' is a (possibly degenerate) *two*-dimensional convex polygon. First, suppose that P' has an empty interior. In this case there must exist another facet F' of P which contains the vertex v and whose normal direction has a nonpositive scalar product with u . In this case the lemma is complete by considering the facets F, F' . Otherwise, we may assume that P' is a convex polygon with nonempty interior. Denote its edges by f_1, f_2, \dots in a manner that f_i and f_{i+1} share a vertex and such that f_1 contains the vertex v . Each edge f_i corresponds to a facet F_i of P , whose normal we denote by n_i .

For all $t \in \mathbb{R}$, let $j(t)$ be the curve starting at v going along P' parametrized by arc length (hence, defined uniquely in a way that $j(t)$ is injective and $\lim_{s \rightarrow t+} \frac{|j(s)-j(t)|}{s-t} = 1$ for t); the direction of which will be chosen promptly out of the two possible directions. We define

$$g(t) := \langle j(t), u_1 \rangle.$$

The assumption (11) implies that for a suitable choice of direction one has that

$$(12) \quad \min(g'_+(0), g'_-(0)) \leq -\sin(\kappa/2)$$

(here $g'_+(0), g'_-(0)$ denote the right- and left-derivatives of g at 0). Define $\ell = \frac{2s}{\sin(\kappa/4)}$. Recalling that $\langle v, \frac{u_1+u_2}{|u_1+u_2|} \rangle \leq s$ and since $\langle v, u_2 \rangle > 0$, this gives $g(0) = \langle v, u_1 \rangle \leq s|u_1 + u_2| \leq 2s$. Thus, since $g(t) \geq 0$ for all t , by the fact that g is piecewise differentiable it follows that there exists $t' \leq \ell$ such that

$$g'(t') \geq -\sin(\kappa/4).$$

The above inequality, combined with (12), implies that the angle between f_1 and f_i is at least $\kappa/4$, where f_i is the facet containing the point $j(t')$. Since u is parallel to H , it follows that the angle between u and n_i is at least $\kappa/4$. Note also that the distance between F and F_i is by definition at most $\frac{2s}{\sin(\kappa/4)}$, and the same is true for the distance between $F_k, F_{k'}$ for all $1 \leq k < k' \leq i$.

At this point the lemma will be concluded given that we find two indices $1 \leq k < k' \leq i$ such that at least one of the following holds: (i) the angle between n_k and $n_{k'}$ is in the range $[\frac{\kappa}{8}, \pi - \frac{\kappa}{8}]$ or (ii) we have $k' = k + 1$, and the angle between n_k and $n_{k'}$ is at least $\frac{\kappa}{8}$. This would be enough to complete the proof since, if we consider the wedge W' spanned by the two facets $F_k, F_{k'}$, then we have that the angle between the facets is at least $\kappa/8$ in both cases, moreover if W' is the wedge spanned by those facets with tip w' , then in case (ii) the distance of both facets to the tip is zero and in case (i) the distance of each facet to the tip is within factor $\frac{1}{\sin(\kappa/8)}$ of the distance between the facets; so that the distance is bounded by $\frac{2s}{\sin(\kappa/4)\sin(\kappa/8)} \leq \frac{128s}{\kappa^2}$.

Suppose now that case (ii) does not hold, hence that the angle between F_k and F_{k+1} is at most $\kappa/8$ for all $1 \leq k \leq i - 1$. In this case either there exists $1 \leq k' \leq i$ such that the angle between $n_{k'}$ and the plane H is at least $\kappa/8$ (in which case we

can take $k = 1$ and case (i) follows), or otherwise it must be that the angles between f_k and f_{k+1} are at most $\kappa/4$ for all $1 \leq k \leq i - 1$. Since the angle between f_1 and f_i is at least $\kappa/4$ and since $\kappa \leq \pi/2$, it follows by a continuity argument that there exists some $k' \leq i$ such that the angle between f_1 and $f_{k'}$ within the range $[\frac{\kappa}{4}, \pi - \frac{\kappa}{4}]$ and case (i) holds again. This completes the proof. \square

3. An upper bound on the probability for discordant facets. For $(r, s) \in \Delta_n \times \Delta_n$, we denote by $\mathbf{t}(r, s) = (t_1, \dots, t_{2n})$ the points of $r \cup s$ in increasing order.

The main proposition of this section is the following one:

PROPOSITION 7. *For every dimension n and every $\kappa > 0$, there exists $C > 0$ such that the following holds, for all $\alpha > C$: Fix $(r, s) \in \Delta_n \times \Delta_n$. Denote $(t_1, \dots, t_{2n}) = \mathbf{t}(r, s)$. Then one has*

$$\mathbb{P}(\tilde{\mathbf{C}}_{\alpha, \kappa}(r, s)) \leq \alpha^{-2n-1} + \alpha^{-2n-\kappa/(16,000n)} \frac{1}{\sqrt{t_1(1-t_{2n})}} \prod_{i=2}^{2n} \frac{1}{t_i - t_{i-1}}.$$

By combining the bound given in the above proposition with Proposition 6, the proof of our main theorem boils down to estimating an explicit integral over $\Delta_n \times \Delta_n$.

The general idea behind the proof of Proposition 7 is to condition on the location of the Brownian motion at the times $\mathbf{t}(r, s)$. Upon this conditioning the event $\tilde{\mathbf{C}}_{\alpha, \kappa}(r, s)$ amounts to the intersection of independent events which correspond to the behavior of the Brownian bridges in the intervals $[0, 1] \setminus (r \cup s)$.

Throughout the section we fix $\kappa > 0$ and $(r, s) \in \Delta_n \times \Delta_n$ and the corresponding times $(t_1, \dots, t_{2n}) = \mathbf{t}(r, s)$. We also fix points $b_0 = 0, b_1, \dots, b_{2n+1} \in \mathbb{R}^n$ and consider the event

$$(13) \quad S := \{B(t_i) = b_i, \forall 1 \leq i \leq 2n + 1\}.$$

Consider the following two properties:

$$(14) \quad |b_{i+1} - b_i| \leq \phi(\alpha)\sqrt{t_{i+1} - t_i} + \alpha^{-2n-1},$$

$$(15) \quad \{\theta(r, s) \geq \kappa/16\} \cap \left\{ \text{dist}(F_r \cup F_s, L(r, s)) \leq \frac{128\phi^2(\alpha)}{\kappa^2\sqrt{\alpha}} \right\}$$

(remark that property (14) is related to the event R_α defined in equation (4) and that property (15) is exactly the event $\mathbf{C}_{\alpha, \kappa}(r, s)$ defined in equation (3)). Our goal will be to replace the event $\tilde{\mathbf{C}}_{\alpha, \kappa}(r, s) \cap R_\alpha$ with an intersection of events which are independent upon conditioning on S . To that end, for all $i, 0 \leq i \leq 2n$, we write

$$(16) \quad H_i := \left\{ \forall t \in [t_i, t_{i+1}], \langle B(t), n_r \rangle \leq \langle B(r_1), n_r \rangle + \frac{\phi(\alpha)^2}{\sqrt{\alpha}} \right. \\ \left. \text{and } \langle B(t), n_s \rangle \leq \langle B(s_1), n_s \rangle + \frac{\phi(\alpha)^2}{\sqrt{\alpha}} \right\},$$

with the convention $t_0 = 0$ and $t_{2n+1} = 1$. Remark that

$$\tilde{E}_\alpha(r) \cap \tilde{E}_\alpha(s) = \bigcap_{i=0}^{2n} H_i,$$

where the events $\tilde{E}_\alpha(\cdot)$ are defined in (8) above. Moreover, by the representation theorem for the Brownian bridge, we have that the events H_i are independent conditioned on S . Recall the events $R_\alpha[a, b]$ defined in equation (4). We thus have

$$\begin{aligned} \mathbb{P}(\tilde{E}_\alpha(r) \cap \tilde{E}_\alpha(s) \cap R_\alpha | S) &\leq \mathbb{P}\left(\tilde{E}_\alpha(r) \cap \tilde{E}_\alpha(s) \cap \left(\bigcap_{i=0}^{2n} R_\alpha[t_i, t_{i+1}]\right) \middle| S\right) \\ (17) \qquad &= \mathbb{P}\left(\bigcap_{i=0}^{2n} (H_i \cap R_\alpha[t_i, t_{i+1}]) \middle| S\right) \\ &= \prod_{i=0}^{2n} \mathbb{P}(H_i \cap R_\alpha[t_i, t_{i+1}] | S). \end{aligned}$$

The following lemma shows that property (14) implies the existence of a “special” interval $[t_j, t_{j+1}]$ among $[t_0, t_1], [t_1, t_2], \dots, [t_{2n}, t_{2n+1}]$ which roughly has the property that the starting point of associated Brownian bridge has distance to the tip of the wedge which is much smaller than the square root of the length of the time interval. For this interval we will be able to derive an improved bound on $\mathbb{P}(H_j | S)$ in the next section.

LEMMA 8. *For every $n \in \mathbb{N}$ and $M > 0$ there exists $C > 0$ such that for all $\alpha > C$, the following holds. Let $t_0 = 0, t_{2n+1} = 1$ and $0 < t_1, \dots, t_{2n} < 1$. Let $b_0 = (0, 0)$ and, for all $1 \leq i \leq 2n + 1$, let $b_i \in \mathbb{R}^2$ so that the condition (14) holds. Let $w_0 \in \mathbb{R}^2$, and suppose that there exists an index $0 \leq j_0 \leq 2n + 1$ for which $|b_{j_0} - w_0| < M \frac{\phi(\alpha)^2}{\sqrt{\alpha}}$. Then there exists an index $0 \leq j \leq 2n$ such that*

$$(18) \qquad t_{j+1} - t_j \geq \alpha^{1/(10n)} \max\left(\min(|b_j - w_0|, |b_{j+1} - w_0|)^2, \frac{1}{\alpha}\right).$$

The proof of this lemma is postponed to the Appendix. Denote by P the orthogonal projection to $sp\{n_r, n_s\}$, and let w_0 be the tip of the wedge created by $P(F_s)$ and $P(F_r)$. Then, condition (15) implies that there exists an index j_0 for which $|P(b_{j_0}) - w_0| < M \frac{\phi(\alpha)^2}{\sqrt{\alpha}}$, where M depends only on κ . We may now invoke the above lemma on the points $(P(b_0), \dots, P(b_{2n}))$ to conclude that whenever $\alpha > C$ where C depends only on n and κ , there exists an index $0 \leq j \leq 2n$ for which

$$(19) \qquad t_{j+1} - t_j \geq \alpha^{1/(10n)} \max\left(\min(|P(b_j) - w_0|, |P(b_{j+1}) - w_0|)^2, \frac{1}{\alpha}\right).$$

The next proposition consists of the core estimates which will be combined in order to yield the bound of Proposition 7. Its proof is based on two-dimensional estimates for exit probabilities for a Brownian motion and a Brownian bridge which will be established in the next section.

PROPOSITION 9. *For every dimension n and every $\theta \geq 0$ and $\varepsilon > 0$, there exists a constant $C_{\theta, \varepsilon}$ such that the following holds. Let $(r, s) \in \Delta_n \times \Delta_n$, and let $(t_0, \dots, t_{2n+1}) = \mathbf{t}(r, s)$. Fix points $b_1, \dots, b_{2n+1} \in \mathbb{R}^n$. Consider the events S and $\{H_i\}_{i=0}^{2n}$ defined in equations (13) and (16). Then, almost surely, we have for all $\alpha > C_{\theta(r,s), \varepsilon}$ the following bounds:*

(i) *For all $1 \leq i \leq 2n - 1$, we have*

$$(20) \quad \mathbb{P}(H_i \cap R_\alpha[t_i, t_{i+1}] | S) \leq \frac{1}{(t_{i+1} - t_i)\alpha} \cdot \alpha^\varepsilon.$$

(ii) *For $i \in \{0, 2n\}$, we have*

$$(21) \quad \mathbb{P}(H_i \cap R_\alpha[t_i, t_{i+1}] | S) \leq \frac{1}{\sqrt{(t_{i+1} - t_i)\alpha}} \cdot \alpha^\varepsilon.$$

(iii) *For all $1 \leq j \leq 2n - 1$ such that condition (19) is satisfied, we have*

$$(22) \quad \mathbb{P}(H_j \cap R_\alpha[t_j, t_{j+1}] | S) \leq \frac{1}{(t_{j+1} - t_j)\alpha} \cdot \alpha^{-\frac{\theta(r,s)}{800n} + \varepsilon}.$$

(iv) *For $j \in \{0, 2n\}$ such that condition (19) is satisfied, we have*

$$(23) \quad \mathbb{P}(H_j \cap R_\alpha[t_j, t_{j+1}] | S) \leq \frac{1}{\sqrt{(t_{j+1} - t_j)\alpha}} \cdot \alpha^{-\frac{\theta(r,s)}{800n} + \varepsilon}.$$

The proof of this proposition is the objective of the next section. Given those bounds, we are finally ready to prove the main result of the section. The proof roughly amounts to writing the event $\tilde{C}_{\alpha, \kappa}(r, s)$ as the product of the events H_i above (upon conditioning on the Brownian motion at the times $\mathbf{t}(r, s)$). Then, we use equations (20) and (21) to bound the probability of each H_i , with the crucial exception that, to one of these events, the improved bounds (22) or (23) can be applied.

PROOF OF PROPOSITION 7. Defining $b_i = B(t_i)$ for $1 \leq i \leq 2n + 1$, let \mathcal{R} be the event that property (14) holds, and remark that the event $C_{\alpha, \kappa}(r, s)$ is equivalent to property (15). Recall that

$$(24) \quad \tilde{C}_{\alpha, \kappa}(r, s) = C_{\alpha, \kappa}(r, s) \cap \tilde{E}_\alpha(r) \cap \tilde{E}_\alpha(s),$$

and observe that

$$(25) \quad R_\alpha \subset \mathcal{R}.$$

Since we know that, under $\mathcal{R} \cap \mathbf{C}_{\alpha,\kappa}(r, s)$, condition (19) is satisfied for at least one index j , we have that one of the terms in the product $\prod_{i=0}^{2n} \mathbb{P}(H_i|S)$ can be bounded by the improved bounds given in equations (22) and (23). Thus, invoking Proposition 9 with $\varepsilon = \frac{\kappa}{2^{15n(2n+1)}}$ gives that under the event $\mathcal{R} \cap \mathbf{C}_{\alpha,\kappa}(r, s)$ we have, almost surely,

$$\begin{aligned}
 & \prod_{i=0}^{2n} \mathbb{P}(H_i \cap R_\alpha[t_i, t_{i+1}] | B(t_1), \dots, B(t_{2n+1})) \\
 (26) \quad & \leq \alpha^{-2n-\theta(r,s)/(800n)+(2n+1)\varepsilon} \frac{1}{\sqrt{t_1(1-t_{2n})}} \prod_{i=2}^{2n} \frac{1}{(t_i - t_{i-1})} \\
 & \leq \alpha^{-2n-\kappa/(16,000n)} \frac{1}{\sqrt{t_1(1-t_{2n})}} \prod_{i=2}^{2n} \frac{1}{(t_i - t_{i-1})},
 \end{aligned}$$

for all $\alpha > C_\kappa$ (where the term $\alpha^{-\theta(r,s)/(800n)}$ appears thanks to existence of the index j given by (19)). Thus, we can calculate

$$\begin{aligned}
 & \mathbb{P}(\tilde{\mathbf{C}}_{\alpha,\kappa}(r, s)) \\
 & = \mathbb{E}[\mathbb{P}(\tilde{\mathbf{C}}_{\alpha,\kappa}(r, s) | B(t_1), \dots, B(t_{2n+1}))] \\
 & \stackrel{(24)}{=} \mathbb{E}[\mathbb{P}(\tilde{E}_\alpha(r) \cap \tilde{E}_\alpha(s) | B(t_1), \dots, B(t_{2n+1})) \mathbf{1}_{\mathbf{C}_{\alpha,\kappa}(r,s)}] \\
 & \stackrel{(25)}{\leq} \mathbb{P}(R_\alpha^C) + \mathbb{E}[\mathbb{P}(\tilde{E}_\alpha(r) \cap \tilde{E}_\alpha(s) \cap R_\alpha | B(t_1), \dots, B(t_{2n+1})) \mathbf{1}_{\mathcal{R} \cap \mathbf{C}_{\alpha,\kappa}(r,s)}] \\
 & \stackrel{(17)}{\leq} \mathbb{P}(R_\alpha^C) + \mathbb{E}\left[\mathbf{1}_{\mathcal{R} \cap \mathbf{C}_{\alpha,\kappa}(r,s)} \prod_{i=0}^{2n} \mathbb{P}(H_i \cap R_\alpha[t_i, t_{i+1}] | B(t_1), \dots, B(t_{2n+1}))\right] \\
 & \stackrel{(26)}{\leq} \mathbb{P}(R_\alpha^C) + \alpha^{-2n-\kappa/(16,000n)} \frac{1}{\sqrt{t_1(1-t_{2n})}} \prod_{i=2}^{2n} \frac{1}{(t_i - t_{i-1})} \\
 & \stackrel{(6)}{\leq} \alpha^{-2n-1} + \alpha^{-2n-\kappa/(16,000n)} \frac{1}{\sqrt{t_1(1-t_{2n})}} \prod_{i=2}^{2n} \frac{1}{(t_i - t_{i-1})}.
 \end{aligned}$$

This completes the proof. \square

4. Estimates for two-dimensional wedges. The objective of this section is to prove Proposition 9 which amounts to new estimates regarding exit probabilities of Brownian bridges from two-dimensional wedges. The needed estimates must have two features that do not seem to be supported by existing bounds in the literature. They need not only to exploit the fact that the path stays inside a wedge but also to fully exploit the fact that both endpoints are close to the boundary of the wedge, and they should be valid upon conditioning on both endpoints.

For $w_0 \in \mathbb{R}^2$ and $\alpha, \beta > 0$ we define

$$W(w_0, \beta) = \{w_0 + t(\cos x, \sin x); t \in [0, \infty), |x| \leq \beta\}.$$

The following lemma is a direct consequence of [8], Theorem 2.

LEMMA 10. *For every $\pi > \theta \geq 0$, there exists $J_\theta > 0$ such that the following holds. Let $t, r > 0$. Let $W = W(0, \pi - \theta)$. Let $B(t)$ be a planar Brownian motion with starting point $B(0) = a$, a point satisfying $a \in W$ and $|a| = r$. Then,*

$$\mathbb{P}(B(s) \in W, \forall s \in [0, t]) \leq J_\theta \left(\frac{r}{\sqrt{t}}\right)^{1+\frac{\theta}{2\pi}}.$$

The next lemma bound is the main technical ingredient of this section. It gives an upper bound for the probability of a Brownian bridge, started close to the tip of a wedge, to remain inside the wedge.

LEMMA 11. *For any $0 \leq \theta < \pi$ and $\varepsilon > 0$, there exists $C > 0$ such that the following holds. Suppose that $\alpha > C$ and $r < 1$ and that $W = W(w_0, \pi - \theta)$ for some $w_0 \in \mathbb{R}^2$. Let $a, b \in \mathbb{R}^2$ be points with $|a - w_0| = r$ and $|b - a| < 2\phi(\alpha)$. Let $X(t)$ be a Brownian bridge with boundary conditions $X(0) = a$ and $X(1) = b$. Then,*

$$(27) \quad \mathbb{P}(X(t) \in W, \forall t \in [0, 1]) \leq \alpha^\varepsilon \max(\alpha^{-1}, r)^{1+\frac{\theta}{20}}.$$

PROOF. By applying a translation, we assume without loss of generality that $a = 0$. Furthermore, it suffices to prove (27) for $r \geq \alpha^{-1}$. Indeed, it is easy to see that the left-hand side of (27) is monotonically increasing with respect to r (by monotonicity of the corresponding events). Define the event

$$E := \{X(t) \in W, \forall t \in [0, 1]\}.$$

We can write

$$(28) \quad X(t) = B(t) - t(B(1) - b),$$

where $B(t)$ is a standard Brownian motion. Consider the event

$$F := \{|B(t)| < t^{1/2}\phi(\alpha) + \alpha^{-2n-1}, \forall t \in [0, 1]\}.$$

Using Lemma 3, we have $\mathbb{P}(F) \geq 1 - \alpha^{-2n-1}$ whenever α is larger than some constant depending only on n . Let k be an integer whose value will be chosen later on. Define

$$u = r\phi(\alpha)^{-1}, \quad t_0 = 0, \quad t_j = u^{2^{-(j-1)}}, \quad \forall j \leq k.$$

Next, consider the events

$$E_j := \{d(B(t), W) \leq 4\phi(\alpha)r^{2^{-(j-1)}}, \forall t \in [t_{j-1}, t_j]\}.$$

For each $1 \leq j \leq k$, let

$$w_j := w_0 - \left(4 \sin\left(\frac{\pi - \theta}{2}\right)^{-1} \phi(\alpha) r^{2^{-(j-1)}}, 0 \right) \quad \text{and} \quad W_j := W(w_j, \pi - \theta),$$

so that W_j is a wedge with containing W , with the same opening angle and whose tip w_j is situated “behind” w_0 such that the distance between their outer boundaries is $4\phi(\alpha)r^{2^{-(j-1)}}$. Clearly, we have

$$E_j \subseteq \{B(t) \in W_j, \forall t \in [t_{j-1}, t_j]\}.$$

Next consider the events

$$E'_j := \{|B(t_{j-1})| \leq \phi(\alpha)\sqrt{t_{j-1}} + \alpha^{-2n-1}\} \cap \{B(t) \in W_j, \forall t \in [t_{j-1}, t_j]\}.$$

First we claim that

$$(29) \quad E \subset \left(\bigcap_{j=1}^k E_j \right) \cup F^C \subseteq \left(\bigcap_{j=1}^k E'_j \right) \cup F^C.$$

Indeed, under the event F we have $|B(1)| \leq \phi(\alpha) + \alpha^{-2n-1} \leq 2\phi(\alpha)$ and, according to equation (28), we have

$$|X(t) - B(t)| \leq 4\phi(\alpha)t_j \leq 4\phi(\alpha)r^{2^{-(j-1)}} \quad \forall t < t_j.$$

By the triangle inequality we have that $X(t) \in W$ implies the event E_j . Under the event F the event E_j implies the event E'_j which establishes (29). Next, by the Markov property of Brownian motion, note that

$$\mathbb{P}(E'_j | \sigma(E'_1, \dots, E'_{j-1}, B(t_{j-1}))) = \mathbb{P}(E'_j | B(t_{j-1})).$$

Remark that, under the event E'_j , we have

$$(30) \quad \begin{aligned} |B(t_{j-1}) - w_j| &\leq |a - w_0| + |B(t_{j-1}) - a| + |w_0 - w_j| \\ &\leq r + \phi(\alpha)\sqrt{t_{j-1}} + \alpha^{-2n-1} + 4 \sin\left(\frac{\pi - \theta}{2}\right)^{-1} \phi(\alpha)r^{2^{-(j-1)}} \\ &\leq 7 \sin\left(\frac{\pi - \theta}{2}\right)^{-1} \phi(\alpha)r^{2^{-(j-1)}}. \end{aligned}$$

We now apply Lemma 10 with respect to the enlarged wedge W_j :

$$\begin{aligned} \mathbb{P}(E'_j | B(t_{j-1})) &\leq J_\theta \left(\frac{|B(t_{j-1}) - w_j|}{\sqrt{t_j - t_{j-1}}} \right)^{1 + \frac{\theta}{2\pi}} \\ &\stackrel{(30)}{\leq} J_\theta \left(\frac{7 \sin(\frac{\pi - \theta}{2})^{-1} \phi(\alpha)r^{2^{-(j-1)}}}{\sqrt{t_j - t_{j-1}}} \right)^{1 + \frac{\theta}{2\pi}} \\ &\leq J_\theta \left(\sqrt{28} \sin\left(\frac{\pi - \delta}{2}\right)^{-1} \phi(\alpha)^2 \right)^{1 + \frac{\theta}{2\pi}} r^{2^{-j}(1 + \frac{\theta}{2\pi})}, \end{aligned}$$

where the last inequality is valid as long as $t_j - t_{j-1} \geq t_j/2$ (for any given k , this holds for all $1 \leq j \leq k$ for sufficiently large α , due to the assumption $r < 1$). Now, because $r \geq \alpha^{-1}$, we can choose k large enough (in a way that depends only on θ, ε), so that one has

$$r^{(2^{-1} + \dots + 2^{-k})(1 + \theta/\pi)} \leq r^{1 + \theta/20}.$$

It now follows that

$$\begin{aligned} \mathbb{P}\left(\bigcap_{j=1}^k E'_j\right) &= \prod_{j=1}^k \mathbb{P}(E'_j | E'_1 \cap \dots \cap E'_{j-1}) \\ &= \prod_{j=1}^k \mathbb{E}(\mathbb{P}(E'_j | E'_1 \cap \dots \cap E'_{j-1}, B(t_{j-1}))) \\ &\leq \prod_{j=1}^k J_\theta \left(\sqrt{28} \sin\left(\frac{\pi - \theta}{2}\right)^{-1} \phi(\alpha)^2 \right)^{1 + \frac{\theta}{2\pi}} r^{2^{-j}(1 + \frac{\theta}{2\pi})} \\ &\leq_{(*)} \frac{1}{2} \alpha^\varepsilon (r^{(2^{-1} + \dots + 2^{-k})})^{1 + \theta/\pi} \leq \frac{1}{2} \alpha^\varepsilon r^{1 + \theta/20}, \end{aligned}$$

where (*) holds for sufficiently large α .

Finally, by a union bound via equation (29) we conclude that for α large enough we have

$$\mathbb{P}(E) \leq \frac{1}{2} \alpha^\varepsilon r^{1 + \theta/20} + \mathbb{P}(F^C) \leq \frac{1}{2} \alpha^\varepsilon r^{1 + \theta/20} + \alpha^{-2n-1} \leq \alpha^\varepsilon r^{1 + \theta/20}. \quad \square$$

We now have, as a direct consequence, the following slight generalization:

LEMMA 12. *Let $0 \leq s_1 < s_2 \leq 1$. For any $0 \leq \theta < \pi$ and $\varepsilon > 0$ there exists $C > 0$ such that the following holds. Suppose that $\alpha > C$, and let $W = W(w_0, \pi - \theta)$. Let $a, b \in \mathbb{R}^2$ be points with $|a - w_0| = r$ and*

$$(31) \quad |b - a| < \phi(\alpha) \sqrt{s_2 - s_1}.$$

Let $X(t)$ be a Brownian bridge with boundary conditions $X(s_1) = a$ and $X(s_2) = b$. Then,

$$\mathbb{P}(X(t) \in W, \forall t \in [s_1, s_2]) \leq \alpha^\varepsilon \max\left(\alpha^{-1}, \frac{r}{\sqrt{s_2 - s_1}}\right)^{1 + \theta/20}.$$

PROOF. This follows directly from Lemma 11 by Brownian scaling. \square

4.1. *Proof of Proposition 9.* The next four lemmas correspond to the bounds of Proposition 9.

Let us introduce some notation that will be used in the proofs. Let $(r, s) \in \Delta_n \times \Delta_n$ be given, with the corresponding $(t_0, \dots, t_{2n+1}) = \mathbf{t}(r, s)$. We fix $0 \leq i \leq 2n$ and denote $s_1 = t_i$ and $s_2 = t_{i+1}$. Let P be the orthogonal projection onto the span of n_r and n_s embedded in \mathbb{R}^2 , and consider the projected Brownian motion $\tilde{B}(t) := P(B(t))$. In what follows we will allow ourselves to use the notation $B(t)$ in place of $\tilde{B}(t)$. Let $d_1 = P(b_i)$ and $d_2 = P(b_{i+1})$, so that $d_1, d_2 \in \mathbb{R}^2$ and consider the events $E_j := \{\tilde{B}(s_j) = d_j\}$, $j = 1, 2$.

Let W be the wedge corresponding to $P(F_s)$ and $P(F_r)$, so that $W \subset \mathbb{R}^2$ is a wedge of opening angle $\pi - \theta$, $\theta := \theta(r, s)$, and tip at some point w_0 , so that d_1, d_2 are on ∂W . Consider the enlarged wedge W' , whose normal directions are n_r, n_s and which contains W in a way that the distance between their respective edges is exactly $\frac{\phi(\alpha)^2}{\sqrt{\alpha}}$. With this notation the event H_i is identical to the event

$$H_i = A := \{B(t) \in W', \forall t \in [s_1, s_2]\}.$$

Finally, let $\mathcal{H}_1, \mathcal{H}_2$ each be one of the half-spaces, which define W' corresponding to d_1 and d_2 respectively, in a way that

$$(32) \quad \text{dist}(d_1, \partial\mathcal{H}_1) = \frac{\phi(\alpha)^2}{\sqrt{\alpha}} \quad \text{if } 1 \leq i \leq 2n$$

and

$$(33) \quad \text{dist}(d_2, \partial\mathcal{H}_2) = \frac{\phi(\alpha)^2}{\sqrt{\alpha}} \quad \text{if } 0 \leq i \leq 2n - 1$$

(remark that \mathcal{H}_1 and \mathcal{H}_2 are not necessarily distinct. Possibly, $\mathcal{H}_1 = \mathcal{H}_2$, in which case W' strictly contains $\mathcal{H}_1 \cap \mathcal{H}_2$. Moreover, note that \mathcal{H}_1 is not defined for the case $i = 0$, and \mathcal{H}_2 is undefined for $i = 2n$).

The following lemma is equivalent to the bound (20). The idea behind it is that if a Brownian bridge both begins and ends close to the boundary of some wedge, then we can condition on its location at the center of the time interval and apply the bound on the probability for a Brownian bridge to stay inside a half-space for both halves separately. Each of the halves contribute a bound of $((s_2 - s_1)\alpha)^{-1/2}$ (this corresponds to the fact that a simple random walk of k steps has probability of the order $1/\sqrt{k}$ to stay positive) and, when combined, this gives a bound of $((s_2 - s_1)\alpha)^{-1}$.

LEMMA 13. *With the above notation, assume that both conditions (32) and (33) hold. For every $\varepsilon > 0$, there exists $C > 0$ such that for any $\alpha > C$:*

$$\mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \leq \frac{1}{(s_2 - s_1)\alpha} \alpha^\varepsilon.$$

PROOF. Denote $r := \frac{s_1+s_2}{2}$ and define $J_1 := [s_1, r]$ and $J_2 := [r, s_2]$. Consider the events

$$D_j := \{B(t) \in \mathcal{H}_j, \forall t \in J_j\}, \quad j = 1, 2.$$

Clearly, we have

$$A \subseteq D_1 \cap D_2.$$

We condition on $B(r)$ and use the independence of increments of Brownian motion to obtain an upper bound:

$$\begin{aligned} \mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) &\leq \mathbb{E}(\mathbb{P}(D_1 \cap R_\alpha[J_1] | E_1, B(r)) \\ &\quad \times \mathbb{P}(D_2 \cap R_\alpha[J_2] | E_2, B(r)) | E_1, E_2). \end{aligned}$$

Note that under the event $R_\alpha[J_1]$, we have $|B(r) - B(s_1)| \leq \sqrt{r - s_1} \phi(\alpha)$ which corresponds to condition (31). We can therefore apply Lemma 12 with respect to the half-space \mathcal{H}_1 (which we think of as a degenerate wedge with opening angle π) and the Brownian bridge on the interval J_1 , taking $\frac{\varepsilon}{3}$ in place of ε . The condition (32) ensures that we may use $r = \frac{\phi(\alpha)^2}{\sqrt{\alpha}}$ as an argument to the lemma. We obtain that, almost surely,

$$\mathbb{P}(D_1 \cap R_\alpha[J_1] | E_1, B(r)) \leq \frac{\alpha^{\frac{\varepsilon}{3}} \max(\alpha^{-1}, \phi(\alpha)^2 \alpha^{-\frac{1}{2}})}{\sqrt{\frac{s_2-s_1}{2}}},$$

whenever $\alpha > C$ for some constant $C > 0$ depending only on ε . In a similar manner this time, relying on the condition (33), we get

$$\mathbb{P}(D_2 \cap R_\alpha[J_2] | E_2, B(r)) \leq \frac{\alpha^{\frac{\varepsilon}{3}} \max(\alpha^{-1}, \phi(\alpha)^2 \alpha^{-\frac{1}{2}})}{\sqrt{\frac{s_2-s_1}{2}}}.$$

By assuming that the constant C is large enough, we have $\max(\alpha^{-1}, \phi(\alpha)^2 \alpha^{-\frac{1}{2}}) = \phi(\alpha)^2 \alpha^{-\frac{1}{2}}$, so combining the last three displays, we get

$$\mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \leq \left(\frac{\alpha^{\frac{\varepsilon}{3}-\frac{1}{2}} \phi(\alpha)^2}{\sqrt{\frac{s_2-s_1}{2}}} \right)^2 \leq \frac{1}{(s_2 - s_1) \alpha} \alpha^\varepsilon,$$

where the final inequality holds for $\alpha > C$ large enough. \square

We now derive an analogous estimate for “edge” intervals, giving the bound (21).

LEMMA 14. *With the above notation, assume that at least one of the two conditions (32), (33) holds. For every $\varepsilon > 0$, there exists $C > 0$, such that for any $\alpha > C$,*

$$\mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \leq \frac{1}{\sqrt{(s_2 - s_1) \alpha}} \alpha^\varepsilon.$$

PROOF. If only the condition (32) holds, then we simply ignore the interval $[(s_1 + s_2)/2, s_2]$, otherwise proceeding as in the previous lemma. Analogously, if (33) holds, then we ignore the other interval. \square

We now move on to the improved bounds (22) and (23) which rely on the additional assumption (19). Thus, in what follows we will also make the assumption

$$(34) \quad s_2 - s_1 \geq \alpha^{1/(10n)} \max\left(\min(|d_1 - w_0|, |d_2 - w_0|)^2, \frac{1}{\alpha}\right).$$

Those lemmas lie in the heart of our argument. This is the only part of the proof where we exploit the fact that the opening angle of the wedge is strictly smaller than π . The next lemma implies the bound (22).

We remark that, as explained above, it is crucial to get a bound whose dependence on α is of the form α^{-1-c} with the constant c being strictly bigger than zero. The term α^{-1} is obtained in a way similar to the previous lemma. One takes advantage of the fact that the Brownian bridge both starts and ends close to the boundary of the wedge, by considering the initial and terminal time intervals and using the bound for a Brownian bridge to stay inside a half-space for each of them. However, instead of partitioning $[s_1, s_2]$ into two parts, here we partition it into three subintervals. The middle interval is where we use the fact that the Brownian bridge is actually contained in a wedge and not only in a half-space. This is done by carefully choosing the partition in a way that in one of the endpoints of the middle interval, the bridge is still close to the tip of the wedge (so that it gives a bound of α^{-c}) but is already quite far from the boundary of the wedge in the sense that the bounds for the first and third intervals can be fully exploited to give a bound of α^{-1} .

LEMMA 15. *With the above notation, assume that the conditions (32), (33) and (34) hold. For every $\varepsilon > 0$, there exists $C > 0$ such that, for any $\alpha > C$,*

$$\mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \leq \frac{\alpha^\varepsilon}{(s_2 - s_1)\alpha} \alpha^{-\frac{\theta}{400n}}.$$

PROOF. Since the distribution of a Brownian bridge is invariant under time reversal (up to the initial conditions), we may assume without loss of generality that $|d_1 - w_0| \leq |d_2 - w_0|$. Denote $S^2 = \max(\frac{1}{\alpha}, |d_1 - w_0|^2)$. Set $\ell := s_1 + S^2$, $r := \frac{s_1 + s_2}{2}$. Remark that

$$A \subset \{B(t) \in \mathcal{H}_1, \forall t \in [s_1, \ell]\} \cap \{B(t) \in W', \forall t \in [\ell, r]\} \\ \cap \{B(t) \in \mathcal{H}_2, \forall t \in [r, s_2]\}.$$

This time we condition on $B(\ell)$ and on $B(r)$. By independence of the conditioned Brownian motion on the disjoint intervals, we get

$$(35) \quad \mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \\ \leq \mathbb{E}(\mathbb{P}(\{B(t) \in \mathcal{H}_1, \forall t \in [s_1, \ell]\} \cap R_\alpha[s_1, \ell] | E_1, B(\ell)))$$

$$(36) \quad \begin{aligned} & \times \mathbb{P}(\{B(t) \in W', \forall t \in [\ell, r]\} \cap R_\alpha[\ell, r] | B(\ell), B(r)) \times \mathbf{1}_G \\ & \times \mathbb{P}(\{B(t) \in \mathcal{H}_2, \forall t \in [r, s_2]\} \cap R_\alpha[r, s_2] | E_2, B(r)) | E_1, E_2), \end{aligned}$$

where

$$G := \{|B(\ell) - B(s_1)| \leq \alpha^{-2n-1} + \phi(\alpha)S\} \supset R_\alpha[s_1, \ell].$$

We invoke Lemma 12 for each one of the three intervals $[s_1, \ell]$, $[\ell, r]$ and $[r, s_2]$. For the first and last intervals we use the half-spaces \mathcal{H}_1 and \mathcal{H}_2 respectively, in place of the wedge W , invoking the lemma with $\theta = 0$. Using $\varepsilon/4$ in place of ε , we get the bounds

$$(37) \quad \begin{aligned} & \mathbb{P}(\{B(t) \in \mathcal{H}_1, \forall t \in [s_1, \ell]\} \cap R_\alpha[s_1, \ell] | E_1, B(\ell)) \\ & \leq \frac{\alpha^{\frac{\varepsilon}{4}} \max(\alpha^{-1}, \alpha^{-\frac{1}{2}}\phi(\alpha)^2)}{S} \\ & \leq \frac{\alpha^{\frac{\varepsilon}{4}} \alpha^{-\frac{1}{2}}\phi(\alpha)^2}{S} \end{aligned}$$

and

$$(38) \quad \mathbb{P}(\{B(t) \in \mathcal{H}_2, \forall t \in [r, s_2]\} \cap R_\alpha[r, s_2] | E_2, B(r)) \leq \frac{\alpha^{\frac{\varepsilon}{4}} \alpha^{-\frac{1}{2}}\phi(\alpha)^2}{\sqrt{\frac{s_2-s_1}{2}}}.$$

In the above note that the condition (31) is verified since, under the event $R_\alpha[s_1, \ell]$, we have $|B(\ell) - B(s_1)| < \phi(\alpha)\sqrt{\ell - s_1}$ and an analogous reasoning holds for the second application of the lemma.

We would now like to invoke the lemma for the middle interval, with the wedge being W' (this is the only time we invoke the lemma using a wedge which is not a half-space). First, remark that under the event G we have

$$\begin{aligned} |B(\ell) - w_0| & \leq |B(s_1 + S^2) - B(s_1)| + |B(s_1) - w_0| \\ & \leq \alpha^{-2n-1} + \phi(\alpha)S + S \\ & \leq 2\phi(\alpha)S. \end{aligned}$$

We invoke Lemma 12 with the choice $r = 2\phi(\alpha)S$. We end up with the almost-sure bound

$$\begin{aligned} & \mathbb{P}(\{B(t) \in W', \forall t \in [\ell, r]\} \cap R_\alpha[\ell, r] | B(\ell), B(r)) \times \mathbf{1}_G \\ & \leq \frac{\alpha^{\frac{\varepsilon}{4}} \max(\alpha^{-1}, 2\phi(\alpha)S)^{1+\frac{\theta}{20}}}{\sqrt{\frac{s_2-s_1}{2} - S^2}} \leq \frac{\alpha^{\frac{\varepsilon}{4}} (2\phi(\alpha)S)^{1+\frac{\theta}{20}}}{\sqrt{\frac{s_2-s_1}{4}}} \end{aligned}$$

(in the last inequality we used the fact that $S \geq \alpha^{-1/2}$ which implies $\max(\alpha^{-1}, 2\phi(\alpha)S) = 2\phi(\alpha)S$ as well as the fact that, by definition, $S^2 \leq \alpha^{-1/(10n)}(s_2 - s_1)$). By combining the last display with (35), (37) and (38), we get

$$\mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \leq \frac{\alpha^{\frac{\varepsilon}{4} - \frac{1}{2}} \phi(\alpha)^2}{S} \times \alpha^{\frac{\varepsilon}{4}} \left(\frac{2\phi(\alpha)S}{\sqrt{\frac{s_2 - s_1}{4}}} \right)^{1 + \frac{\theta}{20}} \times \frac{\alpha^{\frac{\varepsilon}{4} - \frac{1}{2}} \phi(\alpha)^2}{\sqrt{\frac{s_2 - s_1}{2}}}.$$

Since for every $\varepsilon > 0$, for sufficiently large α , one has $16\phi(\alpha)^6 \leq \alpha^{\varepsilon/4}$, we get

$$\mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \leq \frac{\alpha^\varepsilon}{\alpha(s_2 - s_1)} \left(\frac{S}{\sqrt{s_2 - s_1}} \right)^{\frac{\theta}{20}}.$$

Finally, the assumption (34) implies

$$\frac{S}{\sqrt{s_2 - s_1}} \leq \alpha^{-\frac{1}{20n}}.$$

The combination of the two above displays finishes the proof. \square

We again have an analogous lemma for “edge” intervals:

LEMMA 16. *With the above notation, assume that one of the two conditions (32) and (33) hold, and assume further that (34) holds. Then, for every $\varepsilon > 0$, there exists $C > 0$ such that, for any $\alpha > C$,*

$$\mathbb{P}(A \cap R_\alpha[s_1, s_2] | E_1, E_2) \leq \frac{\alpha^\varepsilon}{\sqrt{(s_2 - s_1)\alpha}} \alpha^{-\frac{\theta}{400n}}.$$

PROOF. We proceed as in the previous lemma. If only the condition (32) holds, then we simply ignore the interval $[r, s_2]$. Analogously, if (33) holds, we change the order $s_1 \leftrightarrow s_2$ and proceed as before. \square

By combining the lemmas of this subsection, we have now completed the proof of Proposition 9.

5. Proof of Theorem 1. In light of Propositions 6 and 7, the proof of the main theorem is reduced to estimating an explicit integral over $\Delta_n \times \Delta_n$. Define

$$(39) \quad Z_a = \{(z_1, \dots, z_{2n}); z_1 \geq a, 1 - z_{2n} \geq a \text{ and } \forall 2 \leq j \leq 2n, z_j - z_{j-1} \geq a\}.$$

We will need the following lemma.

LEMMA 17. *For $(r, s) \in \Delta_n \times \Delta_n$, recall that $\mathbf{t}(r, s) = (t_1(r, s), \dots, t_{2n}(r, s))$ are the elements of $r \cup s$ in increasing order. For every dimension n there exists $C > 0$ such that for any $0 < a < e^{-1}$ we have*

$$\int_{\Delta_n \times \Delta_n} \frac{\mathbf{1}_{\{\mathbf{t}(r,s) \in Z_a\}}}{\sqrt{t_1(r,s)(1 - t_{2n}(r,s))}} \prod_{i=2}^{2n} \frac{1}{t_i(r,s) - t_{i-1}(r,s)} dr ds \leq C |\log(a)|^C$$

and

$$\int_{\Delta_n \times \Delta_n} \mathbf{1}_{\{t(r,s) \notin Z_a\}} dr ds \leq Ca.$$

The proof of this lemma is postponed to the [Appendix](#). We are ready to finish the proof of our main theorem.

PROOF OF THEOREM 1. Let Z_a be defined as in equation (39). Fix $\kappa > 0$, and let α be large enough so that the bound given by Proposition 7 is valid. In other words,

$$\begin{aligned} &\mathbb{P}(\tilde{C}_{\alpha,\kappa}(r, s)) \\ &\leq \alpha^{-2n-1} + \alpha^{-2n} \alpha^{-\kappa/(16,000n)} \frac{1}{\sqrt{t_1(r, s)(1 - t_{2n}(r, s))}} \prod_{i=2}^{2n} \frac{1}{t_i(r, s) - t_{i-1}(r, s)}. \end{aligned}$$

Together with the bounds given by Lemma 17, we have

$$\begin{aligned} &\alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{P}(\tilde{C}_{\alpha,\kappa}(r, s)) dr ds \\ &\leq \alpha^{-1} + \alpha^{-\kappa/(16,000n)} \int_{\Delta_n \times \Delta_n} \frac{\mathbf{1}_{\{t(r,s) \in Z_{\alpha^{-2n-1}}\}}}{\sqrt{t_1(r, s)(1 - t_{2n}(r, s))}} \\ &\quad \times \prod_{i=2}^{2n} \frac{1}{t_i(r, s) - t_{i-1}(r, s)} dr ds \\ &\quad + \alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbf{1}_{\{t(r,s) \notin Z_{\alpha^{-2n-1}}\}} dr ds \\ &\leq \alpha^{-1} + C(2n + 1)\alpha^{-\kappa/(16,000n)} \log(\alpha)^C + C\alpha^{-1}, \end{aligned}$$

for some constant $C > 0$ which depends only on n, κ . We conclude that

$$\alpha^{2n} \int_{\Delta_n \times \Delta_n} \mathbb{P}(\tilde{C}_{\alpha,\kappa}(r, s)) dr ds \xrightarrow{\alpha \rightarrow \infty} 0.$$

An application of Proposition 6 finishes the proof. \square

APPENDIX: PROOFS OF TECHNICAL RESULTS

PROOF OF LEMMA 3. Define

$$N_\alpha := \{\forall t \in [0, 1], \Lambda_\alpha \cap [t - \alpha^{-1}\phi(\alpha), t + \alpha^{-1}\phi(\alpha)] \neq \emptyset\}$$

and

$$Y_\alpha := \{M_\delta \leq \delta^{1/2}\phi(\alpha) + \alpha^{-2n-1}, \forall \delta \leq 1\},$$

where

$$M_\delta := \sup_{t_1, t_2 \in [0, 1], |t_1 - t_2| \leq \delta} |B(t_1) - B(t_2)|$$

and remark that $R_\alpha = N_\alpha \cap Y_\alpha$. We will bound $\mathbb{P}(Y_\alpha^C)$ and $\mathbb{P}(N_\alpha^C)$ separately, and then apply a union bound. We start with $\mathbb{P}(Y_\alpha^C)$, for which we use the following inequality whose proof can be found in [5]:

$$\mathbb{E}(M_\delta^p) \leq K_p (\delta \log(1/\delta))^{p/2},$$

for some constant K_p , valid for every $p > 1$ and $\delta < 1/e$. We take $\delta = \alpha^{-6n-3}$, $p = 4$ and conclude that by Markov's inequality we have

$$\begin{aligned} \mathbb{P}(M_{\alpha^{-6n-3}} > \alpha^{-2n-2}) &= \mathbb{P}(M_{\alpha^{-6n-3}}^4 > \alpha^{-8n-8}) \\ &\leq \frac{K_4 (\alpha^{-6n-3} \log(\alpha^{6n+3}))^2}{\alpha^{-8n-8}} = K_4 \alpha^{-4n+2} \log(\alpha^{6n+3})^2. \end{aligned}$$

So clearly $\mathbb{P}(M(\alpha^{-6n-3}) > \alpha^{-2n-2}) = o(\alpha^{-2n-1})$. Therefore $\mathbb{P}(M(\delta) \leq \delta^{1/2} \phi(\alpha) + \alpha^{-2n-1}, \forall \delta < \alpha^{-6n-3}) = 1 - o(\alpha^{-2n-1})$. Now let $g_\alpha = \lceil \alpha^{6n+4} \rceil$, and consider the collection of points $I_\alpha = \{ \frac{k}{g_\alpha} \mid k \in \mathbb{Z}, 0 \leq k \leq g_\alpha \}$. For any pair of points $\frac{k_1}{g_\alpha}, \frac{k_2}{g_\alpha} \in I_\alpha$ we have

$$\mathbb{P}\left(\left| B\left(\frac{k_1}{g_\alpha}\right) - B\left(\frac{k_2}{g_\alpha}\right) \right| > \log(\alpha) \sqrt{\frac{|k_1 - k_2|}{g_\alpha}} \right) = \mathbb{P}(X > \log(\alpha)),$$

where $X \sim N(0, 1)$. By a standard estimate for Gaussian random variables we have

$$\mathbb{P}(X > \log(\alpha)) \leq \frac{1}{\sqrt{2\pi} \log(\alpha)} \exp\left(\frac{-\log(\alpha)^2}{2}\right),$$

therefore, by a union bound, we have

$$\begin{aligned} &\mathbb{P}\left(\exists k_1, k_2 \in I_\alpha \text{ s.t. } \left| B\left(\frac{k_1}{g_\alpha}\right) - B\left(\frac{k_2}{g_\alpha}\right) \right| > \log(\alpha) \sqrt{\frac{|k_1 - k_2|}{g_\alpha}} \right) \\ &\leq g_\alpha^2 \frac{1}{\sqrt{2\pi} \log(\alpha)} \exp\left(\frac{-\log(\alpha)^2}{2}\right) = o(\alpha^{-2n-1}). \end{aligned}$$

For any $x \in [0, 1]$, let $I_\alpha(x)$ be a point in I_α of minimal distance to x . Clearly, $|x - I_\alpha(x)| \leq \alpha^{-6n-4}$. Thus, we have, with probability at least $1 - o(\alpha^{-2n-1})$, that for any $s, t \in [0, 1]$ such that $|s - t| > \alpha^{-6n-3}$,

$$\begin{aligned} |B(s) - B(t)| &\leq |B(s) - B(I_\alpha(s))| + |B(I_\alpha(s)) - B(I_\alpha(t))| + |B(I_\alpha(t)) - B(t)| \\ &\leq 2\alpha^{-2n-2} + \log(\alpha) \sqrt{|I_\alpha(s) - I_\alpha(t)|} \\ &\leq \alpha^{-2n-1} + 2 \log(\alpha) \sqrt{|s - t|} \leq \alpha^{-1} + \phi(\alpha) |s - t|, \end{aligned}$$

where the final two inequalities hold for α sufficiently large. We conclude by a union bound that $\mathbb{P}(Y) = o(\alpha^{-2n-1})$. As for $\mathbb{P}(N_\alpha^C)$, let $f_\alpha = \lceil \frac{\alpha}{\phi(\alpha)} \rceil^{-1}$, and consider the points $0, f_\alpha, 2f_\alpha, \dots, (f_\alpha^{-1} - 1)f_\alpha, 1$. For each $1 \leq j \leq f_\alpha^{-1}$, we have that with probability $1 - e^{-\alpha f_\alpha} = 1 - o(\alpha^{-2n-2})$, the Poisson process Λ_α has at least one point in the interval $[(j - 1)f_\alpha, jf_\alpha]$, so by a union bound with probability at least $1 - o(\alpha^{-2n-1})$, Λ_α intersects all of these intervals, in which case N_α clearly holds. \square

PROOF OF LEMMA 8. Assume towards contradiction that for all $0 \leq j \leq 2n - 1$, one has that $t_{j+1} - t_j \leq \alpha^{1/(10n)} \max(|b_j - w_0|^2, \frac{1}{\alpha})$. Assume without loss of generality that $t_{j_0} \leq 1/2$ (otherwise we can make the transformation $t \leftrightarrow 1 - t$), and consider the sequence $a_1 = t_{j_0+1} - t_{j_0}, a_2 = t_{j_0+2} - t_{j_0}, \dots, a_m = 1 - t_{j_0} \geq 1/2$, where $m \leq 2n$. For $1 \leq i \leq m$, we have the inequalities

$$a_{i+1} - a_i = t_{i+1} - t_i \leq \alpha^{1/10n} \max\left(|b_i - w_0|^2, \frac{1}{\alpha}\right)$$

and

$$\begin{aligned} |b_i - w_0|^2 &\leq (|b_i - b_{j_0}| + |b_{j_0} - w_0|)^2 \\ &\leq_{(1)} \left(\phi(\alpha)(t_i - t_{j_0})^{1/2} + \alpha^{-2n-1} + M \frac{\phi(\alpha)^2}{\sqrt{\alpha}}\right)^2 \\ &\leq_{(2)} 16M^2\phi(\alpha)^4 \max((t_i - t_{j_0}), (1/\alpha)), \end{aligned}$$

where the inequalities (1) and (2) follow from the property (14) and the elementary inequality $(a^{1/2} + b^{1/2})^2 \leq 4 \max(a, b)$ respectively. Thus we have, for all $0 \leq i \leq m$,

$$a_{i+1} - a_i \leq 16M^2\alpha^{1/10n}\phi(\alpha)^4 \max(a_i, 1/\alpha).$$

Furthermore, by assumption we have

$$\begin{aligned} a_1 = t_{j_0+1} - t_{j_0} &\leq \alpha^{1/10n} \max\left(|b_{j_0} - w_0|^2, \frac{1}{\alpha}\right) \\ &\leq \alpha^{1/10n} M^2 \frac{\phi(\alpha)^4}{\alpha}. \end{aligned}$$

We claim that for α larger than some constant, this implies that $a_i < 1/2$ for all i which is a contradiction. Indeed, if $a_i \leq 1/\alpha$ for all i then we're done, assuming $\alpha > 2$. Otherwise, let $1 \leq k \leq m$ be the minimal index such that $a_k > 1/\alpha$. We then have

$$a_k \leq a_1 + \frac{(k - 1)16M^2\alpha^{1/10n}\phi(\alpha)^4}{\alpha} \leq 32nM^2\alpha^{1/10n}\frac{\phi(\alpha)^4}{\alpha},$$

and by induction

$$\begin{aligned} a_m &\leq a_k(16M^2\alpha^{1/10n}\phi(\alpha)^4 + 1)^{m-k} \\ &\leq 32nM^2\alpha^{1/10n}\frac{\phi(\alpha)^4}{\alpha}(16M^2\alpha^{1/10n}\phi(\alpha)^4 + 1)^{2n} \\ &< \alpha^{-1+1/5n+2/5} < 1/2, \end{aligned}$$

where the above inequalities hold for α larger than some constant. \square

PROOF OF LEMMA 17. In what follows we will allow ourselves to abbreviate $t_1 = t_1(r, s)$, $t_2 = t_2(r, s)$, etc. To prove the first bound, we first note that the integrand is independent of the ordering of the elements in $r \cup s$. Furthermore, $\Delta_n \times \Delta_n$ can be written as the union of the following $\binom{2n}{n}$ sets which are disjoint up to sets of measure zero:

$$\Delta_n \times \Delta_n = \bigcup_{1 \leq k_1 < \dots < k_n \leq 2n} \{(x_1, \dots, x_{2n}) \in \Delta_n \times \Delta_n; x_1 = t_{k_1}, \dots, x_n = t_{k_n}\}.$$

Noting that taking $k_1 = 1, \dots, k_n = n$ results in the domain $\Delta_{2n} \subseteq \Delta_n \times \Delta_n$, wherein we have $x_i = t_i$ for all $1 \leq i \leq 2n$, we get that

$$\begin{aligned} &\int_{\Delta_n \times \Delta_n} \frac{\mathbf{1}_{\{(t_1, \dots, t_{2n}) \in Z_a\}}}{\sqrt{t_1(1-t_{2n})}} \prod_{i=2}^{2n} \frac{1}{t_i - t_{i-1}} dr ds \\ &= \binom{2n}{n} \int_{\Delta_{2n}} \frac{\mathbf{1}_{\{(x_1, \dots, x_{2n}) \in Z_a\}}}{\sqrt{x_1(1-x_{2n})}} \prod_{i=2}^{2n} \frac{1}{x_i - x_{i-1}} dx_1 \dots dx_{2n} \\ &\leq \binom{2n}{n} \int_{[0,1]^{2n}} \frac{\mathbf{1}_{\{(x_1, \dots, x_{2n}) \in Z_a\}}}{\sqrt{x_1(1-x_{2n})}} \prod_{i=2}^{2n} \frac{1}{|x_i - x_{i-1}|} dx_1 \dots dx_{2n} \\ &\leq_{(*)} \frac{1}{2} \binom{2n}{n} \int_{[0,1]^{2n}} \frac{\mathbf{1}_{\{(x_1, \dots, x_{2n}) \in Z_a\}}}{x_1} \prod_{i=2}^{2n} \frac{1}{|x_i - x_{i-1}|} dx_1 \dots dx_{2n} \\ &\quad + \frac{1}{2} \binom{2n}{n} \int_{[0,1]^{2n}} \frac{\mathbf{1}_{\{(x_1, \dots, x_{2n}) \in Z_a\}}}{1-x_{2n}} \prod_{i=2}^{2n} \frac{1}{|x_i - x_{i-1}|} dx_1 \dots dx_{2n}, \end{aligned}$$

where $(*)$ follows from the arithmetic-geometric mean inequality. We will prove the required bound for the first summand with the second one being completely analogous. We make the change of variables $y_1 = x_1, \forall 2 \leq j \leq 2n, y_j = x_j -$

x_{j-1} :

$$\int_{[0,1]^{2n}} \frac{\mathbf{1}_{\{(x_1, \dots, x_{2n}) \in Z'_a\}}}{x_1} \prod_{i=2}^{2n} \frac{1}{|x_i - x_{i-1}|} dx_1 \cdots dx_{2n}$$

$$= \int_Q \mathbf{1}_{\{(y_1, \dots, y_{2n}); \forall 1 \leq j \leq 2n, y_j \geq a\}} \prod_{i=1}^{2n} \frac{1}{y_i} dy_1 \cdots dy_{2n},$$

where $Q := \{(y_1, \dots, y_{2n}); \forall 1 \leq j \leq 2n, 0 \leq \sum_{1 \leq i \leq j} y_i \leq 1\}$. We can again upper bound this by replacing Q with $[0, 1]^{2n}$, and finally, by Fubini's theorem, we have

$$\int_{[0,1]^{2n}} \mathbf{1}_{\{(y_1, \dots, y_{2n}); \forall 1 \leq j \leq 2n, y_j \geq a\}} \prod_{i=1}^{2n} \frac{1}{y_i} dy_1 \cdots dy_{2n} = |\log(a)|^{2n},$$

as required. To prove the second inequality, we note by the same reasoning that

$$\int_{\Delta_n \times \Delta_n} \mathbf{1}_{\{(t_1, \dots, t_{2n}) \notin Z_a\}} dr ds = \binom{2n}{n} \int_{\Delta_{2n}} \mathbf{1}_{\{(x_1, \dots, x_{2n}) \notin Z_a\}} dx_1 \cdots dx_{2n}.$$

Now,

$$\mathbf{1}_{\{(x_1, \dots, x_{2n}) \notin Z_a\}} \leq \mathbf{1}_{\{x_1 \leq a\}} + \mathbf{1}_{\{1-x_{2n} \leq a\}} + \sum_{j=2}^{2n} \mathbf{1}_{\{x_j - x_{j-1} \leq a\}},$$

and we can again focus on proving the bound for one summand, say $\mathbf{1}_{\{x_1 \leq a\}}$, with all the others being analogous. To do this, we simply apply Fubini's theorem obtaining that

$$\int_{\Delta_n \times \Delta_n} \mathbf{1}_{\{x_1 \leq a\}} dx_1 \cdots dx_{2n} \leq \int_{[0,1]^{2n}} \mathbf{1}_{\{x_1 \leq a\}} dx_1 \cdots dx_{2n} = a. \quad \square$$

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