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UNIQUENESS OF GIBBS MEASURES FOR CONTINUOUS HARDCORE MODELS

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We formulate a continuous version of the well-known discrete hardcore (or independent set) model on a locally finite graph, parameterized by the so-called activity parameter $\lambda > 0$. In this version the state or "spin value" x_u of any node u of the graph lies in the interval $[0, 1]$, the hardcore constraint $x_u + x_v \le 1$ is satisfied for every edge (u, v) of the graph, and the space of feasible configurations is given by a convex polytope. When the graph is a regular tree, we show that there is a unique Gibbs measure associated to each activity parameter $\lambda > 0$. Our result shows that, in contrast to the standard discrete hardcore model, the continuous hardcore model does not exhibit a phase transition on the infinite regular tree. We also consider a family of continuous models that interpolate between the discrete and continuous hardcore models on a regular tree when $\lambda = 1$ and show that each member of the family has a unique Gibbs measure, even when the discrete model does not. In each case the proof entails the analysis of an associated Hamiltonian dynamical system that describes a certain limit of the marginal distribution at a node. Furthermore, given any sequence of regular graphs with fixed degree and girth diverging to infinity, we apply our results to compute the asymptotic limit of suitably normalized volumes of the corresponding sequence of convex polytopes of feasible configurations. In particular this yields an approximation for the partition function of the continuous hard core model on a regular graph with large girth in the case $\lambda = 1$.

1. Introduction.

1.1. *Background and motivation.* The (discrete) hardcore model, also commonly called the independent set model, is a widely studied model in statistical mechanics, as well as combinatorics and theoretical computer science. The model defines a family of probability measures on configurations on a finite or infinite (but locally finite) graph $\mathbb{G}$, parameterized by the so-called *activity* $\lambda > 0$. On a finite graph $\mathbb{G}$ with node set V and edge set E, the hardcore probability measure with parameter $\lambda > 0$ is supported on the collection of independent sets of the

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graph $\mathbb{G}$, and the probability of an independent set $I \subset V$ is proportional to $\lambda^{|I|}$, where $|I|$ denotes the size of the independent set. Equivalently, the hardcore probability measure can be thought of as being supported on the set of configurations $\mathbf{x} = (x_u, u \in V) \in \{0, 1\}^V$ that satisfy the hardcore constraint $x_u + x_v \le 1$ for every edge $(u, v) \in E$, with the probability of any feasible configuration $\mathbf{x}$ being proportional to $\lambda^{\sum_u x_u}$. The equivalence between the two formulations follows from the observation that given any hardcore configuration $\mathbf{x}$, the set $I = \{u \in V : x_u = 1\}$ is an independent set, and $\sum_{u \in V} x_u = |I|$. The constructed probability measure is a Gibbs measure in the sense that it satisfies a certain spatial Markov property [2, 8, 17]. On an infinite graph $\mathbb{G}$ the definition of the hardcore Gibbs measure is no longer explicit. Instead, it is defined implicitly as a measure that has certain specified conditional distributions on finite subsets of the graph, given the configuration on the complement. Thus, in contrast to the case of finite graphs, on infinite graphs neither existence nor uniqueness of a Gibbs measure is a priori guaranteed. While existence can be generically shown for a large class of models, uniqueness may fail to hold. When there are multiple Gibbs measures for some parameter, the model is said to exhibit a phase transition [8, 17].

The standard discrete hardcore model on a regular tree is known to exhibit a phase transition. Indeed, it was shown in [11, 19, 21] that there is a unique hardcore Gibbs measure on an infinite $(\Delta + 1)$-regular tree $\mathbb{T}_\Delta$ (i.e., a tree in which every node has degree $\Delta + 1$) if and only if $\lambda \le \lambda_c(\Delta) := \Delta^\Delta/(\Delta - 1)^{\Delta+1}$. In particular for λ in this range, the model exhibits a certain correlation decay property, whereas when $\lambda > \lambda_c(\Delta)$ the model exhibits long-range dependence. Roughly speaking, the correlation decay property says that the random variables X_u and X_v, distributed according to the marginal of the Gibbs measure at nodes u and v, respectively, become asymptotically independent as the graph-theoretic distance between u and v tends to infinity. This property is known to be equivalent to uniqueness of the Gibbs measure [8, 17]. The phase transition result above was recently extended to a generalization of the hardcore model which is defined on configurations $\mathbf{x} \in \{0, 1, \dots, M\}^V$, for some integer M, that satisfy the hardcore constraint $x_u + x_v \le M$ for $(u, v) \in E$; the usual hardcore model is recovered by setting $M = 1$. Specifically, it was shown in [6, 15] that the model on $\mathbb{T}_\Delta$ exhibits phase coexistence for all sufficiently high λ, and the point of phase transition was identified asymptotically, as Δ tends to infinity. The original model and its recent generalizations are also motivated by applications in the field of communications [11, 13, 15] in addition to the original statistical physics motivation.

The phase transition property on the infinite tree is known to be related to the algorithmic question of computing the partition function (or normalizing constant) associated with a Gibbs measure on a finite graph. Although the latter computation problem falls into the so-called #P-complete algorithmic complexity class for many models (including the standard hardcore model), there exist polynomial time approximation algorithms, at least for certain models and corresponding ranges of parameters. More precisely, when the underlying parameters are such that the corresponding Gibbs measure is unique, a polynomial time approximate computation of the corresponding partition function has been shown to be possible for several discrete models including the hardcore model [1, 20], matching model [3, 10], coloring model [1, 7] and some general binary models (models with two spin values) [12]. For some models, including the standard hardcore and matching models, approximate computation has been shown to be feasible whenever the model is in the uniqueness regime. For some other problems, including counting the number of proper colorings of a graph, an approximation algorithm has been constructed only for a restricted parameter range, although it is conjectured to exist whenever the model is in the uniqueness regime. Furthermore, the converse has also been established for the hardcore model and some of its extensions. Specifically, it was shown in [18] that for certain parameter values for which there are multiple Gibbs measures, approximate computation of the partition function in polynomial time becomes impossible, unless P = NP. This link between the phase transition property on the infinite regular tree $\mathbb{T}_\Delta$ and hardness of approximate compution of the partition function on a graph with maximum degree $\Delta + 1$ is conjectured to exist for general models.

1.2. *Discussion of results.* In light of the connection between phase transitions and hardness of computation mentioned above, an interesting problem to consider is the problem of computing the volume of a (bounded) convex polytope obtained as the intersection of finitely many half-spaces. It is known that, while this volume computation problem is #P-hard [5], it admits a randomized polynomial time approximation scheme [4] regardless of the parameters of the model. In fact such an algorithm exists for computing the volume of an arbitrary convex body, subject to minor regularity conditions. This motivates the investigation of this problem from the phase transition perspective by considering a model in which the partition function is simply the volume of a polytope. Toward this goal we introduce the *continuous hardcore model* on a finite graph $\mathbb{G}$, which defines a measure that is supported on the following special type of polytope:

$$\mathcal{P}(\mathbb{G}) = \{\mathbf{x} = (x_u, u \in V) : x_u \geq 0, x_u + x_v \leq 1, \forall u \in V, (u, v) \in E\}, \tag{1.1}$$

where V and E are, respectively, the vertex and edge set of the graph $\mathbb{G}$. $\mathcal{P}(\mathbb{G})$ is the linear programming relaxation of the independent set polytope of the graph, and we refer to it as the linear programming (LP) polytope of the graph $\mathbb{G}$. The continuous hardcore model with parameter $\lambda = 1$ is simply the uniform measure on $\mathcal{P}(\mathbb{G})$, and the associated partition function is equal to the volume of the convex polytope $\mathcal{P}(\mathbb{G})$.

As in the discrete case, the continuous hardcore model defines a one-parameter family of probability measures indexed by the activity $\lambda > 0$ (see Section 2 for a precise definition). We consider this model on an infinite regular tree $\mathbb{T}_\Delta$. Our main result (Theorem 3.1) is that, unlike the standard hardcore model, the continuous hardcore model on an infinite regular tree *never* exhibits a phase transition. Namely, for every choice of Δ and λ, there is a unique Gibbs measure for the continuous hardcore model on $\mathbb{T}_\Delta$ with activity λ. This result provides support for the conjecture that the link between the phase transition property and hardness of approximate computation of the partition function is indeed valid for general models, including those in which the spin values, or states of vertices, are continuous rather than discrete. Moreover, in Theorem 3.3 we characterize the cumulative distribution function of the marginal at any node of the continuous hardcore Gibbs measure (with parameter $\lambda = 1$) on the infinite regular tree as the unique solution to a certain ordinary differential equation (ODE). An analogous result is conjectured to hold for general $\lambda > 0$ (see Conjecture 5.2).

We extend our result further by considering a natural interpolation between the standard two-state hardcore and the continuous hardcore models when $\lambda = 1$. Here, in addition to the hardcore constraint, the spin values x_u are further restricted to belong to $[0, \epsilon] \cup [1 - \epsilon, 1]$ for some fixed parameter $\epsilon \in (0, 1/2)$. In a sense made precise in Section 1.2, when $\epsilon \to 1/2$, one obtains the continuous hardcore model and, as $\epsilon \to 0$, it more closely resembles the two-state hardcore model. We establish, perhaps surprisingly, that the model has a unique Gibbs measure for any positive value $\epsilon > 0$ (see Theorem 3.4), even when the discrete-hard core model (formally corresponding to $\varepsilon = 0$) has multiple Gibbs measures. The same argument does not easily extend to the case of general λ, and we leave this case open for further exploration.

Our last result (Theorem 3.5) concerns the computation of the volume of the LP polytope of a regular locally tree-like graph in the limit, as the number of nodes and girth of the graph goes to infinity. This result parallels some of the developments in [1], where it is shown that the partition functions associated with the standard hardcore model defined on a sequence of increasing regular locally tree-like graphs, with growing girth and after appropriate normalization, have a limit, and this limit coincides for all regular locally tree-like graphs with degree $\Delta + 1$ when the model is in the uniqueness regime for the tree $\mathbb{T}_\Delta$, namely, when $\lambda < \lambda_c(\Delta)$. We establish a similar result here, showing that the sequence of partition functions associated with the continuous hardcore model on a sequence of increasing regular graphs with large girth, after appropriate normalization, has a well-defined limit. We establish a corresponding approximation result for the continuous hardcore model, which is valid for all $\lambda > 0$, since, as shown in Theorem 3.1, the continuous hardcore model has a unique Gibbs measure for every λ. For the case $\lambda = 1$, when combined with our characterization of the Gibbs measure in Theorem 3.3, this provides a fairly explicit approximation of the normalized volume of the LP polytope of a regular graph with large girth.

We now comment on the proof technique underlying our result. To establish uniqueness of the Gibbs measure, we establish the correlation decay property. Unlike for the discrete hardcore model, establishing correlation decay for continuous models is significantly more challenging technically, since it involves analyzing recursive maps on the space of absolutely continuous (density) functions rather than one-dimensional or finite-dimensional recursions, and the function obtained as the limit of these recursive maps is characterized as the solution to a certain nonlinear second-order ordinary differential equation (ODE) with boundary conditions rather than as the fixed point of a finite-dimensional map. The direct approach of establishing a contraction property, which is commonly used in the analysis of discrete models, appears unsuitable in our case. Instead, establishing existence, uniqueness and the correlation decay property entails the analysis of this ODE. A key step that facilitates this analysis is the identification of a certain Hamiltonian structure of the ODE. This can be exploited, along with certain monotonicity properties, to establish uniqueness of the Gibbs measure. Characterization of the unique marginal distribution at a node requires additional work, which is related to establishing uniqueness of the solution to this ODE with suitable boundary conditions, and involves a detailed sensitivity analysis of a related parameterized family of ODEs.

1.3. *Outline of paper and common notation.* The remainder of the paper is organized as follows. In Section 2 we precisely define the continuous hardcore model and a family of related models. Then, in Section 3 we state our main results. In Section 4 we prove our main results, Theorems 3.1 and 3.4 on correlation decay (and hence uniqueness of the Gibbs measure) for the continuous hardcore model and its ε-interpolations for $\varepsilon \in (0, 1/2]$. In Section 5.1 we characterize the marginal distribution of the unique Gibbs measure for the continuous hardcore model with $\lambda = 1$ as the unique solution to a certain nonlinear ODE. The conjectured characterization for $\lambda \neq 1$ is described in Section 5.2. In Section 6 we prove our result regarding the volume of the LP polytope of a regular graph with large girth.

In what follows, given a set A, we let $\mathbb{I}_A$ denote the indicator function of the set A—$\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$, otherwise and when A is finite, let $|A|$ denote its cardinality. For $a \in [0, 1]$, let δ_a denote the Dirac delta measure at a, let dx denote one-dimensional Lebesgue measure, and given $\mathbf{x} = \{x_u, u \in V\}$, let $d\mathbf{x}$ denote $|V|$-dimensional Lebesgue measure. Also, for any subset $A \subset V$, let $\mathbf{x}_A$ represent the vector $(x_u, u \in A)$. Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real and nonnegative real numbers, respectively. Given any subset S of J-dimensional Euclidean space $\mathbb{R}^J$, let $\mathcal{B}(S)$ represent the collection of Borel subsets of S. For conciseness, given a measure μ on $\mathcal{B}(\mathbb{R})$, for intervals $[a, b]$, we will use $\mathcal{B}[a, b]$ and $\mu[a, b]$ to represent $\mathcal{B}([a, b])$ and $\mu([a, b])$, respectively.

2. A family of hardcore models. Let $\mathbb{G}$ be a simple undirected graph with finite node set $V = V(\mathbb{G})$ and edge set $E = E(\mathbb{G})$, and recall the associated LP polytope defined in (1.1). We now introduce the continuous hardcore model on the finite graph $\mathbb{G}$ associated with any parameter $\lambda > 0$. In fact we will introduce a more general family of hardcore models that will include both the discrete and continuous hardcore models in a common framework and allow us to also interpolate between the two. Any model in this family is specified by a finite Borel measure μ on $[0, 1]$, which we refer to as the "free spin measure" for the model. The free spin measure μ represents the weights the model puts on different states or spin values when the graph $\mathbb{G}$ is a single isolated vertex; specific examples are provided below. Given a free spin measure μ on the Borel sets $\mathcal{B}[0, 1]$ of $[0, 1]$ and $k \in \mathbb{N}$, let $\mu^{\otimes k}$ represent the product measure on $\mathcal{B}([0, 1]^k)$ with identical marginals equal to μ.

DEFINITION 2.1. The hardcore model corresponding to the graph $\mathbb{G} = (V, E)$ and free spin measure μ is the probability measure $\mathbb{P} = \mathbb{P}_{\mathbb{G},\mu}$ given by

$$\mathbb{P}(A) := \frac{1}{Z}\mu^{\otimes |V|}(A), \quad A \in \mathcal{B}(\mathcal{P}), \tag{2.1}$$

where $\mathcal{P} = \mathcal{P}(\mathbb{G})$ is the LP polytope defined in (1.1), and Z is the partition function or normalization constant given by

$$Z := \mu^{\otimes |V|}(\mathcal{P}). \tag{2.2}$$

The measure $\mathbb{P}$ is well defined as long as $Z > 0$. Since the hypercube $\{\mathbf{x} : 0 \leq x_u \leq 1/2, \forall u \in V\}$ is a subset of $\mathcal{P}(\mathbb{G})$ for every graph $\mathbb{G}$, a simple sufficient condition for this to hold is that the free spin measure satisfies $\mu[0, 1/2] > 0$. This will be the case in all the models we study.

We now describe the free spin measure associated with specific models. For $\lambda > 0$, the free spin measure of the two-state hardcore model with activity λ is given by $\mu = \mu_\lambda^{(2)}$,

$$\mu_\lambda^{(2)}(B) := \lambda\delta_1(B) + \delta_0(B), \quad B \in \mathcal{B}[0, 1]. \tag{2.3}$$

The measure $\mu_\lambda^{(2)}$ in (2.3) is discrete, supported on $\{0, 1\}$ and gives weights λ and 1 to the values 1 and 0, respectively, and the corresponding $\mathbb{P}_{\mathbb{G},\mu_\lambda^{(2)}}$ defines the standard (discrete) hardcore model with parameter $\lambda > 0$. This model was generalized to an $(M + 1)$-state hardcore model, for some integer $M \geq 1$, in [6, 15]. Given a parameter $\lambda > 0$, the free spin measure associated with a rescaled version of the latter model (that has support $[0, 1]$) is

$$\mu_\lambda^{(M+1)}(B) = \sum_{i=0}^{M} \lambda^i \delta_{\frac{i}{M}}(B), \quad B \in \mathcal{B}[0, 1]. \tag{2.4}$$

The case $M = 1$ then recovers the standard (two-state) hardcore model.

We now define the continuous hardcore model on $\mathbb{G}$ with parameter $\lambda > 0$ to be the measure $\mathbb{P}_{\mathbb{G}}^{\lambda} := \mathbb{P}_{\mathbb{G},\nu_\lambda}$, where the free spin measure ν_λ takes the form

$$\nu_\lambda(B) := \int_B \lambda^x \, dx, \quad B \in \mathcal{B}[0, 1]. \tag{2.5}$$

Despite the similarity in the definitions in (2.5) and (2.4), an important difference is that while $\mu_\lambda^{(2)}$ is discrete, ν_λ in (2.5) is absolutely continuous with respect to Lebesgue measure. In fact, when $\lambda = 1$, the free spin measure is just the uniform distribution on $[0, 1]$; the corresponding Gibbs measure is simply the uniform measure on the polytope $\mathcal{P}$, and Z is the volume of the polytope $\mathcal{P}$, as already mentioned in Section 1.2. For each activity parameter $\lambda > 0$, we also introduce a family of models, indexed by $\varepsilon \in (0, 1/2)$, which we refer to as the ε-continuous hardcore model that interpolate between the discrete and continuous hardcore models with the same activity parameter. For $\lambda > 0$ and $\varepsilon \in (0, 1/2)$, the free spin measure of the ε-continuous hardcore model with activity parameter λ is given by

$$\nu_\lambda^\varepsilon(B) := \frac{1}{2\varepsilon}\int_B \lambda^x \big(\mathbb{I}_{[0,\varepsilon]}(x) + \mathbb{I}_{[1-\varepsilon,1]}(x)\big)\,dx, \quad B \in \mathcal{B}[0, 1]. \tag{2.6}$$

We now clarify the precise sense in which this interpolates between the discrete and continuous models. Given probability measures $\{\pi_\varepsilon\}$ and π on $\mathcal{B}[0, 1]$, recall that π_ε is said to converge weakly to π as $\varepsilon \to \varepsilon_0$, if for every bounded continuous function f on $[0, 1]$, $\int_{[0,1]} f(x)\pi_\varepsilon(dx) \to \int_{[0,1]} f(x)\pi(dx)$ as $\varepsilon \to \varepsilon_0$. For any $\lambda > 0$, when $\varepsilon \uparrow 1/2$, ν_λ^ε converges weakly to ν_λ, the free spin measure of the continuous hardcore model with parameter $\lambda > 0$, as in (2.5), whereas as $\varepsilon \downarrow 0$, ν_λ^ε converges weakly to $\mu_\lambda^{(2)}$, the corresponding free spin measure of the two-state hardcore model as in (2.3).

Given any hardcore model on a finite graph $\mathbb{G}$ with free spin measure μ, we let $\mathbf{X} = (X_u, u \in V)$ denote a random element distributed according to $\mathbb{P}_{\mathbb{G},\mu}$ and refer to X_u as the spin value at u. Recall that given a subset S of nodes in $V(\mathbb{G})$, we use $\mathbf{X}_S = (X_u, u \in S)$ to denote the natural projection of $\mathbf{X}$ to the coordinates corresponding to S. The constructed hardcore probability distributions $\mathbb{P}_{\mathbb{G},\mu}$ are Markov random fields, or Gibbs measures, in the sense that they satisfy the following spatial Markov property. Given any subset $S \subset V$, let ∂S denote the set of nodes u in $V \setminus S$ that have neighbors in S, that is, for which $(u, v) \in E$ for some $v \in S$. Then, for every vector $\mathbf{x} = (x_u, u \in V) \in \mathcal{P}(\mathbb{G})$ that lies in the support of $\mathbb{P} = \mathbb{P}_{\mu,\mathbb{G}}$, we have

$$\mathbb{P}(\mathbf{x}_S|\mathbf{x}_{V\setminus S}) = \mathbb{P}(\mathbf{x}_S|\mathbf{x}_{\partial S}).$$

Namely, the joint probability distribution of spin values X_u associated with nodes $u \in S$, conditioned on the spin values at all other nodes of the graph, is equal to the joint distribution obtained on just conditioning on spin values at the boundary of S. Of course such a conditioning should be well defined which is easily seen to be the case for the hardcore models we consider.

3. Main results. We now turn to the setup related to the main results in the paper. We first recall some standard graph-theoretic notation. For every node $u \in V$, $\mathcal{N}(u) = \mathcal{N}_{\mathbb{G}}(u)$ denotes the set of neighbors of u, namely, the set $\{v : (u, v) \in$

$E\}$. The cardinality of $\mathcal{N}(u)$ is called the degree of the node u and is denoted by $\Delta(u)$. A leaf is a node with degree 1. Given a positive integer Δ, a graph is called Δ-regular if $\Delta(u) = \Delta$ for all nodes of the graph. The graph theoretic distance between nodes u and v is the length of a shortest path from u to v measured in terms of the number of edges on the path. Namely, it is the smallest m such that there exist nodes $u_0 = u, u_1, \dots, u_m = v$ such that (u_i, u_{i+1}), $i = 0, 1, \dots, m-1$ are edges. A cycle is a path $u_0 = u, u_1, \dots, u_m$ such that $m \geq 3$, $u_m = u_0$, and all $u_1, \dots, u_m$ are distinct. The girth $g = g(\mathbb{G})$ of the graph $\mathbb{G}$ is the length of a shortest cycle.

Let $\mathbb{T}_{n,\Delta}$ denote a rooted regular tree with degree $\Delta + 1$ and depth n, which is a finite tree with a special vertex called the root node, in which every node has degree $\Delta + 1$ except for the root node, and the leaves which is the collection of nodes that are at a graph-theoretic distance n from the root node and denoted $\partial\mathbb{T}_{n,\Delta}$. Each leaf has degree 1, and the root has degree Δ. Note that $\partial\mathbb{T}_{n,\Delta}$ is also the boundary of the remaining nodes of $\mathbb{T}_{n,\Delta}$ (which we refer to as internal nodes). Fix $\lambda > 0$ and let $\mathbb{P}_{n,\Delta,\lambda}$ represent the (continuous) hardcore distribution on $\mathbb{T}_{n,\Delta}$ with parameter $\lambda > 0$, corresponding to the free spin measure ν_λ in (2.5). We denote the (cumulative) distribution function of the marginal of $\mathbb{P}_{n,\Delta,\lambda}$ at the root node by $F_{n,\Delta,\lambda}(\cdot)$. Clearly, $F_{n,\Delta,\lambda}(\cdot)$ is absolutely continuous, and we denote its density by $f_{n,\Delta,\lambda}(\cdot)$. Given an arbitrary realization of spin values at the boundary $\mathbf{x}_{\partial\mathbb{T}_{n,\Delta}}$, we also let $F_{n,\Delta,\lambda}(\cdot|\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}})$ denote the cumulative distribution function of the conditional distribution of $\mathbb{P}_{n,\Delta,\lambda}$ at the root given $\mathbf{x}_{\partial\mathbb{T}_{n,\Delta}}$. It can be shown (see (4.2) with $\mu = \nu_\lambda$) that for $n \geq 2$, $F_{n,\Delta,\lambda}(\cdot|\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}})$ has a density, which we denote by $f_{n,\Delta,\lambda}(x|\mathbf{x}_{\partial\mathbb{T}_{n,\Delta}})$. In particular for $x \in [0, 1]$,

$$F_{n,\Delta,\lambda}(x) = \int_0^x f_{n,\Delta,\lambda}(t)\,dt, \qquad F_{n,\Delta,\lambda}(x|\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}}) = \int_0^x f_{n,\Delta,\lambda}(t|\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}})\,dt.$$

We now state our first main result which is proved in Section 4.5. For any absolutely continuous function F we let $\dot{F}$ denote the derivative of F which exists almost everywhere. Also, for any real-valued function g on $[0, \infty)$ and compact set $\mathcal{K} \subset [0, \infty)$ we let $\|g(\cdot)\|_{\mathcal{K}} := \sup_{x\in\mathcal{K}} |g(x)|$.

THEOREM 3.1. *For every $\Delta \geq 1$ and $\lambda > 0$, there exists a nondecreasing function $F_{\Delta,\lambda}$ with $F_{\Delta,\lambda}(0) = 0$ that is continuously differentiable on $(0, \infty)$ and satisfies, for any compact subset $\mathcal{K} \subset [0, 1]$,*

$$\lim_{n\to\infty} \sup \| F_{n,\Delta,\lambda}(\cdot|\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}}) - F_{\Delta,\lambda}(\cdot)\|_{[0,1]} = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \sup \| \dot{F}_{n,\Delta,\lambda}(\cdot|\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}}) - \dot{F}_{\Delta,\lambda}(\cdot)\|_{\mathcal{K}} = 0, \tag{3.2}$$

where the supremum is over all boundary conditions $\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}} \in [0, 1]^{|\partial\mathbb{T}_{\Delta,n}|}$.

REMARK 3.2. The relation (3.1) of Theorem 3.1 implies that the cumulative distribution function of the marginal distribution at the root is asymptotically independent from the boundary condition. In particular the model exhibits the correlation decay property regardless of the values of Δ and λ (which implies no phase transition). In fact it follows from Theorem 3.1 that there exists a unique Gibbs measure on the infinite $(\Delta + 1)$-regular tree, that this measure is translation invariant and its marginal distribution function at any node is equal to $F_{\Delta,\lambda}$. Relation (3.2) shows that the decay of correlations property extends to the marginal density.

Next, we provide a more explicit characterization of the marginal distribution function $F_{\Delta,\lambda}$ in the special case $\lambda = 1$ which is the quantity of interest for computing the volume of the polytope $\mathcal{P}(\mathbb{G})$. We show that this limit is the unique solution to a certain first-order ODE.

THEOREM 3.3. *For $\lambda = 1$ and $\Delta \geq 1$, there exists a unique $C = C_{\Delta,1} > 0$ such that the ODE*

$$\dot{F}(z) = C\big(1 - F^{\Delta+1}(z)\big)^{\Delta/(\Delta+1)}, \quad z \in (0, \infty), \tag{3.3}$$

with boundary conditions

$$F(0) = 0 \quad \textit{and} \quad \inf\{t > 0 : F(t) = 1\} = 1, \tag{3.4}$$

has a solution. Moreover, the ODE (3.3)–(3.4) with $C = C_{\Delta,1}$ has a unique solution $\bar{F}_{\Delta,1}$. Furthermore, $\bar{F}_{\Delta,1} = F_{\Delta,1}$, where $F_{\Delta,1}$ is the limit distribution function of Theorem 3.1.

The proof of Theorem 3.3 is given in Section 5.1. In fact we believe a generalization is possible to all $\lambda > 0$. Specifically, as stated in Conjecture 5.2 at the end of Section 5.1, we believe $F_{\Delta,\lambda}$ also admits a characterization in terms of a differential equation, although a more complicated second-order nonlinear differential equation, but we defer the validation of such a conjecture to future work.

The behavior of the continuous hardcore model described above should be contrasted with that of the discrete hardcore model for which, as discussed in the Introduction, the phase transition point on a $(\Delta + 1)$-regular tree is $\lambda_c = \Delta^\Delta/(\Delta - 1)^{\Delta+1}$. In particular when $\Delta \geq 5$, $\lambda_c < 1$ and so the discrete hardcore model on the tree with $\lambda = 1$ admits multiple Gibbs measures. This raises the natural question as to what happens for the ε-interpolated model, with free spin measure ν_1^ε, as in (2.6). It is natural to expect that this model would behave just like the standard hardcore model with $\lambda = 1$ for sufficiently small ε. Somewhat surprisingly, we show that this is not the case. By establishing a correlation decay property similar to that described in Remark 3.2, in Theorem 3.4 we show that there is a unique Gibbs measure for the ε-interpolated model for every positive ε, no matter how small.

THEOREM 3.4. *For every $\Delta \geq 1$, and $\varepsilon \in (0, 1/2)$, let $F_{n,\Delta}^{(\varepsilon)}$ denote the cumulative distribution of the marginal of the Gibbs measure $\mathbb{P}_{\mathbb{T}_{\Delta,n},\nu_1^\varepsilon}$ at the root of $\mathbb{T}_{\Delta,n}$. Then there exists a nondecreasing continuous function $F_\Delta^{(\varepsilon)}$ with $F_\Delta^{(\varepsilon)}(z) = 0$ for $z \leq 0$, $F_\Delta^{(\varepsilon)}(z) = 1$ for $z \geq 1$, that satisfies*

$$\lim_{n\to\infty} \sup \| F_{n,\Delta}^{(\varepsilon)}(\cdot|\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}}) - F_\Delta^{(\varepsilon)}(\cdot)\|_{[0,1]} = 0, \tag{3.5}$$

where the supremum is over all boundary conditions $\mathbf{x}_{\partial\mathbb{T}_{\Delta,n}} \in [0,1]^{|\partial\mathbb{T}_{\Delta,n}|}$.

We now turn to the implications of our results for volume computation. Specifically, applying Theorem 3.1, we are able to compute asymptotically the volume of the LP polytope associated with any regular graph that is locally tree-like (i.e., with large girth). The proof of Theorem 3.5 is given in Section 6.2.

THEOREM 3.5. *Fix $\lambda > 0$ and $\Delta \geq 1$, and let $F_{\Delta,\lambda}$ be as in Theorem 3.1. Let $\mathbb{G}_n$, $n \geq 1$, be a sequence of Δ-regular graphs with $g(\mathbb{G}_n) \to \infty$, and let $Z_{\mathbb{G}_n,\lambda}$ be the associated partition function as defined by (2.2) with $\mu = \mu_\lambda$ in (2.1) and $\mathcal{P} = \mathcal{P}(\mathbb{G}_n)$. Then*

$$\begin{aligned}\lim_{n\to\infty} \frac{\ln Z_{\mathbb{G}_n,\lambda}}{|V(\mathbb{G}_n)|} = {} & -\ln \int_{0\leq x\leq 1} \lambda^x F_{\Delta-1,\lambda}^{\Delta}(1-x)\,dx \\ & - \frac{\Delta}{2} \ln \int_{0\leq x \leq 1} \dot{F}_{\Delta-1,\lambda}(x) F_{\Delta-1,\lambda}(1-x)\,dx.\end{aligned} \tag{3.6}$$

Combining Theorem 3.5 with Theorem 3.3, we see that in the special case $\lambda = 1$, the volume $Z_{\mathbb{G}_n,1}$ of the polytope $\mathcal{P}(\mathbb{G}_n)$ satisfies

$$\lim_{n\to\infty} \frac{\ln Z_{\mathbb{G}_n,1}}{|V(\mathbb{G}_n)|} = \gamma,$$

where γ, which stands for the right-hand side of (3.6) with $\lambda = 1$, takes the form

$$\begin{aligned}\gamma = {} & -\ln \int_0^1 \bar{F}_{\Delta-1,1}^{\Delta}(1-x)\,dx \\ & - \frac{\Delta}{2} \ln \int_0^1 \bigl(1 - \bar{F}_{\Delta-1,1}^{\Delta-1}(x)\bigr)^{\frac{\Delta-1}{\Delta}} \bar{F}_{\Delta-1,1}(1-x)\,dx,\end{aligned}$$

where $\bar{F}_{\Delta-1,1}$ is the unique solution to (3.3)–(3.4) with $C = C_{\Delta,1}$, as identified in Theorem 3.3

This result provides a fairly explicit expression for the exponential limit of the volume of such a polytope, via the solution $\bar{F}_{\Delta-1,1}$ of the ODE which can be computed, for example, numerically. A similar expression for general λ would be obtained if Conjecture 5.2 were shown to be valid.

4. Analysis of continuous hardcore models. For ease of exposition, we fix $\Delta \geq 1$ and for each $n \geq 1$ use the notation $\mathbb{T}_n$ and $\mathcal{P}_n$ in place of $\mathbb{T}_{n,\Delta}$ and $\mathcal{P}(\mathbb{T}_{n,\Delta})$, respectively. Also, in order to present a unified proof of Theorems 3.1 and 3.4 to the extent possible, we will first fix any spin measure μ that is absolutely continuous with respect to Lebesgue measure, let m denote its density, and let F_n and f_n, respectively, denote the cumulative distribution function and density of the marginal at the root node of the hardcore model on $\mathbb{T}_n$ with free spin measure μ. Also, in analogy with the definitions in Section 3, let $F_n(\cdot|\mathbf{x}_{\partial\mathbb{T}_n})$ and, for $n \geq 2$, $f_n(\cdot|\mathbf{x}_{\partial\mathbb{T}_n})$ denote the corresponding conditional distribution functions and density given the boundary condition $\mathbf{x}_{\partial\mathbb{T}_n} \in [0,1]^{|\partial\mathbb{T}_n|}$. Also, let Z_n denote the corresponding hardcore partition function (2.2).

The proof of Theorem 3.1 entails several steps. First, in Section 4.1 we establish a monotonicity result, which allows one to only consider the cases when the boundary condition $\mathbf{x}_{\partial\mathbb{T}_n}$ is the vector of zeros or is the vector of ones. Then, in Sections 4.2 and 4.3, we derive iterative formulas for F_{2n} and F_{2n+1} and show that each of these sequences is pointwise monotonic in n and thus converge to limiting functions F_e and F_o, respectively. In Section 4.4 we characterize F_e and F_o in terms of certain ODEs and also identify a certain Hamiltonian structure that leads to an invariance property in the particular case of the continuous and ε-interpolated models. Finally, in Section 4.5 we use this invariance property to prove Theorems 3.1 and 3.4.

4.1. *Monotonicity property.* Given a spin measure μ, let $\mathbf{0}_{\partial\mathbb{T}_n}$ and $\mathbf{1}_{\partial\mathbb{T}_n}$, respectively, be the boundary condition corresponding to setting the values for the leaves of $\mathbb{T}_n$ to be all zeros and all ones. In Lemma 4.1 we state a monotonicity property for general models having discrete or continuous free spin measure. This property is well known for the special case of the standard (two-state) hardcore model and was further extended in [6], Lemma 2.2, to the multi-state hardcore model with free-spin measure ν_λ^{M+1} in (2.4) for any integer $M \geq 1$. For completeness, the proof of Lemma 4.1 is provided in Appendix A.

LEMMA 4.1. *For $n \geq 1$, every boundary condition $\mathbf{x}_{\partial\mathbb{T}_n}$ and every $z \in [0,1]$,*

$$F_n(z|\mathbf{0}_{\partial\mathbb{T}_n}) \geq F_n(z|\mathbf{x}_{\partial\mathbb{T}_n}) \geq F_n(z|\mathbf{1}_{\partial\mathbb{T}_n}),$$

when n is even, and

$$F_n(z|\mathbf{0}_{\partial\mathbb{T}_n}) \leq F_n(z|\mathbf{x}_{\partial\mathbb{T}_n}) \leq F_n(z|\mathbf{1}_{\partial\mathbb{T}_n}),$$

when n is odd.

4.2. *A recursion for the marginal distribution functions.* We now derive iterative formulas for the functions F_n. Let $\mathbb{T}_0$ denote the trivial tree consisting of an isolated vertex. Then, from (2.5), $Z_0 := \mu[0,1]$, where μ is the free spin measure, and the associated distribution function F_0 takes the form

$$F_0(z) = \frac{\mu[0,z]}{\mu[0,1]}, \quad z \in [0,1]. \tag{4.1}$$

LEMMA 4.2. *Given any free spin measure μ, for every $n \geq 1$, $F_n(z) = 0$ for $z \leq 0$, $F_n(z) = 1$ for $z \geq 1$, F_n is nondecreasing on $(0,1)$ and the following properties hold*:

1. *For $z \in [0,1]$, $F_n(z|\mathbf{0}_{\partial\mathbb{T}_n}) = F_{n-1}(z)$, and $F_{n+1}(z|\mathbf{1}_{\partial\mathbb{T}_n}) = F_{n-1}(z)$.*
2. *Moreover,*

$$F_n(z) = \frac{Z_{n-1}^{\Delta}}{Z_n} \int_{[0,z]} F_{n-1}^{\Delta}(1 - x_{u_0}) \mu(dx_{u_0}), \quad z \in [0,1]. \tag{4.2}$$

3. *Furthermore*,

$$\int_{[0,1]} F_{n-1}^{\Delta}(1-t)\mu(dt) = \frac{Z_n}{Z_{n-1}^{\Delta}} \tag{4.3}$$

and

$$\liminf_{n\to\infty} \frac{Z_{n-1}^{\Delta}}{Z_n} > 0. \tag{4.4}$$

PROOF. The values of F_n on $(-\infty, 0]$ and $[1, \infty)$ and the monotonicity of F_n follow immediately from the fact that F_n is the cumulative distribution function of a random variable with support in $[0,1]$. Next, given the boundary condition $\mathbf{0}_{\partial\mathbb{T}_n}$, the hardcore constraints $x_u + x_v \leq 1$ for every leaf node u and its parent v, reduces to the vacuous constraint $x_v \leq 1$. Thus, the boundary condition $\mathbf{0}_{\partial\mathbb{T}_n}$ translates to a free boundary (no boundary) condition on the tree $\mathbb{T}_{n-1}$. Similarly, the boundary condition $\mathbf{1}_{\partial\mathbb{T}_n}$ forces x_v to be zero for every parent v of a leaf of the tree $\mathbb{T}_n$ which in turn translates into a free boundary condition for the tree $\mathbb{T}_{n-2}$. This proves the first assertion of the lemma.

We now establish the second part of the lemma. Let u_0 denote the root of the tree $\mathbb{T}_n$ and note that for every $n \geq 1$, letting $\mathbf{x} = (x_u, u \in V(\mathbb{T}_n))$, we have for every $z \in [0,1]$,

$$F_n(z) = \frac{1}{Z_n} \bigl(\mu^{\otimes |V(\mathbb{T}_n)|}\bigr)\{\mathbf{x} \in \mathcal{P}_n : x_{u_0} \leq z\}. \tag{4.5}$$

Now, let $u_1, \ldots, u_\Delta$ denote the children of the root u_0. Each child u_i is the root of a tree $\mathbb{T}_{n-1}^i$ that is an isomorphic copy of $\mathbb{T}_{n-1}$. The constraint $\mathbf{x} \in \mathcal{P}_n$ translates into the constraints $x_{u_0} + x_{u_i} \leq 1$, $i = 1, 2, \ldots, \Delta$ plus the condition that the natural restriction $\mathbf{x}_{\mathbb{T}^i_{n-1}}$ of $\mathbf{x}$ to the subtree $\mathbb{T}^i_{n-1}$ lies in $\mathcal{P}^i_{n-1} := \mathcal{P}(\mathbb{T}^i_{n-1})$. Since these subtrees are nonintersecting, we obtain

$$\begin{aligned}&(\mu^{\otimes|V(\mathbb{T}_n)|})\{\mathbf{x} \in \mathcal{P}_n : x_{u_0} \le z\}\\ &\quad = \int_0^z d\mu(x_{u_0}) \prod_{1\le i\le\Delta} (\mu^{\otimes |V(T^i_{n-1})|})\{\mathbf{x} \in \mathcal{P}^i_n : x_{u_i} \le 1 - x_{u_0}\}.\end{aligned} \tag{4.6}$$

Now, for each $1 \le i \le \Delta$, we recognize the identity

$$\frac{1}{Z_{n-1}} (\mu^{\otimes |V(T^i_{n-1})|})\{\mathbf{x} \in \mathcal{P}^i_n : x_{u_i} \le 1 - x_{u_0}\} = F_{n-1}(1 - x_{u_0}).$$

Combined with (4.6) and (4.5), this yields (4.2).

Setting $F_n(1) = 1$ in (4.2), we obtain (4.3). Furthermore, since F_{n-1} is bounded by 1 and μ is a finite Borel measure, (4.3) implies that $\sup_n \frac{Z_n}{Z^\Delta_{n-1}} \le \mu[0, 1] < \infty$ which yields (4.4). $\square$

Combining Lemma 4.1 and the first part of Lemma 4.2, we now obtain a different monotonicity result along certain subsequences.

COROLLARY 4.3. *For every free spin measure μ, for $n \ge 1$ and $z \in [0, 1]$, $F_{2n+1}(z) \le F_{2n-1}(z)$ and $F_{2n}(z) \ge F_{2n-2}(z)$. Furthermore, for every $n_1, n_2 \in \mathbb{Z}_+$, with $F_{2n_1+1}(z) \ge F_{2n_2}(z)$.*

PROOF. Once again, let u_0 denote the root of the tree $\mathbb{T}_n$, and label its children as $u_1, \dots, u_\Delta$. Consider the random vector $\mathbf{X}$ chosen according to the hardcore measure $\mathbb{P} = \mathbb{P}_{\mathbb{T}_n,\mu}$, and let $\mathbb{P}_{\partial\mathbb{T}_n}$ denote the marginal of $\mathbb{P}$ on the leaves $\partial\mathbb{T}_n$. Then, F_n is the cumulative distribution function of the marginal at the root, that is, $F_n(z) = \mathbb{P}(X_{u_0} \le z)$. Thus, for every odd $n \ge 3$, using Lemma 4.1 for the inequality and Lemma 4.2(1) for the last equality below, we have

$$\begin{aligned}F_n(z) = \mathbb{P}(X_{u_0} \le z) &= \int_{\mathbf{x}_{\partial\mathbb{T}_n} \in [0,1]^{|\partial\mathbb{T}_n|}} \mathbb{P}(X_{u_0} \le z | \mathbf{X}_{\partial\mathbb{T}_n} = \mathbf{x}_{\partial\mathbb{T}_n}) \mathbb{P}_{\partial\mathbb{T}_n}(d\mathbf{x}_{\partial\mathbb{T}_n})\\ &\le \mathbb{P}(X_{u_0} \le z | \mathbf{1}_{\partial\mathbb{T}_n})\\ &= F_{n-2}(z).\end{aligned}$$

Similarly, for every even n we obtain $F_n(z) \ge F_{n-2}(z)$ for every z. Finally, to establish the last inequality, suppose first that $n_1 \ge n_2$. Then, since the first assertion of the lemma implies $F_{2n_1}(z) \ge F_{2n_2}(z)$, by a similar derivation we have

$$\begin{aligned}F_{2n_1+1}(z) &= \int_{\mathbf{x}_{\partial\mathbb{T}_{2n_1+1}}} \mathbb{P}(X_{u_0} \le z | \mathbf{X}_{\partial\mathbb{T}_{2n_1+1}} = \mathbf{x}_{\partial\mathbb{T}_{2n_1+1}}) \mathbb{P}_{\partial\mathbb{T}_{2n_1+1}}(d\mathbf{x}_{\partial\mathbb{T}_{2n_1+1}})\\ &\ge \mathbb{P}(X_{u_0} \le z | \mathbf{0}_{\partial\mathbb{T}_{2n_1+1}})\end{aligned}$$

$$
\begin{aligned}
&= F_{2n_1}(z) \\
&\geq F_{2n_2}(z).
\end{aligned}
$$

Conversely, if $n_1 < n_2$, then $2n_1 + 1 \leq 2n_2 - 1$, and we use instead

$$
\begin{aligned}
F_{2n_2}(z) &= \int_{\mathbf{x}_{\partial\mathbb{T}_{2n_2}}} \mathbb{P}(X_{u_0} \leq z | \mathbf{X}_{\partial\mathbb{T}_{2n_2}} = \mathbf{x}_{\partial\mathbb{T}_{2n_2}}) \mathbb{P}_{\partial\mathbb{T}_{2n_2}}(d\mathbf{x}_{\partial\mathbb{T}_{2n_2}}) \\
&\leq \mathbb{P}(X_{u_0} \leq z | \mathbf{0}_{\partial\mathbb{T}_{2n_2}}) \\
&= F_{2n_2-1}(z) \\
&\leq F_{2n_1+1}(z). \qquad \square
\end{aligned}
$$

4.3. *A convergence result*. The monotonicity result of Corollary 4.3 allows us to argue the existence of the following pointwise limits: for $z \in [0, 1]$,

$$
F_o(z) := \lim_{n\to\infty} F_{2n+1}(z), \qquad F_e(z) := \lim_{n\to\infty} F_{2n}(z). \tag{4.7}
$$

Also, note that by Corollary 4.3, for $z \in [0, 1]$,

$$
1 \geq F_{2n+1}(z) \geq F_o(z) \geq F_e(z) \geq F_{2n}(z) \geq 0. \tag{4.8}
$$

Clearly, F_o and F_e are measurable and bounded. So, we can define

$$
C_o := \left(\int_{[0,1]} F_o^{\Delta}(1-t)\mu(dt) \right)^{-1} \tag{4.9}
$$

and

$$
C_e := \left(\int_{[0,1]} F_e^{\Delta}(1-t)\mu(dt) \right)^{-1}. \tag{4.10}
$$

Note that by (4.7), since μ is a finite Borel measure, the dominated convergence theorem, (4.3) and (4.4) imply

$$
C_o = \lim_{n\to\infty} \left(\int_{[0,1]} F_{2n+1}^{\Delta}(1-t)\mu(dt) \right)^{-1} = \lim_{n\to\infty} \frac{Z_{2n+1}^{\Delta}}{Z_{2n+2}} > 0. \tag{4.11}
$$

Moreover, by (4.8) the dominated convergence theorem and (4.3) we have

$$
C_e = \lim_{n\to\infty} \left(\int_{[0,1]} F_{2n}^{\Delta}(1-t)\mu(dt) \right)^{-1} = \lim_{n\to\infty} \frac{Z_{2n}^{\Delta}}{Z_{2n+1}}. \tag{4.12}
$$

The first equality above, together with (4.8) and (4.1), also show that

$$
C_e^{-1} \geq \frac{1}{(\mu[0, 1])^{\Delta}} \int_{[0,1]} (\mu[0, 1-t])^{\Delta} \mu(dt) \geq \frac{(\mu[0, 1/2])^{\Delta+1}}{(\mu[0, 1])^{\Delta}}. \tag{4.13}
$$

We now derive an analogue of (4.2) for the limits F_o and F_e and strengthen the convergence in (4.7).

COROLLARY 4.4. *Suppose the free spin measure* μ *satisfies* $\mu[0, 1/2] > 0$. *Then*, $C_e, C_o \in (0, \infty)$, $F_o(0) = F_e(0) = 0$, $F_o(1) = F_e(1) = 1$ *and for* $z \in [0, 1]$,

$$F_o(z) = C_e \int_0^z F_e^\Delta(1 - t)\mu(dt), \tag{4.14}$$

$$F_e(z) = C_o \int_0^z F_o^\Delta(1 - t)\mu(dt). \tag{4.15}$$

Moreover, we also have

$$\lim_{n\to\infty} \|F_{2n+1}(\cdot) - F_o(\cdot)\|_{[0,1]} = 0,$$

$$\lim_{n\to\infty} \|F_{2n}(\cdot) - F_e(\cdot)\|_{[0,1]} = 0.$$

Finally, suppose μ *has density* m *and that* $\mathcal{I}$ *is an open set of continuity points of* m. *Then* m, F_n, F_o *and* F_e *are continuously differentiable on* $\mathcal{I}$ *and, for every compact subset* $\mathcal{K} \subset \mathcal{I}$,

$$\lim_{n\to\infty} \|\dot{F}_{2n+1}(\cdot) - \dot{F}_o(\cdot)\|_{\mathcal{K}} = 0,$$

$$\lim_{n\to\infty} \|\dot{F}_{2n}(\cdot) - \dot{F}_e(\cdot)\|_{\mathcal{K}} = 0.$$

PROOF. The values of F_o and F_e at 0 and 1 follow directly from the corresponding values of F_n from Lemma 4.2 and (4.7). Since (4.8) implies $C_o \le C_e$, the estimates (4.11) and (4.13) imply that as long as $\mu[0, 1/2] > 0$, both C_o and C_e lie in $(0, \infty)$. For $z \in [0, 1]$, let $F_o^*(z)$ and $F_e^*(z)$ equal the right-hand sides of (4.14) and (4.15), respectively. Taking limits on both sides of (4.2) along odd n and using (4.7), (4.12) and the dominated convergence theorem, we obtain (4.14). The relation (4.15) is obtained analogously, using (4.11) instead of (4.12). The latter relations show that F_o and F_e are continuous. Since they are also pointwise monotone limits of the sequences F_{2n+1} and F_{2n}, respectively (see Corollary 4.3 and (4.7)), by Dini's theorem, the convergence is in fact uniform.

We now prove the last property of the lemma, even though we do not use it in the sequel. Suppose μ has density m that is continuous on $\mathcal{I}$. Then, (4.1) and (4.2) show that for every n, F_n is absolutely continuous and $\dot{F}_n(z) = Z_n^{-1} Z_{n-1}^\Delta F_{n-1}^\Delta(1 - z)m(z)$, from which it follows that $\dot{F}_n$ is continuous on $\mathcal{I}$. Likewise, the continuous differentiability of F_o and F_e on $\mathcal{I}$ can be deduced from (4.14) and (4.15). The uniform convergence of the derivatives on any compact subset $\mathcal{K} \subset \mathcal{I}$ is a direct consequence of (4.2), (4.14)–(4.15) and the uniform convergence of $\{F_{2n+1}\}$ to F_o and $\{F_{2n}\}$ to F_e. $\square$

REMARK 4.5. We now claim (and justify below) that to prove the correlation decay property in Theorems 3.1 and 3.4, it suffices to show (for the respective models) that $C_e = C_o$. Indeed, by Lemma 4.1, Lemma 4.2(1) and (4.7), to show correlation decay is equivalent to showing $F_o = F_e$. Now, by (4.8) we have $F_o(z) \ge$

$F_e(z)$ for every $z \in [0, 1]$. Hence, if $C_e = C_o = C$, then by (4.14)–(4.15), we have for $z \in [0, 1]$,

$$F_o(z) = C \int_{[0,z]} F_e^{\Delta}(1-t)\mu(dt) \le C \int_{[0,z]} F_o^{\Delta}(1-t)\mu(dt) = F_e(z).$$

Together with the observation that $F_0(z) \le F_z(z)$, this implies $F_o = F_e$.

4.4. *Differential equations for F_o and F_e.* To show $C_e = C_o$, we first derive some differential equations for the functions F_o and F_e. The first result of this section is as follows.

PROPOSITION 4.6. *Suppose the free spin measure μ is absolutely continuous with density m and satisfies $\mu[0, 1/2] > 0$. Let $\mathcal{I}$ be any nonempty open set in $[0, 1]$ that is symmetric in the sense that $x \in \mathcal{I}$ implies $1 - x \in \mathcal{I}$. If m is continuously differentiable and strictly positive on $\mathcal{I}$, then on $\mathcal{I}$ the function F_o, defined in (4.7), is twice continuously differentiable and satisfies*

$$\ddot{F}_o(z) = \frac{\dot{m}(z)}{m(z)} \dot{F}_o(z) - C_o C_e^{\frac{1}{\Delta}} \Delta \big(m(z)\big)^{\frac{1}{\Delta}} m(1-z) \big(\dot{F}_o(z)\big)^{\frac{\Delta-1}{\Delta}} \big(F_o(z)\big)^{\Delta}, \tag{4.16}$$

PROOF. Relations (4.14) and (4.15) of Corollary 4.4 imply that F_o and F_e are absolutely continuous with density $C_e F_e^{\Delta}(1-\cdot)m(\cdot)$ and $C_o F_o^{\Delta}(1-\cdot)m(\cdot)$, respectively. Now, if m is continuous on $\mathcal{I}$, then clearly these densities are continuous, and so F_o and F_e are continuously differentiable on $\mathcal{I}$. If $\mathcal{I}$ is symmetric, then $F_o(1-\cdot)$ and $F_e(1-\cdot)$ are also continuously differentiable and so, if m is continuously differentiable on $\mathcal{I}$, then F_o and F_e are twice continuously differentiable on $\mathcal{I}$ and for $z \in \mathcal{I}$,

$$\ddot{F}_o(z) = C_e \dot{m}(z) F_e^{\Delta}(1-z) - C_e m(z) \Delta F_e^{\Delta-1}(1-z) \dot{F}_e(1-z).$$

Applying (4.14) and (4.15) again, we also have

$$F_e(1-z) = \big(\dot{F}_o(z)\big)^{\frac{1}{\Delta}} \big(C_e m(z)\big)^{\frac{-1}{\Delta}},$$
$$\dot{F}_e(1-z) = C_o m(1-z) F_o^{\Delta}(z).$$

Substituting these identities into the previous expression for $\ddot{F}_o$, we obtain the following second-order ODE for F_o on $\mathcal{I}$:

$$\begin{aligned}\ddot{F}_o(z) &= \frac{\dot{m}(z)}{m(z)} \dot{F}_o(z) - C_e m(z) \Delta \big(C_e m(z)\big)^{-\frac{\Delta-1}{\Delta}} \big(\dot{F}_o(z)\big)^{\frac{\Delta-1}{\Delta}} C_o m(1-z) \big(F_o(z)\big)^{\Delta} \\ &= \frac{\dot{m}(z)}{m(z)} \dot{F}_o(z) - C_o C_e^{\frac{1}{\Delta}} \Delta \big(m(z)\big)^{\frac{1}{\Delta}} m(1-z) \big(\dot{F}_o(z)\big)^{\frac{\Delta-1}{\Delta}} \big(F_o(z)\big)^{\Delta},\end{aligned}$$

for $z \in \mathcal{I}$. $\square$

We now fix $\lambda > 0$ and $\varepsilon \in (0, 1/2]$ and specialize to the case when the density m of the free spin measure has the form

$$m(z) = \lambda^z \mathbb{I}_{(0,\varepsilon]\cup[1-\varepsilon,1)}(z), \quad z \in [0, 1]. \tag{4.17}$$

Note that the case $\varepsilon = 1/2$ corresponds to the continuous hardcore model. Define

$$\theta_o := \big(\lambda C_o^{\frac{1}{\Delta}} C_e\big)^{\frac{\Delta}{\Delta^2-1}} \quad \text{and} \quad \theta_e := \big(\lambda C_e^{\frac{1}{\Delta}} C_o\big)^{\frac{\Delta}{\Delta^2-1}}. \tag{4.18}$$

We then have the following result:

PROPOSITION 4.7. *Suppose the free spin measure has a density m of the form* (4.17) *for some $\varepsilon \in (0, 1/2]$ and $\lambda > 0$. Then, F_o is twice continuously differentiable on the intervals $(0, \varepsilon)$ and $(1 - \varepsilon, 1)$ and the function*

$$\begin{aligned} R_\lambda(z) \doteq{}& \lambda^{-z}\big(\theta_e F_o(z)\big)^{\Delta+1} + \lambda^{-\frac{z}{\Delta}}\big(\theta_e \dot{F}_o(z)\big)^{\frac{\Delta+1}{\Delta}} \\ &- (\ln\lambda)\lambda^{-\frac{z}{\Delta+1}}\theta_e^{\frac{\Delta+1}{\Delta}} F_o(z)\big(\dot{F}_o(z)\big)^{\frac{1}{\Delta}}, \end{aligned}$$

is constant on each of the intervals $(0, \varepsilon)$ and $(1 - \varepsilon, 1)$. Moreover, F_o satisfies

$$\dot{F}_o(0+) = C_e, \qquad \dot{F}_o(1-) = 0, \tag{4.19}$$

and

$$\inf\big\{t > 0 : F_o(t) = 1\big\} = 1, \tag{4.20}$$

and R_λ satisfies the boundary conditions

$$R_\lambda(0+) = (\theta_e C_e)^{\frac{\Delta+1}{\Delta}}, \qquad R_\lambda(1-) = \lambda^{-1}\theta_e^{\Delta+1}. \tag{4.21}$$

PROOF. Since the the density m in (4.17) is continuously differentiable on the intervals $(0, \varepsilon)$ and $(1-\varepsilon, 1)$, and the corresponding free spin measure puts strictly positive mass on $[0, 1]$, it follows from Proposition 4.6 that F_o is twice continuously differentiable and satisfies (4.16) on each of those intervals. The proof of the first assertion of the proposition proceeds in three steps.

Step 1. We first recast the second-order ODE for F_o in (4.16) as a system of nonautonomous first-order ODEs. Consider $g(z) = (g_1(z), g_2(z)) := (F_o(z), \dot{F}_o(z))$ which lies in $\mathbb{R}_+^2$ since F_o is nonnegative and nondecreasing. Let $\mathcal{I} = (0, \varepsilon) \cup (1 - \varepsilon, 1)$ if $\varepsilon < 1/2$, and let $\mathcal{I} = (0, 1)$ if $\varepsilon = 1/2$. Since m is continuously differentiable and $m(z) = (\ln\lambda)\lambda^z$ on $\mathcal{I}$, F_o satisfies the second-order ODE in (4.16) which is equivalent to saying that g satisfies the following system of *nonautonomous* first-order ODEs on $\mathcal{I}$:

$$\dot{g}(z) = G\big(g(z), z\big) := \big(G_1\big(g_1(z), g_2(z), z\big), G_2\big(g_1(z), g_2(z), z\big)\big), \tag{4.22}$$

where for $i=1,2$, $G_i:\mathbb{R}_+^3\to\mathbb{R}$ are defined by

$$G_1(y_1,y_2,z):=y_2, \tag{4.23}$$

$$G_2(y_1,y_2,z):=(\ln\lambda)y_2-C_oC_e^{\frac{1}{\Delta}}\Delta\lambda^{\frac{z}{\Delta}}\lambda^{1-z}y_2^{\frac{\Delta-1}{\Delta}}y_1^{\Delta}. \tag{4.24}$$

Step 2. Next, we reparametrize the system of ODEs above to eliminate the explicit dependence of G_2 on z in (4.24). Namely, we reformulate the system of ODEs as an *autonomous* system. Consider the transformation $\Lambda:(g_1,g_2)\mapsto(h_1,h_2)$ defined by

$$h_1(z)=\lambda^{-\frac{z}{\Delta+1}}\theta_e g_1(z), \tag{4.25}$$

$$h_2(z)=\lambda^{-\frac{z}{\Delta(\Delta+1)}}\theta_e^{\frac{1}{\Delta}}\bigl(g_2(z)\bigr)^{\frac{1}{\Delta}}. \tag{4.26}$$

We now claim that $(h_1(\cdot),h_2(\cdot))$ satisfies the following system of ODEs:

$$\dot h(z)=H\bigl(h(z)\bigr),\qquad z\in(0,1), \tag{4.27}$$

where $H:\mathbb{R}_+^2\to\mathbb{R}^2$ is defined by $H(y_1,y_2):=(H_1(y_1,y_2),H_2(y_1,y_2))$, with

$$H_1(y_1,y_2):=-\frac{\ln\lambda}{\Delta+1}y_1+y_2^{\Delta}, \tag{4.28}$$

$$H_2(y_1,y_2):=\frac{\ln\lambda}{\Delta+1}y_2-y_1^{\Delta}. \tag{4.29}$$

The proof is obtained using a fairly straightforward verification. For $z\in(0,1)$, we have using (4.25)–(4.26), and $\dot g_1(z)=g_2(z)$ from (4.22)–(4.23),

$$\begin{aligned}\dot h_1(z)&=-\frac{\ln\lambda}{\Delta+1}\lambda^{-\frac{z}{\Delta+1}}\theta_e g_1(z)+\lambda^{-\frac{z}{\Delta+1}}\theta_e g_2(z)\\&=-\frac{\ln\lambda}{\Delta+1}h_1(z)+\bigl(h_2(z)\bigr)^{\Delta}.\end{aligned}$$

This verifies (4.28). Similarly, applying (4.22) and (4.24) together with (4.25)–(4.27) and (4.29), we obtain

$$\begin{aligned}\dot h_2(z)={}&-\frac{\ln\lambda}{\Delta(\Delta+1)}\lambda^{-\frac{z}{\Delta(\Delta+1)}}\theta_e^{\frac{1}{\Delta}}\bigl(g_2(z)\bigr)^{\frac{1}{\Delta}}\\&+\lambda^{-\frac{z}{\Delta(\Delta+1)}}\theta_e^{\frac{1}{\Delta}}\Delta^{-1}\bigl(g_2(z)\bigr)^{\frac{(1-\Delta)}{\Delta}}(\ln\lambda)g_2(z)\\&-\lambda^{-\frac{z}{\Delta(\Delta+1)}}\theta_e^{\frac{1}{\Delta}}\Delta^{-1}\bigl(g_2(z)\bigr)^{\frac{(1-\Delta)}{\Delta}}C_oC_e^{\frac{1}{\Delta}}\Delta\lambda^{\frac{z}{\Delta}}\lambda^{1-z}\bigl(g_2(z)\bigr)^{\frac{(\Delta-1)}{\Delta}}\bigl(g_1(z)\bigr)^{\Delta}\\={}&-\frac{\ln\lambda}{\Delta(\Delta+1)}h_2(z)+\frac{\ln\lambda}{\Delta}h_2(z)-\theta_e^{\frac{1}{\Delta}}C_oC_e^{\frac{1}{\Delta}}\lambda\theta_e^{-\Delta}\bigl(h_1(z)\bigr)^{\Delta}\\={}&\frac{\ln\lambda}{\Delta+1}h_2(z)-\bigl(h_1(z)\bigr)^{\Delta},\end{aligned}$$

where the last equality uses definition (4.18) of θ_e. This verifies (4.29).

Step 3. Next, we show that the system (4.27)–(4.29) is a Hamiltonian system of ODEs in the sense that if $h(\cdot) = (h_1(\cdot), h_2(\cdot))$ is a solution of (4.27)–(4.29) on some interval, then the function

$$\Phi \circ h(z) = \big(h_1(z)\big)^{\Delta+1} + \big(h_2(z)\big)^{\Delta+1} - (\ln\lambda)h_1(z)h_2(z)$$

is constant on that interval, where $\Phi : (y_1, y_2) \mapsto \mathbb{R}$ is defined by

$$\Phi(y_1, y_2) := y_1^{\Delta+1} + y_2^{\Delta+1} - (\ln\lambda)y_1 y_2.$$

Indeed, note that on substituting the expressions for $\dot{h}_1$ and $\dot{h}_2$ obtained above, we have on this interval,

$$\begin{aligned}\dot{\Phi} \circ h &= (\Delta+1)h_1^{\Delta}\dot{h}_1 + (\Delta+1)h_2^{\Delta}\dot{h}_2 - (\ln\lambda)(\dot{h}_1 h_2 + h_1\dot{h}_2) \\ &= (\Delta+1)h_1^{\Delta}\Big(-\frac{\ln\lambda}{\Delta+1}h_1 + h_2^{\Delta}\Big) + (\Delta+1)h_2^{\Delta}\Big(\frac{\ln\lambda}{\Delta+1}h_2 - h_1^{\Delta}\Big) \\ &\quad - (\ln\lambda)\Big(-\frac{\ln\lambda}{\Delta+1}h_1 + h_2^{\Delta}\Big)h_2 - (\ln\lambda)\Big(\frac{\ln\lambda}{\Delta+1}h_2 - h_1^{\Delta}\Big)h_1 \\ &= 0.\end{aligned}$$

The first assertion of the proposition then follows on substituting the definition of h_i and g_i, $i = 1, 2$, from Steps 1 and 2 into the expression for $\Phi \circ h$ in Step 3.

Next, note that the boundary conditions in (4.19) follows on substituting the form (4.17) of μ into (4.14). When combined with the boundary condition $F_o(0+) = F_o(0) = 0$ and $F_o(1-) = F_o(1) = 1$ from Corollary 4.4, this implies (4.21). Finally, define $\tau = \inf\{t > 0 : F_o(t) = 1\}$. Then, $F_o(1) = 1$ implies that $\tau \leq 1$, but one must have $\dot{F}_o(z) = 0$ for $z > \tau$. Thus, to prove (4.20), it suffices to show that $\dot{F}_o(z) > 1$ for all $z < 1$. Now, by (4.14) for $z \in (0, 1)$, $\dot{F}_o(z) = C_e F_e^{\Delta}(1-z)\lambda^z$. However, this is strictly positive because by symmetry and (4.19) it follows that $\dot{F}_e(0) = C_o > 0$, and hence, $F_e(1-z) > 0$ for all $0 < z < 1$. This establishes (4.20) and concludes the proof. □

4.5. *Proof of uniqueness of Gibbs measures.* By Remark 4.5, to prove Theorems 3.1 and 3.4, it suffices to show that the constants C_o and C_e in (4.9) and (4.10), respectively, are equal when m is given by (4.17) with $\varepsilon = 1/2$ and $\varepsilon \in (0, 1/2)$, respectively. In each case we will use the invariance property in Proposition 4.7 to establish this equality.

PROOF OF THEOREM 3.1. Set $\varepsilon = 1/2$. Then, m is continuously differentiable on $(0, 1)$, and the function R in Proposition 4.7 is constant on the entire interval $(0, 1)$. Thus, setting $R(0+) = R(1-)$ in (4.21), we conclude that $\theta_e^{\frac{\Delta^2-1}{\Delta}} = \lambda C_e^{\frac{\Delta+1}{\Delta}}$. Substituting the value of θ_e from (4.18) into this equation, one concludes that $C_o = C_e$ which completes the proof. □

PROOF OF THEOREM 3.4. Now suppose $\varepsilon \in (0, 1/2)$. Then, m is continuously differentiable on the intervals $(0, \varepsilon)$ and $(1-\varepsilon, 1)$, and so Proposition 4.7 implies

$$R(0+) = R(\varepsilon-), \quad \text{and} \quad R\big((1-\varepsilon)+\big) = R(1-). \tag{4.30}$$

On the other hand, since m is zero on $(\varepsilon, 1-\varepsilon)$, it follows from (4.14)–(4.15) that both F_o and F_e are constant on $(\varepsilon, 1-\varepsilon)$. In turn this implies that

$$\dot{F}_o(\varepsilon-) = \frac{C_e}{2\varepsilon} F_e^{\Delta}(1-\varepsilon)\lambda^{\varepsilon} = \frac{C_e}{2\varepsilon} F_e^{\Delta}(\varepsilon)\lambda^{\varepsilon} = \dot{F}_o\big((1-\varepsilon)+\big)\lambda^{2\varepsilon-1}.$$

Now, if $\lambda = 1$, then these identities and the definition of R imply that $R(\varepsilon+) = R((1-\varepsilon)-)$. Together with (4.30) and (4.21) this implies

$$(\theta_e C_e)^{\frac{\Delta+1}{\Delta}} = \theta_e^{\Delta+1} \quad \Leftrightarrow \quad (\theta_e)^{\frac{\Delta^2-1}{\Delta}} = \lambda C_e^{\frac{\Delta+1}{\Delta}}.$$

When combined with (4.18), this shows that $C_e = C_o$. □

5. Marginal distributions of the continuous hardcore model.

5.1. *The case* $\lambda = 1$: *Proof of Theorem* 3.3. Note that the problem concerns the one-parameter family of ODEs

$$\dot{F}_C(z) = b\big(C, F_C(z)\big), \tag{5.1}$$

where the parameterized family of drifts b, $[0, \infty) \times [0, 1] \mapsto \mathbb{R}_+$ is given by

$$b(C, y) := C\big(1 - y^{\Delta+1}\big)^{\frac{\Delta}{\Delta+1}}. \tag{5.2}$$

For any fixed $C > 0$, the function $y \mapsto b(C, y)$ is a Lipschitz continuous function on $(0, 1-\delta)$ for any $\delta \in (0, 1)$. Thus, there exists a unique solution F_C to the ODE (5.1) with boundary condition

$$F_C(0) = 0, \tag{5.3}$$

on the interval $[0, \tau_C - \delta)$, where

$$\tau_C := \inf\big\{t > 0 : F_C(t) = 1\big\}. \tag{5.4}$$

Here, the infimum over an empty set is taken to be infinity. Since (3.3) implies that F_C is constant after τ_C (if $\tau_C < \infty$), by continuity there is a unique continuous solution F_C to (5.1) and (5.3) on $[0, \infty)$.

We now show existence of a $C > 0$ for which the unique solution F_C to (5.1) and (5.3) also satisfies the boundary condition

$$\tau_C = 1. \tag{5.5}$$

We fix $\lambda = 1$ and $\Delta \geq 1$ and consider the continuous hardcore model with parameter λ and Δ. From the proof of Theorem 3.1, it follows that the constants

$C_o, C_e \in (0, \infty)$ defined in (4.9) and (4.10) respectively, are equal. We denote the common value by $C_{\Delta,1}$ and let $\Theta_{\Delta,1}$ denote the corresponding common value of $\theta_e = \theta_o$ in (4.18). Further, let $F_{\Delta,1}$ denote the corresponding F_e which coincides with F_o by Remark 4.5. By Proposition 4.7 we have

$$\begin{aligned}(\Theta_{\Delta,1}C_{\Delta,1})^{\frac{\Delta+1}{\Delta}} &= R_1(0)\\ &= R_1(z)\\ &= (\Theta_{\Delta,1}(\Theta_{\Delta,1}F_{\Delta,1}(z))^{\Delta+1} + (\Theta_{\Delta,1}\dot{F}_{\Delta,1}(z))^{\frac{\Delta+1}{\Delta}},\end{aligned}$$

for every $z \in (0, 1)$. Noting from (4.18) that $\Theta_{\Delta,1} = C_{\Delta,1}^{\frac{1}{\Delta-1}}$ and rearranging terms above, this implies that $F_{\Delta,1}$ satisfies the ODE (3.3) when $C = C_{\Delta,1}$. Furthermore, $F_{\Delta,1}(0) = 0$ by Corollary 4.4 and hence, $F_{\Delta,1}$ is the unique solution $F_{C_{\Delta,1}}$ to (5.1) and (5.3). Furthermore, it follows from (4.20) that $\tau_{C_{\Delta,1}} = 1$, and thus we have shown that (5.1), (5.3) and (5.5) are satisfied when $C = C_{\Delta,1}$.

To prove Theorem 3.3, it only remains to prove that there is a unique constant C (equal to $C_{\Delta,1}$) for which the unique solution F_C to (5.1) and (5.3) also satisfies (5.5). Our next result shows that this is the case.

PROPOSITION 5.1. *Given $C > 0$, let F_C be the unique solution to* (5.1) *and* (5.3) *on* $[0,\infty)$, *and define τ_C as in* (5.4). *The function* $[0,\infty) \ni C \mapsto \tau_C$ *is strictly decreasing and continuous with range* $\mathbb{R}_+$. *In particular there exists a unique* $C^* > 0$ *such that* $\tau_{C^*} = 1$.

PROOF. The proof entails two main steps.

Step 1. We show $\lim_{C\downarrow 0}\tau_C = \infty$ and $\lim_{C\uparrow\infty}\tau_C = 0$.

First, observe that $\tau_C \geq 1/C$ since $F(0) = 0$ and $\dot{F}(z) \leq C$ for all z. Thus, $\tau_C \to \infty$ as $C \to 0$. Next, set $\sigma_C(0) := 0$ and for $\delta > 0$, define

$$\sigma_C(\delta) := \inf\{z > 0 : F_C(z) = 1 - \delta\}. \tag{5.6}$$

Observe that the set of such z is nonempty since for every z such that $F_C(z) < 1-\delta$ we have the uniform lower bound $\dot{F}_C(z) > C(1-(1-\delta)^{\Delta+1})^{\frac{\Delta}{\Delta+1}} > 0$. Now, for $z \in [\sigma_C(1/(n-1)), \sigma_C(1/n)]$, $\frac{n-2}{n-1} \leq F_C(z) \leq \frac{n-1}{n}$, and hence,

$$\dot{F}_C(z) = C(1 - F_C^{\Delta+1}(z))^{\Delta/(\Delta+1)} = \frac{C(\Delta+1)^{\frac{\Delta}{\Delta+1}}}{n^{\frac{\Delta}{\Delta+1}}} + o\left(\frac{1}{n^{\frac{\Delta}{\Delta+1}}}\right),$$

where $o(\varepsilon)$ represents a quantity that vanishes as $\varepsilon \to 0$. Using the identity

$$\frac{1}{n-1} - \frac{1}{n} = F_C(\sigma_C(1/(n-1)) - F_C(\sigma_C(1/n)) = \int_{\sigma_C(1/(n-1))}^{\sigma_C(1/n)} \dot{F}_C(z)\,dz,$$

we obtain the estimate

$$\sigma_C\left(\frac{1}{n}\right) - \sigma_C\left(\frac{1}{n-1}\right) = \left(C(\Delta+1)^{\frac{\Delta}{\Delta+1}} n^{\frac{\Delta+2}{\Delta+1}}\right)^{-1} + o\left(\frac{1}{n^{\frac{\Delta+2}{\Delta+1}}}\right). \tag{5.7}$$

In turn, since $\tau_C = \sum_{n=1}^{\infty}[\sigma_C(\frac{1}{n}) - \sigma_C(\frac{1}{n-1})]$, this implies that

$$C\tau_C = (\Delta+1)^{-\frac{\Delta}{\Delta+1}} \sum_{n=1}^{\infty} n^{-\frac{\Delta+2}{\Delta+1}} + o(1) < \infty$$

which shows that $\tau_C \to 0$ as $C \to \infty$. This concludes the proof of Step 1.

Before proceeding to Step 2, observe that in a similar fashion, for $\delta \in [1/n, 1/(n-1)]$, we have

$$\sum_{m \geq n}\left[\sigma_C\left(\frac{1}{m}\right) - \sigma_C\left(\frac{1}{m-1}\right)\right] \leq \tau_C - \sigma_C(\delta)$$
$$\leq \sum_{m \geq n-1}\left[\sigma_C\left(\frac{1}{m}\right) - \sigma_C\left(\frac{1}{m-1}\right)\right].$$

Combining this with the estimate (5.7) and the fact that for both $k = n$ and $k = n+1$ the sum $\sum_{m \geq k} m^{-\frac{\Delta+2}{\Delta+1}}$ is of the order $O(\frac{\Delta+1}{\Delta+2} n^{-\frac{1}{\Delta+1}})$, we conclude that

$$\tau_C - \sigma_C(\delta) = \frac{(\Delta+1)^{\frac{1}{\Delta+1}}}{C(\Delta+2)} \delta^{\frac{1}{\Delta+1}} + o\big(\delta^{\frac{1}{\Delta+1}}\big). \tag{5.8}$$

Step 2. We show that $C \mapsto \tau_C$ is strictly decreasing and continuous.

First, note that for b given in (5.2) we have

$$\frac{\partial b}{\partial C}(C, y) = \big(1 - y^{\Delta+1}\big)^{\frac{\Delta}{\Delta+1}}, \qquad \frac{\partial b}{\partial y} = -C\Delta\big(1 - y^{\Delta+1}\big)^{-\frac{1}{\Delta+1}} y^{\Delta}. \tag{5.9}$$

Thus, b is continuously differentiable with bounded partial derivatives on $[0, \infty) \times [0, 1 - \frac{1}{n}]$ for every $n \in \mathbb{N}$. Then, by standard sensitivity analysis for parameterized ODEs we know that for every $n \in \mathbb{N}$, on $[0, \sigma_C(\frac{1}{n})]$, $R_C(z) := \partial F_C(x)/\partial C$ exists and satisfies

$$\frac{\partial R_C}{\partial z}(z) = \frac{\partial^2 F_C}{\partial C \partial z}(z) = \frac{\partial}{\partial C} b\big(C, F_C(z)\big)$$
$$= \frac{\partial b}{\partial C}\big(C, F_C(z)\big) + \frac{\partial b}{\partial y}\big(C, F_C(z)\big)\frac{\partial F_C}{\partial C}(z)$$

which yields the following first-order inhomogeneous linear ODE for R_C:

$$\frac{\partial R_C}{\partial z}(z) = \frac{\partial b}{\partial C}\big(C, F_C(z)\big) + \frac{\partial b}{\partial y}\big(C, F_C(z)\big)R_C(z). \tag{5.10}$$

Moreover, since $F_C(0) = 0$ for all C, R_C satisfies the boundary condition

$$R_C(0) = 0. \tag{5.11}$$

Solving the linear ODE (5.10)–(5.11), we obtain

$$R_C(z) = \int_0^z e^{\int_x^z \frac{\partial b}{\partial y}(C, F_C(t))\,dt} \frac{\partial b}{\partial c}\big(C, F_C(x)\big)\,dx.$$

Substituting the partial derivatives of b from (5.9), we have for $z \in (0, \tau_C)$,

$$R_C(z) = \int_0^z e^{-C\Delta \int_x^z (1-(F_C(t))^{\Delta+1})^{-\frac{1}{\Delta+1}} (F_C(t))^{\Delta}\, dt} \times \bigl(1 - \bigl(F_C(x)\bigr)^{\Delta+1}\bigr)^{\frac{\Delta}{\Delta+1}}\, dx > 0. \tag{5.12}$$

Now, fix $n \in \mathbb{N}$, and recall from (5.6) that $F_C(\sigma_C(1/n)) = 1 - 1/n$. Since $(C, z) \mapsto F_C(z)$ is continuously differentiable and, by (5.1)–(5.2) for any fixed $C_0 > 0$, $\frac{\partial F_{C_0}}{\partial z}(\sigma_{C_0}(1/n)) > 0$, it follows from the implicit function theorem that $C \mapsto \sigma_C(1/n)$ is continuously differentiable and (5.12) then implies that

$$\frac{d(\sigma_C(1/n))}{dC} < 0. \tag{5.13}$$

Now, for any $C < \infty$, fix $C_- < C < C_+$. Then, for all sufficiently large n, it follows from (5.8) that

$$\tau_{C_+} - \sigma_{C_+}(1/n) \le \tau_C - \sigma_C(1/n) \le \tau_{C_-} - \sigma_{C_-}(1/n).$$

Since (5.13) implies that $\sigma_{C+}(1/n) < \sigma_C(1/n) < \sigma_{C-}(1/n)$, this shows that $\tau_{C-} > \tau_C > \tau_{C+}$, namely, $C \mapsto \tau_C$ is strictly decreasing on $(0, \infty)$. Finally, to show that τ is continuous, fix $C > 0$, and given $\varepsilon > 0$, note that (5.8) shows that there exists a sufficiently large n, such that for all $\eta < C$ and $\tilde{C} \in [C - \eta, C + \eta]$,

$$\bigl|\tau_{\tilde{C}} - \sigma_{\tilde{C}}(1/n) - \bigl(\tau_C - \sigma_C(1/n)\bigr)\bigr| \le \frac{\varepsilon}{2}.$$

Since, as shown above, $C \mapsto \sigma_C(1/n)$ is continuous (in fact, continuously differentiable), there exists $\delta < 1$ such that whenever $|\tilde{C} - C| < \delta$, $|\sigma_{\tilde{C}}(1/n) - \sigma_C(1/n)| < \frac{\varepsilon}{2}$, and hence, $|\tau_{\tilde{C}} - \tau_C| < \varepsilon$. This shows that $C \mapsto \tau_C$ is continuous and concludes the proof of Step 2.

Finally, we note that by Step 1 and the continuity of τ_C established in Step 2, $\{\tau_C, C \in (0, \infty)\} = (0, \infty)$. Since $C \mapsto \tau_C$ is a strictly decreasing continuous function by Step 2, this implies the existence of a unique C^* with $\tau_{C^*} = 1$. This completes the proof of Proposition 5.2. □

5.2. *A conjecture for general* $\lambda > 0$. Fix $\lambda > 0$, $\Delta \ge 1$, and let $F = F_{\Delta,\lambda} = F_o$ and $C = C_o = C_e$ be the limiting function and constant, respectively, from Theorem 3.1. Then, the free spin measure with density $m(z) = \lambda^z$ satisfies the conditions of Proposition 4.6 with $\mathcal{I} = (0, 1)$ and so it follows from (4.16) that F satisfies the second-order ODE

$$\ddot{F}(z) = (\ln \lambda)\dot{F}(z) - C^{\frac{1}{\Delta}+1}\Delta\lambda^{(1-z)}\lambda^{z/\Delta}\bigl(\dot{F}(z)\bigr)^{1-\frac{1}{\Delta}} F^{\Delta}(z), \tag{5.14}$$

for $z \in (0, 1)$. Moreover, Corollary 4.4, (4.19) and (4.20) show that F also satisfies the boundary conditions

$$\begin{cases} F(0) = 0, \\ \inf\{t > 0 : F(t) = 1\} = 1, \\ \dot{F}(0+) = C, \\ \dot{F}(1) = 0. \end{cases} \tag{5.15}$$

We conjecture the following generalization of Theorem 3.3 holds but defer investigation of its validity to future work.

CONJECTURE 5.2. *There exists a unique $C_{\Delta,\lambda} > 0$ for which the ODE* (5.14)–(5.15) *admits a solution, and $F_{\Delta,\lambda}$ is a twice continuously differentiable function that is the unique solution to* (5.14)–(5.15) *with $C = C_{\Delta,\lambda}$.*

6. Graphs with large girth. We now switch our focus to the problem of computing the volume of the LP polytope $\mathcal{P}(\mathbb{G})$ of a Δ-regular graph $\mathbb{G}$ with large girth and specifically prove Theorem 3.5 in Section 6.2. The proof approach we use follows closely the technique used in [1] for the problem of counting the asymptotic number of independent sets in regular graphs with large girth. First, in Section 6.1 we discuss a certain *rewiring* technique that allows one to construct $(N-2)$-node regular graph with large girth from an N-node regular graph with large girth by deleting and adding only a constant number of (specific) nodes and edges.

6.1. *Rewiring.* Here, we summarize relevant results from [1], Section 4.3. Given an N-node Δ-regular graph $\mathbb{G}$, fix any two nodes u_1, u_2 such that the graph theoretic distance between u_1 and u_2 is at least four. The latter ensures that there are no edges between the nonoverlapping neighbor sets of u_1 and u_2, which we denote by $u_{1,1}, \dots, u_{1,\Delta}$ and $u_{2,1}, \dots, u_{2,\Delta}$, respectively. Consider a modified graph $\mathbb{H}$ obtained from $\mathbb{G}$ by deleting the nodes u_1 and u_2 and adding an edge between $u_{1,i}$ and $u_{2,i}$ for every $i = 1, \dots, \Delta$; see Figure 1. The resulting graph $\mathbb{H}$ is a Δ-regular graph with $N-2$ nodes. We call this operation a "rewiring" or "rewire" operation. In our application the rewiring step will be applied only to pairs of nodes with distance at least four. Rewiring was used in [14] and [16] in the context of random regular graphs, and it was performed on two nodes selected randomly from the graph. Here, as in [1], we will instead rewire on nodes u_1 and u_2 that are farthest from each other. As shown in the next result, this will enable us to preserve the large girth property of the graph for many rewiring steps.

Recall that $g(\mathbb{G})$ denotes the girth of the graph $\mathbb{G}$. We now state Lemma 2 of [1]. For completeness, we include the proof of this lemma in Appendix B.

LEMMA 6.1. *Given an arbitrary N-node Δ-regular graph $\mathbb{G}$, consider any integer $4 \le g \le g(\mathbb{G})$. If $2(2g+1)\Delta^{2g} < N$, then the rewiring operation can be*

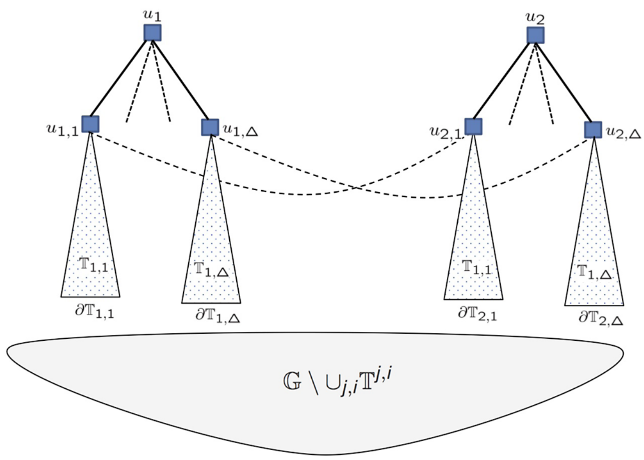

FIG. 1. *Rewiring on nodes u_1 and u_2.*

performed for at least $(N/2) - (2g+1)\Delta^{2g}$ steps on pairs of nodes that are a distance at least $2g+1$ apart. After every rewiring step the resulting graph is Δ-regular with girth at least g.

REMARK 6.2. If the same fixed $g \in \{4, \dots, g(\mathbb{G})\}$ is used at each step, since every rewiring step reduces the graph by *two* nodes, we see that after $(N/2) - (2g+1)\Delta^{2g} = N/2 - O(1)$ rewiring steps, the resulting graph is of constant $O(1)$ size which will have a negligible contribution to the asymptotic formula for the volume of $\mathcal{P}(\mathbb{G})$.

6.2. *Proof of Theorem 3.5.* Fix $\lambda > 0$, $\Delta \geq 1$ and let $F_{\Delta,\lambda}$ be the distribution function from Theorem 3.1. We fix an arbitrary sequence $\mathbb{G}_n$, $n \in \mathbb{N}$, of Δ-regular graphs with diverging girth: $\lim_{n\to\infty} g(\mathbb{G}_n) = \infty$. In what follows we adopt the shorthand notation $x \lesseqgtr (1 \pm \epsilon)y$ to mean $(1-\epsilon)y \leq x \leq (1+\epsilon)y$. The main technical result underlying our proof of Theorem 3.5 is as follows.

THEOREM 6.3. *For every $\Delta \geq 2$, $\epsilon > 0$ and $\lambda > 0$, there exists a large enough integer $g = g(\epsilon, \Delta, \lambda)$ such that if the rewiring is performed on any Δ-regular graph $\mathbb{G}$ with girth $g(\mathbb{G}) \geq g$ on two nodes that are at least $2g+1$ distance apart, then for the resulting graph $\mathbb{H}$, we have*

$$\begin{aligned} \frac{Z_{\mathbb{G},\lambda}}{Z_{\mathbb{H},\lambda}} &\lesseqgtr (1 \pm \epsilon)\left(\int_0^1 \lambda^t F^{\Delta}_{\Delta-1,\lambda}(1-t)\,dt\right)^{-2} \\ &\quad \times \left(\int_0^1 \dot{F}_{\Delta-1,\lambda}(t) F_{\Delta-1,\lambda}(1-t)\,dt\right)^{-\Delta}. \end{aligned} \tag{6.1}$$

We first show how this result implies Theorem 3.5.

PROOF OF THEOREM 3.5. We fix $\epsilon > 0$ and $g = g(\epsilon, \Delta, \lambda) \geq 4$ as described in Theorem 6.3. Since $g(\mathbb{G}_n) \to \infty$, we have $g(\mathbb{G}_n) \geq g$ for all sufficiently large n. For $t = 1, \dots, L_n := |V(\mathbb{G}_n)|/2 - (2g+1)\Delta^{2g}$, let $\mathbb{G}_{n,t}$ be the graph obtained from $\mathbb{G}_{n,0} := \mathbb{G}_n$ after t rewiring steps. Then, for any $\lambda > 0$, trivially we have

$$Z_{\mathbb{G}_n,\lambda} = \left(\prod_{1 \leq t \leq L_n} \frac{Z_{\mathbb{G}_{n,t-1},\lambda}}{Z_{\mathbb{G}_{n,t},\lambda}}\right) Z_{\mathbb{G}_{n,L_n},\lambda}.$$

For conciseness we introduce the notation

$$\Gamma(\Delta, \lambda) := \left(\int_0^1 \lambda^t F_{\Delta-1,\lambda}^{\Delta}(1-t)\,dt\right)^{-2} \left(\int_0^1 \dot{F}_{\Delta-1,\lambda}(t) F_{\Delta-1,\lambda}(1-t)\,dt\right)^{-\Delta}$$

and note that by Lemma 6.1 and Theorem 6.3, for $1 \leq t \leq L_n$,

$$\frac{Z_{\mathbb{G}_{n,t-1},\lambda}}{Z_{\mathbb{G}_{n,t},\lambda}} \lesseqgtr (1 \pm \epsilon)\Gamma(\Delta, \lambda).$$

Therefore, we obtain

$$Z_{\mathbb{G}_n,\lambda} \lesseqgtr (1 \pm \epsilon)^{L_n} \Gamma^{L_n}(\Delta, \lambda) Z_{\mathbb{G}_{n,L_n},\lambda}.$$

Now, recall from Remark 6.2 that the number of nodes, and hence edges, of $\mathbb{G}_{n,L_n}$ is bounded by a constant that does not depend on n. In turn this implies that $Z_{\mathbb{G}_{n,L_n},\lambda}$ is also bounded by a constant that does not depend on n. Therefore, taking the natural logarithm of both sides of the last display, dividing by $|V(\mathbb{G}_n)|$, recalling that $L_n = |V(\mathbb{G}_n)|/2 - O(1)$ and taking limits, first as $n \to \infty$ and then as $\epsilon \to 0$, we obtain (3.6). $\square$

The remainder of this section is devoted to proving Theorem 6.3. Fix an integer g, consider an arbitrary Δ-regular graph $\mathbb{G}$ with girth $g(\mathbb{G}) \geq 2g+1$ and fix any two nodes u_1 and u_2 in $\mathbb{G}$ that are at least a distance $2g+1$ apart. Fix $\lambda > 0$, and let $\mathbb{P} = \mathbb{P}_{\mu_\lambda,\mathbb{G}}$ be the continuous hardcore measure on $\mathcal{P}(\mathbb{G})$, and, for any induced subgraph $\tilde{\mathbb{G}}$, let $\mathbb{P}_{\tilde{\mathbb{G}}}$ represent the continuous hardcore measure on $\tilde{\mathbb{G}}$, and let $\mathbb{E}$ and $\mathbb{E}_{\tilde{\mathbb{G}}}$ represent the corresponding expectations. Also, as usual, let $\mathbf{X}$ be the random vector representing spin values at nodes. Moreover, let $\mathbb{H}$ be the graph obtained on rewiring on u_1 and u_2 and omitting the dependence on λ for conciseness, let $Z_{\mathbb{G}}$, $Z_{\mathbb{G}\setminus\{u_1,u_2\}}$ and $Z_{\mathbb{H}}$, respectively, be the partition functions associated with the continuous hardcore model (with parameter λ) on $\mathbb{G}$, $\mathbb{G} \setminus \{u_1, u_2\}$ and $\mathbb{H}$, respectively. Next, for $j = 1, 2$, we denote by $u_{j,1}, \dots, u_{j,\Delta}$ the neighbors of u_j in $\mathbb{G}$ and let $\mathbb{T}^j$ be the subtree of depth g rooted at u_j, and for $j = 1, 2$, let $\mathbb{T}^{j,i}$ denote the subtree of $\mathbb{G} \setminus \{u_1, u_2\}$ rooted at $u_{1,i}$. Note that (since u_j has been removed) $u_{j,i}$ has $\Delta - 1$ children, as do each of the other internal nodes of $\mathbb{T}^{j,i}$. Thus, for every i, j, $\mathbb{T}^{j,i}$ is isomorphic to $\mathbb{T}_{g-1,\Delta-1}$, denoted $\mathbb{T}^{j,i} \sim \mathbb{T}_{g-1,\Delta-1}$. Finally, recall the definition of $F_{n,\Delta} = F_{n,\Delta,\lambda}$ given in Section 4.2. The proof of Theorem 6.3 relies on two preliminary estimates stated in Lemmas 6.4 and 6.5 below. We first show that these estimates imply Theorem 6.3 and only then prove the estimates.

LEMMA 6.4.

$$\frac{Z_{\mathbb{G}\setminus\{u_1,u_2\}}}{Z_{\mathbb{G}}} = \mathbb{E}\left[\prod_{j=1}^{2}\left(\int_{t\in[0,1]} \lambda^t \prod_{i=1}^{\Delta} F_{g-1,\Delta-1}(1-t|\mathbf{X}_{\partial\mathbb{T}^{j,i}})\,dt\right)^{-1}\right]. \tag{6.2}$$

LEMMA 6.5.

$$\begin{aligned}\frac{Z_{\mathbb{H}}}{Z_{\mathbb{G}\setminus\{u_1,u_2\}}} \\ = \mathbb{E}\left[\prod_{i=1}^{\Delta}\int_{t\in[0,1]} dF_{g-1,\Delta-1}(t|\mathbf{X}_{\partial\mathbb{T}^{1,i}})F_{g-1,\Delta-1}(1-t|\mathbf{X}_{\partial\mathbb{T}^{2,i}})\right].\end{aligned} \tag{6.3}$$

PROOF OF THEOREM 6.3. We first use Theorem 3.1 to approximate the right-hand sides of (6.2) and (6.3) for large g. Let $F_{\Delta-1} = F_{\Delta-1,\lambda}$ be the limit function in Theorem 3.1. Then, given $\epsilon > 0$, by Theorem 3.1 and the bounded convergence theorem, for sufficiently large g and every boundary condition $\mathbf{x}_{\partial\mathbb{T}^{j,i}} \in \mathcal{P}(\partial\mathbb{T}^{j,i})$, $1 \le i \le \Delta$, $j = 1, 2$,

$$1 - \frac{\epsilon}{8} \le \frac{\int_{t\in[0,1]} \lambda^t F_{\Delta-1}^{\Delta}(1-t)\,dt}{\int_{t\in[0,1]} \lambda^t \prod_{i=1}^{\Delta} F_{g-1,\Delta-1}(1-t|\mathbf{x}_{\partial\mathbb{T}^{j,i}})\,dt} \le 1 + \frac{\epsilon}{8}. \tag{6.4}$$

Next, we observe that Theorem 3.1 implies that the probability measure $dF_{g-1,\Delta-1}(\cdot|\mathbf{x}_{\partial\mathbb{T}^{1,i}})$ converges to the probability measure $dF_{\Delta-1}(\cdot)$ in the Kolmogorov distance (and therefore the Lévy distance, which induces weak convergence on $\mathbb{R}$) uniformly with respect to all feasible boundary conditions (see [9], Chapter 2, for definitions of the Kolmogorov and Lévy distances and the relation between them). Since, again by Theorem 3.1, $F_{g-1,\Delta-1}(\cdot|\mathbf{x}_{\partial\mathbb{T}^{2,i}})$, $g \in \mathbb{N}$ is a sequence of bounded continuous functions that converges uniformly to $F_{\Delta-1}(\cdot)$ and also uniformly with respect to the boundary condition $\mathbf{x}_{\partial\mathbb{T}^{2,i}} \in \mathcal{P}(\partial\mathbb{T}^{2,i})$; this shows that there exists g large enough such that

$$\begin{aligned}1 - \frac{\epsilon}{8} &\le \frac{(\int_{t\in[0,1]} dF_{\Delta-1}(t)F_{\Delta-1}(1-t))^{\Delta}}{\prod_{i=1}^{\Delta}\int_{t\in[0,1]} dF_{g-1,\Delta-1}(t|\mathbf{x}_{\partial\mathbb{T}^{1,i}})F_{g-1,\Delta-1}(1-t|\mathbf{x}_{\partial\mathbb{T}^{2,i}})} \\ &\le 1 + \frac{\epsilon}{8},\end{aligned} \tag{6.5}$$

for all boundary conditions $\mathbf{x}_{\bigcup_{j=1}^{2}\partial T^j}$. Now, fix $\varepsilon > 0$ sufficiently small such that $(1-\varepsilon) \le (1-\varepsilon/8)^3$ and $(1+\varepsilon/8)^3 \le (1+\varepsilon)$, and $g = g(\varepsilon, \Delta, \lambda)$ sufficiently large such that (6.4) and (6.5) hold. Given the uniformity in these estimates with respect to boundary conditions, (6.4) and (6.5) also hold when $\mathbf{x}$ is replaced by $\mathbf{X}$. Now, taking the product of the middle term in (6.4) for $j = 1, 2$ and then taking expectations of the denominator in (6.5), combining this with (6.2)–(6.3), and using the fact that $F_{\Delta-1} = F_{\Delta-1,\lambda}$ is absolutely continuous to write $dF_{\Delta-1}(t) = \dot{F}_{\Delta-1}(t)\,dt$, we obtain (6.1). This completes the proof of Theorem 6.3. $\square$

We now turn to the proofs of the lemmas, starting with Lemma 6.4.

PROOF OF LEMMA 6.4. For notational conciseness set $\tilde{\mathbb{G}} := \mathbb{G} \setminus \{u_1, u_2\}$. Then, since u_1, u_2 are not neighbors, for $z_1, z_2 \geq 0$,

$$\begin{aligned}
&\mathbb{P}(X_{u_1} \leq z_1, X_{u_2} \leq z_2) \\
&\quad = Z_{\mathbb{G}}^{-1} \int_{x_{u_1} \in [0,z_1]} \lambda^{x_{u_1}} \int_{x_{u_2} \in [0,z_2]} \lambda^{x_{u_2}} \int_{\mathbf{x}_{\tilde{\mathbb{G}}} \in \mathcal{P}(\tilde{\mathbb{G}}): x_{u_{j,i}} + x_{u_j} \leq 1 \forall i,j} \prod_{u \in V(\tilde{\mathbb{G}})} \lambda^{x_u}\, d\mathbf{x},
\end{aligned}$$

where the range of i, j above is $i = 1, \ldots, \Delta$ and $j = 1, 2$. Then,

$$\begin{aligned}
\frac{\partial^2}{\partial z_1 \partial z_2} \mathbb{P}(X_{u_1} \leq z_1, X_{u_2} \leq z_2)\Big|_{z_1 \downarrow 0, z_2 \downarrow 0} &= Z_{\mathbb{G}}^{-1} \int_{\mathbf{x}_{\tilde{\mathbb{G}}} \in \mathcal{P}(\tilde{\mathbb{G}})} \prod_{u \in V(\tilde{\mathbb{G}})} \lambda^{x_u}\, d\mathbf{x}_{\tilde{\mathbb{G}}} \\
&= Z_{\mathbb{G}}^{-1} Z_{\tilde{\mathbb{G}}}.
\end{aligned} \tag{6.6}$$

Since the trees $\mathbb{T}^j$, $j = 1, 2$ are nonintersecting and each u_j lies in $\mathbb{T}^j$, using the spatial Markov property in the second equality below, we have

$$\begin{aligned}
\mathbb{P}(X_{u_1} \leq z_1, X_{u_2} \leq z_2) &= \mathbb{E}[\mathbb{P}(X_{u_1} \leq z_1, X_{u_2} \leq z_2 | \mathbf{X}_{\bigcup_{j=1}^2 \partial \mathbb{T}^j})] \\
&= \mathbb{E}\Bigg[\prod_{j=1}^2 \mathbb{P}(X_{u_j} \leq z_j | \mathbf{X}_{\partial \mathbb{T}^j})\Bigg].
\end{aligned}$$

Now, fix $j \in \{1, 2\}$. Then, $\mathbb{T}^j$ is not isomorphic to $\mathbb{T}_{g,\Delta-1}$, but each of the disjoint trees $\mathbb{T}^{j,i}$, $i = 1, \ldots, \Delta$, rooted at the corresponding neighbor $u_{j,i}$ of u_j are isomorphic to $\mathbb{T}_{g-1,\Delta-1}$. Thus, another application of the spatial Markov property shows that

$$\begin{aligned}
\mathbb{P}(X_{u_j} \leq z_j | \mathbf{X}_{\partial \mathbb{T}^j}) &= \int_{x_{u_j} \in [0,z_j]} \lambda^{x_{u_j}} \prod_{i=1}^{\Delta} \mathbb{P}(X_{u_{j,i}} \leq 1 - x_{u_j} | \partial \mathbf{X}_{\partial \mathbb{T}^{j,i}})\, dx_{u_j} \\
&\qquad \times \Bigg(\int_{x_{u_j} \in [0,1]} \lambda^{x_{u_j}} \prod_{i=1}^{\Delta} \mathbb{P}(X_{u_{j,i}} \leq 1 - x_{u_j} | \partial \mathbf{X}_{\partial \mathbb{T}^{j,i}})\, dx_{u_j}\Bigg)^{-1}.
\end{aligned}$$

Taking the derivative with respect to z_j, we get

$$\frac{d}{dz_j}\mathbb{P}(X_{u_j} \le z_j|\mathbf{X}_{\partial\mathbb{T}^j})\Big|_{z_j\downarrow 0} = \left(\int_{t\in[0,1]} \lambda^t \prod_{i=1}^{\Delta} \mathbb{P}(X_{u_{j,i}} \le 1-t|\partial\mathbf{X}_{\partial\mathbb{T}^{j,i}})\,dt\right)^{-1}$$
$$= \left(\int_{t\in[0,1]} \lambda^t \prod_{i=1}^{\Delta} F_{g-1,\Delta-1}(1-t|\mathbf{X}_{\partial\mathbb{T}^{j,i}})\,dt\right)^{-1},$$

where the last equality uses the fact that $\mathbb{T}^{j,i} \sim \mathbb{T}_{g-1,\Delta-1}$ and $u_{j,i}$ is its root. The last four displays, together with the dominated convergence theorem (to justify interchange of $\mathbb{E}$ and differentiation d/dz_j) and (6.6), yield (6.2). □

PROOF OF LEMMA 6.5. Recall that $\mathbb{H}$ is the graph obtained from $\mathbb{G}\setminus\{u_1,u_2\}$ by adding edges between $u_{1,i}$ and $u_{2,i}$ for every $i=1,\dots,\Delta$. Thus,

$$\mathcal{P}(\mathbb{H}) = \{\mathbf{x}\in\mathcal{P}(\mathbb{G}\setminus\{u_1,u_2\}) : x_{u_{1,i}} + x_{u_{2,i}} \le 1, 1\le i\le\Delta\},$$

and hence,

$$\frac{Z_{\mathbb{H}}}{Z_{\mathbb{G}\setminus\{u_1,u_2\}}} = \mathbb{P}_{\mathbb{G}\setminus\{u_1,u_2\}}(X_{u_{1,i}} + X_{u_{2,i}} \le 1, 1\le i\le\Delta). \tag{6.7}$$

The right-hand side of (6.7) above can be rewritten as

$$\begin{aligned}&\mathbb{P}_{\mathbb{G}\setminus\{u_1,u_2\}}(X_{u_{1,i}} + X_{u_{2,i}} \le 1, 1\le i\le\Delta)\\ &\quad= \mathbb{E}_{\mathbb{G}\setminus\{u_1,u_2\}}\big[\mathbb{P}_{\mathbb{G}\setminus\{u_1,u_2\}}(X_{u_{1,i}} + X_{u_{2,i}} \le 1, 1\le i\le\Delta|\mathbf{X}_{\bigcup_{i=1}^{\Delta}\bigcup_{j=1}^{2}\partial\mathbb{T}^{j,i}})\big].\end{aligned}$$

Since u_1 and u_2 are more than a distance $2g+1$ apart, the trees $\mathbb{T}^{j,i}$, $j=1,2$, $i=1,\dots,\Delta$ are nonintersecting and disconnected in $\mathbb{G}\setminus\{u_1,u_2\}$ (see Figure 1). Therefore, by the spatial Markov property,

$$\begin{aligned}&\mathbb{P}_{\mathbb{G}\setminus\{u_1,u_2\}}(X_{u_{1,i}} + X_{u_{2,i}} \le 1, 1\le i\le\Delta|\mathbf{X}_{\bigcup_{j=1}^{2}\bigcup_{i=1}^{\Delta}\partial\mathbb{T}^{j,i}})\\ &\quad= \prod_{i=1}^{\Delta}\mathbb{P}_{\bigcup_{j=1}^{2}\partial\mathbb{T}^{j,i}}(X_{u_{1,i}} + X_{u_{2,i}} \le 1|\mathbf{X}_{\bigcup_{j=1}^{2}\partial\mathbb{T}^{j,i}}) \qquad (6.8)\\ &\quad= \prod_{i=1}^{\Delta}\int_{x_{u_{1,i}}\in[0,1]}\mathbb{P}_{\mathbb{T}^{1,i}}(dx_{u_{1,i}}|\mathbf{X}_{\partial\mathbb{T}^{1,i}})\mathbb{P}_{\mathbb{T}^{2,i}}(X_{u_{2,i}} \le 1-x_{u_{1,i}}|\mathbf{X}_{\partial\mathbb{T}^{2,i}}).\end{aligned}$$

Now, each tree $\mathbb{T}^{j,i}$ is a Δ-regular rooted tree (recall that the nodes u_1 and u_2 have been removed) with each node (other than the leaves) having $(\Delta-1)$ children and is thus isomorphic to $\mathbb{T}_{g-1,\Delta-1}$. Therefore, recalling the definition of $F_{n,\Delta}(\cdot|\mathbf{x}_{\partial\mathbb{T}_{n,\Delta}}) = F_{n,\Delta,\lambda}(\cdot|\mathbf{x}_{\partial\mathbb{T}_{n,\Delta}})$ from Section 3, for every $\mathbf{x}_{\bigcup_{j=1}^{2}\partial\mathbb{T}^{j,i}} \in$

$\mathcal{P}(\bigcup_{j=1}^{2} \partial\mathbb{T}^{j,i})$, we have

$$\begin{aligned}
&\mathbb{P}_{\bigcup_{j=1}^{2} \partial\mathbb{T}^{j,i}}(X_{u_{1,i}} + X_{u_{2,i}} \le 1 | \mathbf{X}_{\bigcup_{j=1}^{2} \partial\mathbb{T}^{j,i}} = \mathbf{x}_{\bigcup_{j=1}^{2} \partial\mathbb{T}^{j,i}}) \\
&\quad = \int_{x_{u_{1,i}} \in [0,1]} dF_{g-1,\Delta-1}(x_{u_{1,i}} | \mathbf{x}_{\partial\mathbb{T}^{1,i}}) F_{g-1,\Delta-1}(1 - x_{u_{1,i}} | \mathbf{x}_{\partial\mathbb{T}^{2,i}}).
\end{aligned}$$

Combining the last three displays with (6.7) we obtain (6.3). $\square$

APPENDIX A: PROOF OF MONOTONICITY

PROOF OF LEMMA 4.1. To prove the lemma, it clearly suffices to establish the following:

CLAIM. *For every n and every two boundary conditions $\mathbf{x}_{\partial\mathbb{T}_n}, \mathbf{y}_{\partial\mathbb{T}_n} \in [0,1]^{\partial\mathbb{T}_n}$ such that $\mathbf{x}_{\partial\mathbb{T}_n} \le \mathbf{y}_{\partial\mathbb{T}_n}$ coordinate-wise, there exist random variables X and Y such that $\mathbb{P}(X \le z) = F_n(z|\mathbf{x}_{\partial\mathbb{T}_n})$, $\mathbb{P}(Y \le z) = F_n(z|\mathbf{y}_{\partial\mathbb{T}_n})$, $z \in [0,1]$ and almost surely $X \le Y$ when n is even and $X \ge Y$ when n is odd.*

We establish the claim by induction on n and repeatedly use the following elementary observation regarding the coupling of two random variables with the same distribution. Given a random variable U with cumulative distribution function F and two real numbers $\theta_1 < \theta_2$, there exists a probability space and a random vector (X_1, X_2) defined on it such that X_i has the distribution of U conditioned on $X \le \theta_i$, $i = 1, 2$, and $X_1 \le X_2$ almost surely. In what follows let U be distributed according to the free spin measure μ. We now prove the claim for $n = 1$. Given $\mathbf{x}_{\partial\mathbb{T}_1}, \mathbf{y}_{\partial\mathbb{T}_1}$, let $\bar{x} = \max_{i \in \partial\mathbb{T}_1} (\mathbf{x}_{\partial\mathbb{T}_1})_i$ and $\bar{y} = \max_{i \in \partial T_1} (\mathbf{y}_{\partial\mathbb{T}_1})_i$. Then, by the definition of the hardcore model, $F_1(\cdot|\mathbf{x}_{\partial\mathbb{T}_1})$ is equal to the conditional distribution of U given $U \le 1 - \bar{x}$ and, likewise, $F_1(\cdot|\mathbf{x}_{\partial T_1})$ is the conditional distribution of U given $U \le 1 - \bar{y}$. Since $\mathbf{x}_{\partial\mathbb{T}_1} \le \mathbf{y}_{\partial\mathbb{T}_1}$ implies $1 - \bar{x} \ge 1 - \bar{y}$, the claim for $n = 1$ follows from the observation made above.

Now, for the induction step assume the claim holds for $n = 1, \dots, m-1$. Suppose m is even. Consider two copies of the tree $\mathbb{T}_m$ with roots u and v, respectively, and label their children as $u_1, \dots, u_\Delta$, and $v_1, \dots, v_\Delta$ respectively. On these two copies consider two arbitrary boundary conditions $\mathbf{x}_{\partial\mathbb{T}_m}$ and $\mathbf{y}_{\partial\mathbb{T}_m}$ respectively, that satisfy $\mathbf{x}_{\partial\mathbb{T}_m} \le \mathbf{y}_{\partial\mathbb{T}_m}$. For $i = 1, \dots, \Delta$, let $\mathbf{x}^i_{\partial\mathbb{T}_m}$ (respy, $\mathbf{y}^i_{\partial\mathbb{T}_m}$) be the natural restriction of the boundary condition $\mathbf{x}_{\partial\mathbb{T}_m}$ (respy, $\mathbf{y}_{\partial\mathbb{T}_m}$) to the subtree corresponding to u_i (respy, v_i), each of which is a copy of the tree $\mathbb{T}_{m-1}$. By the inductive assumption, since $m-1$ is odd, for each $i = 1, \dots, \Delta$, there exist two coupled random variables X_i and Y_i distributed according to $F_m(\cdot|\mathbf{x}^i_{\partial\mathbb{T}_m})$ and $F_m(\cdot|\mathbf{y}^i_{\partial\mathbb{T}_m})$ respectively, such that $X_i \ge Y_i$ almost surely. Generate pairs

(X_i, Y_i) independently across $i = 1, \dots, \Delta$ in this way. Now, let U be a random variable distributed according to the free spin measure μ. Then, the random variable X distributed according to $F_m(\cdot|\mathbf{x}_{\partial\mathbb{T}_m})$ has the conditional distribution of U given $U \le 1 - \max_{1\le i\le\Delta} X_i$, integrated over the joint distribution of $X_1, \dots, X_\Delta$. Similarly, Y distributed according to $F_m(\cdot|\mathbf{x}_{\partial\mathbb{T}_m})$ is distributed as the conditional distribution of U, given $U \le 1 - \max_{1\le i\le\Delta} Y_i$, integrated over the joint distribution of $Y_1, \dots, Y_\Delta$. Since by construction we have $X_i \ge Y_i$, then $1 - \max_{1\le i\le\Delta} X_i \le 1 - \max_{1\le i\le\Delta} Y_i$. Thus, there exists a coupling of X and Y such that $X \le Y$ almost surely. The case of odd m is analyzed similarly, using that the result holds for all even $n < m$. Hence, the details are omitted. The claim then follows by induction. $\square$

APPENDIX B: PROOF OF THE REWIRING LEMMA

PROOF OF LEMMA 6.1. In every step of the rewiring, we delete two nodes in the graph. Thus, when we perform $t \le (N/2) - (2g + 1)\Delta^{2g}$ rewiring steps sequentially, in the end we obtain a graph with at least $N - 2((N/2) - (2g + 1)\Delta^{2g}) = 2(2g + 1)\Delta^{2g}$ nodes. Suppose that at step $t \le (N/2) - (2g + 1)\Delta^{2g}$ we have a graph $\mathbb{G}_t$ that is Δ-regular and has girth at least g. We claim that the diameter of this graph is at least $2g + 1$. Indeed, if the diameter is smaller than $2g+1$, then for any given node v any other node is reachable from v by a path with length at most $2g$, and thus the total number of nodes is at most $\sum_{0\le k\le 2g} \Delta^k < (2g + 1)\Delta^{2g}$, which is a contradiction, and the claim is established.

Now, given any $t \le (N/2) - (2g + 1)\Delta^{2g}$, suppose the rewiring was performed at least t steps on pairs of nodes with distance at least $2g + 1$ apart. Select any two nodes u_1, u_2 in the resulting graph $\mathbb{G}_t$, which are at the distance equal to the diameter of $\mathbb{G}_t$, and thus are at least $2g + 1$ edges apart. We already showed that the graph $\mathbb{G}_{t+1}$ obtained by rewiring $\mathbb{G}_t$ on v_1, v_2 is Δ-regular. It remains to show it has girth at least g. Suppose, for the purposes of contradiction, $\mathbb{G}_{t+1}$ has girth $\le g - 1$. Denote by $u_{j,1}, \dots, u_{j,\Delta}$ the Δ neighbors of u_j, $j = 1, 2$. Suppose $k \ge 1$ is the number of newly created edges which participate in creating a cycle with length $\le g - 1$. If $k = 1$ and $u_{1,j}, u_{2,j}$ is the pair creating the unique participating edge, then the original distance between $u_{1,j}$ and $u_{2,j}$ was at most $g - 2$ by following a path on the cycle that does not use the new edge. But then the distance between u_1 and u_2 is at most $g < 2g + 1$, which gives a contradiction. Now suppose $k > 1$, then there exists a path of length at most $(g - 1)/k \le (g - 1)/2$ which uses only the original edges (the edges of the graph $\mathbb{G}_t$) and connects a pair v, v' of nodes from the set $u_{1,1}, \dots, u_{1,\Delta}, u_{2,1}, \dots, u_{2,\Delta}$. If the pair is from the same set, say $u_{1,1}, \dots, u_{1,\Delta}$, then, since these two nodes are connected to u_1, we obtain a cycle in $\mathbb{G}_t$ with length $(g-1)/2+2 < g$ leading to a contradiction (since by assumption $g > 3$). If these two nodes are from different sets, for example, $v = u_{1,j}$, $v' = u_{2,l}$, then we obtain that the distance between u_1 and u_2 in G_t is at most $(g-1)/2+2 < 2g+1$ which also leads to a contradiction. So, we conclude that $\mathbb{G}_{t+1}$ must have girth at least g, as stated. $\square$

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