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EXTREMAL THEORY FOR LONG RANGE DEPENDENT INFINITELY DIVISIBLE PROCESSES

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We prove limit theorems of an entirely new type for certain long memory regularly varying stationary infinitely divisible random processes. These theorems involve multiple phase transitions governed by how long the memory is. Apart from one regime, our results exhibit limits that are not among the classical extreme value distributions. Restricted to the one-dimensional case, the distributions we obtain interpolate, in the appropriate parameter range, the α-Fréchet distribution and the skewed α-stable distribution. In general, the limit is a new family of stationary and self-similar random sup-measures with parameters $\alpha \in (0, \infty)$ and $\beta \in (0, 1)$, with representations based on intersections of independent β-stable regenerative sets. The tail of the limit random sup-measure on each interval with finite positive length is regularly varying with index $-\alpha$. The intriguing structure of these random sup-measures is due to intersections of independent β-stable regenerative sets and the fact that the number of such sets intersecting simultaneously increases to infinity as β increases to one. The results in this paper extend substantially previous investigations where only $\alpha \in (0, 2)$ and $\beta \in (0, 1/2)$ have been considered.

1. Introduction. Given a stationary process $(X_n)_{n\in\mathbb{N}}$, we are interested in the asymptotic behavior of the maximum

$$M_n := \max_{i=1,\dots,n} X_i.$$

After appropriate normalization, what distributions may arise in the limit? This is a classical question in probability theory with a very long history. In the case that $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) random variables, all possible limits of the weak convergence in the form of

$$\frac{M_n - a_n}{b_n} \Rightarrow Z \tag{1.1}$$

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have been known since Fisher and Tippett (1928) and Gnedenko (1943): these form the family of extreme-value distributions, consisting of Fréchet, Gumbel and Weibull types. Furthermore, the functional extremal limit theorem in the form of

$$\left(\frac{M_{\lfloor nt \rfloor} - a_n}{b_n}\right)_{t\geq 0} \Rightarrow \big(Z(t)\big)_{t\geq 0} \tag{1.2}$$

in an appropriate topological space has also been known since Dwass (1964) and Lamperti (1964). The limit process Z, when nondegenerate, is known as the *extremal process*.

If a stationary process $(X_n)_{n\in\mathbb{N}}$ is not a sequence of i.i.d. random variables, the extremes can cluster, and this can affect the extremal limit theorems for such processes. Research along this line has started since the 1960s. A common feature of many results in the literature on this topic is the important role of the so-called *extremal index* $\theta \in (0, 1]$. When this index exists, it affects the limit theorems through the fact that, asymptotically, the limit law of M_n is the same as that of $\widetilde{M}_{\lfloor \theta n \rfloor}$, the maximum of $\lfloor \theta n \rfloor$ i.i.d. copies of X_1, when one uses the same normalization in both cases. This reflects the following picture of extremes of such processes: extreme values of the process occur in finite random clusters, the smaller θ indicates larger, on average, cluster size. It is also worth noting that for all $\theta \in (0, 1]$, the order of the normalization and the limit laws in (1.1) and (1.2) are the same as in the i.i.d. case. Therefore, one can view processes with extremal index $\theta \in (0, 1]$ as having, in the appropriate sense, *short memory* (the reasons for this terminology can be found in Samorodnitsky (2016)). Standard references for extreme value theory on i.i.d. and weakly dependent sequences include de Haan and Ferreira (2006), Leadbetter, Lindgren and Rootzén (1983), Resnick (1987). Point-process techniques are fundamental and powerful when investigating such problems.

There are situations that for the limit theorems of the types (1.1) and (1.2) to hold, the normalization needs to be of a different order, and even the limit may be different, from the short memory case. We refer to the dependence in such examples as strong or *long range dependence*. See the recent monograph Samorodnitsky (2016) for more background and recent developments on long range dependence in terms of limit theorems (not necessarily extremal ones). The first example of long range dependence in extreme value theory is for stationary Gaussian processes: Mittal and Ylvisaker (1975) showed that when the correlation r_n satisfies $\lim_{n\to\infty} r_n \log n = \gamma \in (0, \infty)$, the limit law of M_n is Gumbel convoluted with a Gaussian distribution, in contrast to the case of $\lim_{n\to\infty} r_n \log n = 0$ where the Gumbel distribution arises in the limit, due to Berman (1964). However, very few examples of extremes of stationary non-Gaussian processes with long range dependence have been discovered since then. One of the known examples is important for us in this paper and we will discuss it below.

A fundamental work is due to O'Brien, Torfs and Vervaat (1990) who, in the process of identifying all possible limits of extremes of a sequence of stationary random variables, pointed out that a more natural and revealing way to investigate extremes is via the *random sup-measures*. In this framework, for each n one investigates the random sup-measure M_n in the form of

$$M_n(B) := \max_{k \in nB \cap \mathbb{N}} X_k, \qquad B \subset \mathbb{R}_+,$$

in an appropriate topological space. Then a limit theorem for M_n entails at least the finite-dimensional convergence part in a functional extremal limit theorem as in (1.2) when restricted to all B in the form of $B = [0, t]$, $t \geq 0$. O'Brien, Torfs and Vervaat (1990) showed that all possible random sup-measures η on $[0, \infty)$ arising as limits starting from a stationary process $(X_n)_{n \in \mathbb{N}}$ are, up to affine transforms, stationary and self-similar, in the sense that

$$\eta(\cdot) \stackrel{d}{=} \eta(\cdot + b), \qquad b > 0 \quad \text{and} \quad \eta(a \cdot) \stackrel{d}{=} a^H \eta(\cdot), \qquad a > 0$$

for some $H > 0$. They also provided examples of such random sup-measures. However, the investigation of O'Brien, Torfs and Vervaat (1990) does not directly help in understanding extremal limit theorems under long range dependence.

In this paper, we investigate the extremes of a general class of stationary infinitely divisible processes whose law is linked to the law of a certain null-recurrent Markov chain. Two crucial numerical parameters impact the properties of such infinitely divisible processes: $\alpha \in (0, \infty)$ and $\beta \in (0, 1)$: the parameter α corresponds to the regular variation index of the tail of the marginal distribution, and β determines the rate of the recurrence of the underlying Markov chain (the larger the β, the faster the rate), and as a result, plays an important role in determining the memory of the infinitely divisible process. The extremes of symmetric α-stable processes in this class have been first investigated in Samorodnitsky (2004), who showed that when $\beta \in (0, 1/2)$, the partial maxima converge weakly to the Fréchet distribution, although under the normalization $b_n = n^{(1-\beta)/\alpha}$ instead of $n^{1/\alpha}$ used in the i.i.d. case. (Since infinitely divisible processes we are considering are heavy-tailed, we take the shift $a_n = 0$ in all extremal limit theorems.) The different order of normalization already indicates long range dependence of the process. Furthermore, it was pointed out in the same paper that when $\beta \in (1/2, 1)$, the dependence was so strong that the partial maxima were likely not to converge to the Fréchet distribution, but an alternative limit distribution was not described.

Further studies of the extrema of this class of processes have appeared more recently, still in the symmetric α-stable case, with $\beta \in (0, 1/2)$ (though in a different notation). In Owada and Samorodnitsky (2015a), it was shown that the limit in the functional extremal theorem as in (1.2) is, up to a multiplicative constant, a time-changed extremal process,

$$\left(Z_\alpha\left(t^{1-\beta}\right)\right)_{t \geq 0},$$

where $(Z_\alpha(t))_{t \geq 0}$ is the extremal process for a sequence of i.i.d. random variables with tail index α (the α-Fréchet extremal process). Subsequently, Lacaux and Samorodnitsky (2016), established a limit theorem in the framework of convergence of random sup-measures, and, up to a multiplicative constant, the limit random sup-measure can be represented as

$$\eta(\cdot) = \bigvee_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{(V_j^{(\beta)}+R_j^{(\beta)})\cap\cdot\neq\varnothing\}}, \tag{1.3}$$

where $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ is a measurable enumeration of the points of a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with intensity $\alpha u^{-\alpha-1}\,du(1-\beta)v^{-\beta}\,dv\,dP_\beta$. Here, $\mathcal{F}(\mathbb{R}_+)$ is the space of closed subsets of $\mathbb{R}_+$ equipped with Fell topology, and P_β is the law of a β-stable regenerative set, the closure of the range of a β-stable subordinator, on $\mathcal{F}(\mathbb{R}_+)$. Then

$$\big(\eta([0,t])\big)_{t\geq 0} \stackrel{d}{=} \big(Z_\alpha\big(t^{1-\beta}\big)\big)_{t\geq 0},$$

but the random sup-measure reveals more structure than the time-changed extremal process.

In this paper, we fill the gaps left in the previous studies. First of all, we move away from the assumption of stability to a more general class of stationary infinitely divisible processes. This allows us to remove the restriction of $\alpha \in (0,2)$ in our limit theorems. Much more importantly, we remove the assumption $\beta \in (0,1/2)$. This allows us to consider the extrema of processes whose memory is very long. Our results confirm that the Fréchet limits obtained in Samorodnitsky (2004) and the subsequent publications disappear when $\beta \in (1/2,1)$. In fact, entirely new limits appear. Even the one-dimensional distributions we obtain as marginal limits have not, to the best of our knowledge, been previously described. The limiting random sup-measure is of a shot-noise type (see, e.g., Vervaat (1979)), and it turns out to be uniquely determined by the random upper semicontinuous function

$$\eta^{\alpha,\beta}(t) := \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{t\in V_j^{(\beta)}+R_j^{(\beta)}\}}, \qquad t \geq 0,$$

with $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ as before. When $\beta \in (0,1/2]$, this is the same random sup-measure as the one in (1.3), as independent β-stable regenerative sets do not intersect for such a β. For $\beta > 1/2$, however, eventual intersections occur almost surely, and the larger the β becomes, more independent regenerative sets can intersect at the same time. As Section 3 below shows, for every $\alpha \in (0,\infty)$, $(\eta^{\alpha,\beta})_{\beta\in(0,1)}$ forms a family of random sup-measures corresponding to the full range of dependence: from independence ($\beta \downarrow 0$) to complete dependence ($\beta \uparrow 1$). Importantly, if $\alpha \in (0,1)$, the marginal distributions, for example, those of $\eta^{\alpha,\beta}([0,1])$, form a family of distributions that interpolate between the α-Fréchet distribution (resulting when $\beta \in (0,1/2]$) and the totally skewed to the right α-stable distribution as $\beta \uparrow 1$.

The paper is organized as follows. In Section 2, we present background information on random closed sets and random sup-measures. In Section 3, we introduce and investigate the limiting random sup-measure. The stationary infinitely divisible process with long range dependence whose extremes we study is introduced in Section 4, and a limit theorem for these extremes in the context of random sup-measures is proved in Section 5.

2. Random closed sets and sup-measures. We first provide background on random closed sets. Our main reference is Molchanov (2005). Let $\mathcal{F}(E)$ denote the space of all closed subsets of an interval $E \subset \mathbb{R}$. In this paper, we only work with $E = [0, 1]$ and $E = [0, \infty)$. The space $\mathcal{F} = \mathcal{F}(E)$ is equipped with the Fell topology generated by

$$\mathcal{F}_G := \{F \in \mathcal{F} : F \cap G \neq \varnothing\} \qquad \text{for all } G \in \mathcal{G},$$

where $\mathcal{G} = \mathcal{G}(E)$ is the collection of all open subsets of E, and

$$\mathcal{F}^K := \{F \in \mathcal{F} : F \cap K = \varnothing\} \qquad \text{for all } K \in \mathcal{K},$$

where $\mathcal{K} = \mathcal{K}(E)$ is the collection of all compact subsets of E. This topology is metrizable, and $\mathcal{F}(E)$ is compact under it. If $\mathcal{F}$ is equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{F})$ induced by the Fell topology, a random closed set is a measurable mapping from a probability space to $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. Given random closed sets $(R_n)_{n\in\mathbb{N}}$ and R, a sufficient condition for weak convergence $R_n \Rightarrow R$ is

$$\lim_{n\to\infty} \mathbb{P}(R_n \cap A \neq \varnothing) = \mathbb{P}(R \cap A \neq \varnothing), \qquad \text{for all } A \in \mathcal{A} \cap \mathfrak{S}_R,$$

where $\mathcal{A}$ is the collection of all finite unions of open intervals, and $\mathfrak{S}_R$ is the collection of all continuity sets of R: the collection of relatively compact Borel sets B such that $\mathbb{P}(R \cap \overline{B} \neq \varnothing) = \mathbb{P}(R \cap B^o \neq \varnothing)$. See Molchanov (2005), Corollary 1.6.9, where the collection $\mathcal{A}$ is called a *separating class*.

We proceed with background on sup-measures and upper semicontinuous functions. Our main reference is O'Brien, Torfs and Vervaat (1990). See also Molchanov and Strokorb (2016) and Sabourin and Segers (2017) for some recent developments. Let E be as above, and $\mathcal{G} = \mathcal{G}(E)$ the collection of open subsets of E. A map $m : \mathcal{G} \to [0, \infty]$ is a sup-measure, if

$$m\Big(\bigcup_\alpha G_\alpha\Big) = \sup_\alpha m(G_\alpha)$$

for all arbitrary collections of open sets $(G_\alpha)_\alpha$. Given a sup-measure m, its sup-derivative, denoted by $d^\vee m : E \to [0, \infty]$, is defined as

$$d^\vee m(t) := \inf_{G \ni t} m(G), \qquad t \in E.$$

The sup-derivative of a sup-measure is an upper semicontinuous function, that is a function f such that $\{f < t\}$ is open for all $t > 0$. Given an $[0, \infty]$-valued upper semicontinuous function f, the sup-integral $i^\vee f : \mathcal{G} \to [0, \infty]$ is defined as

$$i^\vee f(G) := \sup_{t \in G} f(t), \qquad G \in \mathcal{G},$$

with $i^\vee f(\varnothing) = 0$ by convention. The sup-integral is a sup-measure. Let $\mathrm{SM} = \mathrm{SM}(E)$ and $\mathrm{USC} = \mathrm{USC}(E)$ denote the spaces of all sup-measures on E and all $[0, \infty]$-valued upper semicontinuous functions on E, respectively. It turns out that $d^\vee$ is a bijection between SM and USC, and $i^\vee$ is its inverse. Every $m \in \mathrm{SM}$ has a canonical extension to all subsets of E, given by

$$m(B) = \sup_{t \in B} \bigl(d^\vee m\bigr)(t), \qquad B \subset E.$$

The space SM is equipped with the so-called sup-vague topology. In this topology, $m_n \to m$ if and only if

$$\limsup_{n \to \infty} m_n(K) \le m(K) \qquad \text{for all } K \in \mathcal{K}$$

and

$$\liminf_{n \to \infty} m_n(G) \ge m(G) \qquad \text{for all } G \in \mathcal{G}.$$

This topology is metrizable and the space SM is compact in this topology. The sup-vague topology on the space USC is then induced by the bijection $d^\vee$, so the convergence of

$$m_n \to m \text{ in SM} \quad \text{and} \quad d^\vee m_n \to d^\vee m \text{ in USC}$$

are equivalent.

A random sup-measure is a random element in $(\mathrm{SM}, \mathcal{B}(\mathrm{SM}))$ with $\mathcal{B}(\mathrm{SM})$ the Borel σ-algebra induced by the sup-vague topology. A random upper semicontinuous function is defined similarly. We will introduce the limiting random sup-measures in our limit theorem through their corresponding random upper semicontinuous functions. When proving weak convergence for random sup-measures we will utilize the following fact: given random sup-measures $(\eta_n)_{n \in \mathbb{N}}$ and η, weak convergence $\eta_n \Rightarrow \eta$ in S is equivalent to the finite-dimensional convergence

$$\bigl(\eta_n(I_1), \dots, \eta_n(I_m)\bigr) \Rightarrow \bigl(\eta(I_1), \dots, \eta(I_m)\bigr)$$

for all $m \in \mathbb{N}$ and all open and η-continuity intervals $I_1, \dots, I_m$ (I is η-continuity if $\eta(I) = \eta(\overline{I})$ with probability one); see O'Brien, Torfs and Vervaat (1990), Theorem 3.2.

We will need the following result on joint convergence of random closed sets.

THEOREM 2.1. *Let* $\{A_k\}_{k=1,\dots,m}$ *and* $\{A_k(n)\}_{n\in\mathbb{N},k=1,\dots,m}$ *be random closed sets in* $\mathcal{F}=\mathcal{F}(\mathbb{R}^d)$, *and set for* $I\subset\{1,\dots,m\}$

$$A_I(n)=\bigcap_{k\in I}A_k(n)\quad\text{and}\quad A_I=\bigcap_{k\in I}A_k,\qquad\text{for all } I\subset\{1,\dots,m\}$$

(*by convention we set* $A_\varnothing(n)=A_\varnothing=\mathbb{R}^d$). *Assume that*

$$\big(A_1(n),\dots,A_m(n)\big)\Rightarrow(A_1,\dots,A_m)\tag{2.1}$$

in $\mathcal{F}^m$ *as* $n\to\infty$.

1. *If* $A_I=\varnothing$ *almost surely, then* $A_I(n)\Rightarrow\varnothing$.
2. *If*

$$A_I(n)\Rightarrow A_I\qquad\text{as } n\to\infty,\ \text{for all } I\subset\{1,\dots,m\}.\tag{2.2}$$

Then

$$\big\{A_I(n)\big\}_{I\subset\{1,\dots,m\}}\Rightarrow\{A_I\}_{I\subset\{1,\dots,m\}},$$

in $\mathcal{F}^{2^m}$.

PROOF. The Fell topology on $\mathcal{F}=\mathcal{F}(\mathbb{R}^d)$ is compact, Hausdorff, and second countable (Salinetti and Wets (1981)). In particular, it is metrizable and separable. Then, by Skorokhod's representation theorem, from (2.1) we can find a common probability space, on which $\{A_k\}_{k=1,\dots,m}$ and $\{A_k(n)\}_{n\in\mathbb{N},k=1,\dots,m}$ are defined, so that the joint law of $\{A_k\}_{k=1,\dots,m}$ is preserved, for each n the joint law of $\{A_k(n)\}_{k=1,\dots,m}$ is preserved as well (we use the same notation), and we have the almost sure convergence

$$A_k(n)\to A_k\qquad\text{as } n\to\infty,\ \text{for all } k=1,\dots,m.\tag{2.3}$$

Fix a nonempty set $I\subset\{1,\dots,m\}$. By the upper semicontinuity in the product Fell topology of the intersection operator (see Appendix D in Molchanov (2005)), we have

$$\limsup_{n\to\infty}A_I(n)\subset A_I,\qquad\text{almost surely.}\tag{2.4}$$

This implies the first part of the theorem immediately.

For the second part of the theorem, from now on we assume that (2.2) holds. We will see that (2.3) and (2.4) imply that

$$A_I(n)\to A_I\qquad\text{as } n\to\infty \text{ in probability.}\tag{2.5}$$

This will, of course, prove that $\{A_I(n)\}_{I\subset\{1,\dots,m\}}\to\{A_I\}_{I\subset\{1,\dots,m\}}$ in probability, and hence, also the desired weak convergence.

We proceed to prove (2.5). Recall that the distance function for a closed set $F\in\mathcal{F}(\mathbb{R}^d)$ is defined as

$$\rho(x,F)=\min\big\{|x-y|:y\in F\big\},\qquad x\in\mathbb{R}^d.$$

Then the weak convergence in the Fell topology of random closed sets $A_I(n) \Rightarrow A_I$, is equivalent to the weak convergence in finite-dimensional distributions of the corresponding distance functions:

$$\rho_n(x) := \rho\big(x, A_I(n)\big) \overset{f.d.d.}{\to} \rho(x) = \rho(x, A_I), \qquad x \in \mathbb{R}^d;$$

see Salinetti and Wets (1986), Theorem 2.5. Note that (2.4) implies that for every $x \in \mathbb{R}^d$,

$$\liminf_{n\to\infty} \rho_n(x) \ge \rho(x) \qquad \text{almost surely,}$$

and we conclude from Lemma 2.2 below that for every $x \in \mathbb{R}^d$,

$$\rho_n(x) \to \rho(x) \qquad \text{in probability as } n \to \infty. \tag{2.6}$$

In order to establish (2.5), it is enough to show that for every subsequence, there exists a further subsequence, say $\{n_k\}_{k\in\mathbb{N}}$, such that $A_I(n_k) \to A_I$ almost surely as $k \to \infty$. For this purpose, we use (2.6) as follows. Convergence in the Fell topology on $\mathbb{R}^d$ is equivalent to the pointwise convergence of the distance functions (see, e.g., Salinetti and Wets (1981), Theorem 2.2(iii). Suppose we show that a further subsequence as above can be found such that

$$\lim_{k\to\infty} \rho_{n_k}(x) = \rho(x) \qquad \text{for all } x \in \mathbb{Q}^d, \text{ almost surely.} \tag{2.7}$$

Since the distance functions are Lipschitz with coefficient 1, on this event of probability 1 we actually have the entire pointwise convergence of the distance functions, hence the convergence in the Fell topology of $A_I(n_k)$ to A_I, as required. It remains to notice that (2.7) follows from (2.6) by the standard diagonalization argument. We have thus proved the theorem. □

LEMMA 2.2. *Let X, $(X_n)_{n\in\mathbb{N}}$ be random variables defined on the same probability space. Suppose that $X_n \Rightarrow X$ and $\liminf_{n\to\infty} X_n \ge X$ a.s. Then $X_n \to X$ in probability.*

PROOF. We may assume without loss of generality that all random variables involved are uniformly bounded (e.g., by applying the arctan function to everything). Then $\mathbb{E}X_n \to \mathbb{E}X$ as $n \to \infty$. Further, by Fatou's lemma,

$$0 \le \limsup_{n\to\infty} \mathbb{E}\big[(X - X_n)_+\big] \le \mathbb{E}\Big[\limsup_{n\to\infty} (X - X_n)_+\Big] \le 0.$$

That is, $\mathbb{E}[(X - X_n)_+] \to 0$. Then also

$$\mathbb{E}\big[(X - X_n)_-\big] = \mathbb{E}\big[(X - X_n)_+\big] - \mathbb{E}(X - X_n) \to 0,$$

and so

$$\mathbb{E}|X - X_n| = \mathbb{E}\big[(X - X_n)_+\big] + \mathbb{E}\big[(X - X_n)_-\big] \to 0.$$

Hence $X_n \to X$ in probability. □

3. A new family of random sup-measures. Recall that for $\beta \in (0,1)$, a β-stable regenerative set is the closure of the range of a strictly β-stable subordinator, viewed as a random closed set in $\mathcal{F}(\mathbb{R}_+)$, and it has Hausdorff dimension β almost surely; see, for example, Bertoin (1999a). We need a result on intersections of independent stable regenerative sets presented below. A number of similar results can be found in literature; see, for example, Hawkes (1976/77), Fitzsimmons, Fristedt and Maisonneuve (1985) and Bertoin (1999b). We could not however find the exact formulation needed, so we included a short proof.

LEMMA 3.1. *Consider $v_1, v_2 \in \mathbb{R}_+$, $v_1 \neq v_2$ and $\beta_1, \beta_2 \in (0,1)$. Let $R_1^{(\beta_1)}$ and $R_2^{(\beta_2)}$ be two independent stable regenerative sets with parameter β_1 and β_2, respectively. Then*

$$\mathbb{P}\big(\big(v_1 + R_1^{(\beta_1)}\big) \cap \big(v_2 + R_2^{(\beta_2)}\big) \neq \varnothing\big) \in \{0,1\}. \tag{3.1}$$

The probability equals one, if and only if $\beta_{1,2} := \beta_1 + \beta_2 - 1 \in (0,1)$, and in this case, the intersection has the law of a shifted $\beta_{1,2}$-stable regenerative set, that is, a random element in $\mathcal{F}(\mathbb{R}_+)$ with a representation

$$V + R^{(\beta_{1,2})},$$

where $R^{(\beta_{1,2})}$ is a $\beta_{1,2}$-stable regenerative set, and $V > \max(v_1, v_2)$ is a random variable independent of $R^{(\beta_{1,2})}$.

PROOF. We may and will assume that $v_1 > v_2 = 0$, and drop the subscript in v_1. For $x > 0$ and $i = 1, 2$ let $B_{x,\beta}$ be the overshoot of the point x by a strictly β-stable subordinator; in particular,

$$B_{x,\beta_i} \stackrel{d}{=} \min\big(R_i^{(\beta_i)} \cap [x,\infty)\big) - x, \qquad x \geq 0.$$

Define a sequence of positive random variables $A_0, A_1, \dots$ by $A_0 = v$, $A_{2n+1} = B^{(2n+1)}_{A_{2n},\beta_2}$, $n = 0, 1, 2, \dots$, $A_{2n} = B^{(2n)}_{A_{2n-1},\beta_1}$, $n = 1, 2, \dots$, where different superscripts correspond to overshoots by independent subordinators. Then, by the strong Markov property, the probability of a nonempty intersection in (3.1) is simply

$$\mathbb{P}\left(\sum_{n=0}^{\infty} A_n < \infty\right). \tag{3.2}$$

The overshoot $B_{x,\beta}$ has the density given by

$$p_B^{(\beta)}(y \mid x) = \frac{1}{\Gamma(\beta)\Gamma(1-\beta)} \left(\frac{x}{y}\right)^{\beta} \frac{1}{x+y}, \qquad y > 0 \tag{3.3}$$

(see, e.g., Kyprianou (2006), Exercise 5.8.) This implies that $B_{x,\beta} \stackrel{d}{=} x B_{1,\beta}$ for $x > 0$. Grouping the terms together, we see that the probability in (3.2) is equal to

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \prod_{j=1}^{n} C_j < \infty\right),$$

where $C_1, C_2, \dots$ are i.i.d. random variables with $C_1 \stackrel{d}{=} B^{(1)}_{1,\beta_1} B^{(2)}_{1,\beta_2}$. An immediate conclusion is that the

$$\mathbb{P}\left(\sum_{n=0}^{\infty} A_n < \infty\right) = \begin{cases} 1 & \text{if } \mathbb{E}\log C_1 < 0, \\ 0 & \text{if } \mathbb{E}\log C_1 \geq 0. \end{cases}$$

However, by (3.3), after some elementary manipulations of the integrals, we have, writing $c(\beta) = 1/(\Gamma(\beta)\Gamma(1-\beta))$

$$\begin{aligned} \mathbb{E}\log C_1 &= c(\beta_1)\int_0^\infty \frac{y^{-\beta_1}\log y}{1+y}\,dy + c(\beta_2)\int_0^\infty \frac{y^{-\beta_2}\log y}{1+y}\,dy \\ &= \varphi(\beta_1) - \varphi(1-\beta_2) \end{aligned}$$

with

$$\varphi(\beta) = \left(\int_0^\infty \frac{y^{-\beta}\log y}{1+y}\,dy\right) \Big/ \left(\int_0^\infty \frac{y^{-\beta}}{1+y}\,dy\right).$$

So if $\beta_2 = 1 - \beta_1$, $\mathbb{E}\log C_1 = 0$, and it is enough to prove that the function $\varphi(\beta)$ is strictly decreasing in $\beta \in (0, 1)$. To see this,

$$\begin{aligned} &\varphi'(\beta_1) \\ &\quad = \left(\int \frac{y^{-\beta_1}}{1+y}\,dy\right)^{-2}\left[\left(\int \frac{y^{-\beta_1}\log y}{1+y}\,dy\right)^2 - \int \frac{y^{-\beta_1}(\log y)^2}{1+y}\,dy \int \frac{y^{-\beta_1}}{1+y}\,dy\right] \\ &\quad = -\mathrm{Var}(\log B_{1,\beta_1}) < 0. \end{aligned}$$

This proves (3.1) together with the criterion for the value of 1. Finally, by the strong Markov property of the stable regenerative sets, if $\beta_1 + \beta_2 > 1$, then

$$\left(v + R_1^{(\beta_1)}\right) \cap R_2^{(\beta_2)} \stackrel{d}{=} \sum_{n=0}^{\infty} A_n + \left(R_1^{(\beta_1)} \cap R_2^{(\beta_2)}\right),$$

where on the right-hand side, the series is independent of the stable regenerative sets. Since it has been shown by Hawkes (1976/77) that

$$R_1^{(\beta_1)} \cap R_2^{(\beta_2)} \stackrel{d}{=} R^{(\beta_1+\beta_2-1)},$$

the proof of the lemma is complete. $\square$

We now proceed with defining a new class of random sup-measures, by first identifying the underlying random upper semicontinuous function. From now on, $\beta \in (0, 1)$ and $\alpha > 0$ are fixed parameters. Consider a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with mean measure

$$\alpha u^{-(1+\alpha)}\,du(1-\beta)v^{-\beta}\,dv\,dP_{R^{(\beta)}},$$

where $P_{R^{(\beta)}}$ is the law of the β-stable regenerative set. We let $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ denote a measurable enumeration of the points of the point process, and

$$\widetilde{R}_j^{(\beta)} := V_j^{(\beta)} + R_j^{(\beta)}, \qquad j \in \mathbb{N}$$

denote the random closed sets $R_j^{(\beta)}$ shifted by $V_j^{(\beta)}$. These are again random closed sets.

Introduce the intersection of such random closed sets with indices from $S \subseteq \mathbb{N}$ by

$$I_S := \bigcap_{j\in S} \widetilde{R}_j^{(\beta)}, \qquad S \neq \varnothing \quad \text{and} \quad I_\varnothing := \mathbb{R}_+. \tag{3.4}$$

Let

$$\ell_\beta := \max\left\{\ell \in \mathbb{N} : \ell < \frac{1}{1-\beta}\right\} \in \mathbb{N}.$$

By Lemma 3.1, we know that

$$I_S \begin{cases} \neq \varnothing & \text{a.s. if } |S| \le \ell_\beta, \\ = \varnothing & \text{a.s. if } |S| > \ell_\beta. \end{cases} \tag{3.5}$$

Furthermore, when $|S| \le \ell_\beta$, I_S is a randomly shifted stable regenerative set with parameter $\beta_{|S|}$, where

$$\beta_\ell := \ell\beta - (\ell - 1) \in (0, 1) \qquad \text{for all } \ell = 1, \dots, \ell_\beta.$$

Let

$$\eta^{\alpha,\beta}(t) := \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{t\in\widetilde{R}_j^{(\beta)}\}}, \qquad t \in \mathbb{R}_+. \tag{3.6}$$

Since a stable regenerative set does not hit fixed points, for every t, $\eta^{\alpha,\beta}(t) = 0$ almost surely. Furthermore, on an event of probability 1, every t belongs to at most ℓ_β different $\widetilde{R}_j^{(\beta)}$, and thus $\eta^{\alpha,\beta}(t)$ is well defined for all $t \in \mathbb{R}_+$. In order to see that it is, on an event of probability 1, an upper semicontinuous function, it is enough to prove its upper semicontinuity on $[0, T]$ for every $T \in (0, \infty)$. Fixing such T, we denote by $U_{(j,T)}^{(\alpha)}$ the jth largest value of $U_j^{(\alpha)}$ for which $V_j^{(\beta)} \in [0, T]$, $j = 1, 2, \dots$. We write for $m = 1, 2, \dots,$

$$\begin{aligned}\eta^{\alpha,\beta}(t) &= \sum_{j=1}^{m} U_{(j,T)}^{(\alpha)} \mathbf{1}_{\{t\in\widetilde{R}_j^{(\beta)}\}} + \sum_{j=m+1}^{\infty} U_{(j,T)}^{(\alpha)} \mathbf{1}_{\{t\in\widetilde{R}_j^{(\beta)}\}} \\ &=: \eta_{1,m}^{\alpha,\beta}(t) + \eta_{2,m}^{\alpha,\beta}(t), \qquad t \in [0, T].\end{aligned}$$

The random function $\eta_{1,m}^{\alpha,\beta}$ is, for every m, upper semicontinuous as a finite sum of upper semicontinuous functions. Furthermore, on an event of probability 1,

$$\sup_{t\in[0,T]} \left|\eta_{2,m}^{\alpha,\beta}(t)\right| \le \sum_{j=m+1}^{m+\ell_\beta} U_{(j,T)}^{(\alpha)} \to 0$$

as $m \to \infty$, whence the upper semicontinuity of $\eta^{\alpha,\beta}$. We define the random sup-measure corresponding to $\eta^{\alpha,\beta}$ by

$$\eta^{\alpha,\beta}(G) := \sup_{t\in G} \eta^{\alpha,\beta}(t), \qquad G \in \mathcal{G}, \text{the collection of open subsets of } \mathbb{R}_+.$$

As usually, one may extend, if necessary, the domain of $\eta^{\alpha,\beta}$ to all subsets of $\mathbb{R}_+$. We emphasize that we use the same notation $\eta^{\alpha,\beta}$ for both the random upper semicontinuous function and the random sup-measure without causing too much ambiguity, thanks to the homeomorphism between the spaces $\mathrm{SM}(\mathbb{R}_+)$ and $\mathrm{USC}(\mathbb{R}_+)$. It remains to show the measurability of $\eta^{\alpha,\beta}$. Recall that the sup-vague topology of $\mathrm{SM} \equiv \mathrm{SM}(\mathbb{R}_+)$ has sub-bases consisting of

$$\{m \in \mathrm{SM} : m(K) < x\}, \qquad \{m \in \mathrm{SM} : m(G) > x\}, \qquad K \in \mathcal{K}, G \in \mathcal{G}, x \in \mathbb{R}_+.$$

See, for example, Vervaat (1997), Section 3. Then, for every $x > 0$,

$$\{\eta^{\alpha,\beta}(K) < x\} = \bigcap_{S\subset\mathbb{N}} \left(\left\{ \sum_{j\in S} U_j^{(\alpha)} < x \right\} \cap \{I_S \cap K \neq \varnothing\} \right)$$

is clearly measurable for $K \in \mathcal{K}$, and so is $\{\eta^{\alpha,\beta}(G) > x\}$ for $G \in \mathcal{G}$. The measurability thus follows.

PROPOSITION 3.2. *The random sup-measure $\eta^{\alpha,\beta}$ is stationary and H-self-similar with $H = (1-\beta)/\alpha$.*

PROOF. To prove the stationarity of $\eta^{\alpha,\beta}$ as a random sup-measure it is enough to prove that the random upper semicontinuous function $\eta^{\alpha,\beta}$ defined in (3.6) has a shift-invariant law. Let $r > 0$ and consider the upper semicontinuous function $(\eta^{\alpha,\beta}(t+r))_{t\in\mathbb{R}_+}$. Note that

$$\eta^{\alpha,\beta}(t+r) = \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{t+r\in\widetilde{R}_j^{(\beta)}\}} = \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{t\in G_r(\widetilde{R}_j^{(\beta)})\}}, \qquad t \in \mathbb{R}_+,$$

where G_r is a map from $\mathcal{F}(\mathbb{R}_+)$ to $\mathcal{F}(\mathbb{R}_+)$, defined by

$$G_r(F) := F \cap [r,\infty) - r.$$

However, by Proposition 4.1 (c) in Lacaux and Samorodnitsky (2016), the map

$$(x, F) \to (x, G_r(F))$$

on $\mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ leaves the mean measure of the Poisson random measure determined by $(U_j^{(\alpha)}, \widetilde{R}_j^{(\beta)})_{j\in\mathbb{N}}$ unaffected. Hence, the law of the random upper semicontinuous function $(\eta^{\alpha,\beta}(t+r))_{t\in\mathbb{R}_+}$ coincides with that of $(\eta^{\alpha,\beta}(t))_{t\in\mathbb{R}_+}$.

Similarly, in order to prove the H-self-similarity of $\eta^{\alpha,\beta}$ as a random sup-measure it is enough to prove that the random upper semicontinuous function $\eta^{\alpha,\beta}$ defined in (3.6) is H-self-similar. To this end, let $a > 0$, and note that by Proposition 4.1(b) in Lacaux and Samorodnitsky (2016)

$$\begin{aligned}\eta^{\alpha,\beta}(at) &= \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{at\in\widetilde{R}_j^{(\beta)}\}} \\ &= \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{t\in a^{-1}V_j^{(\beta)}+a^{-1}R_j^{(\beta)}\}} \\ &\stackrel{d}{=} a^{(1-\beta)/\alpha} \sum_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{t\in\widetilde{R}_j^{(\beta)}\}}\end{aligned}$$

jointly in t. Therefore, $(\eta^{\alpha,\beta}(at))_{t\in\mathbb{R}_+}$ and $(a^{(1-\beta)/\alpha}\eta^{\alpha,\beta}(t))_{t\in\mathbb{R}_+}$ have the same law as the random upper semicontinuous functions. $\square$

If we restrict the random upper semicontinuous functions and random measures above to a compact interval, we can use a particularly convenient measurable enumeration of the points of the Poisson process. Suppose, for simplicity, that the compact interval in question is the unit interval $[0, 1]$. The Poisson random measure $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ restricted to $\mathbb{R}_+ \times [0, 1] \times \mathcal{F}(\mathbb{R}_+)$ can then be viewed as a Poisson point process $(U_j^{(\alpha)})_{j\in\mathbb{N}}$ on $\mathbb{R}_+$ with mean measure $\alpha u^{-(1+\alpha)}\,du$ marked by two independent sequences $(V_j^{(\beta)})_{j\in\mathbb{N}}$ and $(R_j^{(\beta)})_{j\in\mathbb{N}}$ of i.i.d. random variables. The sequence $(R_j^{(\beta)})_{j\in\mathbb{N}}$ is as before, while $(V_j^{(\beta)})_{j\in\mathbb{N}}$ is a sequence of random variables taking values in $[0, 1]$ with the common law $\mathbb{P}(V_j^{(\beta)} \le v) = v^{1-\beta}$, $v \in [0, 1]$. Furthermore, we can enumerate the points of so-obtained Poisson random measure according to the decreasing value of the first coordinate, and express $(U_j^{(\alpha)})_{j\in\mathbb{N}}$ as $(\Gamma_j^{-1/\alpha})_{j\in\mathbb{N}}$ with $(\Gamma_j)_{j\in\mathbb{N}}$ denoting the arrival times of the unit rate Poisson process on $(0, \infty)$. This leads to the following representation:

$$(\eta^{\alpha,\beta}(t))_{t\in[0,1]} \stackrel{d}{=} \left(\sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{t\in\widetilde{R}_j^{(\beta)}\}}\right)_{t\in[0,1]}. \tag{3.7}$$

To conclude this section, we would like to draw the attention of the reader to the fact that for every fixed $\alpha \in (0, \infty)$, the family of random sup-measures $(\eta^{\alpha,\beta})_{\beta\in(0,1)}$ interpolates certain familiar random sup-measures. On one hand, as $\beta \downarrow 0$, the limit is well known and simple. To see this, notice first that for $(U_j^{(\alpha)}, V_j^{(\beta)}, R_j^{(\beta)})_{j\in\mathbb{N}}$ representing the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with mean measure $\alpha u^{-(1+\alpha)}(1-\beta)v^{-\beta}\, dv\, dP_{R^{(\beta)}}$, one can extend the range of parameters to include $\beta = 0$ by setting $P_{R^{(0)}} := \delta_{\{0\}}$ as a probability distribution (unit point mass at $\{0\}$) on $(\mathcal{F}(\mathbb{R}_+), \mathcal{B}(\mathcal{F}(\mathbb{R}_+))$. This is natural as $R^{(\beta)} \Rightarrow \{0\}$ in $\mathcal{F}(\mathbb{R}_+)$ as $\beta \downarrow 0$, which follows, for example, from Kyprianou (2006), Exercise 5.8 (the "zero-stable subordinator" can be thought of as a process staying an exponentially distributed amount of time at zero and then "jumping to infinity"). It then follows that

$$\eta^{\alpha,\beta}(\cdot) \Rightarrow \eta^{\alpha,0}(\cdot) := \bigvee_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{V_j^{(0)} \cap \cdot \neq \varnothing\}}$$

as $\beta \downarrow 0$.

The random sup-measure $\eta^{\alpha,0}$ above is the independently scattered (a.k.a. completely random) α-Fréchet max-stable random sup-measure on $\mathbb{R}_+$ with Lebesgue measure as the control measure (see Stoev and Taqqu (2005) and Molchanov and Strokorb (2016)). Furthermore, $(\eta^{\alpha,\beta}([0,t]))_{t\ge 0}$ corresponds to the extremal process $(Z_\alpha(t))_{t\ge 0}$ in (1.2) for a sequence of i.i.d. random variables with tail index α. The extremal process Z_α also belongs to the class of α-Fréchet max-stable processes (see, e.g., de Haan (1984), Kabluchko (2009)).

In the range $\beta \in (0, 1/2]$, the structure of $\eta^{\alpha,\beta}$ can also be simplified. As there are no intersections among independent shifted β-stable regenerative sets, the random sup-measure on the positive real line becomes

$$\eta^{\alpha,\beta}(\cdot) = \bigvee_{j=1}^{\infty} U_j^{(\alpha)} \mathbf{1}_{\{\widetilde{R}_j^{(\beta)} \cap \cdot \neq \varnothing\}}, \qquad \beta \in (0, 1/2].$$

This random sup-measure was first studied in Lacaux and Samorodnitsky (2016). This is an α-Fréchet max-stable random sup-measure, belonging to the class of the so-called Choquet random sup-measures introduced in Molchanov and Strokorb (2016). It is also known that for $\beta \in (0, 1/2]$, $(\eta^{\alpha,\beta}([0,t]))_{t\ge 0}$ has the same distribution as the time-changed extremal process $(Z_\alpha(t^{1-\beta}))_{t\ge 0}$; see Owada and Samorodnitsky (2015a) and Lacaux and Samorodnitsky (2016).

On the other hand, as soon as $\beta > 1/2$, the random sup-measure $\eta^{\alpha,\beta}$ is no longer an α-Fréchet random sup-measure, due to the appearance of intersections. As $\beta \uparrow 1$, the sets $\widetilde{R}^{(\beta)}$ become larger and larger in terms of Hausdorff dimension, and more and more $U_j^{(\alpha)}$s enter the sums defining the random measure due to intersections of more and more $\widetilde{R}_j^{(\beta)}$. In the limit, $\widetilde{R}^{(\beta)} \Rightarrow [0,\infty)$ in $\mathcal{F}(\mathbb{R}_+)$ as $\beta \uparrow 1$ (the "one-stable subordinator" is just the straight line). In the limit, therefore, all $U_j^{(\alpha)}$s contribute to the sum determining the random sup-measure, but for the infinite sum to be finite, restricting ourselves to the case $\alpha \in (0,1)$ is necessary. In this case, we have

$$\eta^{\alpha,\beta}(\cdot) \Rightarrow \eta^{\alpha,1}(\cdot) := \left(\sum_{j=1}^{\infty} U_j^{(\alpha)}\right)\mathbf{1}_{\{\cdot\cap\mathbb{R}_+\neq\varnothing\}}$$

as $\beta \uparrow 1$. In words, the limit is a random sup-measure with *complete dependence* that takes the same value $\sum_{j=1}^{\infty} U_j^{(\alpha)}$ on every open interval. Note that this random series follows the totally skewed α-stable distribution.

In particular, for every $\alpha \in (0,1)$, the distributions of random variables $(\eta^{\alpha,\beta}((0,1)))_{\beta\in[0,1]}$ interpolate between the α-Fréchet distribution ($\beta = 0$) and the totally skewed α-stable distribution ($\beta = 1$). These distributions, to the best of our knowledge, have not been described before. Their properties will be the subject of future investigations. See Simon (2014) for a recent result on comparison between totally skewed stable and Fréchet distributions.

The tail behavior of $\eta^{\alpha,\beta}((0,1))$ is, however, clear, and it is described in the following simple result.

PROPOSITION 3.3. *For all $\alpha \in (0,\infty)$, $\beta \in (0,1)$,*

$$x^{\alpha}\mathbb{P}\big(\eta^{\alpha,\beta}\big((0,1)\big) > x\big) \to 1$$

as $x \to \infty$.

PROOF. Consider the representation (3.7). Since $\mathbb{P}(\widetilde{R}^{(\beta)} \cap (0,1) \neq \varnothing) = 1$, with probability one

$$\Gamma_1^{-1/\alpha} \le \eta^{\alpha,\beta}\big((0,1)\big) \le \Gamma_1^{-1/\alpha} + (\ell_\beta - 1)\Gamma_2^{-1/\alpha}.$$

Note that $\mathbb{P}(\Gamma_1^{-1/\alpha} > x) \sim x^{-\alpha}$ as $x \to \infty$, and that for $\delta \in (0,\alpha)$,

$$\mathbb{P}\big(\Gamma_2^{-1/\alpha} > x\big) \le \frac{\mathbb{E}\Gamma_2^{-(\alpha+\delta)/\alpha}}{x^{\alpha+\delta}} = \frac{\Gamma(1-\delta/\alpha)}{x^{\alpha+\delta}}, \qquad x > 0,$$

where $\Gamma(x)$ is the Gamma function, hence the result. $\square$

As we shall see below, for each α, β the random sup-measure $\eta^{\alpha,\beta}$ arises in the limit of the extremes of stationary processes: while α indicates the tail behavior, β indicates the length of memory. The limiting case $\beta = 0$ corresponds to the short memory case already extensively investigated in the literature, and the case $\beta \in (0,1)$ corresponds to the long range dependence regime. The larger the β is, the longer the memory becomes.

4. A family of stationary infinitely divisible processes. We consider a discrete-time stationary symmetric infinitely divisible process whose function space Lévy measure is based on an underlying null-recurrent Markov chain. Similar models have been investigated in the symmetric α-stable (SαS) case in Resnick, Samorodnitsky and Xue (2000), Samorodnitsky (2004) Owada and Samorodnitsky (2015a), Owada and Samorodnitsky (2015b), Owada (2016) and Lacaux and Samorodnitsky (2016), which can be consulted for various background facts stated below. We first describe the Markov chain. Consider an irreducible aperiodic null-recurrent Markov chain $(Y_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$. Fix a state i_0, and let $(\pi_i)_{i\in\mathbb{Z}}$ be the unique invariant measure on $\mathbb{Z}$ such that $\pi_{i_0} = 1$. Consider the space $(E,\mathcal{E}) = (\mathbb{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}_0}))$. We denote each element of E by $x \equiv (x_0, x_1, \dots)$. Let P_i denote the probability measure on $(E,\mathcal{E})$ determined by the Markov chain starting at $Y_0 = i$, and introduce an infinite σ-finite measure on $(E,\mathcal{E})$ defined by

$$\mu(B) := \sum_{i\in\mathbb{Z}} \pi_i P_i(B), \qquad B \in \mathcal{E}.$$

Consider

$$A_0 := \{x \in E : x_0 = i_0\},$$

and the first entrance time of A_0

$$\varphi_{A_0}(x) := \inf\{n \in \mathbb{N} : x_n = i_0\}, \qquad x \in E.$$

The key assumption is that, for some $\beta \in (0,1)$ and a slowly varying function L,

$$\overline{F}(n) \equiv P_{i_0}(\varphi_{A_0} > n) = n^{-\beta} L(n). \tag{4.1}$$

This assumption can also be expressed in terms of the so-called *wandering rate sequence* defined by

$$w_n := \mu\left(\bigcup_{k=0}^{n-1} \{x \in E : x_k = i_0\}\right), \qquad n \in \mathbb{N}.$$

Then

$$w_n \sim \mu(\varphi_{A_0} \le n) \sim \sum_{k=1}^{n} P_{i_0}(\varphi_{A_0} \ge k),$$

and the key assumption becomes $w_n \in RV_{1-\beta}$. Here and in the sequel, $RV_{-\alpha}$ stands for the family of functions on $\mathbb{N}_0$ that are regularly varying at infinity with index $-\alpha$. For technical reasons, we will assume additionally that with $p_n := \mathbb{P}(Y_1 = n)$,

$$\sup_{n\in\mathbb{N}} \frac{np_n}{\overline{F}(n)} < \infty. \tag{4.2}$$

If T denotes the shift operator $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$, then μ is T-invariant: $\mu(\cdot) = \mu(T^{-1}\cdot)$ on $(E,\mathcal{E})$. Furthermore, T is conservative and ergodic with respect to μ on $(E, \mathcal{E})$. Next, we shall consider nonnegative functions from $L^\infty(\mu)$ supported by A_0. Fix $\alpha > 0$. For a fixed $f \in L^\infty(\mu)$, write

$$b_n := \left(\int \max_{k=0,\dots,n} \left(f \circ T^k(x)\right)^\alpha \mu(dx)\right)^{1/\alpha}, \qquad n \in \mathbb{N}. \tag{4.3}$$

The sequence (b_n) satisfies

$$\lim_{n\to\infty} \frac{b_n^\alpha}{w_n} = \|f\|_\infty. \tag{4.4}$$

Given a Markov chain as above and $f \in L^\infty(\mu)$ supported by A_0, we define a stationary symmetric infinitely divisible process as a stochastic integral

$$X_n := \int_E f_n(x) M(dx) \qquad \text{with } f_n := f \circ T^n, n \in \mathbb{N}_0, \tag{4.5}$$

where M is a homogeneous symmetric infinitely divisible random measure on $(E, \mathcal{E})$ with control measure μ and a local Lévy measure ρ, symmetric and satisfying

$$\rho\big((z, \infty)\big) = az^{-\alpha} \qquad \text{for } z \geq z_0 > 0. \tag{4.6}$$

We refer the reader to Chapter 3 in Samorodnitsky (2016) for more details on integrals with respect to infinitely divisible random measures and, in particular, for the fact that the stochastic process in (4.5) is a well-defined stationary infinitely divisible process (Theorem 3.6.6 therein). In particular, this process satisfies

$$\mathbb{P}(X_0 > x) \sim a\|f\|_\alpha^\alpha x^{-\alpha}$$

as $x \to \infty$; see Rosiński and Samorodnitsky (1993). We will use the value of α defined by (4.6) in (4.3). Below we will work with a more explicit and helpful series representation, (5.2) of the processes of interest.

We would like to draw the attention of the reader to the fact that we are assuming in (4.6) that the tail of the local Lévy measure has, after a certain point, exact power-law behavior. This is done purely for clarity of the presentation. There is no doubt whatsoever that limiting results similar to the one we prove in the next section hold under a much more general assumption of the regular variation of the tail of ρ. However, the analysis in this case will involve additional layers of approximation that might obscure the nature of the new limiting process we will obtain (note, however, that the assumption (4.6) already covers the SαS case when $\alpha \in (0, 2)$). In a similar vein, for the sake of clarity, we will assume in the next section that f is simply the indicator function of the set A_0.

Other types of limit theorems for this and related class of processes have been investigated for the partial sums (by Jung, Owada and Samorodnitsky (2017), Owada and Samorodnitsky (2015b)) and for the sample covariance functions (by Owada (2016), Resnick, Samorodnitsky and Xue (2000)). In all cases, nonstandard normalizations, or even new limit processes, show up in the limit theorems, indicating long range dependence in the model. Properties of stationary infinitely divisible processes have intrinsic connections to infinite ergodic theory (see Kabluchko and Stoev (2016), Rosiński (1995), Samorodnitsky (2005)), and the family of processes we are considering are said to be *driven by a null-recurrent flow*. The mixing properties of such processes (in the SαS case with $\alpha \in (0, 2)$) were investigated in Rosiński and Samorodnitsky (1996).

5. A limit theorem for stationary infinitely divisible processes. Consider the stationary infinitely divisible process introduced in (4.5). For $n = 1, 2, \dots$ we define a random sup-measure by

$$M_n(B) := \max_{k \in nB} X_k, \qquad B \subset [0, \infty).$$

The main result of this paper is the following theorem.

THEOREM 5.1. *Consider the stationary infinitely divisible process* $(X_n)_{n \in \mathbb{N}_0}$ *defined in the previous section. Let* $f = \mathbf{1}_{A_0}$ *with* $A_0 = \{x \in E : x_0 = i_0\}$. *Under the assumptions* (4.1) *and* (4.2) *and with* b_n *as in* (4.3),

$$\frac{1}{b_n} M_n \Rightarrow a^{1/\alpha} \eta^{\alpha,\beta}$$

as $n \to \infty$ *in the space* $\mathrm{SM}(\mathbb{R}_+)$, *where* a *is as in* (4.6).

We start with some preparation. Note that by (4.4), $b_n^\alpha \in RV_{1-\beta}$. By stationarity it suffices to prove convergence in the space $\mathrm{SM}([0, 1])$. We start by decomposing the process $(X_n)_{n \in \mathbb{N}_0}$ into the sum of two independent stationary symmetric infinitely divisible processes:

$$X_n = X_n^{(1)} + X_n^{(2)}, \qquad n \in \mathbb{N}_0,$$

with

$$X_n^{(i)} := \int_E f_n(x) M^{(i)}(dx), \qquad n \in \mathbb{N}_0, i = 1, 2,$$

with f_n as in (4.5), and $M^{(1)}$ and $M^{(2)}$ two independent homogeneous symmetric infinitely divisible random measures on $(E, \mathcal{E})$, each with control measure μ. The local Lévy measure for $M^{(1)}$ is the measure ρ restricted to the set $\{|z| \geq z_0\}$, while the local Lévy measure for $M^{(2)}$ is the measure ρ restricted to the set $\{|z| < z_0\}$. The first observation is that random variables $(X_n^{(2)})_{n \in \mathbb{N}_0}$ have Lévy measures supported by a bounded set, hence they have exponentially fast decaying tails; see, for example, Sato (1999). Therefore,

$$\frac{1}{b_n} \max_{k=0,1,\dots,n} |X_k^{(2)}| \to 0$$

in probability as $n \to \infty$. Therefore, without loss of generality we may assume that, in addition to (4.6), the local Lévy measure ρ is, to start with, supported by the set $\{|z| \geq z_0\}$.

For each $n \in \mathbb{N}$, the random vector $(X_0, \dots, X_n)$ admits a series representation that we will now describe. For $x > 0$, let

$$G(x) := \begin{cases} a^{1/\alpha} x^{-1/\alpha} & 0 < x < a z_0^{-\alpha}, \\ 0 & x \geq a z_0^{-\alpha}. \end{cases} \tag{5.1}$$

It follows from Theorem 3.4.3 in Samorodnitsky (2016) that the following representation in law holds:

$$(X_k)_{k=0,\dots,n} \stackrel{d}{=} \left(\sum_{j=1}^{\infty} \varepsilon_j G\big(\Gamma_j / 2 b_n^\alpha\big) \mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}} \right)_{k=0,\dots,n}, \tag{5.2}$$

where $(\Gamma_j)_{j \in \mathbb{N}}$ are as in (3.7), $(\varepsilon_j)_{j \in \mathbb{N}}$ are i.i.d. Rademacher random variables and $(U_j^{(n)})_{j \in \mathbb{N}}$ are i.i.d. E-valued random variables with common law μ_n, determined by

$$\frac{d\mu_n}{d\mu}(x) = \frac{1}{b_n^\alpha} \mathbf{1}_{\{T^k(x)_0 = i_0 \text{ for some } k = 0, 1, \dots, n\}}, \qquad x \in E.$$

All three sequences are independent. Here and in the sequel, for $x \in E \equiv \mathbb{Z}^{\mathbb{N}_0}$ we write $T^k(x)_0 \equiv [T^k(x)]_0 \in \mathbb{Z}$.

Our argument consists of coupling the series representation of $\eta^{\alpha,\beta}$ in (3.7) with the series representation of the process in (5.2). Note that the point process $(\Gamma_j^{-1/\alpha})_{j \in \mathbb{N}, \varepsilon_j = 1}$ is a Poisson point process with mean measure $2^{-1} \alpha u^{-(1+\alpha)}\, du$, $u > 0$, and it can be represented in law as the point process $(2^{-1/\alpha} \Gamma_j^{-1/\alpha})_{j \in \mathbb{N}}$. Therefore, we may and will work with a version of $\eta^{\alpha,\beta}$ given by

$$\big(\eta^{\alpha,\beta}(t)\big)_{t \in [0,1]} = \left(2^{1/\alpha} \sum_{j=1}^{\infty} \mathbf{1}_{\{\varepsilon_j = 1\}} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{t \in \widetilde{R}_j^{(\beta)}\}} \right)_{t \in [0,1]}.$$

We proceed through a truncation argument. Introduce for $\ell = 1, 2, \dots$

$$M_{\ell,n}(B) := \max_{k \in nB} \sum_{j=1}^{\ell} \varepsilon_j G\big(\Gamma_j / 2 b_n^\alpha\big) \mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}, \qquad n \in \mathbb{N},$$

and

$$\eta_\ell^{\alpha,\beta}(t) := 2^{1/\alpha} \sum_{j=1}^{\ell} \mathbf{1}_{\{\varepsilon_j = 1\}} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{t \in \widetilde{R}_j^{(\beta)}\}}, \qquad t \in [0, 1]. \tag{5.3}$$

We also let $\eta_\ell^{\alpha,\beta}$ denote the corresponding truncated random sup-measure. The key steps of the proof of Theorem 5.1 are Propositions 5.2 and 5.3 below.

PROPOSITION 5.2. *Under the assumptions of Theorem 5.1, for all* $\ell \in \mathbb{N}$,

$$\frac{1}{b_n} M_{\ell,n} \Rightarrow a^{1/\alpha} \eta_\ell^{\alpha,\beta}$$

as $n \to \infty$ *in the space of* $\mathrm{SM}([0,1])$.

PROPOSITION 5.3. *Under the assumptions of Theorem 5.1, for all* $\delta > 0$,

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\max_{k=0,\dots,n} \frac{1}{b_n} \left| \sum_{j=\ell+1}^{\infty} \varepsilon_j G(\Gamma_j/2b_n^\alpha) \mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}} \right| > \delta \right) = 0.$$

We start with several preliminary results needed for the proof of Proposition 5.2. First of all, we establish convergence of simultaneous return times of independent Markov chains. Introduce

$$\widehat{R}_{j,n}^{(\beta)} := \frac{1}{n}\{k \in \{0,\dots,n\} : T^k(U_j^{(n)})_0 = i_0\},$$

$$\widehat{I}_{S,n} := \bigcap_{j\in S} \widehat{R}_{j,n}^{(\beta)}, \qquad S \subset \mathbb{N}, S \neq \varnothing \quad \text{and} \quad \widehat{I}_{\varnothing,n} := \frac{1}{n}\{0,1,\dots,n\}.$$

Recall the definition of I_S in (3.4).

THEOREM 5.4. *Assume that* (4.1) *and* (4.2) *hold. Then for all* $\ell \in \mathbb{N}$,

$$(\widehat{I}_{S,n})_{S\subset\{1,\dots,\ell\}} \Rightarrow \bigl(I_S \cap [0,1]\bigr)_{S\subset\{1,\dots,\ell\}}$$

as $n \to \infty$ *in* $\mathcal{F}([0,1])^{2^\ell}$, *where for each* n *the law in the left-hand side is computed under* μ_n.

PROOF. By the second part of Theorem 2.1, it suffices to show the marginal convergence for $S = \{1,\dots,\ell\}$, for all $\ell \in \mathbb{N}$. First, we have seen in (3.5) that if

$$\beta_\ell^* := \ell\beta - \ell + 1 \in (0,1) \tag{5.4}$$

is *violated*, then the desired limit is the deterministic empty set. The convergence then follows from the first part of Theorem 2.1. From now on, we assume (5.4). For the simultaneous return times $\widehat{I}_{S,n}$ of independent Markov chains indexed by S, by introducing

$$V_{S,n}^{(\beta)} := \frac{1}{n} \min\{k = 0,\dots,n : T^k(U_j^{(n)})_0 = i_0, \text{ for all } j \in S\},$$

$$R_{S,n}^{(\beta)} := \frac{1}{n}\{k = 0,\dots,n : T^{nV_{I,n}^{(\beta)}+k}(U_j^{(n)})_0 = i_0, \text{ for all } j \in S\},$$

we have the decomposition $\widehat{I}_{S,n} = V_{S,n}^{(\beta)} + R_{S,n}^{(\beta)}$. Applying Corollary B.3 to I_S in (3.4), we have that $I_S = \widetilde{V}^{(\ell)} + R^{(\beta_\ell^*)}$ where $\widetilde{V}^{(\ell)}$ satisfies (B.9) with $\beta_j = \beta$, $j =$

$1, \ldots, \ell$ and $R^{(\beta_\ell^*)}$ is a β_ℓ^*-stable regenerative set, and the two are independent. In summary, the desired convergence now becomes

$$(V_{S,n}^{(\beta)} + R_{S,n}^{(\beta)}) \cap [0,1] \Rightarrow (\widetilde{V}^{(\ell)} + R^{(\beta_\ell^*)}) \cap [0,1]. \tag{5.5}$$

We first show the convergence of $V_{S,n}^{(\beta)}$ to $\widetilde{V}^{(\ell)}$, and we start with the last visit decomposition

$$\mu_n(V_{S,n}^{(\beta)} \le x) = \sum_{k=1}^{\lfloor nx \rfloor} \mu_n(\text{last simultaneous return to } i_0 \text{ before } \lfloor nx \rfloor \text{ is at time } k).$$

Denoting by $\overline{F^*}$ the tail of the time between two successive simultaneous visits to i_0 by ℓ i.i.d. Markov chains, we have

$$\begin{aligned}\mu_n(V_{S,n}^{(\beta)} \le x) &= \sum_{k=1}^{\lfloor nx \rfloor} \mu_n(T^k(U_j^{(n)})_0 = i_0 \text{ for all } j \in S)\overline{F^*}(\lfloor nt \rfloor - k) \\ &= \frac{1}{w_n^\ell} \sum_{k=1}^{\lfloor nx \rfloor} \overline{F^*}(\lfloor nx \rfloor - k) = \frac{1}{w_n^\ell} \sum_{k=0}^{\lfloor nx \rfloor - 1} \overline{F^*}(k).\end{aligned}$$

By Lemma A.1 in the Appendix and Karamata's theorem,

$$\begin{aligned}\mu_n(V_{S,n}^{(\beta)} \le x) &\sim \frac{(1-\beta)^\ell}{L(n)^\ell} n^{\beta_\ell^* - 1} L^*(\lfloor nx \rfloor) \frac{\lfloor nx \rfloor^{1-\beta_\ell^*}}{1-\beta_\ell^*} \\ &\to x^{1-\beta_\ell^*} \frac{(\Gamma(\beta)\Gamma(2-\beta))^\ell}{\Gamma(\beta_\ell^*)\Gamma(2-\beta_\ell^*)} = \mathbb{P}(\widetilde{V}^{(\ell)} \le x)\end{aligned}$$

as $n \to \infty$ for $x \in [0,1]$ (comparing with (B.9)).

Furthermore, the law of $nR_{S,n}^{(\beta)}$ is that of a renewal process with inter-renewal times distributed as F^*; see Appendix A. Therefore, by Giacomin (2007), Theorem A.8, $R_{S,n}^{(\beta)} \Rightarrow R^{(\beta^*)}$ in $\mathcal{F}(\mathbb{R}_+)$ as $n \to \infty$. The claim (5.5) now follows by an application of the continuous mapping theorem: the map

$$\mathbb{R}_+ \times \mathcal{F}([0,1]) \ni (x, F) \mapsto (x + F) \cap [0,1] \in \mathcal{F}([0,1])$$

is continuous, except at the point $\{(x,F) : (x+F) \cap [0,1] = \{1\}\}$ (e.g., Molchanov (2005), Appendix B). The probability that the latter point is hit by $\widetilde{V}^{(\ell)} + R^{(\beta_\ell^*)} \cap [0,1]$ is, however, equal to zero. This proof is thus complete. $\square$

Next, we show that for each open interval T, outside an event $A_n(T)$ to be defined below, of which the probability tends to zero as $n \to \infty$, the following key identity holds:

$$\max_{k\in nT}\sum_{j=1}^{\ell}\varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)\mathbf{1}_{\{T^k(U_j^{(n)})_0=i_0\}} \tag{5.6}$$
$$=\max_{S\subset\{1,\dots,\ell\}}\mathbf{1}_{\{\widehat{I}_{S,n}\cap T\neq\varnothing\}}\sum_{j\in S}\varepsilon_j G\big(\Gamma_j/2b_n^\alpha\big)$$
$$=\max_{S\subset\{1,\dots,\ell\}}\mathbf{1}_{\{\widehat{I}_{S,n}\cap T\neq\varnothing\}}\sum_{j\in S}\mathbf{1}_{\{\varepsilon_j=1\}}G\big(\Gamma_j/2b_n^\alpha\big),$$

with the convention that $\sum_{j\in\varnothing}=0$.

To establish this, we take a closer look at the simultaneous returns of Markov chain to i_0. We say that the chain indexed by j returns to i_0 at time k, if $T^k(U_j^{(n)})_0=i_0$. Note that if

$$\frac{k}{n}\in\widehat{I}_{S,n}\cap T=\Big(\bigcap_{j\in S}\widehat{R}_{j,n}^{(\beta)}\Big)\cap T,$$

then there might be another $j'\in\{1,\dots,\ell\}\setminus S$, such that the chain indexed by j' returns to i_0 at the same time k as well. We need an exact description of simultaneous returns of multiple chains. For this purpose, introduce

$$\widehat{I}^*_{S,n}:=\widehat{I}_{S,n}\cap\Big(\bigcup_{j\in\{1,\dots,\ell\}\setminus S}\widehat{R}_{j,n}^{(\beta)}\Big)^c,$$

the collection of all time points (divided by n) at which all chains indexed by S, and only these chains, return to i_0 simultaneously. We define the event

$$A_n(T):=\bigcup_{S\subset\{1,\dots,\ell\}}\big(\{\widehat{I}_{S,n}\cap T\neq\varnothing\}\cap\{\widehat{I}^*_{S,n}\cap T=\varnothing\}\big). \tag{5.7}$$

In words, on the complement of $A_n(T)$, if $\widehat{I}_{S,n}\cap T\neq\varnothing$ for some nonempty set S, then at some time point $k\in nT$, exactly those chains indexed by S return to i_0.

LEMMA 5.5. *For every open interval T, (5.6) holds on $A_n(T)^c$, and* $\lim_{n\to\infty}\mathbb{P}(A_n(T))=0$.

PROOF. We first prove the first part of the lemma. Noticing that $S=\varnothing$ is also included in the union above, and

$$\widehat{I}^*_{\varnothing,n}=\Big(\bigcup_{j=1,\dots,\ell}\widehat{R}_{j,n}^{(\beta)}\Big)^c,$$

we see that $A_n(T)$ includes the event that at every time k at least one of the ℓ chains returns to i_0. So on $A_n(T)^c$, the first two terms in (5.6), which are, clearly, always equal, are nonnegative. Furthermore, when $\widehat{I}_{S,n} \cap T \neq \varnothing$ for some nonempty S, then for $S' := \{j \in S : \varepsilon_j = 1\} \subset S$, $\widehat{I}_{S,n} \cap T \neq \varnothing$ implies $\widehat{I}_{S',n} \cap T \neq \varnothing$ and, therefore, restricted to the event $A_n(T)^c$ we have $\widehat{I}^*_{S',n} \cap T \neq \varnothing$. It follows that the second equality in (5.6) also holds on $A_n(T)^c$.

For the second part of the lemma, in view of (5.7), it suffices to show for all S,

$$\lim_{n\to\infty} \mathbb{P}\big(\{\widehat{I}_{S,n} \cap T \neq \varnothing\} \cap \{\widehat{I}^*_{S,n} \cap T = \varnothing\}\big) = 0.$$

The case $S = \varnothing$ is trivial. So without loss of generality, assume $S = \{1, \dots, \ell'\}$ for some $\ell' \in \{1, 2, \dots, \ell - 1\}$. Introduce

$$K_n := n \min(\widehat{I}_{S,n} \cap T),$$

the first time in nT that all chains indexed by S return to i_0 simultaneously. Then

$$\{\widehat{I}_{S,n} \cap T \neq \varnothing\} \cap \{\widehat{I}^*_{S,n} \cap T = \varnothing\} \subset \bigcup_{j=\ell'+1}^{\ell} \{T^{K_n}(U_j^{(n)})_0 = i_0, \widehat{I}_{S,n} \cap T \neq \varnothing\}.$$

The probability of each event in the union on the right-hand side is bounded from above by

$$\mathbb{P}\big(T^{K_n}(U_j^{(n)})_0 = i_0 \mid \widehat{I}_{S,n} \cap T \neq \varnothing\big) \leq \max_{k=0,\dots,n} \mathbb{P}\big(T^k(U_1^{(n)})_0 = i_0\big) = b_n^{-\alpha}$$

by the i.i.d. assumption on the chains. Since $b_n \to \infty$, the proof is complete. $\square$

Now we are ready to prove the main result.

PROOF OF PROPOSITION 5.2. By Theorem 3.2 in O'Brien, Torfs and Vervaat (1990) and the fact that the stable regenerative sets do not hit points, it suffices to show, for all $m \in \mathbb{N}$ and all disjoint open intervals $T_i = (t_i, t_i') \subset [0, 1]$, $i = 1, \dots, m$,

$$\left(\frac{1}{b_n} M_{\ell,n}(T_i)\right)_{i=1,\dots,m} \Rightarrow \big(a^{1/\alpha} \eta_\ell^{\alpha,\beta}(T_i)\big)_{i=1,\dots,m} \tag{5.8}$$

as random vectors in $\mathbb{R}^m$. The expression (5.1) and the fact that $b_n \to \infty$ tell us that the event $B_n := \{\Gamma_\ell / 2b_n^\alpha < a z_0^{-\alpha}\}$ has probability going to 1 as $n \to \infty$, and on B_n we have $G(\Gamma_j / 2b_n^\alpha) = (2a)^{1/\alpha} \Gamma_j^{-1/\alpha} b_n$. In particular, on the event B_n we have

$$\frac{1}{b_n} M_{\ell,n}(T_i) = \max_{k \in nT_i} \sum_{j=1}^{\ell} \varepsilon_j (2a)^{1/\alpha} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}.$$

Therefore, proving (5.8) is the same as proving that

$$\left(\max_{k \in nT_i} \sum_{j=1}^{\ell} \varepsilon_j 2^{1/\alpha} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}\right)_{i=1,\dots,m} \Rightarrow \big(\eta_\ell^{\alpha,\beta}(T_i)\big)_{i=1,\dots,m}. \tag{5.9}$$

The first part of Lemma 5.5 yields that on $A_n(T_i)^c \cap B_n$,

$$\max_{k\in nT_i}\sum_{j=1}^{\ell}\varepsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_{\{T^k(U_j^{(n)})_0=i_0\}}=\max_{S\subset\{1,\dots,\ell\}}\mathbf{1}_{\{\widehat{I}_{S,n}\cap T_i\neq\varnothing\}}\sum_{j\in S}\mathbf{1}_{\{\varepsilon_j=1\}}\Gamma_j^{-1/\alpha}.$$

Since by Lemma 5.5, $\mathbb{P}(A_n(T_i)^c\cap B_n)\to 1$ as $n\to\infty$, the statement (5.9) will follow once we prove that

$$\Bigl(\max_{S\subset\{1,\dots,\ell\}}\mathbf{1}_{\{\widehat{I}_{S,n}\cap T_i\neq\varnothing\}}\sum_{j\in S}2^{1/\alpha}\mathbf{1}_{\{\varepsilon_j=1\}}\Gamma_j^{-1/\alpha}\Bigr)_{i=1,\dots,m}\Rightarrow\bigl(\eta_\ell^{\alpha,\beta}(T_i)\bigr)_{i=1,\dots,m}.$$

This is, however, an immediate consequence of Theorem 5.4 and the fact that $\eta_\ell^{\alpha,\beta}(T_i)$ can be written in the form (recalling (5.3))

$$\eta_\ell^{\alpha,\beta}(T_i)=\max_{S\subset\{1,\dots,\ell\}}\mathbf{1}_{\{I_S\cap T_i\neq\varnothing\}}2^{1/\alpha}\sum_{j\in S}\mathbf{1}_{\{\varepsilon_j=1\}}\Gamma_j^{-1/\alpha},\qquad i=1,\dots,m.\qquad\square$$

PROOF OF PROPOSITION 5.3. For $M>0$, let $D_\ell^M:=\{\Gamma_{\ell+1}\ge M\}$. It is clear that $\lim_{\ell\to\infty}\mathbb{P}(D_\ell^M)=1$. We have

$$\begin{aligned}&\mathbb{P}\Biggl(\Biggl\{\max_{k=0,\dots,n}\frac{1}{b_n}\Biggl|\sum_{j=\ell+1}^{\infty}\varepsilon_j G\bigl(\Gamma_j/2b_n^\alpha\bigr)\mathbf{1}_{\{T^k(U_j^{(n)})_0=i_0\}}\Biggr|>\delta\Biggr\}\cap D_\ell^M\Biggr)\\&\quad\le\sum_{k=0}^{n}\mathbb{P}\Biggl(\Biggl\{\Biggl|\sum_{j=\ell+1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_{\{\Gamma_j\le 2ab_n^\alpha z_0^{-\alpha}\}}\mathbf{1}_{\{T^k(U_j^{(n)})_0=i_0\}}\Biggr|>\frac{\delta}{(2a)^{1/\alpha}}\Biggr\}\cap D_\ell^M\Biggr).\end{aligned}$$

Note that on the right-hand side above, the summand takes the same value for all $k=0,1,\dots,n$. Write $\delta':=\delta/(2a)^{1/\alpha}$. We shall show that, for all $\delta'>0$, one can choose M depending on α, β and δ' only, such that for all ℓ,

$$\limsup_{n\to\infty}n\mathbb{P}\Biggl(\Biggl\{\Biggl|\sum_{j=\ell+1}^{\infty}\varepsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_{\{\Gamma_j\le 2ab_n^\alpha z_0^{-\alpha}\}}\mathbf{1}_{\{T^k(U_j^{(n)})_0=i_0\}}\Biggr|>\delta'\Biggr\}\cap D_\ell^M\Biggr)=0.$$

The desired result then follows. To show the above, first observe that the probability of interest is bounded from above by

$$\mathbb{P}\Biggl(\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{M\le\Gamma_j\le 2ab_n^\alpha z_0^{-\alpha}\}}\mathbf{1}_{\{T^k(U_j^{(n)})_0=i_0\}}>\delta'\Biggr).\tag{5.10}$$

Observe that the restriction to $(0,\infty)$ of the point process with the points

$$\bigl(b_n\Gamma_j^{-1/\alpha}\mathbf{1}_{\{T^k(U_j^{(n)})_0=i_0\}}\bigr)_{j\in\mathbb{N}}$$

represents a Poisson random measure on $(0,\infty)$ with intensity $\mu(A_0)\alpha u^{-(\alpha+1)}\,du$, $u>0$, and another representation of the same Poisson random measure is

$$\bigl(\mu(A_0)^{1/\alpha}\Gamma_j^{-1/\alpha}\bigr)_{j\in\mathbb{N}}.$$

By definition of the Markov chain, $\mu(A_0) = 1$. Therefore, (5.10) becomes

$$\mathbb{P}\left(b_n^{-1}\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{M/b_n^\alpha \le \Gamma_j \le 2az_0^{-\alpha}\}} > \delta'\right)$$

$$\le \mathbb{P}\left(b_n^{-1}\sum_{j=j_M+1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{\Gamma_j \le 2az_0^{-\alpha}\}} > \delta'/2\right) \tag{5.11}$$

by taking $j_M := \lfloor M^{1/\alpha}\delta'/2\rfloor$, so that $b_n^{-1}\sum_{j=1}^{j_M}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{M/b_n^\alpha \le \Gamma_j \le 2az_0^{-\alpha}\}} \le \delta'/2$ with probability one. By Markov inequality, we can further bound (5.11) by, up to a multiplicative constant depending on δ',

$$b_n^{-p}\mathbb{E}\left(\sum_{j=j_M+1}^{\infty}\Gamma_j^{-1/\alpha}\mathbf{1}_{\{\Gamma_j \le 2az_0^{-\alpha}\}}\right)^p.$$

If we choose $p > 1/(1-\beta)$, then $b_n^{-p} = o(n^{-1})$. Since choosing M, and hence, j_M large enough, we can ensure finiteness of the above expectation, and this completes the proof. $\square$

PROOF OF THEOREM 5.1. As in the proof of Proposition 5.2, it suffices to show, for all $m \in \mathbb{N}$ and all disjoint open intervals $T_i = (t_i, t_i') \subset [0,1]$, $i = 1, \dots, m$,

$$\left(\max_{k\in nT_i}\frac{1}{b_n}\sum_{j=1}^{\infty}\varepsilon_j G(\Gamma_j/2b_n^\alpha)\mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}\right)_{i=1,\dots,m} \Rightarrow \left(a^{1/\alpha}\eta^{\alpha,\beta}(T_i)\right)_{i=1,\dots,m}.$$

We will use Theorem 3.2 in Billingsley (1999). By Proposition 5.2 and the obvious fact that

$$\left(\eta_\ell^{\alpha,\beta}(T_i)\right)_{i=1,\dots,m} \to \left(\eta^{\alpha,\beta}(T_i)\right)_{i=1,\dots,m}$$

a.s. as $\ell \to \infty$, it only remains to check that for any $i = 1, \dots, m$,

$$\lim_{\ell\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\frac{1}{b_n}\left|\max_{k\in nT_i}\sum_{j=1}^{\infty}\varepsilon_j G(\Gamma_j/2b_n^\alpha)\mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}\right.\right.$$

$$\left.\left.-\max_{k\in nT_i}\sum_{j=1}^{\ell}\varepsilon_j G(\Gamma_j/2b_n^\alpha)\mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}\right| > \varepsilon\right) = 0 \tag{5.12}$$

for any $\varepsilon > 0$. However, the above probability dos not exceed

$$\mathbb{P}\left(\frac{1}{b_n}\left|\max_{k\in nT_i}\sum_{j=\ell+1}^{\infty}\varepsilon_j G(\Gamma_j/2b_n^\alpha)\mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}\right| > \varepsilon\right)$$

$$\le \mathbb{P}\left(\frac{1}{b_n}\max_{k=0,\dots,n}\left|\sum_{j=\ell+1}^{\infty}\varepsilon_j G(\Gamma_j/2b_n^\alpha)\mathbf{1}_{\{T^k(U_j^{(n)})_0 = i_0\}}\right| > \varepsilon\right),$$

and (5.12) follows from Proposition 5.3. □

APPENDIX A: ELEMENTS OF RENEWAL THEORY

Consider an $\mathbb{N}$-valued renewal process $S_0 = 0$, $S_n := Y_1 + \cdots + Y_n$, $n = 1, 2, \ldots$ whose inter-renewal times $\{Y_n\}_{n\in\mathbb{N}}$ have a tail distribution $\overline{F}(n) := \mathbb{P}(Y_1 > n)$. The renewal function is defined by $u(n) := \sum_{k=0}^{\infty} \mathbb{P}(S_k = n)$, and we denote $U(n) := \sum_{j=0}^{n} u(j)$. The following two assumptions are equivalent for all $\beta \in (0, 1)$: as $n \to \infty$,

$$\overline{F}(n) \sim n^{-\beta} L(n), \tag{A.1}$$

$$U(n) \sim \frac{n^{\beta}}{\Gamma(1+\beta)\Gamma(1-\beta)L(n)}. \tag{A.2}$$

See, for example, Bingham, Goldie and Teugels (1987), Theorem 8.7.3. By Karamata's theorem,

$$u(n) \sim \frac{n^{\beta-1}}{\Gamma(\beta)\Gamma(1-\beta)L(n)} \tag{A.3}$$

implies (A.2). Furthermore, for $p_n := \mathbb{P}(Y_1 = n)$, under the assumption

$$\sup_{n\in\mathbb{N}} \frac{np_n}{\overline{F}(n)} < \infty, \tag{A.4}$$

it is known that (A.2) and (A.3) are equivalent; see Doney (1997).

Let now $\{Y_n^{(j)}\}_{n\in\mathbb{N}}$, $j = 1, \ldots, m$ be independent $\mathbb{N}$-valued renewal processes satisfying (A.1) with with parameters $\beta_1, \ldots, \beta_m \in (0, 1)$, respectively, such that

$$\beta^* := \sum_{q=1}^{m} \beta_q - m + 1 \in (0, 1), \tag{A.5}$$

and define $S_0^{(j)} := 0$, $S_n^{(j)} := Y_1^{(j)} + \cdots + Y_n^{(j)}$, $n \geq 1$. Set

$$Y_1^* := \min\{\ell \in \mathbb{N} : \ell = S_{n_k}^{(k)} \text{ for some } n_k \in \mathbb{N}, \forall k = 1, \ldots, m\},$$

and iteratively

$$Y_{m+1}^* := \min\{\ell \in \mathbb{N} : Y_1^* + \cdots + Y_m^* + \ell = S_{n_k}^{(k)} \text{ for some } n_k \in \mathbb{N}, \forall k = 1, \ldots, m\}.$$

That is, $\{Y_n^*\}_{n\in\mathbb{N}}$ are the simultaneous renewal times of $\{Y_n^{(j)}\}_{n\in\mathbb{N}}$, $j = 1, \ldots, m$, and they form another renewal process, which we refer to as the *intersection renewal process* of m independent renewal processes. We denote by F^*, u^* and U^* the corresponding functions defined at beginning of this section.

LEMMA A.1. *Assume that for every* $j = 1, \dots, m$,

$$\overline{F}^{(j)}(n) = n^{-\beta_j} L_j(n), \qquad \beta_j \in (0,1)$$

for some slowly varying at infinity function $L_j(n)$, *and that* (A.4) *holds. If* $\beta_1, \dots, \beta_m$ *satisfy* (A.5), *then the intersection renewal process satisfies, as* $n \to \infty$,

$$\overline{F^*}(n) \sim n^{-\beta^*} L^*(n) \qquad \text{with } L^*(n) = \frac{\prod_{q=1}^m [\Gamma(\beta_q)\Gamma(1-\beta_q)L_q(n)]}{\Gamma(\beta^*)\Gamma(1-\beta^*)}.$$

PROOF. Let $u^{(q)}$ denote the renewal function of the renewal process $\{Y_n^{(q)}\}_{n\in\mathbb{N}}$, $q = 1, \dots, m$. Because of (A.4) we know that (A.3) holds for each q. By independence,

$$u^*(n) = \prod_{q=1}^m u^{(q)}(n) \sim \frac{n^{\beta^*-1}}{\prod_{q=1}^m [\Gamma(\beta_q)\Gamma(1-\beta_q)L_q(n)]}.$$

Since (A.3) implies (A.1), the desired result follows. □

APPENDIX B: FIRST INTERSECTION TIME OF INDEPENDENT SHIFTED STABLE REGENERATIVE SETS

This paper uses certain results on the intersections of shifted stable regenerative sets. Some of these results may be known, but we could not find an appropriate reference. So we present them in this section.

Let $B_{a,\beta}$ denote the overshoot distribution of a β-stable subordinator over $a > 0$. Recall that the density function of $B_{a,\beta}$ in (3.3), and that the closure of image of a β-stable subordinator is known as a β-stable regenerative set. Consider two independent stable regenerative sets $R^{(\beta_1)}$ and $R^{(\beta_2)}$, with indices β_1, β_2, respectively. We know that if $\beta_{1,2} := \beta_1 + \beta_2 - 1 > 0$, then $R^{(\beta_1)} \cap R^{(\beta_2)}$ is again a stable regenerative set, with parameter $\beta_{1,2}$.

We will derive an explicit formula for the cumulative distribution function of the first intersection time of the two independent stable regenerative sets, with the second one shifted by $a > 0$. This random time is defined as

$$D_{a,\beta_1,\beta_2} := \min\{R^{(\beta_1)} \cap (a + R^{(\beta_2)})\} - a = \min\{(R^{(\beta_1)} - a) \cap R^{(\beta_2)}\}. \tag{B.1}$$

THEOREM B.1. *For all* $\beta_1, \beta_2 \in (0,1)$ *such that* $\beta_1 + \beta_2 - 1 > 0$,

$$D_{1,\beta_1,\beta_2} \stackrel{d}{=} B_{1,\beta_1}(1 + D_{1,\beta_2,\beta_1}) \tag{B.2}$$

and

$$\frac{D_{a,\beta_1,\beta_2}}{a} \stackrel{d}{=} D_{1,\beta_1,\beta_2} \qquad \text{for all } a > 0. \tag{B.3}$$

Moreover,

$$\mathbb{P}(D_{1,\beta_1,\beta_2} \le x) = P_D^{\beta_1,\beta_2}(x \mid 1), \qquad x > 0, \tag{B.4}$$

where for $a > 0$,

$$P_D^{\beta_1,\beta_2}(x \mid a) = \frac{1}{\Gamma(\beta_{1,2})\Gamma(1-\beta_{1,2})} \int_0^1 \left(\frac{a}{x}+y\right)^{\beta_1-1} y^{\beta_2-1}(1-y)^{-\beta_{1,2}}\,dy. \tag{B.5}$$

The main ingredient of the proof is the following proposition.

PROPOSITION B.2. *For all* $\beta_1, \beta_2 \in (0,1)$ *such that* $\beta_1+\beta_2-1>0$,

$$P_D^{\beta_1,\beta_2}(x \mid 1) = \int_0^x p_B^{(\beta_1)}(y \mid 1) P_D^{\beta_2,\beta_1}(x-y \mid y)\,dy, \qquad x > 0. \tag{B.6}$$

We first show how to derive Theorem B.1 from this proposition.

PROOF OF THEOREM B.1. It follows from Lemma 3.1 that

$$D_{1,\beta_1,\beta_2} \stackrel{d}{=} B_{1,\beta_1,0} \sum_{n=0}^{\infty} \left(\prod_{q=1}^{n} B_{1,\beta_1,q} B_{1,\beta_2,q} + B_{1,\beta_2,1} \prod_{q=1}^{n} B_{1,\beta_1,q} B_{1,\beta_2,q+1} \right),$$

with the convention $\prod_{q=1}^{0} = 1$, where on the right-hand side $\{B_{1,\beta_i,n}\}_{n\in\mathbb{N}_0}$ are i.i.d. copies of B_{1,β_i}, $i=1,2$, $\{B_{1,\beta_1,n}\}_{n\in\mathbb{N}_0}$ and $\{B_{1,\beta_2,n}\}_{n\in\mathbb{N}}$ are independent, and the series converges almost surely. This implies the recursive relation (B.2). Furthermore, (B.3) follows from (B.1) and the scaling invariance of the regenerative sets.

It remains to prove (B.4). It follows from (B.2) that

$$D_{1,\beta_1,\beta_2} \stackrel{d}{=} B_{1,\beta_1}\big[1 + B_{1,\beta_2}(1 + D_{1,\beta_1,\beta_2})\big]. \tag{B.7}$$

By the Letac principle (Letac (1986)) applied to the recursion

$$D_n = B_{1,\beta_1,n}\big[1 + B_{1,\beta_2,n}(1 + D_{n-1})\big], \qquad n \in \mathbb{N},$$

the law of D_{1,β_1,β_2} satisfying (B.7) is uniquely determined. Therefore, it suffices to show that a random variable whose law is given by the right-hand side of (B.4) satisfies (B.2), that is,

$$P_D^{\beta_1,\beta_2}(x \mid 1) = \int_0^x p_B^{(\beta_1)}(y \mid 1) P_D^{\beta_2,\beta_1}\left(\frac{x}{y} - 1 \,\Big|\, 1\right) dy, \qquad \text{for all } x > 0.$$

By the scaling property (B.3) this is exactly (B.6). □

PROOF OF PROPOSITION B.2. Recall that the hypergeometric function ${}_2F_1$ is defined as

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$$

(for c that is not a nonpositive integer) with $(a)_0 = 0$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$. By the Euler integral representation,

$$B(b, c-b)\,{}_2F_1(a, b; c; z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx,$$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $z \in \mathbb{C} \setminus [1, \infty)$ (B is the beta function), we have

$$P_D^{\beta_1,\beta_2}(x \mid a) = \frac{\Gamma(\beta_2)}{\Gamma(\beta_{1,2})\Gamma(2-\beta_1)}\left(\frac{a}{x}\right)^{\beta_1-1} {}_2F_1\left(1-\beta_1, \beta_2; 2-\beta_1; -\frac{x}{a}\right).$$

Therefore, the right-hand side of (B.6) is

$$\frac{1}{\Gamma(\beta_{1,2})\Gamma(2-\beta_2)\Gamma(1-\beta_1)} \times \int_0^x \frac{1}{1+y}\frac{1}{y^{\beta_1}}\left(\frac{y}{x-y}\right)^{\beta_2-1} {}_2F_1\left(1-\beta_2, \beta_1; 2-\beta_2; 1-\frac{x}{y}\right)dy.$$

Changing the variable $u = x/y - 1$, the above becomes

$$\frac{x^{1-\beta_1}}{\Gamma(\beta_{1,2})\Gamma(2-\beta_2)\Gamma(1-\beta_1)} \times \int_0^\infty \frac{u^{1-\beta_2}}{u+x+1}(1+u)^{\beta_1-1}{}_2F_1(1-\beta_2, \beta_1; 2-\beta_2; -u)\,du.$$

By the Euler transformation,

$$(1+u)^{\beta_1-1}{}_2F_1(1-\beta_2, \beta_1; 2-\beta_2; -u) = F_1(1, 1-\beta_{1,2}; 2-\beta_2; -u),$$

this can be written as

$$\frac{x^{1-\beta_1}}{\Gamma(\beta_{1,2})\Gamma(2-\beta_2)\Gamma(1-\beta_1)} \int_0^\infty \frac{u^{1-\beta_2}}{u+x+1}{}_2F_1(1, 1-\beta_{1,2}; 2-\beta_2; -u)\,du. \tag{B.8}$$

Using the table of integrals of Prudnikov, Brychkov and Marichev (1990), 2.21.1.16,

$$\begin{aligned}\int_0^\infty \frac{x^{c-1}}{(x+z)^\rho}{}_2F_1(a, b; c; -wx)\,dx &= w^{a-c}\frac{\Gamma(c)\Gamma(a-c+\rho)\Gamma(b-c+\rho)}{\Gamma(a+b-c+\rho)} \\ &\quad \times {}_2F_1(a-c+\rho, b-c+\rho; a+b-c+\rho; 1-wz)\end{aligned}$$

(provided $\mathrm{Re}(a+\rho), \mathrm{Re}(b+\rho) > \mathrm{Re}(c) > 0, |\arg w|, |\arg z| < \pi$)), (B.8) becomes

$$\frac{\Gamma(\beta_2)}{\Gamma(\beta_{1,2})\Gamma(2-\beta_1)}x^{1-\beta_1}{}_2F_1(\beta_2, 1-\beta_1; 2-\beta_1; -x) = P_D^{\beta_1,\beta_2}(x \mid 1).$$

This completes the proof. $\square$

COROLLARY B.3. *Let $\ell \in \mathbb{N}$, $\ell \geq 2$ and $\beta_1, \dots, \beta_\ell \in (0,1)$ be such that*

$$\beta_\ell^* := \sum_{j=1}^{\ell} \beta_j - \ell + 1 > 0.$$

For each $j = 1, \dots, \ell$, let $R_j^{(\beta_j)}$ be a β_j-stable regenerative set and $V_j^{(\beta_j)}$ a random variable with $\mathbb{P}(V_j^{(\beta_j)} \leq x) = x^{1-\beta_j}$, $x \in (0,1)$. Assume that all $R_j^{(\beta_j)}, V_j^{(\beta_j)}$, $j = 1, \dots, \ell$ are independent. Then

$$\bigcap_{j=1}^{\ell} \bigl(V_j^{(\beta_j)} + R_j^{(\beta_j)}\bigr) \stackrel{d}{=} \widetilde{V}^{(\ell)} + R^{(\beta_\ell^*)},$$

where $R^{(\beta_\ell^*)}$ is a β_ℓ^*-stable regenerative set, independent of a nonnegative random variable $\widetilde{V}^{(\ell)}$, whose law satisfies

$$\mathbb{P}\bigl(\widetilde{V}^{(\ell)} \leq x\bigr) = \frac{x^{1-\beta_\ell^*}}{\Gamma(\beta_\ell^*)\Gamma(2-\beta_\ell^*)} \prod_{j=1}^{\ell} \bigl(\Gamma(\beta_j)\Gamma(2-\beta_j)\bigr) \qquad \text{for } x \in [0,1]. \tag{B.9}$$

PROOF. The proof is by induction in ℓ. For $\ell = 1$ the claim is trivial. We proceed with the case $\ell = 2$. We already know that the intersection interest has the law of a β_2^*-stable regenerative set shifted by an independent random variable, so it suffices to check that the law of the shift satisfies (B.9). We will use (B.5) in the form (obtained by a change of variables)

$$P_D^{\beta_1,\beta_2}(x \mid a) = \frac{1}{\Gamma(\beta_2^*)\Gamma(1-\beta_2^*)} \int_0^x (a+y)^{\beta_1 - 1} y^{\beta_2 - 1} (x-y)^{-\beta_{1,2}}\, dy.$$

Fix $x \in (0,1)$. Then

$$\begin{aligned}
&\mathbb{P}\bigl(\widetilde{V}^{(2)} \leq x\bigr)\\
&\quad = \int_0^x (1-\beta_1) y_1^{-\beta_1} \int_0^{y_1} (1-\beta_2) y_2^{-\beta_2} P_D^{\beta_2,\beta_1}(x - y_1 \mid y_1 - y_2)\, dy_2\, dy_1\\
&\qquad + \int_0^x (1-\beta_1) y_1^{-\beta_1} \int_{y_1}^x (1-\beta_2) y_2^{-\beta_2} P_D^{\beta_1,\beta_2}(x - y_2 \mid y_2 - y_1)\, dy_2\, dy_1.
\end{aligned} \tag{B.10}$$

Denote

$$c_{\beta_1,\beta_2} := \frac{(1-\beta_1)(1-\beta_2)}{\Gamma(\beta_2^*)\Gamma(1-\beta_2^*)}.$$

Then the first double integral in (B.10) can be reduced, after a change of variable, to

$$c_{\beta_1,\beta_2} \int_0^x \int_0^{y_1} \int_{y_1}^x y_1^{-\beta_1} y_2^{-\beta_2} (z-y_2)^{\beta_2 - 1} (z-y_1)^{\beta_1 - 1} (x-z)^{-\beta_2^*}\, dz\, dy_2\, dy_1,$$

while the second double integral in (B.10) reduces, similar to

$$c_{\beta_1,\beta_2}\int_0^x\int_{y_1}^x\int_{y_2}^x y_1^{-\beta_1}y_2^{-\beta_2}(z-y_1)^{\beta_1-1}(z-y_2)^{\beta_2-1}(x-z)^{-\beta_2^*}\,dz\,dy_2\,dy_1.$$

By Fubini's theorem, for any nonnegative function f,

$$\int_0^x\int_0^{y_1}\int_{y_1}^x f\,dz\,dy_2\,dy_1+\int_0^x\int_{y_1}^x\int_{y_2}^x f\,dz\,dy_2\,dy_1=\int_0^x\int_0^z\int_0^z f\,dy_1\,dy_2\,dz.$$

Therefore,

$$\begin{aligned}&\mathbb{P}(\widetilde{V}^{(2)}\le x)\\&\quad=c_{\beta_1,\beta_2}\int_0^x\int_0^z\int_0^z y_1^{-\beta_1}y_2^{-\beta_2}(z-y_1)^{\beta_1-1}(z-y_2)^{\beta_2-1}(x-z)^{-\beta_2^*}\,dy_1\,dy_2\,dz\\&\quad=c_{\beta_1,\beta_2}B(1-\beta_1,\beta_1)B(1-\beta_2,\beta_2)\int_0^x(x-z)^{-\beta_2^*}\,dz,\end{aligned}$$

which is the same as (B.9) with $\ell=2$.

Next, suppose that the claim holds for some $\ell\ge 2$, and consider the intersection of $\ell+1$ independent shifted stable regenerative sets. By the assumption of the induction,

$$\bigcap_{j=1}^{\ell+1}\big(V_j^{(\beta_j)}+R_j^{(\beta_j)}\big)\stackrel{d}{=}\big(\widetilde{V}^{(\ell)}+R^{(\beta_\ell^*)}\big)\cap\big(V_{\ell+1}^{(\beta_{\ell+1})}+R_{\ell+1}^{(\beta_{\ell+1})}\big),$$

where the four random elements on the right-hand side above are assumed to be independent. Again the intersection on the right-hand side above, by strong Markov property, is a shifted stable regenerative set with index $\beta_\ell^*+\beta_{\ell+1}-1=\beta_{\ell+1}^*$, and it suffices to identify the law of the shift $\widetilde{V}^{(\ell+1)}$.

Let $x\in(0,1)$. We have

$$\begin{aligned}\mathbb{P}(\widetilde{V}^{(\ell+1)}\le x)&=\mathbb{P}(\widetilde{V}^{(\ell+1)}\le x,\widetilde{V}^{(\ell)}\le 1)\\&=\mathbb{P}(\widetilde{V}^{(\ell+1)}\le x\mid\widetilde{V}^{(\ell)}\le 1)\mathbb{P}(\widetilde{V}^{(\ell)}\le 1).\end{aligned}$$

By the assumption of the induction, the conditional cumulative distribution function of $\widetilde{V}^{(\ell)}$ given $\widetilde{V}^{(\ell)}\le 1$ is $x^{1-\beta_\ell^*}$, $0\le x\le 1$. Therefore, we are in the situation of the intersection of 2 independent shifted stable regenerative sets, which has already been considered. Using once again the assumption of the induction, we obtain

$$\begin{aligned}\mathbb{P}(\widetilde{V}^{(\ell+1)}\le x)&=\frac{x^{1-\beta_{\ell+1}^*}}{\Gamma(\beta_{\ell+1}^*)\Gamma(2-\beta_{\ell+1}^*)}\big(\Gamma(\beta_\ell^*)\Gamma(2-\beta_\ell^*)\Gamma(\beta_{\ell+1})\Gamma(2-\beta_{\ell+1})\big)\\&\qquad\times\frac{1}{\Gamma(\beta_\ell^*)\Gamma(2-\beta_\ell^*)}\prod_{j=1}^{\ell}\big(\Gamma(\beta_j)\Gamma(2-\beta_j)\big)\end{aligned}$$

$$= \frac{x^{1-\beta^*_{\ell+1}}}{\Gamma(\beta^*_{\ell+1})\Gamma(2-\beta^*_{\ell+1})} \prod_{j=1}^{\ell+1} \big(\Gamma(\beta_j)\Gamma(2-\beta_j)\big),$$

as required. $\square$

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